Assessing Conceptual Understanding of Arithmetic Structure and Language

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Abstract

We contend that the primary role of an illustration or physical manipulable for teaching mathematics is to help the learner understand the language of the mathematics by providing the learner with a referential semantics. Having taught this to subjects, we address the question of how to assess their understanding. Problem-solving performance, we show, is insufficient by itself. An assessment of students' memory for the original problem statement, and their ability to use cues within the referential semantics is demonstrated as a potential method.

Fourth graders (n=24) solved word algebra problem after (a) training with a designed referential semantics from a computer tutor called the Planner, (b) training with symbolic manipulatives, or (c) receiving no training (control). Although pretest-posttest gains were only moderately better for the Planner group than the symbol group, the former showed reliably better ability to reconstruct the problem statements after a 5-day delay. A particular advantage for recall of algebraic relations (as compared to assignments) was evident. Mental representation of relations has been singled out as a major obstacle to successful word problem solving. The support that a well-designed referential semantics plays in the formation and retrieval of appropriate mental structures for problem solving are discussed, as are methods for assessing probiem comprehension and conceptual change.

Assessing Conceptual Understanding of Arithmetic Structure and Language

Research on the growth of the concepts of number and their operators has shown that while important segments of mathematical knowledge have their origins in everyday experience, learning that is wholly dependent on this approach can be limited (e.g., learning about negative numbers, or the symbols of mathematics notation). In the early stage of protoquantitative reasoning, children learn about non-quantified relations from talk and direct engagement with physical materials and illustrations (Resnick & Greeno, 1990). However, the meaning conveyed by manipulables (through their manipulation properties) and illustrations may not be sufficient to support the learning of vital pre-algebraic concepts and procedures (e.g., Resnick and Omanson, 1987). Ultimately, one must reason without immediate reference to counted or measured materials. Several researchers (Nesher, 1989; Schwarz, Kohn, & Resnick, in press; White, in press) propose the use of intermediate models for acquiring high level concepts that cannot be discovered informally (e.g., negative numbers). To be helpful, intermediate models must behave as the mathematical objects they exemplify do, and their functioning must be self-evident. Their construction is based on an epistemological analysis of the mathematical knowledge they exemplify. Competency stems from engagement in situations exemplified by the objects of a system and talk about those objects. Later, the child develops a mathematical language (i.e., algebra) to record and model actions in the system.

As in a natural language, the study of the semantics of arithmetic and algebraic symbols leads to a distinction between sense and reference (cf. Nesher, 1989; Ohlsson, 1987). The addition operator, for example, has several senses (Carpenter, Moser, & Romberg, 1982; Nesher & Katriel, 1977): Combining or joining two quantities, changing or increasing the value of a quantity, and comparing quantities. Different senses of subtraction and multiplication also exist.

Assessing understanding of arithmetic structures. In performing our assessment, we are asking, in the ideal, "What are the mental structures formed by students learning arithmetic and pre-algebra?" Although problem-solving performance is the most common measure, we believe that as a sole measure it is incomplete. Students' abilities to blindly apply mathematical procedures without regard for their applicability, meaning or the meaning of the solution are well-documented (e.g., Paige & Simon, 1966; Wertheimer, 1982/1945). Furthermore, because of the complexity of many problem-solving tasks (which include understanding the instructions, the problem statement, etc.) students can obtain wrong answers even when their understanding of the mathematics is high. Consequently, we propose to look at students' memory for the problems they solve in addition to their performance. Memory for problems taps directly into students' mental representations for the problems and provides a measure of their problem comprehension in a manner similar to measures of reading comprehension (Dellarosa et al., 1985). Memory for problems also serves as an effective method for uncovering students' problem-solving difficulties. In Mayer's (1982) study, for example, the impoverished memory that students had for relational statements found in word algebra problems was related directly to their problem-solving difficulties.

Research applying reading comprehension to mathematics problem solving has shown that word problem comprehension can be understood within the general theory of discourse processing (e.g., Cummins et al., 1988; Kintsch & Greeno, 1985).

To evaluate students' problem comprehension, their memory for a problem is compared with the original problem text. The comparison is performed at a propositional level (though the propositions are at a coarse level, as used by Mayer, 1982), as this level of analysis has been shown to best capture the regularities of a reader's text representation (e.g., van Dijk & Kintsch). Recall is first prompted by the title of a problem. If the recall protocol is only a partial representation of the original problem statement, then a prompt is given at the level of the underlying problem model. Recently, problem comprehension has been described as the extent to which a solver's representation for a problem situation (the situation model) is related to the conceptual relations of the problem solution (the problem model, Nathan et al., 1992). The extent to which problem model-based cues help subjects to reconstruct the original problem situation will be an added measure of problem comprehension.

