Fuzzy Implication Formation in Distributed Associative Memory

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An analysis is presented of the emergence of implicational relations within associative memory systems. Implication is first formulated within the framework of Zadeh's theory of approximate reasoning. In this framework, implication is seen to be a fuzzy relation holding between linguistic variables, that is, variables taking linguistic terms (e.g., "young", "very old") as values. The conditional expressions that obtain from this formulation may be naturally cast in terms of vectors and matrices representing the membership functions of the fuzzy sets that, in turn, represent the various linguistic terms and fuzzy relations. The resulting linear algebraic equations are shown to directly correspond to those that specify the operation of certain distributed associative connectionist memory systems. In terms of this correspondence, implication as a fuzzy relation can be seen to arise within the associative memory by means of the operation of standard unsupervised learning procedures. That is, implication emerges as a simple and direct result of experience with instances of events over which the implicational relationship applies. This is illustrated with an example of emergent implication in a natural coarsely coded sensory system. The percepts implied by sensory inputs in this example are seen to exhibit properties that have, in fact, been observed in the system in nature. Thus, the approach appears to have promise for accounting for the induction of implicational structures in cognitive systems.

The nature of implication is a contentious topic in the philosophy and psychology of reasoning. Much of the difficulty arises from attempts to make implication be truth functional and, thus, can be avoided by recognizing that, in most instances, conditional propositions are atomic. Nevertheless, like any other bit of knowledge, such propositions must arise from experience somehow. The present paper discusses how the induction of implication may be accomplished through self-organizing processes in a distributed associative memory.

Linguistic variables, fuzzy relations and the compositional rule of inference

It will be useful to consider the very general form of implication developed by Zadeh (1975) as part of a system for approximate reasoning. This system is based on the concept of linguistic variables. A linguistic variable takes on values that are words or phrases rather than numbers. An often cited example is that of the linguistic variable *age* which may have as values "old", "very old", "not very old", "young", etc. Each such value is actually a fuzzy set defined over the underlying base variable of years lived. That is, each linguistic value is fuzzy in that it may be appropriately applied to various years to different degrees. For any given underlying value (number of years), there will be some degree to which it belongs to the fuzzy set. Figure 1 gives illustrative membership or compatibility functions for "young", for "old" and for "very old" (see Oden, 1978, 1984).

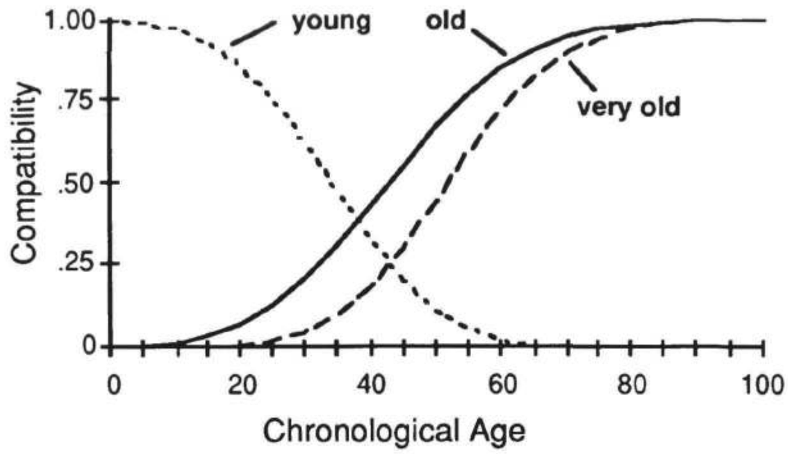


Fig. 1. Compatibility of chronological ages with various linguistic values.

Another way to view the notion of linguistic variables is by means of a hierarchical representation such as that shown in Figure 2.

