Effects of visual representations on fraction arithmetic learning

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Abstract

Two common visual representations of fractions are circular area models and the number line. The present study examined effects of these visual representations on acquisition of fraction knowledge. In Experiment 1, elementary school students learned aspects of fraction arithmetic with a visual representation or with standard symbolic notation alone. Results found no advantage for the inclusion of a visual representation. In Experiment 2, elementary and middle students were tested on their ability to recognize, discriminate, and construct area models of fractions and number line representations of fractions. The results show higher accuracy for area model questions than for number line representation questions. Taken together these findings suggest that for fractions less than 1, simple area models may have advantages over the number line for recognition and discrimination of fractions representations. However, the incorporation of area models into instruction on fractions arithmetic provided no benefit over instruction with symbolic notation alone.

Keywords: Mathematics; Fractions; Visual Representations; Learning.

Introduction

Fractions are an important part of elementary school mathematics. Standard curricula, such as the Common Core Curriculum (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; see also Ohio's Learning Standards for Mathematics, Ohio Department of Education, 2017), state that students should be able to recognize and use visual models of fractions such as area models (e.g. fractions as proportions of circles) and the number line (i.e. fractions as locations on the real number line).

The number line is a particularly important visual representation. It is a generic representation of the real numbers that visually captures the density of the real numbers. The number line is part of standard curriculum from elementary through high school ((National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010).

There is recent evidence of an advantage of the number line representation over a common area model representation for improving students' fraction magnitude understanding (Hamdan & Gunderson, 2016). Elementary students who were given training to represent fractions on the number line showed improved fraction magnitude knowledge in comparison to students who were given training to represent fractions as proportions of circles.

However, it is not clear exactly how visual representations should be incorporated into instruction to best promote more advanced aspects of fraction knowledge, such as arithmetic. Should instruction with visual representations focus on mapping standard symbolic fractions to and from visual representations, while teaching topics like fraction arithmetic primarily with standard symbols? Or should visual models be presented along with standard symbolic notation when teaching fraction arithmetic? Moreover, what types of visual models should be used? These are important questions because students have difficulty acquiring various aspects of fractions, including magnitude and arithmetic (Braithwaite, Tian, & Siegler, 2017; Gabriel, Coché, Szucs, Carette, Rey, & Content, 2013; Kouba, Zawojewski, & Strutchens, 1997; Lortie-Forgues, Tian, & Siegler, 2015).

Intuitively, it may seem that instruction on topics such as fraction arithmetic should include visual representations because they perceptually ground the operands and result of operation. For example, the addition $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$ may be more understandable to students when they can see a visual analog that shows one $\frac{1}{3}$ part plus another $\frac{1}{3}$ part yields two $\frac{1}{3}$ parts, which is $\frac{2}{3}$ (see Figure 1). However, the ultimate goal of learning fractions is to have knowledge of fraction magnitude and arithmetic procedures that can be applied to situations in absence of the visual representations that may have been used during instruction.

The inclusion of visual representations along with standard symbolic notation during instruction adds additional information to which students must attend. It is unclear whether young students have the attentional capacity to attend to both the visual representation and the standard symbols. The added visual representation may divert attention from the symbolic notation and consequently hinder learning. Previous research has shown that the inclusion of extraneous, often irrelevant information in visual representations can hinder learning, transfer, and spontaneous responding on various mathematical tasks including basic fraction tasks (Kaminski & Sloutsky, 2011, 2012, 2013; Kaminski, Sloutsky, & Heckler, 2008, 2013; McNeil, Uttal, Jarvin, & Sternberg, 2009; Mix, 1999; Son, Smith, & Goldstone, 2011). For example, teaching children to label proportions with fractions using collections of colorful objects (e.g. flowers) hindered learning in comparison to using collections of simple monochromatic circles (Kaminski & Sloutsky, 2012).

However, simple visual representations such as the number line and simple area models communicate no glaringly irrelevant information (such as irrelevant information communicated by collections of colorful flowers to represent fractions) and little extraneous information. Moreover, perceptual information communicated by such visual representations is correlated with the mathematical relations. For example, the size of shaded proportions of circles is correlated with the magnitude of fractions that equal the proportions. Similarly, the absolute magnitude of real numbers is correlated with their distance from 0. As such, providing students with a simple, generic visual representation may reduce demands on working memory (Zhang & Norman, 1995) freeing up resources for learning the new information and procedures.

