Young Children's Understanding of the Successor Function

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Abstract

This study examined 4-year-old children's understanding of the successor function, the concept that for every positive integer there is a unique next integer. Children were tested in the context of cardinal numbers and ordinal numbers. The results suggest that knowledge of the successor for cardinal numbers precedes that for ordinal numbers. In addition, for both cardinal and ordinal numbers, children generally failed to demonstrate understanding that the successor of a given number is unique.

Keywords: Counting; Natural Numbers; Cardinal Number; Ordinal Number; Successor Function.

Introduction

A large amount of research has examined children's understanding of natural numbers (i.e. positive integers) and counting (e.g. Baroody, 1987; Fuson, 1988; Gelman & Gallistel, 1978; Piaget, 1952; Schaeffer, Eggleston, & Scott, 1974; Wynn, 1990, 1992). One interesting phenomenon that has been observed is that children aged two to three years may appear at first glance to know how to count; they may correctly recite the count list ("one, two, three, four, …") and even point to items in a collection and tag them with a number (e.g. Briars & Siegler, 1984; Fuson, 1988). However, when asked "how many" items are in a collection they just "counted", these children often do not know (Schaeffer, Eggleston, & Scott, 1974; Wynn, 1990, 1992). Moreover, many children who can correctly answer the "how many" question are unable to produce a collection of objects of a specific cardinality (Le Corre & Carey, 2007; Wynn, 1992). For example, an experimenter might ask a child to "give three toys to a puppet" and the child would give an incorrect number. Failure on these basic tasks suggests that young children (typically those under four years of age) do not understand fundamental aspects of natural numbers, including the Cardinality Principle. (see Gelman and Gallistel, 1978 for principles of counting) This principle states that when counting a collection of items, the number associated with the final item counted represents the cardinality of the collection of items (i.e. the number of items).

Accurate performance on such "Give-*N*" tasks has been taken as evidence that children have knowledge of the cardinality principle (Le Corre & Carey, 2007; Wynn, 1992). This knowledge emerges in a series of stages. First children respond by giving an arbitrary or an idiosyncratic number of items (e.g. always a handful or always one). By age 2 ½ to 3 years, children typically respond correctly when asked to give one item, but give an incorrect number for numbers larger than one. Then children become "two-knowers", correctly responding to requests for one or two items, but incorrectly for larger numbers. Subsequently, they become "three-knowers" and possibly later "four-knower". After several months as a "subset-knower" (for collections less than five), a child typically reaches a stage where they can respond correctly to a request for any number of items (i.e. cardinal-principle knower).

Some researchers have suggested that children transition from subset knowers to cardinal-principle knowers through a bootstrapping process (Carey, 2004). Children who have knowledge of cardinality for collections of one, two, and three items extend this knowledge to larger collections of items by the inductive inference "If the number *N* represents the cardinality of a set of *N* items, then the next number in the count list, *N*+1, represents the cardinality of a set of *N* +1 items. Sarnecka and Carey (2008) describe this notion of a "next number" representing cardinality of a set one unit larger than currently under consideration as knowledge of the mathematical concept of successor. They suggest that cardinal-principle knowers have implicit knowledge of the successor function that enables them to make the induction from knowledge of cardinalities of small sets (e.g. less than four) to cardinalities of larger sets. They support their argument by demonstrating that cardinal-principle knowers, and not subset-knowers, recognize (1) that adding one item and not subtracting one item to a set of cardinality five produces a set of cardinality six and (2) that adding one item to a set of cardinality four results in five while adding two items results in six.

While it is clear from these results that children who understand the cardinality principle, as evidence by accurate performance on "Give-*N*" tasks, perform better on other numerical tasks than children who don't understand the cardinality principle, these results do not necessarily imply that cardinal-principle knowers have knowledge of the successor function. The interpretation of successor that Sarnecka and Carey (2008) employ is essentially a notion of "there is a next". This interpretation is inconsistent with a formal mathematical definition of successor function, and more importantly it is insufficient to generate the natural numbers and hence explain the acquisition of the natural numbers. As Lance, Asmuth, and Bloomfield (2005) point out, the simple notion that there is a "next" number can describe sets other than the natural numbers. For example, the set of integers with addition modulo 12 (i.e. equivalent to telling time on a standard clock face) has a successor function; the successor of one is two, the successor of two is three, etc. However, unlike the natural numbers, the successor of twelve is one.