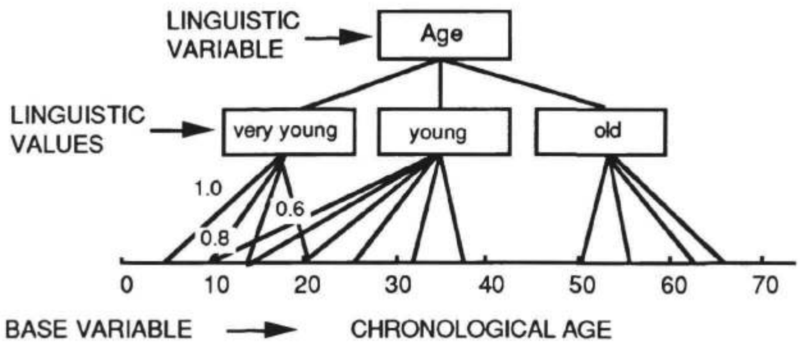


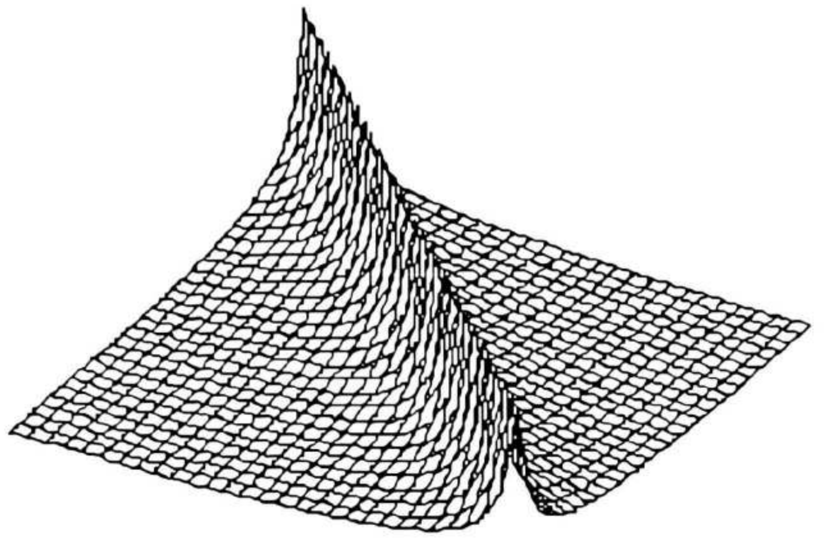
Fig. 2. Hierarchical structure of a linguistic variable.

This shows that the linguistic variable "age" can take on any of several linguistic terms as its value and that each term, such as "very young", is a fuzzy set containing each of the possible values of the base variable (chronological age) to various degrees. Again, the degree of membership of a given chronological age in the fuzzy set "very young" can be thought of as the compatibility of that age with the concept of a very young person. It is sometimes useful to represent fuzzy sets in terms of the pairings of each base variable value with its associated compatibility value

$$\text{very young} = \{(5, 1.0), (10, 0.8) \ldots (120, 0)\},$$

which is often more compactly represented in vector notation, $A = [\ 1.0\ \ 0.8\ \ \ldots\ \ 0.0\ ]$. These representations require a discretized view of the base variable, but, of course, one can have as small a grain size as one wishes.

A second component of Zadeh's extension of implication involves the set-theoretic notion of relations holding between objects. For example, "less than" is a relation that holds between numbers. A *fuzzy* relation R holding between X and Y is a fuzzy subset of the Cartesian product $X \times Y$, which is often represented as a matrix having entries that are the respective membership degrees, denoted by $\mu_R(x,y)$. A useful way of visualizing a large complicated matrix, such as is required to represent a fuzzy relation, is to plot the matrix as a surface. Figure 3 illustrates a resemblance relation plotted as a surface and also in tabular form[1].



1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429
0.5000	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667
0.3333	0.5000	1.0000	0.5000	0.3333	0.2500	0.2000
0.2500	0.3333	0.5000	1.0000	0.5000	0.3333	0.2500
0.2000	0.2500	0.3333	0.5000	1.0000	0.5000	0.3333
0.1667	0.2000	0.2500	0.3333	0.5000	1.0000	0.5000
0.1429	0.1667	0.2000	0.2500	0.3333	0.5000	1.0000

Fig. 3. Relation $\mu_R(x,y)$ shown as a surface together with the upper left portion of its matrix.