If students have the attentional capacity to attend to both the visual representation and the symbol representation, then including visual representations in fraction instruction may reduce demands on working memory and facilitate learning in comparison to instruction using only symbolic notation. However, it is not clear that student can attend to both the visual and symbolic representations and integrate them into a coherent internal representation. In this case, inclusion of visual representations may place added demands on attention and as a result hinder learning. The goal of Experiment 1 was to examine the effects on fraction acquisition of incorporating visual representations into instruction. Elementary students' were given instruction on fractions with an area model (i.e. fractions as proportions of circles), with the number line, or with symbolic notation alone.

The concepts covered were aspects of basic fraction knowledge that are generally part of standard 4th grade curriculum (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; Ohio's Learning Standards for Mathematics, Ohio Department of Education, 2017). These topics include addition and subtraction of fractions with common denominators, fraction additive decomposition, mixed numbers, and magnitude comparison to 1 (see Table 1). Students had two sessions of instruction over two days and were tested after a one-week delay and after a one-month delay.

Experiment 1

Method

Participants Participants were 127 third-grade students from suburban, small town, and rural schools in Ohio (57 girls and 70 boys, M = 9.2 years, SD = .78 years).

Materials and Design Participation in the experiment involved four sessions on different days: (1) prerequisite test of fraction labeling, pretest, and part 1 of instruction, (2) part 2 of instruction, (3) posttest 1, and (4) posttest 2. Groups of participants were randomly assigned to one of three conditions (Circle, Number Line, or Number Only) that specified the format of instruction.

Table 1: Components of Fraction Knowledge covered in Experiment 1.

Part 1	Part 2
1. Addition /subtraction of fractions with common denominators $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$	4. Fractions of form *m/n* = such that *m>n* implies that *m/n > 1* $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3} > 1$
2. Additive decomposition $\frac{2}{3} = \frac{1}{3} + \frac{1}{3}$	5. Conversion of improper fractions to mixed numbers $\frac{4}{3} = 1\frac{1}{3}$
3. Fractions of form *m/m* = 1 $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$	6. Addition /subtraction of mixed numbers $1\frac{1}{3} + 2\frac{1}{3}$ $= 1 + 2 + \frac{1}{3} + \frac{1}{3}$ $= 3 + \frac{2}{3} = 3\frac{2}{3}$

Participants were initially given a 10-question multiple-choice test of fraction labeling; each question presented a pictorial example of a proportion (e.g. a proportion of blue cars out of all cars shown as pictures) and participants were asked to choose a fraction that describes the proportion. The purpose of this test was to insure that participants had prior knowledge of basic fractions that is prerequisite for learning the material covered in the instruction. The pretest and posttests consisted of 24 questions: 9 open-response fraction addition or subtraction problems, 3 open-response fraction decomposition questions, 3 multiple-choice magnitude comparison questions, 3 improper fraction/ mixed number conversion questions, 3 mixed number arithmetic questions, and 3 fraction word problems. Identical pretest and posttests were administered to participants in all conditions. Pretest and posttest 1 were identical; posttest 2 was isomorphic to the pretest and posttest 1, but consisted of novel questions. The following are example question from the pretest and posttest1.

1) $\frac{2}{5} + \frac{2}{5} =$ _____
2) $\frac{5}{6} - \frac{3}{6} =$ _____
3) $\frac{3}{5} =$ ____ + ____ + ____
4) Which fraction below is <u>larger</u> than 1?
   $\frac{3}{3}$ $\frac{3}{4}$ $\frac{2}{3}$ $\frac{3}{2}$
5) Write a mixed number that is equal to the fraction $\frac{5}{3}$
6) Suppose you walk $\frac{3}{10}$ of a mile to your friend's house, and then you walk $\frac{5}{10}$ of a mile to school. How far did you walk altogether?