The problem with the Sarnecka and Carey interpretation is that they omitted two necessary characteristics of the successor function. First, that the number one is not the successor of any number (if we do not include zero in the set of natural numbers). Second, the successor of a number is unique. For example, the successor of two is only three; and three is the successor only of two. Specifically, the natural numbers can be defined by the Dedekind/ Peano's Axioms that state the following (Dedekind, 1901).

1. One is a natural number.
2. Every natural number has a successor that is a natural number.
3. One is not the successor of any natural number.
4. Two natural numbers for which the successors are equal are themselves equal (i.e. the successor of any natural number is unique).
5. If a set, *S*, of natural numbers contains one and for every *k* in *S*, the successor of *k* is also in *S*, then every natural number is in *S.*

These axioms provide necessary and sufficient criteria to generate the set of natural numbers and therefore knowledge of these axioms could explain the acquisition of natural numbers.

The interpretation of successor used in the Sarnecka and Carey (2008) study does not demonstrate uniqueness of the successor function. Specifically, demonstrating that a child recognizes five as the cardinality of a set that contained four and then another item was added, does not imply that the child knows that cardinality of five cannot be achieved by adding one to another set size. In our everyday lives, as adults we take this implication for granted, but we cannot assume that children do. Therefore demonstrating that children have knowledge of the successor without demonstrating its uniqueness cannot fully explain how children acquire knowledge of natural numbers.

Additionally, the concept of successor is an ordinal concept, yet is has been tested in the context of cardinality.

The goal of the present study was to examine the conditions under which young children demonstrate knowledge of the successor function. If children who understand the cardinality principle demonstrate knowledge of a unique successor for cardinal numbers, then perhaps, as Sarnecka and Carey suggest, knowledge of the successor function "turns a subset knower into a cardinal principle knower". Furthermore, if children demonstrate this knowledge in the context of both cardinal number and ordinal numbers, then knowledge of the successor may explain the acquisition of natural number and their ordinal properties.

However, if children fail to appreciate the uniqueness of the successor, then we cannot conclude that children's understanding of the successor is sufficient to give rise to knowledge of the cardinality principle and to the natural numbers.

Mathematically, uniqueness can be proven by demonstrating that for two natural numbers, *a* and *b*, if the successor of *a* = the successor of *b*, then *a* = *b*. Knowledge of a unique successor can be tested in an analogous way by stating a number and asking the participant what number immediately precedes it. Two testing contexts were created, one cardinal and one ordinal. Participants were tested on their ability to state a next number when given a number (i.e. successor). These questions are denoted as +1 questions in the Method section. Participants were also tested on their ability to state a preceding number when given a number (i.e. uniqueness of the successor). These questions are denoted as -1 questions in the Method section. The cardinal context involved collections of circular disks that were placed under a cup. Participants needed to state how many disks were under the cup when a disk was added or removed. The ordinal context involved a simple board game in which participants were asked the number associated with locations before or after the location of a game pawn (see Figure 1).

Experiment

Method

Participants Participants were 35 preschool children (21 girls, 14 boys, *M* = 3.86 years, *SD* = 0.15) recruited from preschools and childcare centers in the Columbus, Ohio area.

Materials and Design Participants were given a series of different tasks, *How-Many task, Give-N task, Cup task,* and *Game task.*

How-Many task. This task consisted of six questions and involved black circular chips approximately 1 inch in diameter. For each question, the experimenter placed a collection of chips on a plastic plate approximately 7 inches in diameter. The experimenter then showed the child the collection of chips and asked, "Can you tell me how many chips are here?" The questions presented collections of size 2, 3, 4, 5, 6, and 8 in a pseudo-random order.

Give-N task. This task consisted of six questions testing the numbers 2, 3, 4, 5, 6, and 8 in a pseudo-random order. The experimenter placed a collection of ten chips in front of the child. The experimenter then gave the child the plastic plate and asked, "Can you give me *N* chips on this plate?"

