HOW DO CHILDREN THINK ABOUT NUMBERS?
Let us count the ways

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Abstract

Children use a wide variety of strategies when dealing with numbers. This diversity has previously been approached through a study of "bugs", strategies that are flawed in various ways. However, we have uncovered a variety of valid strategies which different children bring to bear when dealing with numbers in simple addition and subtraction tasks. By observing these strategies in computer-based estimation games, we have identified some of the components of expert mathematical knowledge.

Introduction

What makes a person an expert mathematical problem solver? The uniformity of the current mathematics curriculum and many of the current psychological models of mathematical problem solving imply that there is one way to represent numbers and one set of processes for solving a given mathematical problem.

Our observations of children dealing with a computer math microworld have highlighted instead the diversity of approaches they bring to bear. Some of the approaches are "buggy", such that the results they produce are systematically incorrect. However, others are quite different from the standard approaches children are taught in school, yet are equally valid representations of number and numerical operations.

Shark Shooting

One computer microworld we have developed and used is called "Moving Shark". In this world, children see the fin of a shark on the video screen, then the fin disappears beneath the water, leaving a set of ripples. The children see a "digital readout" at the top of the screen which tells how far the shark has moved underwater. They are to type in a number saying where to throw a harpoon to hit the hidden shark.

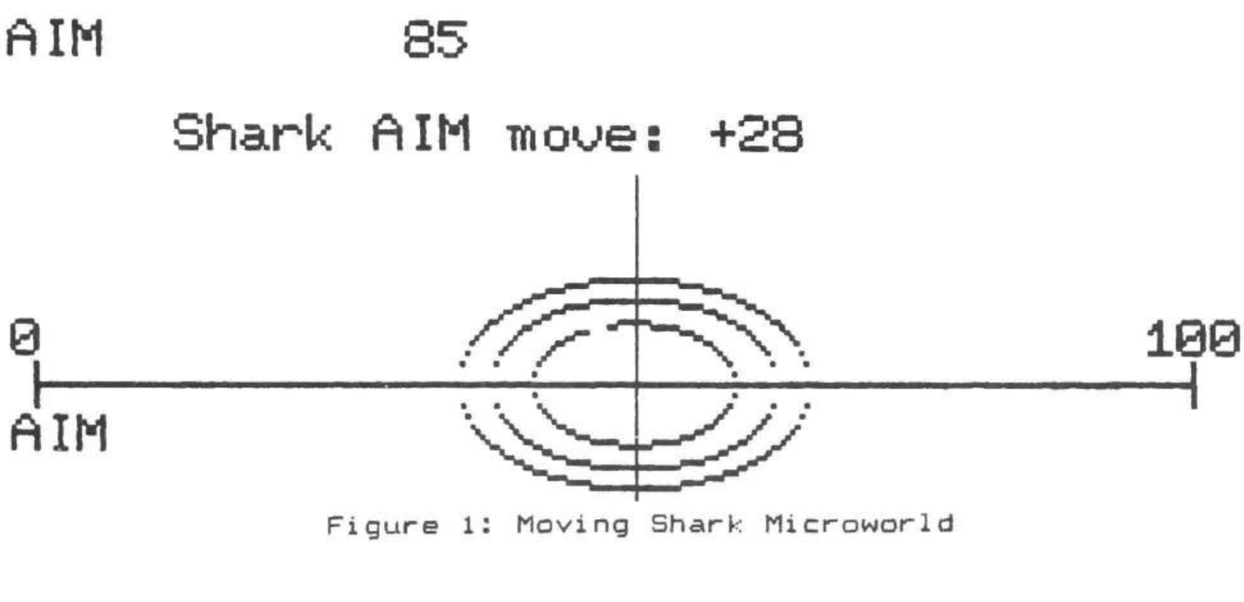

Figure 1: Moving Shark Microworld

This example is an addition task: the player has to add the movement amount to the initial position in order to compute the new position. One common approach, which most adults employ, goes like this: 1) estimate the number specifying the initial position (in this case, perhaps 55), 2) add to that number the movement number, and 3) type in the sum.

