Promoting Children's Relational Understanding of Equivalence

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Abstract

Deep understanding of mathematical equivalence is critical for later mathematical understandings. However, research studies and national test results have repeatedly demonstrated that many students fail to develop adequate understanding of equivalence. Recent work from McNeil and colleagues proposes that this failure is partly due to the format of traditional instruction and practice with highly similar problems. Specifically, the change-resistance account (McNeil & Alibali, 2005) proposes that students struggle with equivalence because they have developed overgeneralized "rules" that affect how they process and approach math problems, (e.g., the operators are always on the left side, the equal sign means to "do something" or "give the answer") and fail to see equations having two separate sides that are being related to one another. Extensive practice with problems in a similar format (e.g., those that present all arithmetic operations on the left side of the equal sign) encourages students to develop ineffective mental models of problem types. We replicate and extend prior work that brings cognitive science research to the classroom. Our findings indicate that applying research-based design principles to arithmetic practice improves student understanding of mathematical equivalence enough to support transfer to novel problem types.

Keywords: Mathematical representations; relational reasoning; mathematics education; randomized control trial

Introduction

Can a research-based, early elementary intervention help students learn key concepts that may prevent later struggles in algebra? Research suggests that understanding mathematical equivalence is a critical component of algebraic reasoning (Carpenter, Franke, & Levi, 2003; Charles, 2005; Knuth, Stephens, McNeil, & Alibali, 2006). However, the majority of US students fail to reason with and apply concepts of equivalence (McNeil & Alibali, 2005), making encoding errors when remembering mathematical equations (e.g., McNeil & Alibali, 2004), and interpreting the equal sign to mean "calculate the total" rather than "two amounts are the same" (e.g., Behr, Erlwanger, & Nichols, 1980).

Why do so many students lack a relational understanding of the equal sign? McNeil and Alibali (2005) proposed a change-resistance account: traditional arithmetic instruction that focuses on procedures (i.e., solving problems such as $3 + 4 = \_$) promotes a misconception of the equal sign as a request for an answer and interferes with the development of relational understanding. The majority of examples of arithmetic problems in early elementary math curricula show operations (e.g., addition and subtraction) on the left of the equal sign and the "answer" on the right (Seo & Ginsburg, 2003). Children detect and extract patterns from these examples and ultimately construct long-term memory representations. Although default representations typically speed computation in the problem-solving contexts that children encounter most frequently, these representations may lead to difficulties when patterns are mistakenly transferred to similar, but non-applicable, problem types (e.g., Bruner, 1957).

McNeil and Alibali characterize the representations that develop in early mathematics as "operational patterns" as they reflect an understanding of arithmetic that focuses on the operators (e.g., $+$, $-$, $\times$, $\div$) rather than the relational nature of mathematical equations. Research has identified three types of operational patterns that represent a distorted view of arithmetic and hinder conceptual understanding of the underlying mathematics. First, children learn to expect math problems to have all operations on the left side of the equal sign, with the equal sign immediately before the answer blank on the right, an "operations = answer" problem format (McNeil & Alibali, 2004). Second, children learn to interpret the equal sign operationally as a symbol to do something (Baroody & Ginsburg, 1983; Behr et al., 1980). Third, children learn to perform operations on all given numbers in a math problem (e.g., add up all the numbers in an addition problem, McNeil & Alibali, 2005).

Once entrenched, children rely on these potentially misleading patterns when encoding, interpreting, and solving novel mathematics problems. Students that expect all problems to have operations on the left fail to correctly encode the problem being asked. For instance, after briefly viewing the problem "7 + 4 + 5 = 7 + __" many children rely on their knowledge of the "operations = answer" problem format and erroneously remember the problem as "7 + 4 + 5 + 7 = __" (McNeil & Alibali, 2004). Students also struggle to interpret what a mathematical problem is asking. When asked to define the equal sign—even in the context of a mathematical equivalence problem—many children treat it like an arithmetic operator (like + or –) that means they should calculate the total (McNeil & Alibali, 2005). Finally, entrenched patterns mislead students to solve the problem "7 + 4 + 5 = 7 + __" by performing all given operations on all given numbers and put 23 (instead of 9) in the blank (McNeil, 2007; Rittle-Johnson, 2006). These findings support the idea that children's difficulties with mathematical equivalence are partially due to inappropriate knowledge derived from overly narrow experience with traditional arithmetic.

