Representation, Agency, and Disciplinarity: Calculus Experts at Work

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Abstract

Differential calculus provides various ways to conceptualize change, any of which can be employed with applied problems. Experts associated with different academic disciplines (chemistry, physics, mathematics) were asked to think out loud while working on a problem requiring a differential equation for its exact solution. These experts used strikingly different representations in solving the problem. Comparisons between their protocols are based on a historical-cognitive approach that ties present-day representational practices of differential calculus to the history and conceptual development of the calculus. Agency, here defined as the task assigned to the problem solver by the representation, is at the heart of this link between past and present practices. Whereas the agency characteristic of the Leibnizian calculus is choice, the agency characteristic of Newtonian calculus is transformation, and that of the modern function-based calculus may, in applied contexts, be characterized as observation and manipulation.

Multiple Representations

Contemporary use of calculus is characterized not only by multiple notations, *dy/dx*, $f'(x)$, $\dot{y}$, and so on, but also by multiple representations. This representational multiplicity of the differential calculus goes beyond the distinction between graphical and symbolical mathematical representation and implicates the central concepts of the differential calculus, in particular, the derivative and its conceptual precursors. In fact, the existence of multiple representations is not unique to the differential calculus but a feature of calculi in general (Kurz, Gigerenzer, & Hoffrage, in press). Representational multiplicity has been characteristic of differential calculus since its very inception in the late 17th century and was already present in the mathematical writings of the two eminent figures credited with its breakthrough, Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716). For a considerable time the nature of the mathematical representation of change and the associated methods were the center of controversy. In the second half of the 19th century these debates were settled and questions concerning rigorous mathematical representation of change became to a large extent merely of historical interest. In another sense, however, calculus never left its history behind.

In the historical literature it is frequently acknowledged that in the course of his work, Newton employed various algebraic and geometric methods to justify the new calculus; moreover, Newton also employed various ways to conceptualize change, using *moments*, *fluxions*, and the *ultimate ratio* (Boyer, 1949; Kitcher, 1973). In the Leibnizian calculus, by contrast, *differentials* were central (Bos, 1993). Although these various ways to conceptualize change employed by Newton, Leibniz, and their respective followers led to equivalent solutions of particular problems, they still entailed different representations of change. The subsequent conceptual development of calculus has added new representations, specifically, representations based on explicit definitions of the mathematical concept of limit, eventually including a purely arithmetic representation--the "epsilon-delta" formulation. However, in applied contexts the historically recent and rigorous $\varepsilon - \delta$ formulation is usually not the representation of choice. In an introductory calculus textbook written by the mathematician Morris Kline (1967), this point was made explicit: "The reader may conclude that the mathematician constantly applies the $\varepsilon - \delta$ definition to decide whether a function has a given number as a limit. [...] The working mathematician and certainly the theoretical physicist and engineer do not apply the rigorous definitions and proofs. They reason on the basis of the geometrical interpretation, physical evidence, intuitive arguments, and relatively loose analytical procedures." In effect, Kline was raising a cognitive question: What *are* the mathematical representations of change used in applied contexts?

Experts associated with different academic disciplines (chemistry, physics, mathematics) were asked to solve a problem requiring a differential equation for its exact solution. These experts used strikingly different representations in solving the problem. In the following I will first characterize the solutions worked out by the expert participants. Then I will suggest systematic comparisons based on a historical-cognitive approach (Kurz, 1997). This approach ties present-day representational practices of differential calculus to its history and conceptual development. Agency, so my argument goes, is at the heart of this link between past and present representational practices.