Empirical Evaluation.

The experimental group of greatest concern was taught to use The Planner, a computer-based tutor which provides intermediate models that behave as the mathematical objects they exemplify. It presents graphical depictions of trains which represent quantities and loading/unloading machines which operate on those quantities and capture for the student the different meanings of the basic operations; i.e., the referential semantics. Students learn to assemble chains of machines which operate on a train and convey the many different senses of an operator, changing its length (+Change), merging trains (Combine), or comparing lengths of trains (Compare). Problems are numerically solved when The Planner runs the train(s) through a series of machines, thus executing an expression.

Method

Fourth grade students (n=24) from two inner-city parochial schools, matched on their mathematical problem-solving abilities, participated in a pretest-posttest control group design. In session 1, subjects took a pretest which contained one addition problem (of the form ?+a=b), a multiplication problem, and a complex, multi-step addition problem. The planner group (12 subjects) then met in pairs and learned over the next three experimental sessions (sessions 2-4) to specify Planner objects and manipulate them as a means to model and solve word algebra problems. The symbol group (6 subjects) received analogous training, with manipulable pads containing the set of necessary algebraic symbols replacing the Planner objects (see Figure 1). The control group (6 subjects) took the pretest and posttest but received no training. On the fifth session, all subjects took a posttest which was identical to the pretest. On session 6 (5 days after the pretest), subjects in each of the groups were asked to recall the test word problems. Pre-determined cues were provided as memory aids for subjects to complete a partial recall. Cue 1 was the story problem title. Cue 2 presented subjects with an ordered series of uninstantiated (i.e., value-less) machines (planner group), or a series of pads with values missing but operators and relations present (symbol and control group). Cue 3 was a full representation of the problem in the form used in either the planner training or the symbol training (for both the symbol and control groups). If recollection was still incomplete, the story was read to the student aloud. After recall, students solved the problem. Further details are presented in Schwarz et al (in press).

Results

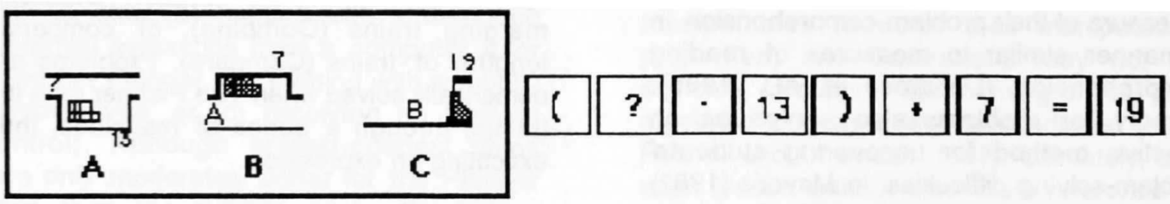
Problem-solving performance. Figure 2 shows the average number of problems correctly solved by each group at the pretest, the posttest, and after problem recall. Subjects in the two experimental groups performed consistently higher than control subjects on the posttest and after recall, with subjects in the planner group performing slightly better than those in the symbol group.

Problem statement recall. From the problem solving results it would appear that no notable underlying differences exist between planner and symbol subjects. Analyses of subjects' recall after a five-day delay, however, show important differences in conceptual formation for the solved problems. Figure 3 shows the proportional recall for all of the 39 propositions of the original three texts after receiving each cue (all subjects received cue 1, the story title). Recall is necessarily monotonically increasing as cues are given. Cues aided subjects in all of the groups. Planner users clearly had superior memory for the problems at every level of cue.

Of particular interest in these data are the relationship between control group subjects and symbol users on the one hand, and symbol and Planner users on the other.

Figure 2 showed marked differences in the problem-solving performance of symbol users and control subjects. After presentation of the titles (cue 1), however, there is no distinction in their relative macro-level based access of the original story information.

Planner users, however, have access to more story information, recalling about half of it after receiving only the title. Planner users benefit even more when they are given cues based on the referential semantics taught during training. After cue 3, when the complete formal relation of the problem is



Symbols used by the Planner group | Symbols used by the Symbol group

Terry Loves Books

Terry loves books. He wants to renew his library. Some books no longer interest him and there are others he wants but doesn't have. So he gives his little brother Al 13 books that are no longer good for his age. Then Terry receives 7 new books from his father. He now has 19 books. How many did Terry have before he changed his library?