The ICUE Intervention

Current math practice seems to promote the development of faulty representations, and the change-resistance account's focus on "operational patterns" offers design principles for instruction to improve students' understanding of equivalence. Initially, researchers hypothesized that greater exposure to "non-traditional" arithmetic practice (e.g., presenting operations on the right side of the equation, "__ = 2 + 4," [McNeil et al., 2011], organizing practice by equivalent sums [McNeil et al., 2012], and using relational phrases such as "is equal to" instead of the equal sign in problems [Chesney, McNeil, Petersen, & Dunwiddie, 2012]). may prevent students from developing operational patterns. Though practice with non-traditional arithmetic in a classroom intervention led to improved outcomes over traditional instruction, a number of students failed to reach proficiency (McNeil, Fyfe, & Dunwiddie, 2015).

To further promote mastery of equivalence, McNeil and colleagues added additional design features beyond non-traditional arithmetic practice. The current version of the materials, dubbed Improving Children's Understanding of Equivalence (ICUE), consists of second grade student activities that reduce reliance on operational patterns and promote deep understanding of mathematical equivalence through four key components that have independently been shown to be effective:

1. Non-traditional arithmetic practice (Chesney et al., 2012; McNeil et al., 2012, 2015, 2011);
2. Lessons that first introduce the equal sign outside of arithmetic contexts (e.g., "28 = 28") before introducing arithmetic expressions (e.g., Baroody & Ginsburg, 1983; McNeil, 2008);
3. Concreteness fading exercises in which concrete, real-world, relational contexts (e.g., sharing stickers, balancing a scale) are gradually faded into the corresponding abstract mathematical symbols (e.g., Fyfe, McNeil, Son, & Goldstone, 2014); and
4. Activities that require students to compare and explain different problem formats and problem-solving strategies (e.g., Carpenter et al., 2003; Rittle-Johnson, 2006).

The Current Study: Improving Children's Understanding of Equivalence

A pilot study found the ICUE intervention was successful in improving student understanding of mathematical equivalence (Byrd, McNeil et al. 2015; McNeil, Hornburg, Brletic-Shipley, & Matthews, under review). The current study sought to replicate the findings with a new population of students and additionally investigate whether the learning transferred to the mathematical practice of generating explanations.

To replicate Byrd et al.'s (2015) pilot study, we compared the full ICUE intervention to a control condition consisting solely of non-traditional mathematical practice and measured students' ability to encode equations, solve problems, and define the relational function of the equal sign.

To test whether the learning transferred to the ability of students to generate mathematical explanations related to arithmetic problems, we gave students performance tasks from the Silicon Valley Mathematics Initiative's (SVMI) Mathematics Assessment Collaborative (MAC). MAC partners with the Mathematics Assessment Resource Service (MARS) to develop tasks that assess core mathematical ideas and practices taught in each grade level. Tasks require students to solve complex math problems as well as give open-ended explanations of their reasoning. For each task, MARS provides scoring rubrics and scorer training procedures, student performance statistics, and examples of common student errors (Foster & Noyce, 2004).

Our research questions were:

1. Does ICUE promote measurable gains in children's understanding of equivalence?
2. Do the benefits of ICUE activities transfer to generating mathematical explanations?

Method

Design

We used a cluster-randomized control trial design to examine the efficacy and generalizability of the ICUE intervention relative to an active control program. Teachers were randomly assigned to use the either the ICUE intervention or Active Control materials. The active control consisted of workbook activities to control for time on task. The active control contained non-traditional arithmetic practice but not the additional components present in ICUE, described above.

Participants. Five second-grade teachers (three treatment, two control) from three California schools used the activities in their classrooms. Class sizes ranged from 21 to 32, and we analyzed data from 81 students who completed the ICUE activities and measures and 49 students who completed the Active Control activities and measures.

Procedure and Materials

The procedure for ICUE Treatment and Active Control conditions were identical, differing only in the content of the materials used by teachers and students. Each teacher received training on the study purpose, features of the activities, and strategies for integrating the activities into their typical mathematics curriculum.

Prior to starting the study, participating teachers completed online surveys assessing their mathematics teaching experience and classroom structure and dynamics.

After administering a pretest, teachers used the study materials for approximately 15 minutes twice each week for 16 weeks. In both conditions, teachers were asked to use the study materials to supplement, rather than replace, current instruction, and to limit session duration to 20 minutes.

After completing the 32 sessions, teachers administered the same pretest measure of mathematical equivalence, a proximal transfer measure, two measures of transfer to mathematical explanations, and the Math Concepts subtest of the Iowa Test of Basic Skills.

Active Control. Teachers in the Active Control condition received a set of student workbooks (see Figure 1) and a teacher guide.