Calculus Experts at Work

Calculus experts, as defined for the purposes of this project, are people who have gone through the kind of mathematical training obligatory, for instance, for most of the natural sciences, and who encounter calculus on a regular basis in their area of specialization. Three such experts were asked to think out loud while solving a mixture problem called the Flask Problem. The problem was adopted from *Problems in differential equations* by Brenner (1963). The following problem statement was presented to the expert participants: "A flask contains 10 liters of water and to it is being added a salt solution that contains 0.3 kilograms of salt per liter. This salt solution is being poured in at a rate of 2 liters per minute. The solution is being thoroughly mixed and drained off, and the mixture is drained off at the same rate so that the flask contains 10 liters at all times. How much salt is in the flask after 5 minutes?" The Flask Problem can be modeled by a first-order linear ordinary differential equation, which stated in Newtonian notation, is: $\dot{x} = 0.6 - 0.2x$. Solving the equation and taking into account that initially there is no salt in the flask the answer to the problem is $x = 3 - 3e^{-0.2\times5}$ or 1.9 kilograms (rounded to one decimal position).

What makes the Flask Problem a suitable problem? A most important feature of the Flask Problem with respect to the objective of this project--namely, the study of the representational practice of calculus--is that the problem requires a conceptualization of instantaneous change. Consider, for example, the following "mutilation" of the problem: "A flask contains 10 liters of water and to it is being added a salt solution that contains 0.3 kilograms of salt per liter. This salt solution is being poured in at a rate of 2 liters per minute. How much salt is in the flask after 5 minutes?" The answer is, of course, 3 kilograms, almost only an exercise in multiplication. In this version the problem still requires use of the rate of change of incoming salt, but there is no need (cognitively speaking!) to operate with the concept of instantaneous change. In its "full" version the Flask Problem makes it necessary to conceptualize instantaneous change and to operate with it. An exciting feature of calculus is that it provides more than one way to do that.

Participants were asked to think out loud, using instructional materials adapted from Ericsson and Simon (1993). The protocols were taped and transcribed (Kurz, 1997). Participants were allowed to use paper and pencil and a hand calculator, but no access to reference books was permitted. Participants were not told that the problem requires calculus for its solution. The obtained protocols were divided into problem solving episodes. This analysis serves as the basis for the following brief descriptions of participants T, U, and S's problem solving processes.

The protocol obtained with participant T was also encoded and represented as a Problem Behavior Graph (PBG; Kurz, 1997). The use of this approach was inspired by Tweney and Hoffner's (1987) application of protocol analysis to the scientific working diary of the British physicist Michael Faraday. In their encoding of portions of Faraday's scientific diary, Tweney and Hoffner (1987) modified the original state-operator scheme (Newell & Simon, 1972) to accommodate the complexity and multiplicity of relevant problem spaces (some of which are best described as being open). Similarly, the coding scheme developed here consisted of three major categories: Plans, Actions, and Evaluations (see also Kilpatrick, cited in Ericsson & Simon, 1993). With this coding scheme an almost complete and highly reliable encoding of T's protocol was achieved (reliability was assessed across two independent coders for part of the data and was in the range of 90%). The encoded protocol was then used to construct a PBG for each problem solving episode. However, these graphs were largely uninformative with respect to the goal of achieving an understanding of the representational use of the calculus. This is not surprising given the task-independence of the coding categories. By contrast, the encoding of Michael Faraday's scientific diaries by Tweney and Hoffner was fruitful in this respect because the experimental manipulations that Faraday performed could be seen as analogous to operators on mental states. No attempt was made to encode and represent the other protocols as PBGs; instead another "content-oriented" (Newell & Simon, 1972) analysis based on the episodic structure of the protocols was pursued.

Participant T: A mathematician

Participant T is a young, highly productive mathematician whose major field is analysis. T is a faculty member in a doctoral level mathematics department. He worked about 25 minutes on the Flask Problem. His protocol consisted of 11 episodes.

After having read the problem statement (Episode I), T started out with a schematized pictorial representation of the flask and the in- and outflow of mixture (Episode II). The flask was depicted as a rectangle, fluid flowing in and out as arrows. This picture was complemented by labeling the rate at which solution flows in and out, and the concentration of the salt solution coming in was noted next to the arrow indicating influx. T proceeded by assigning variables (specifically x_t for "the concentration of kilograms per liter at any given time") and briefly considered a rule of three-like approach (Episode III). But because he could not find another time t for which he knew the concentration, besides time t = 0, he abandoned the approach.

