Interactivity, Expertise and Individual Differences in Mental Arithmetic

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Abstract

Participants completed long single digit sums in two interactivity contexts. In a low interactivity condition sums were solved with hands down. In a second, high interactivity condition participants used moveable tokens. As expected accuracy and efficiency was greater in the high compared to the low interactivity condition. In addition, participants were profiled in terms of working memory capacity, numeracy, math anxiety and expertise in math. All of these measures predicted calculation errors in the low interactivity conditions; however, in the high interactivity condition, participants' performance was not determined by any of these variables. We also developed a scale to measure task engagement: Participants were significantly more engaged with the task when they completed the sums in the high interactivity condition. However engagement level did not correlate with calculation error, suggesting improvement in performance with tokens was not the result of greater task engagement. Interactivity transformed the deployment of arithmetic skills, ameliorated performance, and helped to reduce the difference in performance between individuals of low and high math expertise.

Keywords: Interactivity; arithmetic; expertise; math anxiety; working memory; task engagement.

Introduction

Mental arithmetic is commonly construed as an operation to be completed in the head by virtue of the word 'Mental'. However, in practice individuals use the world around them to complete simple math tasks. Scripture (1891, p. 2) explained how the eminent "calculators" of the day used artefacts when learning the fundamentals of math. These math prodigies described learning arithmetic from pebbles, peas, marbles, shot and dominoes, at times without any awareness of rudimentary terms such as multiply. Their expertise in completing large calculations using simple times tables was acquired through the physical arrangement of these manipulables.

Children appear to learn to calculate by using their fingers in conjunction with repeating the names of the numbers aloud (Butterworth, 2005). In addition Alibali and DiRusso (1999) found that while gesturing is an aid to encouraging counting accuracy in children, touching items when counting facilitates more accurate performance than simply pointing to countable items.

Mental arithmetic pervades everyday life, with simple sums frequently performed without the use of artefacts. However, as the problem increases in complexity we may resort to pen and paper or in counting a handful of change we may lay out the coins, grouping common coins together during the tallying. Classrooms frequently use interactive instruction when introducing mathematical concepts to children (Martin & Schwarz, 2005; Fyfe, McNeil, Son & Goldstone, 2014). The world is saturated with number-based artefacts, and it is thought that paleolithic artefacts, such as calendars and clay tokens, contributed to the evolution of human numerical concepts (De Smedt & De Cruz, 2011; Malafouris, 2013). Even low interactivity mathematics, such as mental arithmetic, appears to rely on internal representations of number lines (Dehaene, Piazza, Pinel, & Cohen, 2003). This reliance on graphical or physical representations suggests that mathematics tasks could be enhanced by the use of interactivity with amenable external artefacts. Thus interactivity has an obvious impact on mathematical problem solving.

Solving a mental arithmetic problem can place high demands on limited working memory storage capacity and processes (DeStefano & LeFevre, 2004; Butterworth, 2006). When internal cognitive resources are strained, people naturally mine their external surroundings in order to augment cognition (Kirsh, 2013). During mathematical calculations, individuals, adults and children alike, may use gestures, and fingers to point and count (Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001; Carlson, Avraamides Cary, & Strasberg, 2007). Experts and novices have been shown to devise shortcuts and procedures drawn from their interaction with the world in order to reduce the load on working memory (Butterworth, 2006; Kirsh, 1995). Thinking does not simply take place 'in the head' but rather emerges from an interaction with artefacts in the world. "Cognition has no location" it is a dynamic interplay between internal and external resources (Malafouris, 2013, p. 85).

Expertise

Expertise in a particular domain is often attributed to innate aptitudes (Ericsson & Charness, 1994.) Galton (1892) proposed that "intellectual powers", along with the enthusiasm for hard work were inherited gifts with these innate abilities almost certainly guaranteeing eminence (p. 16). In response to his cousin's opinion, Darwin maintained, "men did not differ much in intellect, only zeal and hard work; I still think this is an *eminently* important difference." (Galton, 1908, p. 290). Ensuing research and theories have indicated that high levels of performance and expertise are mediated by ongoing acquisition and consolidation of skills (Eriscsson &

Charness, 1994; Sternberg, 1999) In the case of mathematical expertise a number of factors have been identified as contributors to exceptional performance including working memory, deliberate practice, intrinsic reward in the success of solving a problem (Butterworth, 2006; Ericsson & Charness, 1994.)