Another way to represent a relation is as weighted graph or network, where $\mu_R(x,y)$ is taken to be the weight on the arc from each x to each y. When viewed in this fashion, one can imagine a complicated cascade of relations represented by a multilayered network architecture. The "glue" that joins relations is the composition operator denoted R o S. The fuzzy set-theoretic definition of composition is of a *union of intersections* over the membership functions $\mu_R(x,y)$ and $\mu_S(y,z)$. The operation is directly analogous to that of matrix multiplication in linear algebra.

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[1] A resemblance relation is defined as being reflexive and symmetric, that is, one for which $\mu_R(x,y) = 1$ and $\mu_R(x,y) = \mu_R(y,x)$. The membership function shown in Figure 1 is $(1 - |x - y|)^{-1}$.

Zadeh draws the connection between fuzzy implication and fuzzy relations in terms of forming inferences based upon linguistic variables. Implication serves as a primitive for a basic rule of inference rather than as a connective derived from a truth table. Consider the conditional expression IF A AND $(A \Rightarrow B)$ THEN B, which defines modus ponens in traditional logic,. Here we infer the truth of proposition B from the truth of A and the truth of the implication $A \Rightarrow B$. Zadeh (1975) proposed an extension of modus ponens, which he called the compositional rule of inference, that uses the composition operator as described in the previous section. The compositional rule of inference is defined as follows: Let X and Y be two universes of discourse with base variables x and y, respectively. Further, let R(x), R(x,y) and R(y) denote fuzzy sets in X, $X \times Y$ and Y. Then the *compositional rule of inference* asserts that the solution of R(x) = A together with R(x,y) = F is given by R(y) = A o F. In this sense, R(y) is inferred from the combination of the linguistic value A and the relation F. For example, for

$$A = \text{small} = [\,1 \quad 0.6 \quad 0.2 \quad 0\,]$$

and

$$F = \text{approximately equal} = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}$$

then $R(y) = A \circ F = \text{small} \circ \text{approximately equal}$

$$= [\,1.0 \quad 0.6 \quad 0.2 \quad 0\,] \circ \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}$$

$$= [\,1 \quad 0.6 \quad 0.5 \quad 0.2\,] = \text{more or less small.}$$

Distributed associative memory

In Zadeh's view, implication is a fuzzy relation and, thus, atomic rather than truth functional. The interpretation of implication as a primitive instead of as a connective, however, is still engineered rather than emergent. That is, the architect of the reasoning model describes the implication primitive in the form of a relation that reflects the designer's intuition. It is natural to wonder, however, how a particular implicational expression might arise naturally as a primitive term. One possible answer is based on the striking similarity of the compositional rule of inference to the distributed associative memory models proposed by Kohonen (1987). The linear recall problem as formulated by Kohonen is: what memory array allows one representation, b, to be recalled in response to another input representation, a, for a number of such associated a–b pairs? Kohonen formulates this issue in linear algebraic terms as: what is the matrix operator M by which a pattern b_k is obtained from a pattern a_k, i.e. $b_k = M \cdot a_k$? In the following discussion, the similarities of this question to compositional inference are explored.

For our purposes, associative memories are heteroassociative: they store associations of data from conjoint pairs: $(a_1 b_1), (a_2 b_2), \ldots (a_P b_P)$, where $a = [a_1,\ldots,a_N]^T$ and $b = [b_1,\ldots,b_N]^T$ are vectors from different N-dimensional vector spaces . These vectors may correspond to crisp sets from binary vector space or fuzzy sets. By using a simple correlational or Hebbian rule, the associations between patterns a and b can be incrementally stored as an optimal orthogonal projection operator M, the memory matrix[2]. To illustrate how self-organization in associative memory systems may effect emergent implication structures, we will examine how coarse coding can be considered to be an implicational process. This discussion will be specifically developed in terms of sensory coding in the auditory system; this is simply because relevant neurophysiological data is available to us and we have developed a filter model of the auditory periphery (Jenison, Kluender, Greenberg, and Rhode, 1990). However, it should be emphasized that this is just a choice of convenience; this analysis can equally well be applied to coarse coding in any sensory or conceptual domain.