Cup task (Cardinal Test). This task was designed to measure knowledge of the successor function in the context of cardinal numbers. Materials for this task included black circular chips and a red, plastic cup. For each question, the experimenter placed a specific number of chips in front of the child and clearly stated the number of chips presented. Next, the experimenter covered the chips with the cup and did one of two things. Either the experimenter added another chip to the hidden amount or took a chip away from the hidden amount. To prevent the child from seeing the adjusted number of chips, the chip was added or removed by lifting a corner of the cup only slightly and sliding a chip in or out. After the adjustment was made, the child's job was to tell the experimenter the number of chips under the cup.

Two practice trials with feedback were given before the test questions. A total of twenty-two questions were presented. There were ten questions in which one chip was added (i.e. Cardinal +1 questions), two for each initial number of 1 through 5. For example, for an initial number of 1, one chip was placed under the cup and then an additional chip was added. There were eight questions in which one chip was removed (i.e. Cardinal -1 questions), two for each initial number of 2 through 5. For example, for an initial number of 2, two chips were placed under the cup and then one chip removed. There were four additional questions in which two chips were added or removed (i.e. 2+2, 3+2, 3-2, and 4-2). Questions were presented in a pseudo-random order.

Board Game task (Ordinal Test). This task was designed to measure knowledge of the successor function in the context of ordinal numbers. Prior to administering the actual successor game task, participants were given a short test of understanding of the board game. This test was administered to ensure that participants understood how to move a game pawn across spaces on a board by counting contiguous spaces.

Materials for this task consisted of a laminated "game board" with ten contiguous, linear spaces and several colored game pawns (see Figure 1). Three arrows were placed on the board game to guide the direction of movement along the spaces on the game board. For half the participants the arrows pointed left, and for the other half the arrows pointed right. The experimenter told the child that they were going to play a number game. Both the child and the experimenter chose a game pawn to use. The experimenter told the child that he/she can count on the board by counting and moving the pawn one space at a time in the direction of the arrows on the board. The experimenter demonstrated this and then told the child, "the funny thing about this game is that you never know where I am going to start my counting". The experimenter showed the child two examples. The following is an excerpt of the script.

I might start counting on any one of these spaces. So for example, maybe I would start here (placing the game piece on the 3rd space on the board). If I count to 5, I would count like this; count 1, 2, 3, 4, 5 (tapping on each subsequent space with the game piece) and stop here, at 5.

After the two examples, the participant was given five questions to measure understanding of the game (i.e. practice game test). For these questions, the experimenter pointed to a specific space on the board, asked the child to count from that space to a given number (2 through 6 in pseudo-random order), and leave his/her pawn on the stopping space. The child was told to use his/her game pawn and tap each passing space as he/ she counted. Each of these trials had a different starting space.

For the ordinal successor test, the experimenter explained to the participant that in the game the child will not know the space on which the experimenter will start counting, the experimenter will count in her head, not out loud, and then tell the child the number that she stopped at,

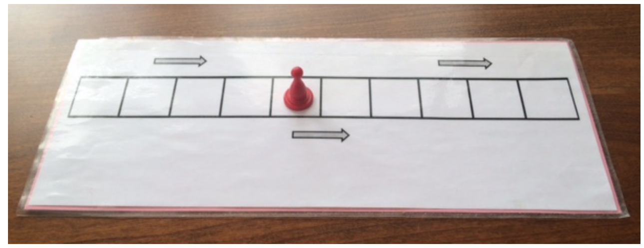

Figure 1. Picture of the game board and pawn used to test knowledge of the successor function in the context of ordinal numbers.

leaving her pawn on the stopping location. The following is an excerpt of the script.

I am counting in my head. You don't know where I started, but I am stopping right here (placing the game pawn on a specific space on the board) at this number, X (where X ranged from 1 to 6). What number did I stop at?

Corrective feedback was given if necessary. Then the experimenter asked one of two questions, either "What number would I count if I kept counting until here (pointing to the next space forward)" or "What number did I count when I was here (pointing to the space before the stopping space)?" Twenty-two questions were presented in a pseudo-random order. These questions were analogous to those in the Cardinal Test (i.e. cup task) and included Ordinal +1 questions, Ordinal -1 questions, and four additional questions in which two chips were added or removed (i.e. 2+2, 3+2, 3-2, and 4-2).

