Modeling Relational Priming and Multiplicative Reasoning with Rational Numbers

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Abstract

Previous research on multiplicative reasoning has shown that for whole numbers, understanding of division is intimately linked to multiplication, as retrieval of division facts is often accomplished through reverse multiplication. We recently extended this research to rational numbers, and found that inverse multiplication problems can serve as primes for one another (e.g., $a \times b/a = a$ primes $b \times a/b = b$) when the second multiplier is expressed as a fraction, but not when it is expressed as a decimal. In the current paper we propose a process model of how such relational priming takes place, and report two experiments that test the limits of this priming effect. The first varies the format of the equations as fractions or a total division equation, and shows that priming is only observed using the fraction format; the second varies the multiplicative complexity of the factors in the equations, and shows that priming requires a common factor linking the successive problems.

Keywords: multiplicative reasoning; relational priming; number concepts; fractions and decimals; mathematics education

Relational and Multiplicative Reasoning

To understand mathematics is to grasp a system of formal relations among numbers and numerical operations. By the time children finish elementary school, they have been exposed to numerous fundamental relations involving both whole numbers and rational numbers (fractions and decimals). They are taught, for example, that addition and subtraction are inverse operations, as are multiplication and division (Nunes & Bryant, 1996; Bisanz & LeFevre, 1990); and that common factors connect multiplication and division (e.g., 4 is a common factor of 8 and 12 because 4 divides evenly into each, which in turn implies that 4 can be multiplied by some whole number to yield either 8 or 12). When rational numbers are introduced, children also are taught that fractions—the first type of number they encounter with a complex internal structure, *a/b*—are intimately related to division. For example, the fraction 2/3 stands for the quantity obtained by dividing 2 by 3; and the fraction 4/6 is equivalent to 2/3 because the numerator and denominator of the former (4 and 6, respectively) share a common factor of 2, and hence can be divided to reduce 4/6 to 2/3.

In the process of learning about such numerical relations, students are typically drilled on arithmetic facts. The term "fact" is perhaps misleading, as it suggests a list of arbitrary pieces of information. However, the successful students are likely to learn that these facts are far from arbitrary. For example, the standard multiplication table is not just a list of associated numbers; rather, it is a relational database in which the position of a number conveys its status as a factor or product relative to other numbers, and the direction of mappings between numbers reflects the symmetry of the multiplication and division operators. Any specific multiplication fact, such as $2 \times 3 = 6$, can potentially be accessed from any of the constituent role bindings (i.e., 2 or 3 as factors, 6 as the product), thereby exhibiting the property of *omnidirectional access* characteristic of relational structures (Halford, Wilson & Philips, 1998; Halford, Wilson, Andrews, & Phillips, 2014). Thus to truly learn the multiplication table is to acquire a *multiplicative schema* that specifies interrelated factors and products and also the dependencies between multiplication and division (see Campbell, 1999; Campbell & Alberts, 2009).

Despite the clear importance of relational understanding in mathematics, relatively little is known about the extent to which students acquire a multiplicative schema that can be flexibly used to solve math problems. Particularly for problems that go beyond simple whole numbers to include fractions and decimals, there is evidence of great individual differences. For example, Siegler and Lortie-Forgues (in press) found that pre-service teachers and middle-school students performed at below-chance levels on estimation problems involving multiplication and division of fractions smaller than 1 (e.g., is $31/56 \times 17/42 > 31/56$?), whereas math and science students from a selective university were consistently correct on such items. Notably, pre-service teachers and middle-school students who correctly executed fraction arithmetic procedures and exhibited accurate knowledge of fraction magnitudes were still unable to solve this type of multiplicative estimation problem, suggesting their difficulty was conceptual in nature.