He then realized that he "should probably use some calculus, in the sense of rates of change" (Episode IV), more specifically, that a "derivative with respect to time" was needed. But when trying to implement such a calculus-based approach T found himself in a tangle of related concepts, occurring in the following order (Episodes IV &V): "rates of change, " ".3 kilograms per liter is the rate of, no!," "the rate of change of salt, " "the derivative with respect to time, " "the rate of change of the concentration, " "the rate of change of the concentration for the whole thing, " and "a rate of 2 liters per minute. " He knew that he was "supposed to write down sort of a derivative" but was "not seeing how to do this right off the bat," a fact that caused him some consternation.

As a way out of the dilemma, he retreated to „time increments" of 1 minute (Episode V). Having computed the amount of salt entering the flask in 1 minute it struck him as too gross an approximation to infer from there an answer to the problem, namely the amount after 5 minutes. He wanted to give an "accurate" answer. Therefore, he re-introduced "the instantaneous rate of change," this time, however, in a procedural interpretation (Episode VI): "Instead I wanna try to figure out what's the instantaneous rate of change of, well what's the saline solution after any given time. So let me go to 30 seconds." His declared conceptual stratagem was to work with decreasing fixed time increments, that is, to choose smaller and smaller time increments. This procedure would eventually lead to increments that are infinitely small, namely, differentials. T never went that far. After having struggled through 1-minute time increments (Episode V) and 30-second time increments (Episodes VI-VIII), it sufficed for him to consider the possibility of "refining" further, as by using 15-second increments (Episode IX).

In fact, T's actual computations never went beyond the first minute of the physical process. Thus, when considering 30 second time intervals his computations were concerned with the first and the second 30 seconds of the physical process. As it turned out, these computations were riddled with difficulty. In particular, for the first 30 seconds he could assume that pure water was drained from the flask. For the second 30 seconds, however, he had to take salt loss into account. These computations were cumbersome, including checking and re-checking of results (Episodes VII-VIII), but they paid off.

After the laborious computations concerned with 30-second increments he felt no particular urge actually to continue such a procedure with 15-second time increments, the declared next step in his stratagem to "refine until nothing" (Episode IX). Instead his computations had prepared him for a new problem conceptualization. He realized that "the rate in is always the same" (Episode X). An insight that led him to the question "now the rate out should be what?" Without much difficulty he was able to determine the rate of outgoing salt and he was very pleased at "coming up with the differential equation." Once represented in this form T quickly solved the formulated differential equation (Episode XI). His algorithms for such solutions were obviously well-practiced. Unfortunately, his differential equation was not entirely correct because the value for the rate out was off by one decimal position; he had not taken into account that the incoming salt was dissolved in 10 liters of fluid. Consequently, his final solution was not meaningfully interpretable and remained unsatisfactory to him. However, at this point, being both frustrated and pressed for time, he was unwilling to "debug" his solution.

Participant U: A chemist

U is a middle-career physical chemist and a faculty member in a doctoral-level chemistry department. She is very active in research and has published many papers in her field. U spent approximately 40 minutes working on the problem. Her protocol consisted of 12 episodes.

U spent considerable time (about 10 minutes) in reading (Episode I) and re-reading the problem (Episode II). Her reading was interspersed by questions (e.g., "Is that what it says here?"), also concerning the phrasing and completeness of the problem statement (e.g., "What do you mean 'drained off'?") and first inferences (e.g., ".6 kilograms of salt going in and .6 kilograms per minute going out"--a misconception she entertained briefly). At the end of her second reading she singled out "the critical sentence here," namely, that "the solution is being thoroughly mixed and drained off" (Episode III). Thus, "the concentration would be increasing over a period of a few minutes, and at some point you'd reach a steady state where you were putting the same amount of salt in as was going out." She continued to elaborate the steady state concept (Episode IV), informally addressing two questions: what the concentration at equilibrium would be and how long it would take to reach this state.