The peripheral auditory system can be thought of as a spectrum analyzer of sorts. That is, in the auditory periphery, frequency components are separated out and the degree to which each component contributes to the overall complex of sound is transmitted to higher processing centers by a pathway of nerve fibers. This mapping from a physical acoustic source to representation in the auditory system is not crisp by any means. A simple model of the acoustic to auditory mapping (derived from Peterson and Boll,1983) is shown schematically in Figure 5. This illustrates the coarse coding of the acoustic signal in terms of 23 auditory channels spanning the frequency range from 0 to 4500Hz (here quantized into 128 frequencies). Each channel is represented in the figure by a receptive field or response area mapped to the base variable of frequency. This is directly analogous to our discussion of linguistic variables for which membership functions specify the meaning of each linguistic value that may be assigned to a linguistic variable.

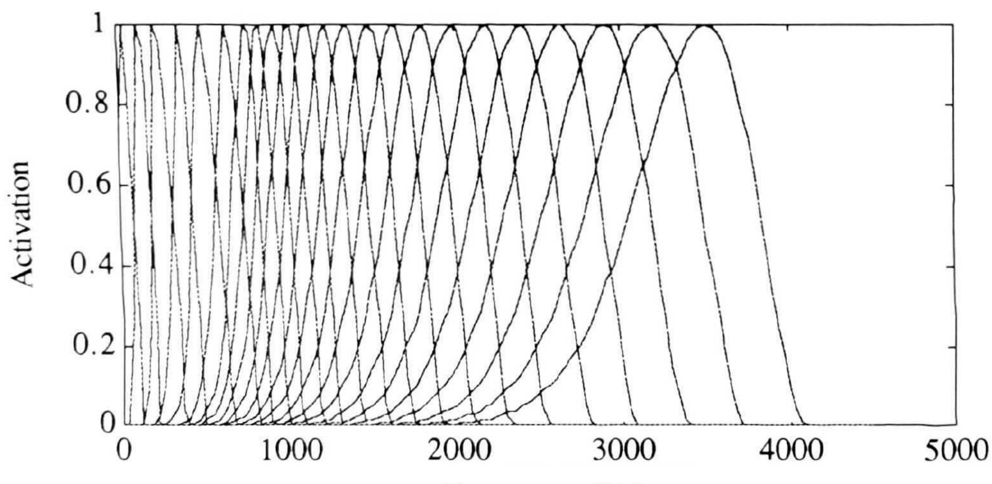

Fig. 5. Canonical auditory periphery response functions.

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[2]Training the network M on the conjoint association of a and b turns out to be equivalent to solving M directly, $M = B \cdot A^+$, where $^+$ denotes the pseudoinverse operator (Kohonen, 1987). $A = [a^{(1)},\ldots, a^{(P)}]$ and $B = [b^{(1)},\ldots, b^{(P)}]$ denote the pattern matrices, where a and b are column vectors.

If we view the auditory response functions as *linguistic values*, how might implications form such that a perception is induced at higher cognitive levels? The auditory response functions shown in Figure 5 can be cast in matrix notation, where each response area corresponds to a row in the matrix. The column vectors, therefore, correspond to the excitation pattern across the 23 nerve fibers generated by a particular frequency component. Similarly, the output of the system is a matrix which corresponds to the identity matrix. The identity matrix is representative of identification tags for each pattern, which is simply a set of unit vectors, uniquely tagging each tone (percept). The mapping of response patterns to unit vectors can be thought of as forming optimal tonotopic maps, given the coarse coding of the auditory periphery.