Relational Priming with Fractions

Previous research has tested the extent to which adults have access to reciprocal relations when performing multiplication with rational numbers. DeWolf and Holyoak (2014) gave college students at a selective university a series of multiplication problems that students' had to decide were either "true" (correct) or "false" (incorrect). Half of the students performed the task with problems that had the format $a \times b/c = d$, and the other half had problems where the b/c term was expressed as an equivalent decimal. Importantly, a quarter of the trials were set up such that two successive problems were inversely related to one another (the other three quarters of the trials were foil problems intended to obscure the relationships between successive problems). For example, the first trial in a pair (the prime) might be 3 × 8/6 = 4, and the following problem (the target) might be 4 × 6/8 = 3. Because 3 × 8/6 = 4 is true, and the two problems are exactly the inverse of one another, the target problem must also be true. DeWolf and Holyoak found that participants showed a significant priming effect, such that the target problem was significantly faster than the prime problem (with appropriate counterbalancing to control for basic problem difficulty). However, no such priming effect was found when the same exact problems were presented with decimals rather than fractions as the second multiplier. In addition, performance on all fraction trials was significantly faster and more accurate than on trials with decimals. These findings suggest that (1) multiplicative reasoning with fractions and decimals is fundamentally different, affording different strategies and processes even when the quantities are matched in magnitude, and (2) college students are sensitive to inverse relations between problems expressed as fractions. We have developed a process model of multiplicative reasoning with fractions to account for this priming effect.

Model of Activation for Primed Pairs

People may employ various strategies to evaluate the correctness of a simple multiplication problem. For problems with fractions, perhaps the most obvious strategy is to evaluate the problem in a manner similar to a whole-numbers division problem. For example, 3 × 8/6 = 4 could be evaluated by first multiplying 3 × 8 and then dividing that product by 6. This strategy will work on any fraction multiplication problem, even if the equation itself is false. However, a limitation of this strategy is that multiplying the whole number and the fraction numerator may lead to a very large product.

For problems in which common factors are available, people may instead solve problems using fractions using a simplification strategy that minimizes the calculations required. One potential simplification strategy would be to simplify the fraction 8/6 to 4/3. The simplified equation is now 3 × 4/3, which reduces to 4. Another possible simplification strategy would be to reduce the whole number multiplier and the denominator of the fraction, (essentially, 3 × 1/6 = 1/2), which would result in 8/2, and note that 8 divided by 2 is simply 4. One consequence of such simplification strategies is that common factor relations between the whole number, fraction numerator, and fraction denominator are activated.

It is important to note that in the fraction problems used by DeWolf and Holyoak (2014) and in the experiments reported here, the relevant pairs of prime and target problems could always be solved by at least one of the simplification strategies, because common factors linked either the whole number and denominator, the numerator and denominator, or both. The false problems (half of the total set) could not be simplified in such a way (e.g., a false problem might be 8 × 9/6 = 9 which does not simplify to 9).

We propose that in the process of evaluating the prime (i.e., the first trial in a successive primed pair), participants will typically activate the common factor between the whole number and denominator, or the numerator and denominator, in order to simplify the problem (see, left panel). Hence, if the prime were 4 × 6/8 = 3, then the numbers 4, 6, 8, and 3 would provide the initial sources of activation. In Figure 1, the arrows connecting numbers represent the relation "is a factor of". These arrows are depicted as unidirectional to reflect the asymmetry of the relational roles (factor and product), but we assume activation can spread in both directions (see Campbell, 1999). We assume for simplicity that the two nearest factors and products of a number are activated (i.e., those numbers depicted in the example networks). (Additional factors/products may also be activated, but two are sufficient for the problems used in our experiments.) In solving the prime problem by simplification, relevant factor relations become highlighted, while irrelevant ones are deactivated. Thus in solving the prime problem (bottom left in Figure 2), the fact that 3 is a factor of 6, and/or 4 is a factor of 8, would become highly active. If the target trial in a successive primed pair (Figure 1, right panel) is the inverse problem 3 X 8/6 = 4, these same factor relations are precisely those necessary for simplification. Hence, the process of activating relevant factor/product relations will have a "head start" in the target problem, relative to the prime problem. Accordingly, priming is expected in that the target problem will be solved more quickly than the prime (where the order of the prime and target are counterbalanced to control for other sources of problem difficulty (i.e., a problem would appear as the prime for one participant but as the target for another).

This process model suggests that when a participant evaluates the target problem, two components determine if priming will be obtained. First, by definition, the connection between the prime and target trials depends on the inverse relation between them. However, in order for this relation to actually facilitate solution of the target trial, there must at least be some implicit recognition of how the two problems relate to one another. The fact that no priming is found when decimals are used in place of fractions (DeWolf & Holyoak, 2014) suggests that when the structural parallel between the prime and target is not apparent (e.g., because the reciprocals are obscured when expressed as decimals), there will be no facilitation of the target problem.

