

# A Two-Stage Model of Solving Arithmetic Problems

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## Abstract

This paper examines a process of solving different types of counterfactual arithmetic problems (problems contradicted a visual experience, an experience of temperature, encyclopedic knowledge, etc.) in comparison with their ‘real’ counterparts by different types of subjects (e.g., educated in math and educated in humanities). As a result, a two-stage model of solving arithmetic problems is outlined in the paper.

**Keywords:** situated cognition; counterfactual reasoning; arithmetical problem; two-stage model; four-level-cognitive-development theory.

## Introduction

This research has been carried out at the junction of two sets of problems. The first one is concerned with counterfactual reasoning<sup>1</sup>. This issue has been discussed intensively in recent decades (e.g., Pearl 2000; Fauconnier & Turner 2002, p. 17–59; Hiddleston 2005; de Vega et al. 2007; de Vega 2008; Ferguson & Sanford 2008; de Vega & Uritta 2011; Rips & Edwards 2013), but some aspects thereof have not been touched so far. ‘Situated cognition’ is a blanket term for the second set (e.g., Clancey 1997; Kirshner & Whitson 1997; Watson & Winbourne 2007; Robbins & Aydede 2009). This paper examines the role of situated cognition in counterfactual reasoning. A distinguished work of A. Luria (1976) was a starting point for that. When investigating cognitive skills of dekchans of Central Asia in 1930<sup>th</sup> he encountered a curious phenomenon. His subjects were not able to solve counterfactual problems, whereas they solved quite easily similar problems consistent with their everyday life. Importantly, trying to solve counterfactual problems subjects addressed their day-to-day experience. Some striking examples thereof are given in Luria’s monograph (1976, p. 131):

A ‘conditional’ problem that conflicts with actual experience is given:

[Exp.] Suppose it were to take six hours to get from here to Fergana on foot and a bicycle was twice as slow?

[Sub.] Then a bicycle would get there in three hours!

Solution on a level corresponding to practical reality.

[Exp.] No, a teacher gave this problem as an exercise – suppose that the bicycle were twice as slow.

[Sub.] If the cyclist makes good time, he will get to Fergana in two and a half or three hours.

According to your problem, though, if the bicycle brakes down on the way, he’ll arrive later, of course. If there’s a breakdown, he’ll be two or three hours late.

From the perspective of cultural psychology (Vygotsky, Luria, Cole, Tulviste, etc.), this and similar facts are usually interpreted as a difference between ‘situational’ thinking and ‘abstract’ thinking. The subjects of Luria’s investigation were people of so-called ‘sympractical’ culture, in which ‘situational’ thinking based on day-to-day experience is supposed to be the only way of reasoning (Cole & Scribner 1974, p. 160-168, 178-179; Luria 1976; Tulviste 1991). In that, Luria’s dekchans endeavored to reinterpret abstract problems as stories from their everyday life.

Indeed, people of modern industrial cultures can solve counterfactual problems; meanwhile, situated cognition is an important part of their everyday life, and, therefore, it may influence their way and speed of solving counterfactual problems. The overall objective of our research was to test some obstacles they may encounter in this process.

It is worth noting that there are two basic models of counterfactual reasoning in contemporary cognitive science (Pearl 2000; Hiddleston 2005; Rips & Edwards 2013). The first model is called ‘pruning theory’. From this perspective, when modeling counterfactual situation, subject changes the only element (in particular, in famed *If Clinton were the Titanic, the iceberg would sink*

<sup>1</sup> There are two basic interpretations of the concept *counterfactual* in cognitive science: ‘contrary to reality’ (Pearl 2000; Hiddleston 2005; Rips & Edwards 2013), and ‘possible, but not implemented in some situation’ (Roese & Olson 1995; Roese 1997). For this paper only the first interpretation is actual.

Clinton replaces iceberg), all others being the same. The second model is named 'minimal network theory'. In its scope, a change of one element entails a number of changes in elements close to this one. Importantly, both models look formal and do not distinguish between situated cognition and abstract knowledge.