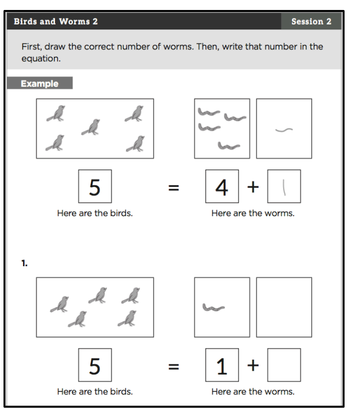

Figure 1. Sample workbook page from the Active Control condition materials featuring non-traditional math practice.

ICUE. Teachers in the ICUE Treatment condition received a set of student workbooks (see Figure 2), a teacher guide, and a set of classroom manipulatives including balance scales and flashcards.

Measures

Pre- and post-test measures of mathematical equivalence. We assessed children's understanding of mathematical equivalence before and after the interventions using similar measures of equation encoding, equation solving, and defining the equal sign used in previous work by McNeil and colleagues (Byrd et al., 2015; McNeil & Alibali, 2005; McNeil et al., 2015).

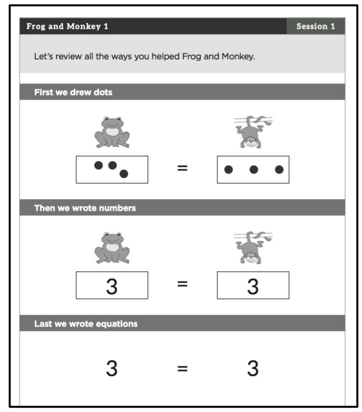

Figure 2. Sample workbook page from the ICUE Treatment condition materials featuring a concreteness fading exercise.

Equation encoding. The encoding measure consisted of recalling four mathematical equivalence problems (e.g., 5 + 4 = 3 + __) presented one at a time. Each equation was visible for five seconds and students were instructed to remember and write down exactly what they saw after the equation was hidden from view. Responses were coded as correct if the student wrote the equation exactly as shown (i.e., the correct numbers and symbols in the correct order).

Equation solving. The equation solving measure consisted of eight equations with operations on both sides of the equal sign (e.g., 3 + 5 + 6 = 3 + __).

Defining the equal sign. The defining the equal sign measure prompted students to write responses to three questions about the equal sign symbol (=): 1) What is the name of this math symbol? 2) What does this math symbol mean? And, 3) Can it mean anything else? Teachers read each question aloud and waited for students to write their responses before moving on to the next question. Responses were coded as relational if the response defined the equal sign as relating two sides of the equation (e.g., two amounts are the same, something is equivalent to another thing).[1]

Measures of knowledge transfer.

Proximal transfer measure. The proximal transfer measure, used by Byrd et al. (2015), consisted of nine more advanced problems of mathematical equivalence, not strictly aligned with the ICUE intervention. The transfer questions included equations with operations on both sides of the equal sign involving subtraction (e.g., 2 + 5 + 3 = 14 – __),

[1] Although many students in this age range have poor spelling, coders did not have trouble determining what a given child had written, even when words were misspelled (e.g., "the toltal", "write the anser next"). Inter-rater agreement between coders on whether a given definition was relational ranged from 95-100%.

larger numbers (e.g., 13 + 18 = __ + 19), "word problem" items featuring story-to-equation translation, and an "explaining equivalence" problem, which asked students to decide whether the same number should appear in two equations and explain their reasoning.

Distal transfer to mathematical explanations. We selected two MARS items that tested students' understanding of mathematical equivalence, described below. Items were scored by project staff following scorer training, calibration, and reliability procedures established by MARS (Foster & Noyce, 2004).

Incredible Equations. In this task, students are asked to fill in the missing parts of equations such as "__ + 8 + __ = 16" and "11 + 5 = __ + 8." Students are asked to explain how they know their answer is correct. When 6,305 students took the task in 2007, the mean score was 6.08 out of 10 with a standard deviation of 2.5 (MARS, 2007).

Agree or Disagree? In this task, students are asked if they agree or disagree with two number sentences: "8 + 5 = 5 + 8" and "6 – 4 = 4 – 6". Students are asked to explain their answers using words, numbers, or pictures. MARS administered this task to 4,585 second graders in 2004 and found the mean score was 3.10 out of 6 with a standard deviation of 1.94 (MARS, 2004).

Iowa Test of Basic Skills. To make sure any gains in understanding of equivalence do not come at the expense of problem-solving fluency, students completed the Math Concepts subtest of Level 8 of the Iowa Tests of Basic Skills (ITBS), which served as a measure of general mathematical reasoning. Participation in the ICUE Treatment neither helped nor hurt students' performance on this measure, relative to the Active Control group ($t(83)=1.48$, *ns*), establishing that the intervention does not improve understanding of equivalence at the expense of general computational fluency.