This approach is so natural and obvious that we might dismiss all others as "buggy" unless we kept an open mind. Another approach, which we've seen children apply to this problem is: 1) estimate the length of movement represented by the movement number, 2) apply that length on the screen to the initial location, 3) estimate the number specifying the new position (in this case, perhaps 85).

In this little microworld, there are two very different representations of numbers: as decimal digits and as position on a number line. The goals embedded in the game require dealing with both of these ways of thinking about the same thing. In fact, simpler versions of this game mainly require translating from position on a number line to digits (a game we developed called Harpoon) or from digits to position on a number line (a game we developed called Sonar). We have found that children differ on their facility in each of these two ways of thinking about number.

With each way of thinking about numbers, there are corresponding numerical processes for manipulating numbers. So, addition of digits is the multi-step right-to-left symbolic algorithm that we all learned as "addition". However, addition of two numbers can be carried out by representing the two numbers as positions on a number line, then translating the line length representing one number to the end of the other. Galton (1907) reported this technique as the way that one of his subjects standardly did addition and subtraction. Similar estimation techniques have been discovered with subjects doing multiplication (Lopes, 1976).

What we have discovered is a variety of "non-standard" (but mathematically valid) ways of representing numbers and numerical operations. Are these non-standard ways just curiosities, illustrating the wonderful perversity of human nature in bucking twelve or more years of the best efforts of the educational establishment?

We find these variations non-trivial for two important reasons:

1) They are consistent with close observations of the ways that adults actually deal with numbers.
2) They may hold the key to the nature of mathematical expertise.

Mathematical expertise

Lave and her associates have been carrying out careful studies of how "just plain folks" in Southern California deal with numbers in their everyday lives (Lave, Murtaugh, & de la Rocha, 1983). For example, they observed housewives doing their grocery shopping. They found little use of the standard multi-place algorithms learned in school. Instead, they found an extensive use of a diverse set of specialized estimation strategies. In many cases, the people were somewhat embarrassed to have these specialized strategies observed, since they felt they "should" use the standard algorithms.

These findings have been reinforced by the studies of Scribner, observing the arithmetic of warehouse workers (Scribner, 1983). Again, these workers use a diverse set of specialized approaches for performing arithmetic operations in service of their work. Before we dismiss these findings with a cavalier conclusion that people are lazy, let us examine some studies of expert physics problem solvers.

Larkin and her associates (Larkin, McDermott, Simon, & Simon, 1980) have been studying the ways that experts and novices differ in solving physics problems. When a physics teacher demonstrates in a class how to solve a problem, s/he often describes the problem, then immediately writes down a set of equations, which, when applied in the correct order, lead to the desired solution. Novices, taking this as a model, start solving the problems given to them by writing equations. They then get stuck in a quagmire of details.

A close examination of expert solvers solving unfamiliar problems shows that there are several preliminary steps that they take which are not often revealed to novices. Experts initially represent the problem to themselves in a qualitative, global way, often by drawing a diagram that abstracts out the major factors. At this point, they can then classify the problem as being an instance of some general type. These preliminary steps then guide the experts in writing the appropriate equations and using them in an order that leads directly to a solution.

The hallmark of an expert is the ability to think about a problem in their domain of expertise in multiple ways, and to draw upon these multiple points of view in a sequence that leads straightforwardly to a solution. This kind of "orchestration" of multiple representations of the problem elements is just what we find distinguishes experts in our estimation microworld from novices.

When a player misses the shark, s/he gets several kinds of information as feedback. First of all, the harpoon "splashes" into the water, leaving a set of ripples to mark the spot. Secondly, the player gets textual feedback, either "Smaller", "Bigger", or "Right On!". Thirdly, an arrow appears that indicates which way to go in making the next guess.