A preliminary hypothesis of this research was as follows. Given some discrepancies between situated cognition and a counterfactual situation in arithmetic problems, people of modern industrial cultures face a number of difficulties when solving counterfactuals. These difficulties would engender an extended period of time needed for solving counterfactual problem in comparison with 'real' one and also more errors in that. Perhaps, the most intriguing issue in this scope is a particular way of how situated cognition is involved in the process of solving. There are two basic options. From the first perspective, situated cognition is actual for the whole period of solving; from the second perspective, it is at work only in the first stage, in which an abstract model of the task is built, whereas in the second stage only formal operations are processed. If the first option is true, difficulties caused by a counterfactual situation can be drawn forth at any moment of reasoning; if the second option works, they are present early in stage, and then there is no difference, other things being equal, in solving counterfactual and 'real' problems.

One of the ways to test these options is to compare mean  $\Delta t_{cr}$  (the difference between the time needed to solve a counterfactual problem and the time needed to solve its 'real' equivalent) for people who solve problems faster (aka 'experts') and slower (aka 'amateurs'). 'Experts' superiority over 'amateurs' may be a result of more developed computational skills (factor *a*), of higher speed of transformation of a task into a system of abstract symbols (factor *b*), and of a combination of these factors. If mean  $\Delta t_{cr}$  for 'experts' is less than for 'amateurs', the factor *b* is important; if the difference between mean  $\Delta t_{cr}$  is not significant, the factor *a* dominates.

The lack of difference between mean  $\Delta t_{cr}$  can be also an argument for the two-stage model with the following interpretation: in the first stage there is no difference between 'experts' and 'amateurs', the process in this stage is determined by situated cognition; in the second stage situated cognition is out of work, subjects only perform computational operations with formal objects. Importantly, it does not exactly mean that perception is also out of work in the second stage; it may also mean that in this stage perception works in quite a specific way, distant from day-to-day experience.

Another intriguing issue is a correlation between mean  $\Delta t_{cr}$  and a type of problem. Counterfactual situation can be concerned with various perceptive channels (vision, hearing, taste, etc.) as well as with some theoretical knowledge. To find out which channel causes maximum  $\Delta t_{cr}$  is useful to clarify the structure of basic elements of situated cognition. The priority of perceptual channels over encyclopedic knowledge was our working hypothesis in this case (e.g., Zacks 2015, p. 95-107).

The framework of the experiments to carry out was determined by a problem field represented above.

## Experiment 1

### Method

**Subjects.** A total of 25 students of Moscow high schools, 15-16-year-old were the subjects of Experiment 1. We argued for this choice by fewer discrepancies in solving arithmetical problems for students than it would be for adult participants.

**Material.** The subjects were suggested to solve 16 simple arithmetical problems divided into 8 groups: a) problems which contradict a social experience, and their 'real' counterparts (e.g., *A 24-page notebook costs 20 roubles more than a 96-page notebook. What is the price of the 24-page notebook if the price of 96-page notebook is 10 roubles?*, and *A 96-page notebook costs 20 roubles more than a 24-page notebook. What is the price of 96-page notebook if the price of 24-page notebook is 10 roubles?*); b) problems which contradict a visual experience, and their 'real' counterparts (e.g., *A cyclist moves 10 times faster than a car driver. Please, work out the speed of the car if the speed of the cyclist is 80 km/h;* and *A cyclist moves 10 times slower than a car driver. Please, work out the speed of the car if the speed of the cyclist is 8 km/h*); c) problems which contradict hearing, and their 'real' counterparts (e.g., *In normal conditions, a shout covers a distance of 10 metres; this is 40 metres less than the distance covered by a whisper. Please work out the distance that a whisper covers,* and *In normal conditions, a shout covers a distance of 50 metres; this is 40 metres more than the distance covered by a whisper. Please work out the distance that a whisper covers*); d) problems which contradict an experience of temperature, and their 'real' counterparts; e) problems which contradict an experience of taste, and their 'real' counterparts; f) problems which contradict laws of biology, and their 'real' counterparts (e.g., *A father is 20 years younger than his son. How old is the son if the father is now 17 years old?*; and *A mother is 20 years older than her daughter. How old is the daughter if the mother is now 37 years old?*); g) problems which contradict encyclopedic knowledge, and their 'real' counterparts; h) problems which contradict an experience of weight, and their 'real' counterparts. Each problem was printed on a special card.