Interactivity in problem solving has been attributed with diminishing the load on working memory as some of the limited internal memory storage is unburdened onto the external world (Kirsh, 1995; Vallée-Tourangeau, 2013). Furthermore, other executive functions and strategy selection may benefit from the dynamic problem configuration enacted through interactivity.

Math Anxiety

The strain on working memory during mental arithmetic may be exacerbated when the individual experiences math anxiety as this anxiety utilises cognitive resources that would otherwise be directed at the problem (Ashcraft & Kirk, 2001; Ashcraft & Ridley, 2005). Math anxiety is typically associated with feelings of tension, uneasiness, confusion and fear when faced with solving math problems either in the classroom, workplace or daily life (Richardson & Suinn, 1975; Ashcraft & Moore, 2009). Math-anxious individuals have repeatedly been shown to perform less well in math than their less anxious counterparts (Hembree, 1990; Ma, 1999; Lyons & Beilock, 2011). Ma (1999) conjectures that those exposed extensively to mathematics may have greater control over their anxiety, even suggesting that these feelings of anxiety may be channeled to an improved level of performance. In a study investigating math anxiety and interactivity Vallée-Tourangeau, Sirota, and Villejoubert (2013) found that math anxiety was highly correlated with calculation error in a low interactivity condition where participants could not modify the problem presentation nor use their hands to point at numbers; however, in a high interactivity condition where participants could shape and reshape the problem presentation, math anxiety was no longer a predictor of calculation error. They argued that in the higher interactivity condition, a dynamic problem presentation wrought through action transforms working memory capacity, not only in terms of storage but also executive function skills, mitigating the impact of performance anxiety.

Task Engagement

The experience of learning and achievement are potentially influenced by active engagement in the performance of academic tasks in the classroom (Shernoff, Csikszentmihalyi, Schneider, & Shernoff, 2003). Students report a greater sense of engagement with perception of control and relevance to the real world (Shernoff et al., 2003; Newmann, Wehlage, & Lamborn, 1992). Affective variables such as enjoyment, interest and challenge have been associated with academic success, thus positive emotions elicited by the task experience contributes to increased problem-solving capacities and improved mathematical performance (Hembree, 1990; Shernoff et al., 2003). In turn difficulty in performing tasks may be experienced as a result of negative affect (Storbeck & Clore, 2007). Increasing the level of interactivity when solving a math problem has been shown to positively impact the level of engagement (Guthrie & Vallée-Tourangeau, 2015). This implies that giving participants control over their environment may directly increase affect and engagement in the task compared to the level of engagement in a low interactivity environment.

The Current Experiment

Highly enactive approaches to math have been shown to increase efficiency and accuracy, while reducing calculation error (Vallée-Tourangeau, 2013). In the current experiment, participants varying in math expertise were invited to complete simple sums. These sums were composed of either 11 or 17 numbers. This task does not challenge the arithmetic knowledge and skills of college-educated participants; nonetheless in the absence of pen and paper, accuracy requires good working memory capacity and executive function skills especially when dealing with longer sums. The aim was to investigate whether any changes in performance were related to the mode of problem solving. We designed two reasoning contexts: In a first low-interactivity context, participants were shown a random configuration of numbers, and were asked to calculate the sum with hands on table. In a second high-interactivity context, the same configurations were presented with number tokens, and participants were free to move them and re-arrange the problem presentation as they calculated an answer. The focus being on the effect of interactivity without and with artifacts, namely the wooden tokens, in facilitating an improvement in performance. A selection of tests and questionnaires were also included in the experimental session to measure individual differences implicated in mental arithmetic such as math anxiety, working memory, numeracy, math expertise and engagement. A dynamic, high interactivity environment using artefacts as opposed to a low interactivity quasi-static one may encourage more efficient calculations, reflecting better skills, through the dynamic reconfiguration of the problem. A high degree of interactivity may improve performance for participants with lower math expertise. In contrast, the performance of participants with a higher degree of math expertise may not vary greatly as a function of interactivity since their well-practiced internal resources may work efficiently and creatively when dealing with numbers.