Prior to the actual ordinal test just described, the experimenter gave the participant four practice questions with corrective feedback. For two of the questions, the experimenter actually indicated her starting position and counted out loud. For two questions, the experimenter did not indicate her starting position and did not count out loud.

Participants were randomly assigned to one of two between-subjects condition that differed in the order of two successor tasks (i.e. the cup task and the game task).

Procedure Participants were tested in a quiet room at their preschool by a female experimenter. The experiment was conducted in two sessions over two days. In the first session, participants completed the *How-many* task, the *Give-me* task, and one of the successor tasks. In the second session, participants completed the second successor task. Responses were recorded on paper by the experimenter.

Results

Each participant was categorized as a cardinal principle knower (CP-knower) or non-cardinal principle knower (Non-CP-knower) based on his/her performance on the "How Many" and "Give-N" questions. A participant was considered to be a CP-knower if he/she achieved scores of at least 83% correct (i.e. 5/6) on both the "How many" and the "Give-*N*" tests. The criteria for knowledge of the cardinality principle were similar to that of previous research (e.g. Sarnecka & Carey, 2008; Wynn, 1992). Seventeen participants were categorized as CP-knowers and 18 participants were categorized as Non-CP-Knowers. Unlike previous research, the present analysis did not examine performance of specific subset-knowers. The present analysis focused on performance on the successor tasks (i.e. Cardinal +1, Cardinal -1, Ordinal +1, and Ordinal -1 questions).

Tasks similar to the cardinal successor task have been used in previous research (e.g. Sarnecka & Carey, 2008). However, the ordinal successor task has not been used in previous research. Therefore, to ensure that any poor performance on this task was due to lack of knowledge of the successor function and not due to failure to understand the task, the primary analysis included only those participants who demonstrated understanding of the context of the game (the ordinal successor task) by scoring at least 4 of 5 correct on the practice game test. Seven children were excluded from the analysis for scoring below 4 (M = 1.00, SD = 1.15). Accuracy on the practice game test was high for the participants who scored above 4 (M = 4.89, SD = .31). There were differences between the seven excluded participants and the remaining participants in the frequency of CP-knowers. Only one of the excluded seven participants met the criteria for CP-knower, while 16 of 28 (57%) of the remaining participants were CP-knowers, this difference in proportion was significant, Fisher Exact Test, p = .05. Also note that for these participants, the mean scores on the Cardinal +1 questions (M = 58.6%, SD = 20.7%) and on the Cardinal -1 questions (M = 64.3%, SD = 26.4%) was not statistically different from those of the participants who did understand the game (M = 72.9%, SD = 19.8% and M = 71.9%, SD = 18.2% for the Cardinal +1 and Cardinal –1 questions respectively), independent samples t(33)s < 1.74, ps > .09. In addition, there was no difference in age between these groups of children (M = 3.8 years SD = .17 and M = 3.9 years SD = .14 for the seven excluded participants and the remaining participants respectively).

The remaining analysis focuses on performance on the four types of successor questions (i.e. Cardinal +1, Cardinal -1, Ordinal +1, and Ordinal -1). No differences in performance between the two between-subject conditions (i.e. Cup 1st and Game 1st) was found for any of these measures, independent samples t(26)s < 1.15, ps > .25. There was also no difference between the two conditions in proportion of CP-knowers (60% of Cup 1st participants and 54% of Game 1st participants), $\chi^2\ (1, N = 28) = .11, p =$ .74. There was no difference in performance on the Ordinal +1 and Ordinal -1 tests for participants who moved the game pawn to the left and those who moved the game pawn to the right, independent samples t(26)s < .13, ps > .89. Therefore, data was collapsed across these conditions.