**Procedure.** Each subject worked out the problems individually. Each subject solved firstly 16 counterfactual problems, randomly given to him/her (2 from each group), and then, a month after, 16 their 'real' counterparts. We endeavored, as seen, to minimize any text difference in each pair of problems in order to avoid 'noise interference'. Before the main procedure the subjects solved two control problems to make sure that they had no difficulties in that. Time from receiving a card to reporting the answer was measured for each problem.

The data was processed as follows. Firstly,  $\Delta t_{cr}$  was counted for each pair of problems. Then, a total of positive and negative  $\Delta t_{cr}$  was calculated, and the significance of the difference between positive and negative  $\Delta t_{cr}$  alongside with the significance of the difference between a total of 'counterfactual' errors and that of 'real' errors was estimated with Pearson's chi-squared test. After that, for more detailed analysis

comparative data for all pairs of groups of problems were processed with one-way ANOVA.

## Results & Discussion

The results of Experiment 1 provided strong evidence for supporting the hypothesis that subjects will encounter more obstacles when solving counterfactual problems than real ones: only 5  $\Delta t_{cr}$  from 400 had a negative value ( $\chi^2(1)=380.25$ ;  $p<0.0001$ ).

Some complementary evidence for supporting general hypothesis was also provided by the analysis of the errors: a total of 34 for counterfactual problems; a total of 10 for 'real' ones ( $\chi^2(1)=380.25$ ;  $p<0.01$ ).

Mean  $\Delta t_{cr}$  and standard deviation for each group of problems are presented in Table 1.

Table 1: Mean  $\Delta t_{cr}$  for the groups of problems (sec.)

a)	b)	c)	d)
8.12±5.89	11.95±15.66	10.62±6.04	8.32±10.93
e)	f)	g)	h)
10.30±12.62	6.30±2.95	6.15±5.25	8.39±12.70

The first remarkable result in this table is a gross standard deviation engendered by a big dispersion of results for different participants. The minimal standard deviation holds for the groups f), g), a), and c). This may be connected with a computational complexity of a particular problem: in particular, other things being equal, multiplication leads to higher dispersion than addition; multiplication by eight – to higher dispersion than multiplication by two, etc.

Because of high dispersion, there is no significant difference in  $\Delta t_{cr}$  for almost all pairs. The only exception is pairs (c, f) and (c, g) ( $p<0.01$ ). However, this information is also useful as an argument for the hypothesis of the perceptual channels priority over encyclopedic knowledge: problems contradictory with hearing need more time to comprehend the task than problems contradictory with biological laws and encyclopedic knowledge.

By and large, the results of Experiment 1 gave clear evidence to support the basic hypothesis of more difficulties in solving counterfactuals than in solving their 'real' counterparts. Meanwhile, they raised a number of significant questions for further research. Firstly, Experiment 1 gave no evidence pro or contra the two-stage-model. The way of how a counterfactual situation matters the process of reasoning needed a fine-grained analysis. Secondly, the hypothesis of the perceptual channels priority over encyclopedic knowledge required a more detailed investigation.

The framework of Experiment 2 was determined by these issues. To examine the two-stage-model by engaging two groups of subjects with different skills of solving arithmetical problems was its objective. In that, students specialized in mathematics (SM) and their peers specialized in humanities (SH) were chosen to participate Experiment 2. As mentioned above, SM were expected to solve both 'real' and counterfactual problems faster than SH. Given that, the comparison of mean  $\Delta t_{cr}$  for SM and

SH was under discussion. If significant difference between them could be interpreted in different ways and demanded further investigations to draw a more minute description, then the lack of such difference would testify for the two-stage model.

Let us take a more detailed look at this. The influence of a counterfactual situation is obviously concerned with situated cognition. The same  $\Delta t_{cr}$  show that such influence does not depend on computational skills being an invariant, at least, for a particular age. Then, if situated cognition is actual for the whole process of working out a problem, its equal influence on SM and SH will be represented in equal  $\epsilon$  (the ratio  $\Delta t_{cr} : t_{real}$ ), but not  $\Delta t_{cr}$ . Equal  $\Delta t_{cr}$  is an argument for the two-stage model.