Instruction was designed to be similar to classroom instruction. Information was presented to groups of students as a PowerPoint presentation by an experimenter. Instruction was completely isomorphic across conditions. In all conditions, fractions, operations, and equations were presented in standard notation. The difference between conditions was the inclusion of visual representations (see Figure 1). In the Circle condition, fractions were represented in standard symbolic format and also as proportions of circles. In the Number Line condition, fractions were represented as standard symbols and as locations on the number line. In the Number Only condition, fractions were only represented as standard symbols.

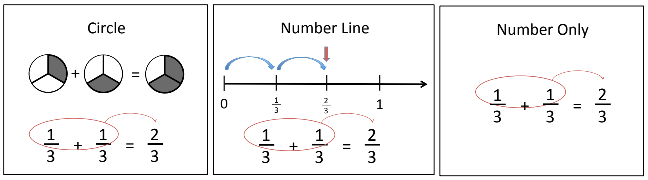

Figure 1: Example from instruction in each of the three conditions.

Part 1 of instruction covered addition and subtraction of fractions with common denominators and additive decomposition (see Table 1). Part 1 consisted of explicit explanations, three examples and fifteen multiple-choice questions with feedback. Following instruction participants were given a 12-question open-response test of learning. Part 2 of instruction covered magnitude comparison of fractions to 1, mixed numbers, and addition and subtraction of mixed numbers (see Table 1). Part 2 consisted of explicit explanations, four examples, and 23 multiple-choice questions with feedback. Instruction was followed by a 16-question test of learning consisting of six multiple-choice magnitude comparison questions, four improper fraction/ mixed number conversions, and six mixed number arithmetic questions. Tests of learning were in the format of instruction. In the Circle condition, questions showed circle representations of fractions along with the standard symbolic notation. In the Number line condition, questions showed number line representations of fractions along with the standard notation. In the Number only condition, questions were presented only as standard notation.

Procedure Instruction and testing were presented to groups of participants in classrooms at their schools. A female experimenter presented PowerPoint slides with the instruction on days 1 and 2. The verbal instruction was scripted and analogous across conditions. Parts 1 and 2 of instruction took approximately 30 minutes each. Participants were given paper test booklets for fraction labeling test, pretest, tests of learning, and posttests. They were asked to write their answer for each question.

On Day 1, participants were given the prerequisite test of fraction labeling, the pretest, part 1 of instruction, followed by the part 1 test of learning. Day 2 occurred two days after Day 1 and presented part 2 of instruction and the part 2 test of learning. On Day 3, one week after Day 2, participants were given posttest 1. Posttest 2 was given on Day 4, approximately one month after Day 2.

Results

Eight participants (one Circle, five Number Line, and two Number Only) were removed from the analysis because they missed one day of instruction. Fourteen participants (five Circle, six Number Line, and three Number Only) were also excluded from the analysis because they scored 60% or less on the prerequisite test of fraction labeling. Additionally, six participants (two from each condition) were removed for scoring more than 2.5 standard deviations below the mean learning score of participants in their condition. Analysis included 99 participants (32 Circle, 35, Number Line, and 32 Number Only).

Figure 1 presents mean accuracy across conditions on the tests of learning, posttest 1, and posttest 2 split by content parts 1 and 2.

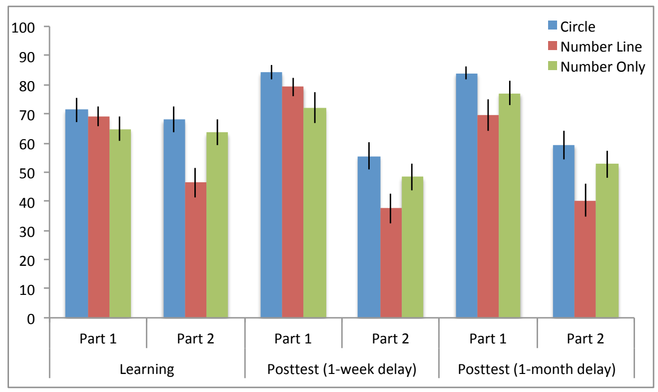

Figure 2: Mean Accuracy (% correct) on Tests of Learning and Posttests in Experiment 1. Error bars represent standard error of the mean.