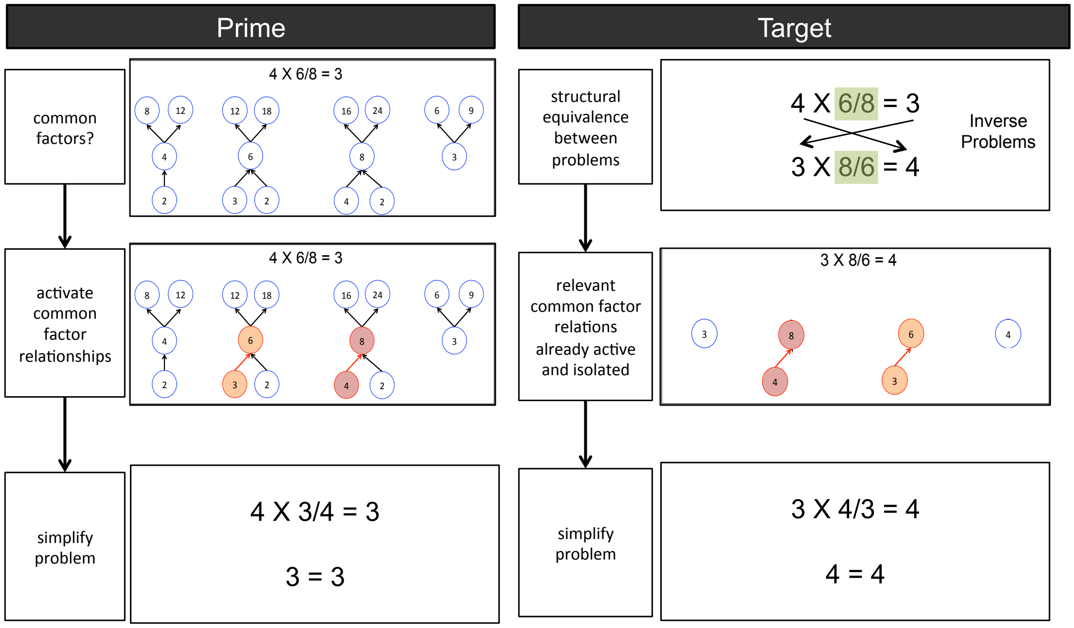

Figure 1. Process model for activation of common factors when solving prime and target

The second component that contributes to successful priming is that the same common factor relations can be activated in the prime as in the target. For example, in the prime trial shown in Figure 1, the fact that 3 is a factor of 6 and/or 4 is a factor of 8 are activated in the evaluation of both the prime and target problems.

Empirical Tests of the Model

In order to evaluate whether these two components are both necessary for successful priming, we conducted two experiments to test these hypotheses. Experiment 1 tested whether the structural equivalence afforded by the fractions is a necessary component for priming. Primarily, the advantage of the fraction is that it highlights the central term that is being inverted when forming the inverse relation. Another format, which is equivalent in magnitude but differs slightly in meaning, is the division format shown in Figure 2. In the division format, the relevant reciprocal relation is obscured, making it harder to recognize the structural similarity across problems. Experiment 1 assessed whether the division format will yield a priming effect similar to that found for the fraction format in the study of DeWolf and Holyoak (2014). Our process model predicts that the priming effect will be reduced or even eliminated when problems are presented in a division format because the division format does not highlight the structural equivalence between problems.

The sets of stimuli tested by DeWolf and Holyoak (2014) and Experiment 1 here were not suitable for testing the "common factor" hypothesis because they all included common factors. Therefore, Experiment 2 was designed to test whether common factor relations are indeed necessary for successful priming, as predicted by our process model. We created a set of problems for which there was no perceptual match between the whole numbers and the fraction components (e.g., $8 \times 6/4 = 12$; $12 \times 2/3 = 8$). These two problems still maintain the inverse relation because 6/4 and 2/3 are reciprocally related to one another (see Figure 4, top). In addition, the components of the fractions maintained the "is a factor" relation (3 is a factor of 6 and 2 is a factor of 4; see Figure 4, bottom). We also included a subset of problems that are reciprocally related but for which the components do *not* share the "is a factor" relation (e.g., $2 \times 21/6 = 7$; $7 \times 4/14 = 2$). Our process model predicts that priming will be obtained when common factor relations are present, but not otherwise.