## Experiment 2

### Method

**Subjects.** A total of 40 students of Moscow high schools, 15-16-year-old – 20 SM students and 20 SH students – were the subjects of Experiment 2. None of them participated Experiment 1.

**Material.** The subjects were suggested to solve ten arithmetical problems: five problems contradicting a visual experience (type b) and five problems contradicting encyclopedic knowledge (type g). The problems suggested were the same as the problems of this type in Experiment 1.

**Procedure.** It was the same as that of Experiment 1. The data was processed in a following way. Firstly, similar to Experiment 1, Pearson's chi-squared test was applied to estimate the significance of the difference between positive and negative  $\Delta t_{cr}$  alongside with the significance of the difference between a total of 'counterfactual' errors and that of 'real' errors. After that, a correlation between a group of subjects (SM and SH) and  $t_{real}$ ,  $t_{cf}$ ,  $\Delta t_{cr}$  was checked with one-way ANOVA. Finally, one-way ANOVA was used to check a correlation between  $\Delta t_{cr}$  and the type of problem (type b vs. type g).

### Results

In accordance with Experiment 1, both SM and SH needed more time to solve counterfactual problems in comparison with their 'real' counterparts ( $p<0.0001$ ).

As predicted, SM solved both 'real' and counterfactual problems faster than SH ( $p<0.001$ ).

The difference between  $\Delta t_{crSM}$  and  $\Delta t_{crSH}$  was not significant. In Table 2 mean  $\Delta t_{crSM}$ ,  $\Delta t_{crSH}$  and standard deviation are presented for each problem (problems 1-5 are concerned with visual experience, problems 6-10 – with encyclopedic knowledge).

Table 2: Mean  $\Delta t_{crSM}$ ,  $\Delta t_{crSH}$ , **P** for each problem

N <sub>o</sub>	$\Delta t_{crSM}$ , sec.	$\Delta t_{crSH}$ , sec.	<b>P</b>
1	5.9±5.9	11.2±10.0	<b>0,037</b>
2	4.5±4.0	4.2±9.6	0,883
3	3.8±4.0	5.8±7.5	0,521
4	7.8±6.2	3.5±3.2	<b>0,010</b>
5	5.2±5.1	8.1±7.2	0,123

6	4.6±3.3	7.0±7.3	0,197
7	4.7±3.6	4.6±7.5	0,970
8	4.6±2.5	5.4±2.6	0,319
9	4.8±4.8	4.5±7.3	0,782
10	4.5±3.7	4.6±4.1	0,951

As seen, only the results for problem 1 and problem 4 are more or less significant; at that, for problem 1  $\Delta t_{crSM} < \Delta t_{crSH}$  and for problem 4  $\Delta t_{crSM} > \Delta t_{crSH}$ .

Mean  $\Delta t_{cr}$  for problems contradicting a visual experience (№ 1-5) is more than that for problems contradicting encyclopedic knowledge (№ 6-10) ( $\Delta t_{cr1-5} = 6.24 \pm 7.07$ ;  $\Delta t_{cr6-10} = 4.91 \pm 5.42$ ;  $p = 0.046$ ).

### Discussion

The results of Experiment 2 provide some evidence to support the two-stage-model. A higher level of computational skills entailing a higher speed to work out a problem does not lead to less  $\Delta t_{cr}$ . As noticed above, constant  $\Delta t_{cr}$  is evidence of the same – at least, for the groups of subjects involved in the experiment – stage of the process. This stage is likely to be connected with the constructing a formal model of the problem, put another way, with the transforming a particular situation into the system of abstract symbols. Situated cognition dominates in this stage, whereas the next stage is concerned with computational operations with such system.

The results of Experiment 2 also support the hypothesis of the priority of perceptual channels over encyclopedic knowledge. The subjects face more difficulties in the situation contradicting their visual experience than in the situation contradicting their encyclopedic knowledge. These data are consistent with some observations in different fields, e.g., with the decisive role of perception in categorization (this idea is represented by the concept of basic level category; see, e.g., Rosch 1978; Lakoff 1987).