Figure 1. Symbols used in the experimental training for the Planner group and the Symbol group. Here, both representations convey the instantiated version of Terry Loves Books, the expression (? - 13) + 7 = 19. For an uninstantiated version, the specific numbers or quantities of boxes shown are omitted.

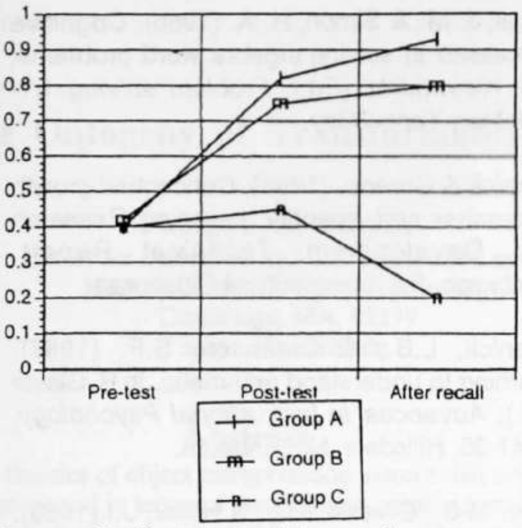


Figure 2. Problem-solving performance on the pretest, posttest, and after recall (5 days later).

given, Planners are still dramatically better than symbol users and control subjects in reconstructing the original problem from its solution. In contrast to the performance data of Figure 2 it is apparent that an important aspect of problem comprehension is not adequately measured by performance. A similar pattern of results are apparent for subjects' recall of relations as a function of treatment (Figure 3b). Relational propositions have been singled out as a major obstacle to word problem comprehension (e.g., Mayer, 1982), yet Planner users show high recall of relations which is nearly perfect after the relation-based cue (cue 2).

Conclusions

The high performance coupled with near-perfect reconstruction of the problem situation suggests that Planner users have a superior understanding of both the structure and the language of algebra to their experimental counterparts. It is not possible from these data to determine whether the Planner helps students to better encode the situation and the conceptual structure of the problem, or if their memory advantage is due to the effect the referential semantics has on retrieval (e.g., greater salience, better indexing).

The objects of The Planner have been shown to be an adequate means of supporting algebra problem solving (consistent with Schwarz et al., in press).

Furthermore, the referential semantics it provides proves to be both an adequate language for representing problem information, and a better method than manipulable algebraic symbol pads for promoting problem comprehension.

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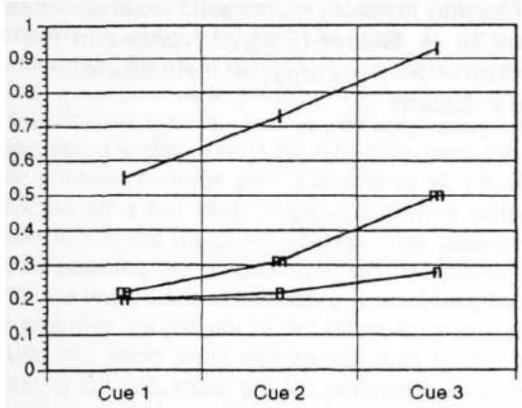

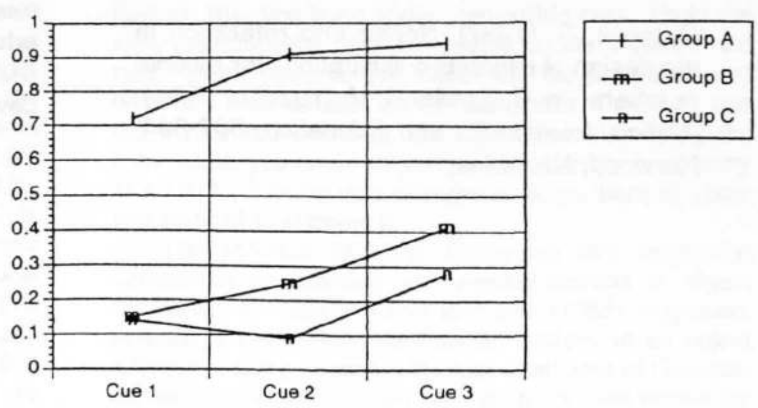

Figure 3. Recall of the test problem statements for subjects in the three experimental conditions after each cue. (a) All propositions in the stories. (b) Relational propositions.

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