Participants in all conditions learned. There were significant differences in pretest and posttest 1 scores, paired sample t-test, *t*s > 6.03, *p*s < .001 ($M_{pretest}$= 39.8%, *SD* =34.9% vs. $M_{posttest1}$= 73.4%, *SD* =15.7% in the Circle condition; $M_{pretest}$= 26.8%, *SD* =27.6% vs. $M_{posttest1}$= 63.6%, *SD* =19.2% in the Number Line condition; $M_{pretest}$= 29.2%, *SD* =19.7% vs. $M_{posttest1}$= 63.2%, *SD* =21.0% in the Number Only condition). Also, the open-response tests of learning parts 1 and 2 were relatively high (see Figure 2). In addition, participants retained their acquired fraction knowledge over the one-month delay. Posttest 2 scores (see Figure 2) were significantly above pretest scores, paired sample t-test, *t*s > 6.69, *p*s < .001.

While participants in all conditions learned, there were differences across conditions on the learning part 2 scores, ANCOVA with pretest scores as a covariate, $F(2,95) = 4.82$, $p < .02$, $\eta_p^2 = .09$. Learning part 2 scores were lower in the Number Line condition than those in the Circle condition and those in the Number Only condition, *p*s <.02. There were no differences in learning part 2 scores in the Circle and Number Only conditions, $p > .63$. No significant differences across condition in learning part 1 scores were found, ANCOVA with pretest scores as a covariate, $F(2,95) = .56$, $p > .57$.

Participants in the Number Line Condition also tended to score lower than those in the other conditions on both posttest 1 and posttest 2. There were differences between conditions on part 2 scores on both posttests (see Figure 2), ANCOVA with pretest part 1 scores as a covariate, $F(2,95)$s > 2.80, *ps*

= .065, η_p^2s = .06. Both posttest part 2 scores were lower in the Number Line condition than those in the Circle (ps < .05) and Number Only conditions (ps < .09). There were moderate differences between conditions on the part 1 scores on both posttests, ANCOVA with pretest part 1 scores as a covariate, $F(2,95)$s > 2.0, $ps < .11$, $\eta_p^2 = .06$. Participants in the Number Line condition also scored lower than those in the Circle condition on part 1 of the 1-month delayed posttest, ANCOVA with pretest part 1 scores as a covariate, $F(2,95) = 4.44$ $p < .04$, $\eta_p^2 = .06$.

The results of Experiment 1 suggest that the inclusion of visual representations in instruction provided no advantage over instruction with symbols alone. Specifically, learning and posttests scores were no different in the Circle and Number Only conditions, and scores were generally lower in the Number Line condition than in the Number Only condition.

In addition, instruction with the number line resulted in lower learning and posttest scores than instruction with circles, particularly on part 2 of the tested content which involved fractions greater than 1. This finding is somewhat surprising because the number line is a coherent model of the real numbers that can accommodate fractions less than 1 and fractions greater than 1 in a single representation. The number line also avoids ambiguity that can occur when representing fractions greater than 1 with area models. For example, using a circular area model, $\frac{3}{2}$ can be represented as three shaded halves of two circles, but this representation could also be interpreted as $\frac{3}{4}$. Once students correctly learn the number line, there is no ambiguity about the location of $\frac{3}{2}$.

Better performance in the Circle condition may be due to easier processing of the visual proportion than the position on a number line. The proportions of circles provide students with two dimensions of perceptual information, while the number line is only one-dimensional. Therefore, it may be easier for students to discriminate differences in proportions and recognize proportions in the circle representation than on the number line.

The goal of Experiment 2 was to examine the effects of a circular area model versus the number line on students' ability to recognize, discriminate, and construct representations of fractions. Participants were third and fourth grade students who, according to standard curricula, are in the course of acquiring the fraction topics considered in Experiment 1, and sixth grade students, who should have a more developed knowledge of fractions.