As mentioned, subjects of Experiment 1 and Experiment 2 were 15-16-year-old students. Such a choice was determined by a higher level of homogeneity in computational skills for that group in comparison with adults. Nevertheless, in order to verify the results on another age group, Experiment 3 was carried out.

## Experiment 3

### Method

**Subjects.** A total of 20 high-educated adults (age 35–60; mean age – 48), half with education in math and physics (EM), and half with education in humanities (EH) were subjects of Experiment 3.

Material and procedure coincided with that of Experiment 2.

### Results and discussion

As it was hypothesized, dispersion for adults was much more significant than for students, because of notable

difference in practice. Nevertheless, the main results of Experiment 1 and Experiment 2 were confirmed.

Subjects solved counterfactual problems longer than ‘real’ ones ( $\chi^2(1) = 26.27$ ;  $p < 0.0001$  for EM;  $\chi^2(1) = 7.19$ ;  $p < 0.01$  for EH).

Although EM subjects solved the problems of both types faster than EH ones ( $p < 0.005$ ) the difference between  $\Delta t_{SM}$  and  $\Delta t_{SH}$  was not significant ( $p = 0.33$ ).

### General discussion

Returning to the issues represented in the introduction it is worth stressing again that the obstacles which Luria's dekchans faced when working out counterfactual problems also characterize people of modern industrial societies in similar situation. These obstacles are not as crucial, however, they lead to longer time needed to solve counterfactual problem in comparison with their ‘real’ counterparts as well as to more solving errors. These results are consistent with some data from other research fields. Thus, works by Frumkina and colleagues (see Frumkina & Mikheev 1996 as a summary) gave clear evidence that ‘complex thinking’, which is, according to Vygotsky, a feature of preschool-age children and people of hunter-gatherer cultures, can also characterize people of modern industrial culture in some situations (e.g., in classification tasks). The only difference is that modern people can change their mind and shift from complex thinking to more abstract cognitive models after some clarifications of an experimenter.

In order to generalize these and similar observations, it is worth addressing the four-level-cognitive-development theory (Glebkin 2015, Glebkin 2015a). This theory singles out four basic cognitive levels, which hold also a framework for cultural-historical typology: Level A characterizes great apes; Level B – prehistoric culture and hunter-gatherer culture; Level C – early theoretical cultures; Level D – Modernity in Europe and modern industrial cultures. Importantly, these levels build on each other, but do not interchange with each other; modern people, guided by circumstances, can operate on all levels. In particular, a majority of everyday skills (swimming, navigation in space, etc.) demand Level A and Level B; Level C is actual for, e.g., working out problems of school geometry; Level D – for abstract algebraic operations. By and large, conceptual systems on Level C operate with objects of natural/social world and their direct representations (historical events, social and political actions, natural objects, etc.). Unlike that, systems on level D operate with abstract objects which have no direct connections with natural/social world (e.g., non-Euclidean geometry, quantum field theory, etc.).

From this perspective the two-stage-model of working out arithmetic problems, developed in this paper, might be interpreted as a shift from Level C, basic in the first stage, to Level D dominating in the second stage. It means that humans can change cognitive levels not only by changing problems but also when solving the same problem.

Finally, it is worth noting that a two-stage model was also suggested by Maruyama et al. (2012) to account for a process of performing nested calculations (e.g.  $8 + (5 - (3$

+ 1)) ). Both an analysis of eye-movements of subjects and magneto-encephalography data give some evidence for that. This means that such a model may work not only for arithmetical problems but also for other types of problems in mathematics.