Method

Participants

Sixty participants (38 women, $M_{\text{age}} = 21.3$, $SD = 2.37$) were recruited from a variety of academic backgrounds. Thirty-two psychology undergraduates participated in exchange for credits, 21 undergraduates from other disciplines volunteered to participate and seven additional participants either working in a highly numerical field (e.g., accounting), or recently graduated with a math discipline degree also participated voluntarily.

Materials and Measures

Arithmetic Task. Each participant was presented with two sets of simple additions, each composed of five 11 and five 17 single-digit numbers. These additions were performed in two interactivity conditions. In the low interactivity condition, participants were given a sheet of A4 paper, with numbers to be summed distributed randomly on the page (see Fig. 1, left panel). While adding the numbers, participants were instructed to keep their hands flat on the table. In the high interactivity condition, participants were given a similar set of sums, with the same distribution, but presented as moveable numbered wooden tokens (1.2 cm in diameter; see Fig. 1, right panel). On completing each sum participants were requested to announce the answer aloud to the researcher.

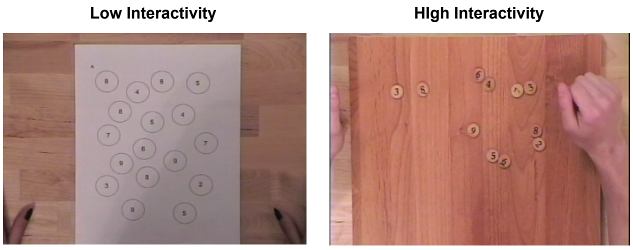

Figure 1: The sum was presented on a sheet of A4 as a random configuration of digits in the low interactivity condition (left panel); participants in that condition kept their hands flat on the table top. In the high interactivity condition, the sums were presented with movable wooden tokens (right panel) which participants touched, moved, grouped, as they saw fit.

Math Anxiety. Participants completed a 25-item Mathematics Anxiety Scale-UK (MAS-UK; Hunt, Clark-Carter & Sheffield, 2011). The questionnaire consisted of a series of situations with participants asked to indicate how anxious they would feel in those situations, on a Likert-style scale, with 1 = "not at all" and 5 = "very much". Items included statements such as "Working out how much your shopping bill comes to" or "Taking a math exam".

Objective Numeracy. A basic arithmetic scale (BAS) was used to test participants' objective numeracy. It consisted of 60 simple arithmetic problems (such as 7x8 = ?). Participants were required to write the answers on the paper provided, in the order presented, completing as many as possible in 60 seconds.

Working Memory. Participants completed two working memory tasks. The computation-span task, testing both processing and storage of numbers, while a non-numerical visuo-spatial task, the Corsi block task, testing the temporary storage of visual and spatial information.

Computation-span task. The computation-span task (Ashcraft & Kirk, 2001) required participants to answer simple arithmetic problems (e.g., 2 + 8 = ?, 12 – 4 = ?), before recalling the second number of these problems (e.g., 8, 4). Sequences of equations ranged from 1 – 7 and participants had to process each sum *and* recall the relevant digit correctly to score a point.

Corsi block task. In this version of the Corsi Block task participants were shown ten sequences of shaded blocks in a 4 x 4 matrix on a computer screen. The number of blocks to be remembered in each sequence increased from 2 to 6 blocks in length. Participants scored one point for each correctly ordered block, thus the maximum score was 40.

Math Expertise. We developed an instrument to evaluate math expertise based on experience. Four questions were related to math grades at school such as, "Have you taken math GSCE (or equivalent)?", this was scored as a binary 'yes' = 1, 'no' = 0; "If yes, please indicate which grade", this was scored as 'A*/A' = 4, 'B' = 3, 'C' =2, '<C' =1 'N/A' = 0. Three questions asked for details on current university degree, any past university degree and current job if applicable. The responses were given a score from 1-4 where 4 = a math-heavy degree or job and 1 = no degree or job. The highest score from these three questions was used to measure math experience in terms of degree and employment. This score was added to the responses on math education to provide a continuous numerical measure of math expertise.