Experiment 2

Method

Participants Participants were 100 students from suburban and small town schools in Ohio, 24 third-grade students (11 girls and 13 boys, *M* = 9.4 years, *SD* = .43 years), 26 fourth graders (13 girls and 13 boys, *M* = 9.8 years, *SD* = .57 years), and 50 sixth graders (25 girls and 25 boys, *M* = 12.2 years, *SD* = .60 years).

Materials and Design The experiment had a 2 (question format: circle and number line) by 3 (question type: recognition, discrimination, and construction) within-subjects design. Participants were given a set of 60 questions, ten of each format/type category. The recognition questions were designed to test participants' ability to recognize proportions and label them with fractions. These questions presented proportions as circles or locations on the number line; participants were asked to write the fraction that described the proportion shown (circle recognition questions) or write the fraction that is located where the arrow pointed on the number line (number line recognition questions).

The discrimination questions were designed to test participants' ability to discriminate between representations of different proportions (circle discrimination questions) or different locations on the number line (number line discrimination questions). Each of these questions presented a fraction along with four different visual representations (of the same format). For the circle questions, participants were asked to choose the circle that showed a proportion that equals the fraction. For the number line questions, participants were asked to choose the number line that has a red arrow pointing to the location of the fraction. The response options included the following: (1) the correct response, (2) a visual representation corresponding to a fraction with a correct numerator, but incorrect denominator as the given fraction, (3) a visual representation corresponding to a fraction with a correct denominator, but incorrect numerator, and (4) a visual representation matching the numerator and denominator of the given fraction, but with unequal parts. Locations of the different types of responses were counterbalanced across questions.

Each construction question presented a fraction and participants were asked to make a representation of the fraction. For the circle construction questions, they were asked to draw a circle (or more than one circle) with a shaded proportion that equals the fraction. For the number line construction questions, they were asked to mark the location of the fraction on the number line with an arrow.

Questions had denominators of 2, 3, 4, or 5. For each question format/type, seven of the ten questions involved fractions less than 1 and three involved fractions greater than 1. Different fractions were used for the recognition, discriminations, and construction questions. However, the same fractions were used for both the circle and number line formats.

Procedure. Participants were tested in groups in their classrooms by a female experimenter. Each participant was given a paper booklet of test questions. Questions were presented in six blocks in the following order: recognition circle, recognition number line, discrimination circle, discrimination number line, construction circle, construction number line. Prior to each block, the experimenter read the

Table 2: Mean Accuracy (% correct) on Fraction Questions of Experiment 2. Standard deviations are in parentheses.

	3rd Grade		4th Grade		6th Grade	
Question Type	Circle	Number Line	Circle	Number Line	Circle	Number Line
Recognition						
fractions <1	83.9 (26.7)	27.3 (27.6) [a]	92.3 (15.2)	30.8 (26.4) [a]	95.1 (12.4)	80.2 (33.5) [a]
fractions >1	40.6 (42.6)	7.3 (20.0) [a]	34.6 (40.5)	6.4 (16.4) [a]	92.2 (24.3)	72.3 (41.3) [b]
Discrimination						
fractions <1	92.6 (18.7)	59.6 (29.4) [a]	91.8 (10.8)	59.9 (35.7) [a]	93.9 (16.5)	90.6 (19.6)
fractions >1	58.0 (41.7)	43.5 (29.2)	52.6 (44.4)	47.4 (36.7)	90.8 (22.7)	87.2 (26.5)
Construction						
fractions <1	60.9 (28.0)	36.7 (34.9) [b]	49.5 (19.5)	40.7 (35.8)	66.0 (23.0)	75.4 (28.4) [b]
fractions >1	31.9 (36.9)	18.8 (28.1) [b]	28.2 (30.8)	18.0 (33.0)	68.1 (34.0)	66.0 (41.4)

[a] significant differences between circle and number line questions at $p < .01$
[b] significant differences between circle and number line questions at $p < .05$

instructions and gave one example. Participants proceeded through the questions at their own pace.

Results

Three participants (one third grader and two sixth graders) were removed from the analysis for scoring more than three standard deviations below the mean of their age group on one or more question format/type categories.