Experiment 1

Experiment 1 assessed whether the fraction format in particular affords recognition of the structural equivalence across inverse problems, or if relational priming can be achieved with equivalent problems written in a division format (see Figure 2).

Method

Participants A total of 74 undergraduate students (mean age = 21.2 ; females = 59) from the University of California, Los Angeles (UCLA) participated in the study for course credit.

Design and Materials There were two conditions: fraction format and division format. The stimuli and materials were adapted from the "non-matching fractions" condition of Experiment 2 reported by DeWolf and Holyoak (2014). Equations were shown either in fraction format or division format (see Figure 2), but were otherwise identical. Half of the participants were assigned to the fraction format condition and half were assigned to the division format condition.

Fraction Format	Division Format
3 X 8/6 = 4	$\frac{3 \times 8}{6} = 4$
4 X 6/8 = 3	$\frac{4 \times 6}{8} = 3$

Figure 2. Examples of fraction and division format used in Experiment 1.

There were a total of 240 trials, with 60 of the trials designed to be true (correct) primed pairs, where the prime had the format exemplified by $3 \times 8/6 = 4$ and the target was the inverse equation, exemplified by $4 \times 6/8 = 3$. The other 180 trials consisted of 60 "false" primed trials and 120 foil trials. The false primed pairs were similar to the true primed pairs in that there was a superficial similarity between successive problems, but the answers for each trials was "false" instead of "true". An example of false primed pairs is $7 \times 10/4 = 8$ followed by $8 \times 4/10 = 7$. The remaining foil trials were not related to each other in any specific way. These were designed to vary the order of trials (true/false, false/true, true/true, false/false) and to obscure the structural similarity between the trials in primed pairs. DeWolf and Holyoak (2014) found no evidence of priming for either false primed pairs or foil problems.

Half of the 240 trials were true and half of the trials were false. Except for the pairing involved in primed trials, problems were shown in random order for every participant.

Procedure The study was administered using Superlab 4.5 (Cedrus Corp., 2004), which was used to collect accuracy and response time data. Participants were told that they would see multiplication (or division) problems. They were told to press the "a" key if the problem was true or the "l" key if the problem was false. Participants were told that the answers were shown rounded to the nearest whole number. As we were particularly interested in potentially subtle response time differences, participants were instructed to respond as quickly as possible while maintaining high accuracy. They were first given four practice trials that used only whole numbers. After the practice trials, they were given a chance to ask questions before starting the test trials.

Results and Discussion

Accuracy Across all trials (including prime, target and foil trials), there was no difference in accuracy for the fraction condition and the division condition (90% vs. 90%; $t(72) = .007$; $p = .994$). There was also no evidence of priming for true prime-target pairs based on the accuracy measure, either for the fraction condition (prime: 88% target: 89%; $t(36) = 1.74$, $p = .09$) or the division condition (prime: 86% target: 87%; $t(36) = 1.85$, $p = .07$).

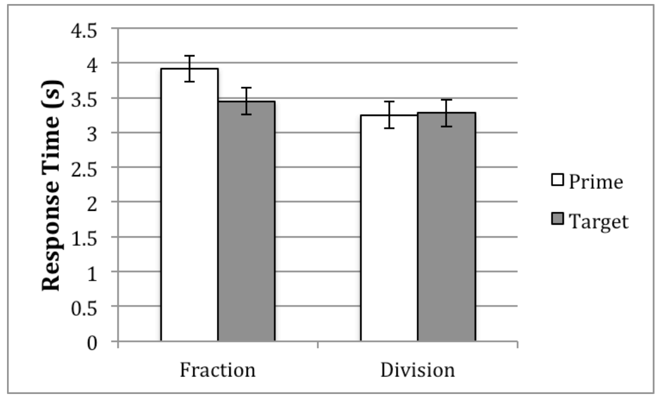

Figure 3. Mean response times for the true prime and target trials by format condition (Experiment 1).

Response Time Across all trials, participants responded more slowly to fraction-format trials than division-format trials (3.57 s vs. 2.79 s; $t(72) = 3.65$, $p = .001$). Figure 3 shows the average response times for the true prime and target trials by format condition. 2 (Fraction vs. Division) X 2 (Prime vs. Target) ANOVA revealed a significant interaction ($F(1, 72) = 4.34$, $p = .04$). There was a significant priming effect for the fraction-format trials (3.92 vs. 3.45, $F(1, 72) = 7.82$, $p = .007$), but no significant priming effect for division-format trials (3.26 vs. 3.28, $F(1, 72) = .02$, $p = .88$).