## References

- Clancey, W. (1997). *Situated cognition: on human knowledge and computer representations*. Cambridge; N.Y.: Cambridge University Press.
- Cole, M. & Scribner, S. (1974). *Culture and thought*. New York: Wiley.
- de Vega, M. (2008). Levels of embodied meaning: From pointing to counterfactuals. In M. de Vega, A. Glenberg & A. Graesser (eds.), *Symbols and embodiment: debates on meaning and cognition* (pp. 285–308). Oxford; New York: Oxford University Press.
- de Vega, M. & Uritta, M. (2011). Counterfactual sentences activate embodied meaning: An action–sentence compatibility effect study. *Journal of Cognitive Psychology*, 23 (8), 962–973.
- de Vega, M., Uritta, M. & Riffio, B. (2007). Canceling updating in the comprehension of counterfactuals embedded in narratives. *Memory and Cognition*, 35, 1410–1421.
- Fauconnier, G. & Turner M. (2002). *The Way We Think. Conceptual Blending and the Mind's Hidden Complexities*. N. Y.: Basic Books.
- Ferguson, H. & Sanford, A. (2008). Anomalies in real and counterfactual worlds: An eye-movement investigation. *Journal of Memory and Language*, 58, 609–626.
- Frumkina, R. & Mikheev, A. (1996). *Meaning and categorization*. New York: Nova Science.
- Gibbs, R. W. Jr. (2006). *Embodiment and cognitive science*. Cambridge; New York: Cambridge University Press.
- Glebkin, V. (2015). The Problem of Cultural-Historical Typology From the Four-Level-Cognitive-Development Theory Perspective. *Journal of Cross-Cultural Psychology*, 46 (8), 1010–1022.
- Glebkin, V. (2015a). A cognitive view on cultural-historical typology. In G. Airenti, B. G. Bara, G. Sandini (Eds.) *Proceedings of the EuroAsianPacific Joint Conference on Cognitive Science*. Torino, Italy, September 25–27, 738–743.
- Hiddleston, E. (2005). A causal theory of counterfactuals. *Nous*, 39, 632–657.
- Kirshner D. & Whitson J. A. (eds.) *Situated cognition: social, semiotic, and psychological perspectives*. Mahwah: L. Erlbaum, 1997.
- Lakoff, G. (1987). *Women, Fire and Dangerous Things*. Chicago and London: The University of Chicago Press.
- Landy, D. & Goldstone, R. L. (2009). How much of symbolic manipulation is just symbol pushing? In N.A. Taatgen & H. van Rijn (eds.), *Proceedings of the 31th Annual Conference of the Cognitive Science Society* (pp. 1318–1323). Austin, TX: Cognitive Science Society.
- Luria, A. (1976). *Cognitive Development: Its Cultural and Social Foundations*. Cambridge, Mass.: Harvard University Press.
- Maruyama, M., Pallier, C., Jobert, A., Sigman, M. & Dehaene, S. (2012). The cortical representation of simple mathematical expressions. *Neuroimage*, 61, 1444–1460.
- Pearl, J. (2000). *Causality*. Cambridge, UK: Cambridge University Press.
- Rips, L. & Edwards, B. (2013). Inference and Explanation in Counterfactual Reasoning. *Cognitive Science*, 37, 1107–1135.
- Robbins, Ph. & Aydede, M. (eds.) *The Cambridge handbook of situated cognition*. Cambridge; N.Y.: Cambridge University Press, 2009.
- Roese, N. (1997). Counterfactual thinking. *Psychological Bulletin*, 121, 133–148.
- Roese, N. & Olson, J. (1995). Counterfactual thinking: A critical overview. In *What might have been: The social psychology of counterfactual thinking* (pp. 1–55). Mahwah: Erlbaum.
- Rosch, E. (1978). Principles of Categorization. In *Cognition and categorization* (pp. 27–48). Hillsdale, N.J.: L. Erlbaum Associates; N.Y.: Halsted Press.
- Shapiro, L. (2008). Symbolism, embodied cognition, and the broader debate. Symbols and embodiment: debates on meaning and cognition. In M. de Vega, A. Glenberg & A. Graesser (eds.), *Symbols and embodiment: debates on meaning and cognition* (pp. 55–74). Oxford; New York: Oxford University Press.
- Tulviste, P. (1991). *The cultural-historical development of verbal thinking*. Commack, N.Y.: Nova Science Publishers.
- Vankov, I. & Kokinov, B. (2011). Embodied Comparison of Functional Relations. In B. Kokinov, A. Karmiloff-Smith & N. J. Nersessian (eds.), *European Perspectives on Cognitive Science*. Sofia: New Bulgarian University Press.
- Watson, A. & Winbourne, P. 2007. (eds.). *New directions for situated cognition in mathematics education*. N.Y.: Springer.
- Zacks, J. (2015). *Flicker. Your Brain on Movies*. Oxford, New York: Oxford University Press.