Through conjoint association, a matrix M forms as a function of Hebbian learning[3]. The derived matrix, which can now be thought of as a fuzzy relational instantiation of emergent implication, is shown in Figure 6. The surface described by the fuzzy relation corresponds to the connection strengths/compatibility values between the linguistic variable quantified by the response areas and the output auditory percepts.

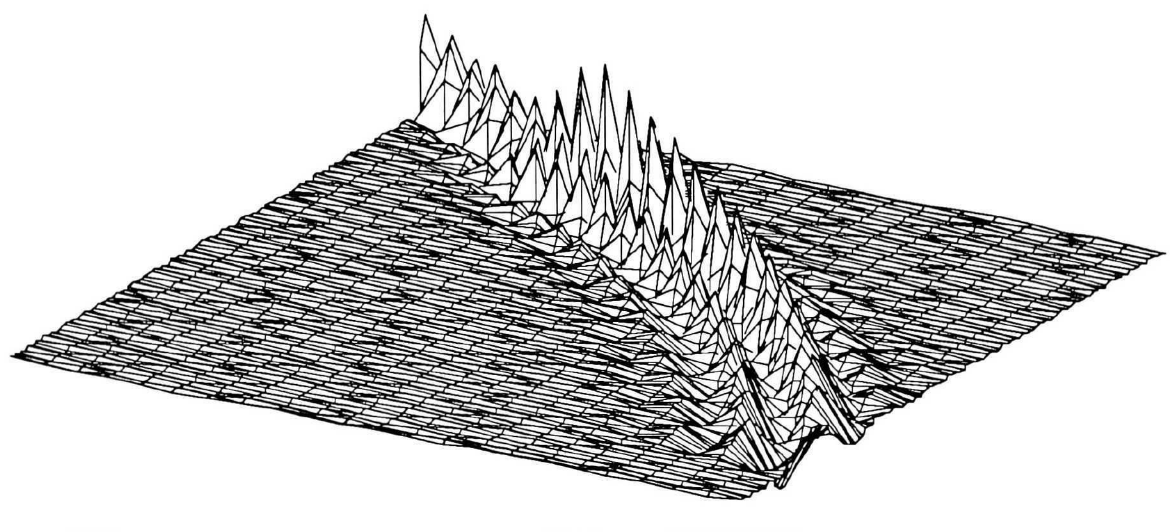

Fig. 6. Fuzzy relation derived from response function vector inputs and output identity matrix.

This exposition of implication in the auditory neural substrate has been strictly illustrative, but it is interesting to consider what this system might "infer" given some acoustic input. That is, what perceptual event will be taken to be implied by given sensory facts? Consider the composite input response function shown in Figure 7, representing the excitation pattern of three tones coded by the response areas in Figure 5.

[3]To achieve the desired independence of coding, one needs to augment a simple Hebbian feedforward net with mutual inhibition at the output nodes (Matsuoka, 1989). In our example, this was accomplished implicitly by means of the application of the pseudoinverse operator (see footnote 2 and the reference therein).