Overall, the results of Experiment 1 indicate that priming between inverse problems depends on the fraction format, which highlights the reciprocal relation between the two inverse problems. When equivalent problems were presented in division format, no facilitation from prime to target was obtained for the true prime-target pairs.

An unanticipated finding was the overall advantage for the division format over the fraction format in response time. Although the division format does not yield facilitation that depends on recognizing similarities across problems, it may facilitate activation of common factors within each individual problem. Each format may afford a different order of operations. The division format may facilitate multiple solution paths (e.g., Landy & Goldstone, 2010), making it easier to simplify either of the numerator numbers with the denominator.

Experiment 2

The goal of Experiment 2 was to test the second hypothesis generated by the process model: that inverse priming requires activation of common-factor relations across the prime and target problems.

Method

Participants Participants were 37 UCLA undergraduates (mean age = 20; 24 females) who received course credit for participating.

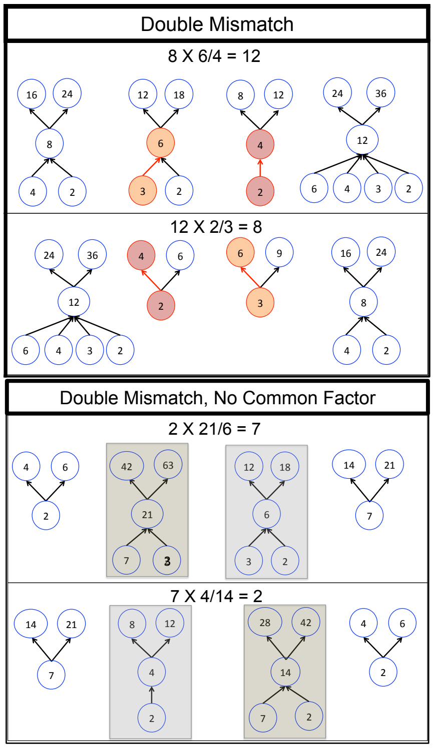

Figure 4: Examples of primed pairs used in Experiment 2. Double-Mismatch pairs share common-factor relations, whereas Double-Mismatch, No CF pairs do not.

Design, Materials, and Procedure A single within-subjects variable was tested, using problems presented with the fraction format. The set of stimuli was identical to the fraction condition of Experiment 1 except that the 60 primed trials comprised two subsets of prime pairs: "Double-Mismatch" trials and "Double-Mismatch, no CF (common factor)" trials. In all Double-Mismatch problems, the whole numbers *within* a single problem do not match the fraction components, and the fraction components *between* the two problems do not match. This type of problem enables tight control of perceptual similarities between the prime and target problems. The top panel of Figure 4 shows an example of a Double-Mismatch primed pair. The two problems are related in that 6/4 is a reciprocal of 2/3. They also share common factor relationships because 3 is a factor of 6 and 2 is a factor of 4. By contrast, the bottom panel of Figure 4 shows an example of a Double-Mismatch, no CF primed pair. The two problems are related by a reciprocal, but not by common factors. In Figure 4, 21/6 and 4/14 are reciprocals (21/6 = 7/2, 4/14 = 2/7). However, in the 2 × 21/6 = 7 equation, the "is a factor" relations that are activated is "7 is a factor of 21" and "2 is a factor of 6". In the 7 × 4/14 = 2 equation, the "is a factor" relations that are activated do not match- instead they are: "2 is a factor of 4" and "7 is a factor of 14".

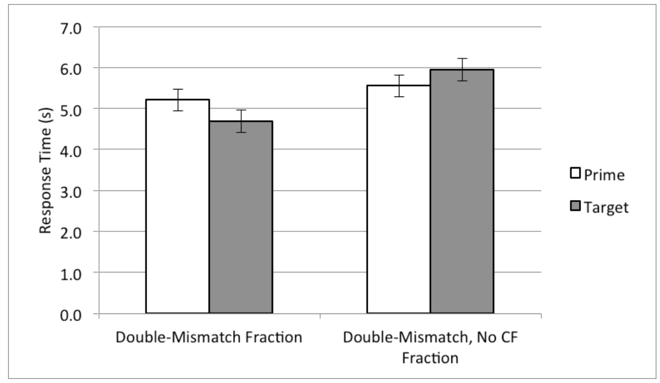

Figure 5. Average response times for the prime and target true primed trials for Double-Mismatch and Double-Mismatch, no CF pairs (Experiment 2).