Task Engagement Scale. The Task Engagement Scale (TES) was developed to gauge a participant's engagement and enjoyment during a task. The 9-item scale was based on Shernoff et al. (2003) who identified three key components of task engagement: concentration, enjoyment and interest. The scale asked participants to rate how anxious they felt; how easy, pleasurable, fun, threatening, stressful, tiresome, or effortful the task was; and how motivated they were to perform well in the task. Each item was scored on an 8-point Likert scale, labeled from zero (definitely not) to seven (definitely yes): The higher the score the more positive the attitude toward the task. Participants completed the TES after the 10 sums in each interactivity condition. The alpha reliability of the nine-item scale for both interactivity conditions indicated that the scale had good reliability (Low interactivity, Cronbach's α = .77; High interactivity, Cronbach's α= .81).

Procedure

The length of the additions (11 or 17-digits) and level of interactivity (high, low) were repeated measures factors yielding four conditions. The presentation order of these conditions was counterbalanced across participants. The sets of sums for each interactivity condition were separated by at least one other task (either the MAS-UK, BAS, Computation-span or Corsi Block). The other tasks were presented at either the beginning or the end of the session and their order was counterbalanced across participants. Each experimental condition was followed by the TES, and the experiment ended with a math experience questionnaire. The working memory tasks were presented on a computer with all other tasks being presented on paper. The experimental session lasted approximately an hour.

Mental arithmetic performance was measured in terms of accuracy (proportion of sums correct), latency to solution, absolute calculation error and efficiency. Absolute calculation error was defined as the absolute error from the correct answer. Therefore, if the participant answered 52 and the correct answer was 50, the absolute calculation error would be 2, if the participant answered 48 the absolute calculation error would also be 2. A participant's efficiency was his or her proportion correct answers over the proportion of time used to calculate a set of sums. This proportion was derived by taking the participant's mean latency divided by the mean latency of the slowest quartile of participants. Thus, if a participant's accuracy for a series of five sums was .6, and her average latency to complete these sums was 40% of the average of the slowest participants, then her efficiency ratio would be .6/.4 or 1.5. Ratios at or above 1 reflect efficient reasoning; ratios below 1 reflect inefficient reasoning.

Results

Overall Arithmetic Performance

Accuracy. The mean percent correct, as reported in the top right panel of Figure 2, was greater in the high interactivity condition than the low interactivity condition for both sum lengths. A 2 (Interactivity: Low and high) x2 (Sum length: 11-digit and 17-digit) repeated measures analysis of variance (ANOVA) indicated a significant main effect of interactivity, $F(1,59) = 30.04$, $p < .001$, $\eta^2 = .34$ and sum length $F(1,59) = 21.23$, $p < .001$, $\eta^2 = .265$; however, the interaction was not significant , $F < 1$.

Absolute Calculation Error. The mean absolute calculation error (Fig. 2, bottom left panel) was lower in the high interactivity condition than in the low interactivity condition regardless of the sum length. A 2x2 repeated measures ANOVA indicated a significant main effect of interactivity, $F(1,59) = 11.01$, $p = .002$, $\eta^2 = .16$ and sum length $F(1,59) = 17.20$, $p < .001$, $\eta^2 = .23$. However, there was no significant interaction, $F < 1$.

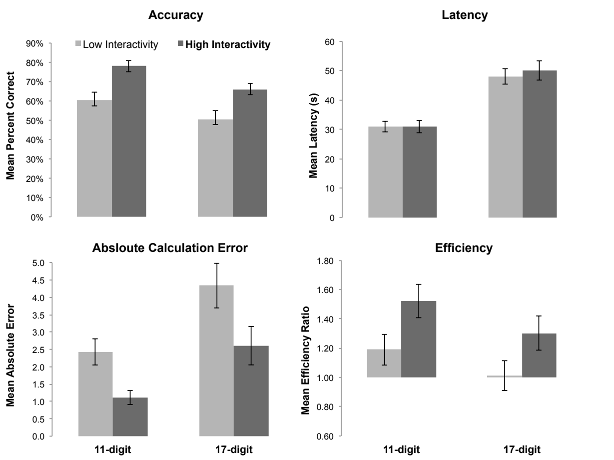

Figure 2: Mean percent correct (top left), mean latency (top right), mean absolute calculation error (bottom left) mean calculation efficiency (bottom right) as a function of sum length (11-digit and 17-digit sums) in the low (light grey bars) and high (dark grey bars) interactivity condition. Error bars are standard errors of the mean.