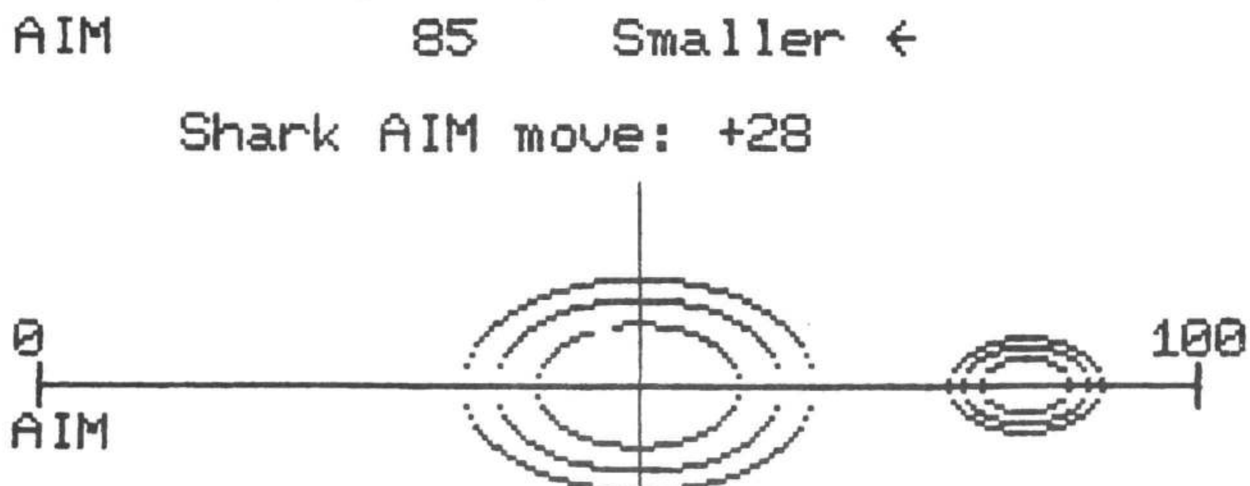

Figure 2: Moving Shark, First Guess

Some of the novice players only pay attention to the textual feedback or to the arrow feedback, effectively turning the game into a bisection strategy game like Hurkle. Other players use this same feedback, but instead of bisection, they increment or decrement their guess by a fixed amount (often some multiple of ten). Still other players use the "splash" feedback to calibrate the previous location of the shark, and then reevaluate their addition estimation. Expert players often use several strategies, within games or between games. Within a game, some experts use different strategies for the first guess and for subsequent guesses. Some even use multiple strategies within a single harpoon throw decision.

Theoretical implications

What are the theoretical implications of this view of expertise? The notion of coordinating multiple simultaneous approaches to a problem is very much in the spirit of the current work on parallel distributed processing (Hinton & Anderson, 1981). But what kind of representation can we have for numbers that allows us to capture very different ways of thinking about numbers and their representation? Most cognitive models have proposed relatively limited representations for numbers, often finessing the entire issue by representing numbers by real numbers. Others have proposed multiple, independent representations for the different aspects. Shepard, Kilpatric & Cunningham (1975), for example, propose several independent multidimensional scaling spaces to represent the numbers less than 10.

We have proposed a "landmark representation" for continua (Levin, 1981; Hutchins & Levin, 1981). In this representation, there are a set of "landmark" concepts for particular discrete numbers. Any particular number is then represented by the differential "activation" of one or more of these landmarks. This framework allows a substantial diversity of representational types, each with its own set of landmark concepts. These different types are then coordinated by the interaction of the landmarks of the types when simultaneously activated.

The particular types of number manipulation processes (such as addition or subtraction) are cognitive processes that are defined to operate on one or more of these types of number landmarks. So, for example, the multi-place right-to-left digit-based algorithm is defined in terms of the landmarks of the digits; the quantity manipulation algorithm is defined in terms of the spatial landmarks for numbers; etc.

Now we have a way of thinking systematically about the diversity that seems to characterize expert functioning in rich environments. So what? Well, even at this preliminary stage, we can derive from this view some design principles for educational software. If expert functioning depends upon the coordination of many different points of view on a problem, then it may be valuable to present to learners different views of a problem simultaneously, so that they can learn how the different points of view coordinate with each other. This is a feature we have built into many of our educational tools and educational games, and this feature seems to have pedagogical power.

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