Table 2 presents mean accuracy across grade level and question category. Question scores were also split for fraction less than and greater than 1. Scores were submitted to a one-way (grade level) analysis of variance followed by post hoc Bonferroni tests. The results indicate that grade level had a significant effect on all scores, $F(2,93)$s > 3.23, ps < .05, η_p^2s > .07, except circle discrimination for fractions less than 1, $F(2,93) = .17$, $p = .84$. Participants at all grade levels were above 90% accurate on circle discrimination questions for fractions less than 1. Sixth graders were more accurate than third and fourth graders, post hoc Bonferroni ps <.01 on all but the following questions. Scores of sixth graders were not significantly better than those of fourth graders on circle recognition questions for fractions less than 1, $p = 1.00$. Also, scores of sixth graders were not significantly better than those of third graders on circle construction questions for fractions less than 1, $p = 1.00$. No significant differences in scores were found between third and fourth graders, ps >.28.

Significant differences were also found as a function of the representation format. Third, fourth, and sixth grade participants were substantially more accurate on circle recognition questions than on number line recognition questions, paired sample t-test, ts > 3.03, ps < .01. Third and fourth graders were also more accurate on circle discrimination questions than on number line discrimination questions, paired sample t-test, ts > 3.62, ps < .01. Third graders were more accurate on circle construction questions less than 1 than number line construction questions less than 1, paired sample t-test, $t(22) = 2.40$, $p < .03$. However, the patter reversed for sixth graders who were more accurate on number line construction questions less than 1 than on the circle construction questions less than 1, paired sample t-test, $t(46) = 2.13$, $p < .04$.

Overall, the results show that construction of correct visual representations is more difficult for all participants than recognition and discrimination of visual representations. Also across all grade levels, participants were very accurate recognizing and discriminating circular area representations of fractions less than 1. However, for third and fourth graders, accuracy was much lower for number line representations than for circle representations. Third and fourth graders also appear to have had difficulty recognizing and discriminating both types of visual representation for fractions greater than 1.

Discussion

The present findings demonstrated that incorporating circular area models or number line representations into instruction on fraction arithmetic provided no advantage over instruction involving only symbolic notation. The results of Experiments 1 and 2, taken together, suggest that circles and number line representations may have affected fraction arithmetic learning in different ways. Elementary students appear to be very familiar with circle representation of fractions, perhaps from an emphasis on such models in school. Circular area models appear to have benefits over number line representations for recognition and discrimination of fractions less than 1. Therefore the inclusion of circle representations did not appear to hinder participants' learning of basic fraction addition (part 1 of instruction). However, the results of Experiment 2 show that third and fourth graders had difficulty interpreting circle representations for fractions greater than 1. Therefore it is unlikely that the circle representation would provide an advantage for learning fractions greater than 1 (part 2). Moreover, the presence of the circle representation may have diverted attention from symbols. The results of Experiment 2 also suggest that third and fourth graders struggle with using and interpreting number line representations of fractions, both less than and greater than 1. Learning in the Number Line condition of Experiment 1 was likely lower than that in the Circle and Number Only conditions because participants could not easily connect number line representations to fractions. The presence of the number line may have diverted attention from the symbols while providing no benefit.

The implications of the present study are not to avoid using visual representations of fractions. Rather, the present findings show the limitations of circular area models of fractions; they may be suitable for early introduction of basic fraction labeling, but they appear to provide little or no benefit for instruction on fraction arithmetic and fractions greater than 1.

Mathematical reasoning may involve the perceptual system, and therefore mathematics learning may benefit from carefully designed integration of visual representations (Goldstone, Landy, & Son, 2010; Marghetis, Landy, & Goldstone, 2016). The number line is a likely candidate to effectively incorporate into instruction because it well represents the real number system. The number line represents all real numbers including negative, rational, and irrational numbers. Furthermore, the Cartesian plane is constructed from two real number lines. Therefore students need to become comfortable with using and interpreting it. However, because the number line is 1-dimensional, it may be more difficult for students to learn to recognize and discriminate fractions on the number line than fractions as proportions of 2-dimensional area models, as suggested by the results of Experiment 2. While the extent to which number line representations should be integrated into fraction arithmetic learning remains unclear, it is clear that students need more instruction and practice with the number line.

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