Latency. While latency to completion was influenced by sum length, interactivity level resulted in very little difference in latency (see Fig. 2, top right panel). In a 2x2 repeated measures ANOVA the main effect of interactivity was not significant, $F(1,59) = 1.42$, $p = .239$, $\eta^2 = .02$. However, there was a significant main effect of sum length $F(1,59) = 201.60$, $p < .001$, $\eta^2 = .78$ and a significant interaction between sum length and condition $F(1,59) = 6.68$, $p = .012$, $\eta^2 = .10$.

Efficiency. Participants were less efficient when calculating the sums in the low interactivity condition than when using tokens across both sets of sums (see Fig. 2, bottom right panel). The efficiency ratio decreased for longer sums, although it was still larger in the high interactivity condition. A 2x2 repeated measures ANOVA indicated a significant main effect of interactivity, $F(1,59) = 22.01$, $p < .001$, $\eta^2 = .27$ and sum length $F(1,59) = 17.11$, $p < .001$, $\eta^2 = .225$; the interaction, however, was not significant, $F < 1$.

Task Engagement Scale

Participants were more engaged in the high interactivity condition ($M = 44.13$, $SD = 9.2$) than in the low interactivity condition ($M = 37.8$, $SD = 8.8$). This difference was significant, $t(59) = -6.16$, $p < .001$. There were no significant correlations between TES and the measures of performance (see Table 1).

Table 1: Correlation matrix for average absolute calculation error, individual difference measures of math anxiety, objective numeracy, working memory (computation-span and Corsi blocks), math expertise and task engagement in both interactivity conditions (*df* = 58).

	1 MAS	2 OBJ-N	3 C-Span	4 Corsi	5 Exp	6 TES-L	7 TES-H	8 ACE-L	9 ACE-H
1	-	-.47 **	-.46 **	-.23	-.68 **	-.13	.07	-.51 **	.10
2		-	.60 **	.30 **	.65 **	.18	-.02	-.48 **	-.23
3			-	.39 **	.59 **	.25	-.05	-.50 **	-.24
4				-	.36 **	-.04	-.08	-.17	-.07
5					-	.14	.02	-.52 **	.04
6						-	.61 **	-.23	-.22
7							-	.04	.14
8								-	.29 *
9									-

Note: * $p < .05$, ** $p < .01$. **MAS** = Math anxiety; **OBJ-N** = Objective numeracy (basic arithmetic skill); **C-span** = Computation-span; **Corsi** = Visuo-spatial working memory; **Exp** = Math expertise (continuous measure); **TES-L** = Task engagement in the low interactivity condition; **TES-H** = Task engagement in the high interactivity condition; **ACE-L** = Absolute calculation error in the low interactivity condition; **ACE-H** = Absolute calculation error in the high interactivity condition.

Expertise, Working Memory and Math Anxiety

Relative to calculation error, accuracy is a course-grained measure; participants were only able to score 0-5 and accuracy does not discriminate between small and large calculation errors. Absolute calculation error was therefore chosen for further analysis above the other three performance measures for its resolution and for capturing the most important aspect of arithmetic performance: the solutions themselves. The mean absolute calculation errors in the 11 and 17-digit sums were averaged for each participant creating two new variables, overall mean absolute calculation error in the low interactivity condition ($M = 3.38$, $SD = 3.49$) and in the high interactivity condition ($M = 1.85$, $SD = 2.27$). The mean absolute calculation error in the low interactivity condition was significantly greater than in the high interactivity condition, $t(59) = 3.31$, $p = .002$.