Figure 2 presents mean accuracy on each Cardinal +1 question and each Ordinal +1 question for Non-CP-knowers and CP-knowers. Figure 3 presents mean accuracy on each

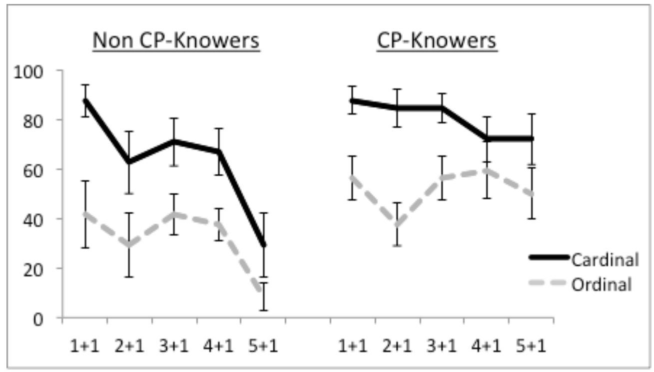

Figure 2. Mean Accuracy (% Correct) on Cardinal and Ordinal +1 Questions. Error bars represent standard error of the mean.

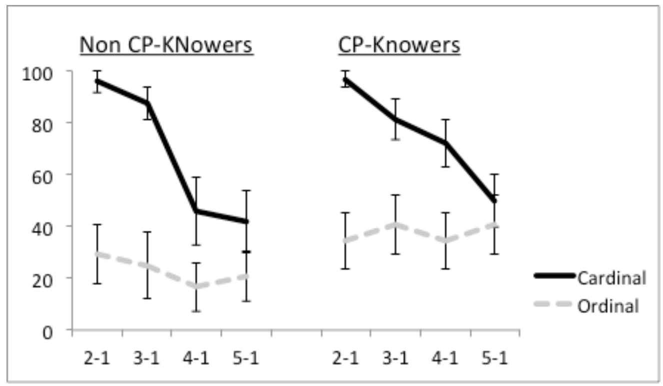

Figure 3. Mean Accuracy (% Correct) on Cardinal and Ordinal -1 Questions. Error bars represent standard error of the mean.

Cardinal -1 question and each Ordinal -1 question for Non-CP-knowers and CP-knowers. Both CP-knowers and Non-CP-knowers were more accurate on cardinal questions than ordinal questions. Scores on Cardinal +1 questions were higher than scores on Ordinal +1 questions, paired sample ts > 4.63, ps < .001; scores on Cardinal -1 questions were higher than scores on Ordinal -1 questions, paired sample ts > 5.63, ps < .001. However, beyond this comparison, different patterns of results emerged for CP-knowers and Non-CP-knowers.

On both Cardinal +1 questions and Ordinal +1 questions (Figure 2), CP-knowers were more accurate than Non-CP-knowers, independent samples t(26)s > 2.06, ps < .05. On Cardinal +1 questions, Non-CP-knowers had a clear drop in accuracy as the test numbers increased. Repeated Measures ANOVA shows significant differences in accuracy on the different test questions, F(4, 44) = 4.99, p < .01, η^2 = .31 and downward trend in accuracy, linear and cubic contrasts F(1, 11)s > 6.40, ps < .03, η^2s = .36. CP-knowers did not have a significant drop in performance on the Cardinal +1 question, repeated measures ANOVA, F(4, 60) = .98 p > .42. On the Ordinal +1 questions for Non-CP-knowers, there was a moderate difference in accuracy across the different test numbers, repeated measures ANOVA F(4, 44) = 2,37 p =.06, η^2 = .18. No significant difference in accuracy on the Ordinal +1 questions was found across the different test numbers for CP-knowers, repeated measures ANOVA $F(4, 60) = 1.97$ $p = .11$.

While CP-knowers were more accurate than Non-CP-knowers on the Cardinal +1 and Ordinal +1 questions, there were no differences in accuracy between these two groups on Cardinal -1 and Ordinal -1 questions (Figure 3), independent samples $t(26)$s < 1.34, ps > .19. Both CP-knowers and Non-CP-knowers had a linear drop in accuracy on Cardinal -1 questions as the test number increased, repeated measures ANOVA Fs > 6.27 ps < .001, η^2s > .29, linear contrasts Fs > 17.9 ps < .01, η^2s > .59. On Ordinal -1 questions, there were no differences in accuracy across questions for either the CP-knowers or Non- CP-knowers, repeated measures ANOVAs Fs < .26 ps > .85.