Results

Does ICUE promote measurable gains in children's understanding of equivalence, relative to an Active Control?

We assessed three critical abilities identified by McNeil and colleagues as necessary for success in reasoning about equivalence (Byrd et al., 2015; McNeil et al., under review):

1. Equation encoding: the ability to accurately encode and recreate an equation after seeing it briefly;
2. Equation solving: the ability to solve equations that feature operations on both sides of the equal sign; and
3. Defining the equal sign (=): the ability to identify "=" as a symbol that signals a relation between two equal numbers or quantities.

Specifically, we examined students' gains in performance on identical pre-intervention and post-intervention tests that assessed the three abilities above. For each of the target abilities, we compared the gains made by students in the ICUE Treatment condition to those of students in the Active Control condition (Figure 3).

There were no reliable differences between pretest scores for each group, and students in the ICUE Treatment condition made substantially greater gains during the intervention than students in the Active Control condition. The proportion of correct responses for Equation solving items increased by 0.65 for ICUE students, compared to only 0.065 for Active Control students ($t(119)=48.8$, $p<.001$; *Cohen's d* > 3); the proportion of correct responses for Equation encoding items increased by 0.34 for ICUE compared to 0.26 for Active Control ($t(52)=5.31$, $p<.001$; *Cohen's d* > 3); and the proportion of correct definitions of the equal sign increased by 0.38 for ICUE compared to 0.02 for Active Control ($t(125)=8.42$, $p<.001$; *Cohen's d* > 3). These results suggest that the ICUE Treatment intervention leads to systematic and measureable gains in children's understanding of and reasoning about mathematical equivalence.

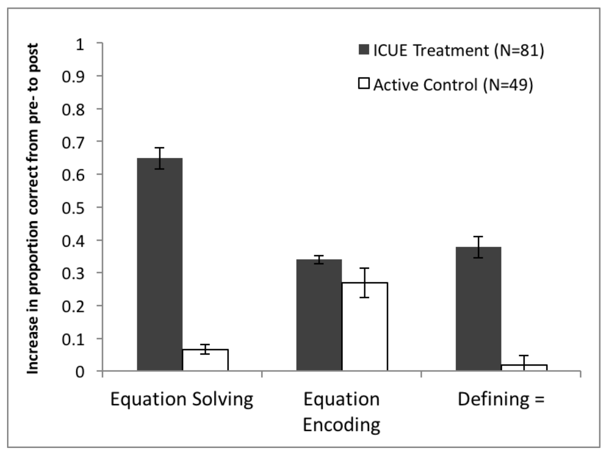

Figure 3. Mean performance gains from pre- to post-test for children in the ICUE and Active Control groups.

Do the benefits of ICUE activities transfer to more challenging material and generating mathematical explanations?

We explored whether the knowledge that children gained from the intervention activities transferred to problem-solving tasks that were not strictly aligned with the content and goals of the ICUE or Active Control interventions. We first examined performance on a proximal researcher-developed measure that included a series of complex equation solving items, word problem items that required translating story content into mathematical equations, and an explaining equivalence item that required students to justify why two sides of an equation were equal (i.e., "Is the number that goes in the □ the same number in the following two equations? Explain your reasoning."). We compared the performance of ICUE and Active Control students on the measure, which was administered after each group completed all intervention activities (Figure 4).

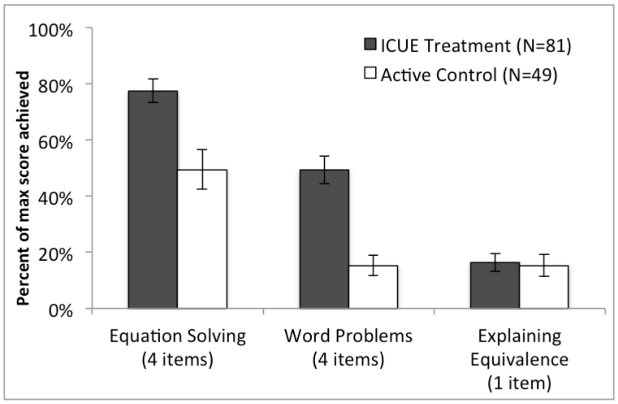

Figure 4. Mean ICUE and Active Control group performance for researcher-developed transfer items.