Such exploration of process characteristics prompted her to "draw a graph in time" in a coordinate system showing salt concentration on the ordinate (Episode V). Figure 1 shows the exact graph. U constructed her graph by first marking the value that the concentration would reach after 1 minute: "Over the first minute I've poured in 6 kilograms so this is gonna be about .6." (This value was, in fact, off by one decimal position; according to the problem statement .6 kilogram are poured in over the first minute, thus resulting in a concentration of .06 kilogram per liter.) Next, going to 2 minutes, she reasoned that the same amount of salt would be poured in except now "some of it is coming out," thus "after 1 minute we'd be a little low." She therefore marked a point slightly lower than 1.2. She determined a third point: "At 3 minutes, we're up here to 1.8, but still again we are low, we'd be even lower." Then, knowing that the steady state concentration had to be .3, she realized that her scale was not right. But she nevertheless established the general asymptotic shape of the graph ("so it's gonna be coming up like this and then it's just gonna be a straight line for the rest of the time"). Then she reminded herself of what the problem statement asked for and inferred a solution (Episode VI). However, given that her scale was wrong and that she had not made a serious attempt at fixing this problem she could not use her graph to "read off" the solution. Instead her solution assumed that at 5 minutes the system had reached steady-state concentration, which made the answer trivial (at equilibrium the concentration is equal to the concentration of incoming salt solution, 0.3 kg/liter), so 3.0 kg was her answer.

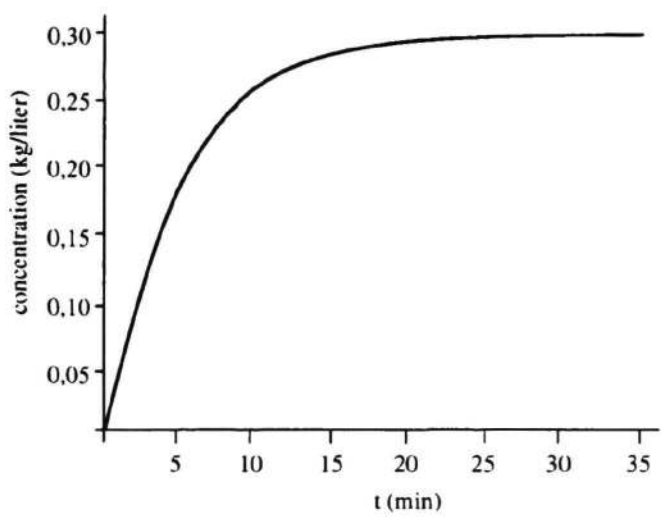

Figure 1: How salt concentration in the flask changes in time.

Somewhat worried that U would declare an end to her problem solving, the experimenter asked her whether she would be able to formulate an equation (Episode VII). She answered in the affirmative and immediately wrote the Leibnizian notation, *dc/dt* (*c* for concentration, *t* for time), denoting the "change in the concentration." She started the right-hand side of the equation reasoning that "first the concentration is 0" so "the intercept is 0." This reasoning was flawed, in a sense confusing the differential equation with its solution; it might have been prompted by her graph, which had, of course, an intercept of zero. Next she added a term denoting "the increase in the concentration" which was 0.6 with the units "kilograms per liter per minute." The "concentration going out, " being the "minus part," created a problem that she approached by making a "linear assumption" (Episode VIII). This assumption, however, created the anomaly of "not getting a steady state," as she quickly realized. Therefore it had to be a nonlinear function, but she was unclear as to its specific "functional form." As a remedy she recapitulated her understanding of the physical process--note the mathematical terminology--(Episode IX): "So it's increasing at a constant rate," or, in other words, "the concentration is increasing linearly." She was certain "that it's not decreasing at a constant rate," and also that "a constant amount"--read volume--of the perfectly mixed solution was poured out. Her understanding was thus fairly complete but not represented in the form of an equation. After having silently reread at least parts of the problem statement she emphasized that she was thinking about the process "in terms of a continuous thing" but that the approximation method she was going to propose next would be a "quickie" way to arrive at a solution. "Then [she] would try to work back so that [she'd] have an instantaneous picture of what was going on."