There were also no significant differences in accuracy between CP-knowers and Non-CP-knowers on the Cardinal and Ordinal +2 and -2 questions (see Table 1), independent samples ts <1.16, ps > .25. Both CP-knowers and Non-CP-knowers were more accurate on Cardinal questions than on Ordinal questions, paired samples ts > 2.34, ps < .04. Additionally, Non-CP-knowers scored higher on Cardinal +2 questions than on Cardinal -2 questions, paired sample $t(11) = 3.00$, $p < .02$. CP-knowers were moderately more accurate on Cardinal +2 questions than on Cardinal -2 questions, $t(15) = 1.81$, $p = .09$. Scores on the Ordinal +2 and Ordinal -2 were quite low with no differences between +2 and -2 questions for either group, paired samples ts < 1.15, ps > .27.

Discussion

The goal of the present study was to investigate young children's understanding of natural numbers by examining the conditions under which they demonstrate knowledge of the successor function. Specifically, do children have knowledge of the successor function that can explain the emergence of the cardinality principle? Do children have an understanding of the successor function in both cardinal and ordinal contexts?

The results indicate that CP-knowers have some understanding of the concept of a successor that Non-CP-knowers do not have. When asked to state a successor of a given number (i.e. +1 questions), CP-knowers outperformed Non-CP-knowers in both cardinal and ordinal contexts. CP-knowers were quite accurate stating a successor for cardinal numbers ($M = 80.0$, $SD = 7.5$ on all Cardinal +1 questions), but their accuracy was much lower for ordinal numbers ($M = 51.9$, $SD = 8.7$ on all Ordinal +1 questions).

However, the results provide no evidence that CP-knowers appreciate uniqueness of the successor function. They could not reliably state a correct preceding number (-1 questions). In both the cardinal and the ordinal contexts, their accuracy on stating a preceding number was no higher than that of the Non-CP-knowers. Therefore, the extent of CP-knowers' knowledge of the successor cannot explain the acquisition of the cardinality principle and of natural numbers.

Table 1: Mean Accuracy (% Correct) for +2 and -2 Questions. Standard deviations are in parentheses.

	Non CP-knower	CP-knower
Cardinal Questions		
+ 2	42 (36)	56 (40)
- 2	79 (26)	78 (26)
Ordinal Questions		
+ 2	8 (19)	22 (36)
- 2	17 (25)	13 (29)

The fact that accuracy on Cardinal -1 questions drops linearly as the test number increases suggests that an awareness of unique successors may emerge in stages with smaller numbers preceding larger numbers. Therefore young children's understanding of unique successors is not an understanding of a general principle, but rather it is isolated and tied to specific numbers.

It is also interesting to note that overall, CP-knowers outperformed Non-CP-knowers only on the Cardinal and Ordinal +1 questions. This suggests that knowledge of the cardinality principle provides no better insight into the uniqueness of the successor (Cardinal and Ordinal -1 questions) and no better accuracy on the +2 and -2 questions than the absence of this knowledge.

The present findings also suggest that acquisition of cardinality precedes that of ordinality (see Colome & Noel, 2012 for similar findings). The fact that young children can correctly recite the count list might suggest that they have an appreciation of order and an understanding of successor in the context of order. However, in the present study, participants could not reliably determine successors in a simple ordinal number task. This indicates that the count list that children recite is simply a memorized sequence that does not reflect the properties of natural numbers. At the same time, both CP-knowers and Non-CP-knowers were more able to determine the successor in the cardinal number task than in the ordinal task. Performance on the +2 and -2 questions provides additional evidence that knowledge of cardinal numbers precedes that of ordinal numbers; both CP-knowers and Non-CP-knowers were markedly more accurate on the cardinal questions than the ordinal questions.

The results of this study demonstrate that young children do have some knowledge of the successor, but the nature of this knowledge is not sufficiently constrained to explain the emergence of the cardinality principle and full understanding of the natural numbers. What knowledge or experience does provide sufficient constraints to account for mature representations of natural numbers is unclear and requires further investigation.

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