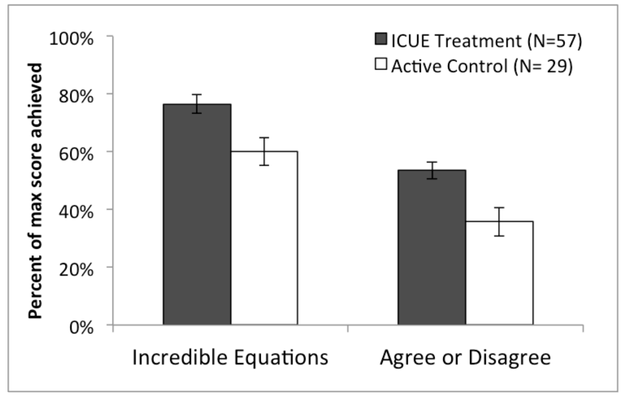

Figure 5. Mean ICUE and Active Control group performance for MARS transfer items.

Students in the ICUE condition scored reliably higher, on average, than students in the Active Control condition on both complex equation solving items ($t(81)=3.44$, $p<.01$; *Cohen's d* = 1.5) and word problem items ($t(129)=5.31$, $p<.001$; *Cohen's d* = 2.5). However, the groups did not differ in their mean performance on the explaining equivalence item ($t(81)=0.15$, *ns*).

We also examined transfer to the MARS items. We measured post-intervention performance on the "Incredible Equations" task (scored out of a possible 10 points) and the "Agree or Disagree?" task (scored out of a possible 6 points). As before, we compared performance by students in the ICUE and Active Control conditions, shown in Figure 5. As one teacher from each condition failed to return the MARS posttest materials, results are reported from two treatment teachers and one control teacher.

Students in the ICUE condition performed reliably better than Active Control students on both the Incredible Equations ($t(54)=2.83$, $p<.05$; *Cohen's d* = 0.32) and Agree or Disagree? tasks ($t(47)=2.36$, $p<.05$; *Cohen's d* = 0.43).

Conclusions

A deep understanding of mathematical equivalence is a key building block for later mathematical understandings. However, research studies and national test results have repeatedly demonstrated that many students fail to develop this understanding. The change-resistance account suggests that traditional instruction that relies on extensive practice with problems in a single format may be contributing to students' difficulties by encouraging students to develop ineffective mental models of problem types.

In the current study, we sought to replicate and extend prior work that brings research from the lab into the classroom. The change-resistance account proposes that students struggle with equivalence because they have developed overgeneralized "rules" that affect how they process and approach math problems, (e.g., the operators are always on the left side, the equal sign means to "do something" or "give the answer") and fail to see equations having two separate sides that are being related to one another.

Overall, our findings indicate that applying research-based design principles in the form of multiple types of practice improved student understanding of the critical concept of mathematical equivalence.

Our findings replicate Byrd et al. (2015), who found that activities that include the use of the equal sign outside of arithmetic contexts, that start with concrete examples and fade to extractions, and that explicitly prompt students to compare and explain different problem formats and strategies improve student understanding of mathematical equivalence beyond non-traditional arithmetic practice alone.

Students receiving the ICUE materials demonstrated improved performance in equation solving, equation encoding, and providing relational definitions of the equal sign. These improvements did not come at the expense of arithmetic problem-solving fluency, as measured by the ITBS. Further, the learning in ICUE transferred to greater student abilities to solve complex equations and word problems.

Students in both conditions struggled with the researcher-developed item that required students to explain equivalence. Their poor performance may reflect confusion with equivalence that persists for more complicated problems with multiple "terms" and different types of operators (both addition and subtraction), a confusion that was reflected in students' explanations of their answers.

The robust improvements on the MARS items supports the possibility that the lack of transfer in the equivalence explanation question was due to confusion regarding multiple terms and operators rather than the ability to generate the explanation. These established items, developed externally, also asked students to explain equivalence, but used blanks, rather than variables, to reflect the unknown entities. On both items, students in the ICUE condition outperformed the students in the active control condition. These findings suggest that the additional practice comparing and explaining different problem formats helped students gain a deeper understanding of not only *whether* different examples were equivalent, but also *why or why not.*

Why is it important to test the synergistic effect of research-based design principles? Instructional designers face a large number of decisions in selecting appropriate activities and tasks for students. Though much research seeks to identify how different facets work independently, if research in cognitive science is to extend meaningfully to practice the cumulative effects of using multiple strategies must be tested. Our small-scale cluster-randomized trial suggests that the multi-component ICUE intervention was more effective than an active control of non-traditional arithmetic practice (which in prior work was also more effective than traditional instruction).

Future work, in progress, will test the efficacy of the ICUE intervention in a large-scale cluster-randomized trial with diverse students across the state of California. This work demonstrates how findings in the lab can be successfully implemented in authentic classroom settings to improve student learning outcomes.

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