There were 42 Double-Mismatch trials and 18 Double-Mismatch, no CF trials. The number of possible problems of the latter type is limited (given the constraint of avoiding problems including large numbers). The remaining 180 trials were the same foil trials used in Experiment 1. The procedure was also identical to that of Experiment 1.

Results and Discussion

Accuracy Average accuracy across participants on all trials (including true prime, target and foils) was 87%. There was no difference in accuracy across all trials on the Double-Mismatch and Double-Mismatch, no CF trials (79% vs. 80%, $t(36) = .68$, $p = .50$). Among the true primed trials, there was also no evidence of priming based on accuracy measure for either the Double-Mismatch pairs (79% vs. 78%, $t(36) = 1.12$, $p = .27$) or the Double-Mismatch, no CF pairs (80% vs. 80%, $t(36) = .41$, $p = .68$).

Response Time Mean response time across participants on all trials was 3.83 s. Response times for the Double-Mismatch true prime-target trials were significantly faster than for the Double-Mismatch, no CF true prime-target trials (4.96 s vs. 5.75 s; $t(36) = 2.37$, $p = .02$). Figure 5 shows the average response times for prime and target trials in the Double-Mismatch and Double-Mismatch, no CF prime conditions. A significant priming effect was obtained for the Double-Mismatch trials (5.21 s vs. 4.69 s, $t(36) = 2.1$, $p = .04$), but not for the Double-Mismatch, no CF trials (5.56 s vs. 5.94 s, $t(36) = .41$).

These response time results support the prediction of our process model, in that a necessary condition for priming is that common factor relations must link the prime and target trials. The Double-Mismatch trials yielded clear inverse priming even though they lack perceptual similarity. In contrast, the Double-Mismatch, no CF trials, which share a reciprocal relation but not common factors, did not yield priming.

Discussion

The results of these two experiments support hypotheses derived from a process model of how activation spreads among common factors during fraction and whole number multiplication. The fraction format seems to highlight or isolate the inverse relationship across related problems, thereby facilitating performance on the target trial. The results of Experiment 1 indicate that the fraction format highlights the relevant inverse relationship, resulting in priming, whereas an equivalent division format does not. The results of Experiment 2 reveal that the fraction format with a shared reciprocal is not sufficient to create the necessary structural similarity across problems. In addition, an "is a factor" relation that connects the prime and target problems must be activated in order for priming to occur. Hence Double-Mismatch, no CF problems, which share a reciprocal relation but not a common factor (e.g., the reciprocals 21/6 and 4/14) do not yield priming.

The process model we propose here has important implications for how multiplication with fractions is performed. Previous research has examined multiplicative reasoning with whole numbers but not with fractions. Our model demonstrates that fractions can be incorporated into a multiplicative schema that connects them with whole numbers. College students seem to prefer to analyze the component parts of fractions, and their relations to whole-number components in the problem, rather than considering the fraction as a single unit.

The process model also explains why the fraction format affords a flexible set of strategies for solving simple multiplication and division problems. When solvers have a deep understanding of how fractions and whole numbers are embedded within a multiplicative schema, they are able to flexibly simplify fraction multiplication problems based on their network of common-factor relations.

The present findings thus add to other evidence of major differences in the procedural and conceptual knowledge associated with different types of rational numbers. Although one might expect that people would solve a multiplication task with fractions by simply estimating the magnitude of the fraction and hence the resulting product, this was not the case in our study. Decimals represent one-dimensional magnitudes, whereas fractions represent two-dimensional relations; hence adults access magnitudes more easily for decimals than for equivalent fractions (DeWolf, Grounds, Bassok & Holyoak, 2014). Accordingly, multiplication is much more likely to be based on magnitude estimation for problems involving decimals rather than fractions. At the same time, the relational structure of fractions is advantageous for reasoning tasks that depend on relations between certain quantities, such as that between the cardinality of a subset and the full set (DeWolf, Bassok & Holyoak, 2015; Rapp, Bassok, DeWolf & Holyoak, 2015). In general, the different formats for rational numbers each provide unique affordances for performing different mathematical tasks.

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