In order to evaluate the influence of individual differences on performance, math anxiety, numeracy, working memory and expertise were correlated with calculation error (see Table 1). There were a number of highly significant correlations observed in the low interactivity condition: Math anxiety, $r(58) = -.51$, $p < .001$; objective numeracy, $r(58) = -.48$, $p < .001$; computation-span, $r(58) = -.50$, $p < .001$; expertise, $r(58) = -.52$, $p < .001$. However, in the high interactivity condition none of these variables predicted absolute calculation error; the largest non-significant correlation involved the computation-span, $r(58) = -.24$, $p = .064$.

Discussion

Participants completed two sets of addition problems: one set was completed with restricted hand movement reducing interactivity; the other using round numbered wooden tokens increasing the opportunity to reconfigure the problem presentation as the sum was calculated. Generally, participants answered more sums accurately, achieved higher efficiency ratios and the calculation error was lower in the high interactivity condition. Latency, however, remained constant across the two levels of interactivity for the short and long additions, suggesting improvements in other measures were related to the mode of problem solving, rather than the time required to complete the addition. This improvement in performance could not be attributed to extraneous between-subject factors, such as individual differences, because of the repeated measures design employed in this experiment: all participants completed the sums in both interactivity conditions. The results support our claim that high degree of interactivity improves the performance of those with less math expertise in these simple arithmetic problems.

The strong correlation between objective numeracy and expertise, $r(58) = .59$, $p < .001$, indicated that our measure of expertise reflected the arithmetic proficiency of an individual. With a static problem presentation and hands down on the table, participants' performance reflected their arithmetic and working memory skills. The lack of correlations with arithmetic performance and expertise in the high interactivity condition implied that the manipulation of number tokens augmented the arithmetic skills of participants with less math expertise such as to render their performance indistinguishable from those with greater expertise.

The influence of math anxiety on performance was dramatically different as a function of interactivity. When interactivity with the world was low, math anxiety had a significant impact on performance: Higher math anxiety scores were related with greater calculation error. In turn, the high interactivity context eliminated the variance in performance explained by math anxiety. This implies that even on these simple math tasks, a dynamic presentation offering a greater level of interactivity may assist in reducing or controlling the impact of math anxiety on mental arithmetic performance.

The enhancements in performance of the lesser skilled individuals in the high interactivity condition may be attributed to off-loading working memory onto the external environment. The two measures of working memory, computation-span and the Corsi block task, were moderately correlated, $r(58) = .30$, $p < .01$. Computation-span correlated highly with numeracy and expertise supporting claims that working memory is a contributing factor to mental arithmetic skill (see Butterworth, 2006). Our computation-span test was designed to reflect a conventional complex span task requiring some numerical skill; unsurprisingly, this correlated with measures of math skill in the low interactivity condition, more interestingly it did not correlate with performance in the high interactivity condition. The Corsi task, as a measure of visuo-spatial working memory was deliberately selected to reduce the reliance on numeracy. Corsi scores did not correlate with performance in either condition of interactivity. The findings here are consistent with previous mental arithmetic research conducted in lower interactivity environments indicating that the short-term storage component of the visuo-spatial sketchpad played a small role in mathematical performance (Lee, Ng, Ng, & Lim, 2004). Span tasks, such as the computation-span assess an individual's working memory in both processing and storage, whereas the Corsi test as designed here gauges storage capacity only. These results suggest high interactivity does not simply function as a means for off-loading working memory storage; rather the manipulation of the tokens scaffolds thinking enhancing the participants' calculations (Kirsh, 2013).

Individuals were more engaged in the task when given the opportunity to use the tokens than when they had to maintain their hands on the table. This pattern in the level of engagement did not change as a function of math expertise. Notably performance, as measured by absolute calculation error, was not influenced by how engaged participants were in the task. Participants might have felt more engaged when completing the task with tokens, but the level of engagement did not in itself explain the improvement in arithmetic performance in the high interactivity condition.

Expertise in the domain of mathematics may be attributable to factors including practice, intrinsic reward and components of working memory. This paper has shown that a systemic perspective on mental arithmetic helps us better understand how resources internal and external to the participants are configured dynamically to reflect expertise and skills at solving simple mathematical problems.

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