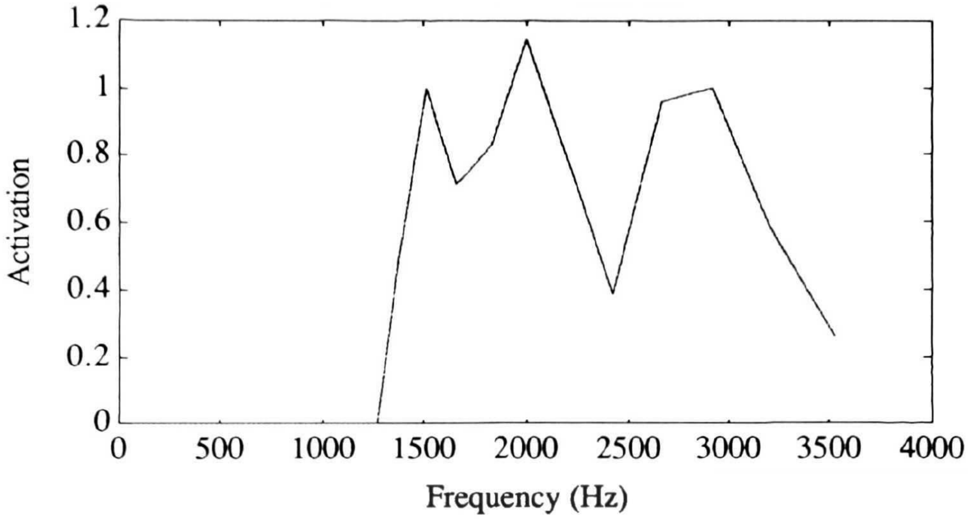

Fig. 7. Excitation pattern generated by 1500, 2000, and 2800 Hz.

The vector for this combination is the value assigned to the linguistic variable A. Now we apply the matrix multiplication form of the composition operator to A together with the fuzzy relation (emergent implication) matrix R shown in Figure 6 to infer some output B, shown in Figure 8. We observe that although the implied output from the linguistic variable A and the fuzzy relation has similarities to its antecedent, such as three prominent peaks, it exhibits certain interesting differences as well. The three frequency components are clearly more distinctive than initially suggested by the coarse coded representation .

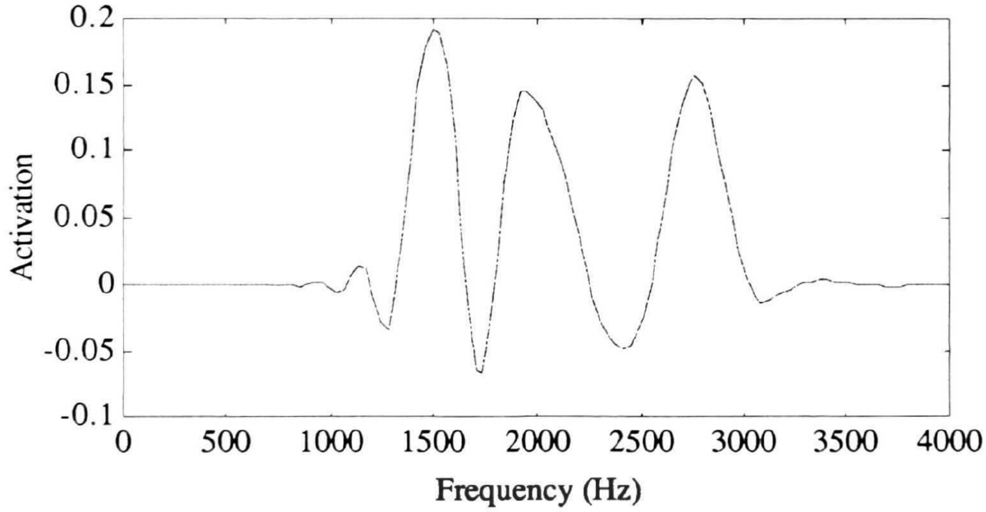

Fig. 8. Implied output vector from compositional inference.

The effect of compositional inference is one of integrating information that is coarse coded to produce an optimal output. In addition, the implied output exhibits a degree of lateral inhibition that suppresses neighboring percepts, as shown by the negative values in Figure 8. This phenomenon has often been described in the auditory system as well as other sensory systems (Sachs and Kiang, 1968) Thus, the implied percept may indeed have perceptual reality.

Conclusion

We have developed a simple account of emergent implication from unsupervised exposure to linguistic variables. The concepts used in constructing the framework of emergent implication were taken from Zadeh's notion of implication as a fuzzy relation and from connectionist models of associative memory. An application of this machinery to the peripheral auditory system illustrated how it may serve as a level of symbolic description linked by means of emergent implication to a subsymbolic description thereby achieving the kind of bistratal account advocated by Oden (1988).

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