Her approximation method (Episode X) worked by considering 1-minute time intervals and matched with how she had constructed her graph. In the first minute 0.6 kilogram of salt was added to the flask and she assumed again that during the first minute no salt was deleted. In the second minute another 0.6 kilogram was added and 1/5 of the amount of salt in the flask after 1 minute was subtracted because the salt was dissolved in 10 liters, 2 of which were withdrawn from the flask during the second minute. With the third minute another 0.6 kilogram of salt was added to the amount in the flask and 1/5 of the amount of salt in the flask after 2 minutes was subtracted. She carried this procedure through to 5 minutes (Episode XI). In the process, U kept checking the obtained values against her expectation concerning the steady state of the system (e.g., "after 3 minutes I am more than halfway there"). She announced her solution as "2.14 kilograms in 10 liters; that's approximate!"

As a final question the experimenter asked what could be done to improve this approximation (Episode XII). She responded that one would have to "take smaller time intervals" and that she had tried "to do it the calculus way" but that she could not "see how to, that way, now." With the calculus, of course, "one could figure out what the concentration was at any second" but she thought that "even eyeballing it here [she] could predict that maybe at around minute 7 or 7 1/2 would be when you'd reach steady state." This estimate overestimated the change in gradient considerably. (After 8 minutes, for example, the amount of salt in the flask is still only 2.4 kilograms, rounded.) However, that was a projection into the future of the system; for the time interval for which she had actually computed the approximation, from zero to 5 minutes, her intuitions were quite refined. When asked for her best guess of the precise solution her answer was "1.9 kilograms"--the value (rounded, of course) that is obtained with the solution of the differential equation!

Participant S: A physicist

Participant S is a theoretical physicist, internationally known for contributions to his field. S teaches undergraduate and graduate physics courses in a masters-level physics department. S spent about 50 minutes with the Flask Problem. His protocol consisted of 18 episodes.

Reading the problem statement (Episode I) led S to assert that "this problem is not such a simple problem. " It was immediately clear to him that the problem implied a "rate idea" and that "it is easy to figure out how much is added" (Episode II). The "complication" was that "some of that salt is being lost because it's being mixed and you probably assume that it's thoroughly mixed and drained off and so you have to figure out essentially how much is being lost because as more is added the concentration increases and more is being lost. " Thus, his task was "to put all this together in some formulas or something and see these relationships."

Although S showed immediate insight into essential aspects of the physical process as specified by the problem statement, his first computations ignored some of its specifications, in particular, that the volume of mixture in the flask is constant. He computed that in 5 minutes 3 kilograms of salt enter the flask (Episode III). But then he proceeded to compute the amount of salt leaving in 5 minutes, basing this computation on the total volume of solution added in 5 minutes (Episode IV): In 5 minutes a total of 10 liters of salt solution is added to the 10 liters of water already in the flask resulting in a total of 20 liters of mixture. Thus, the 3 kilograms of salt entering in 5 minutes are, so he reasoned, dissolved in 20 liters. In 5 minutes a total of 10 liters of mixture is withdrawn from the flask. Because 10 liters are half of the total of 20 liters, half of the 3 kilograms of salt dissolved in 20 liters is withdrawn from the flask. This leaves 1.5 kilograms of salt in the flask after 5 minutes. He emphasized that this was "a guess," and he (correctly) suspected it to be wrong. Fearing that participant S might at this point declare an end to his problem solving, the experimenter asked "Can you come up with an equation?" "A good question," he agreed, because equations had been "implicit" in what he had been thinking but now the challenge was to "find what they are." It was clear to him that it would need to be "sort of a rate equation" (Episode V). He started to wonder whether he had missed something before and therefore thought it best to "read this again" (Episode VI).

After rereading the problem he determined x to be "the amount of salt in tank" (Episode VII), thus specifying the dependent variable (although he did not use this terminology). The left-hand side of the equation had to be "the time derivative of x" which was noted in Newtonian notation. The right-hand side of the equation was "rate at which it's added minus the amount that is leaving" (Episode VIII). The rate of incoming salt was noted as "0.6 kg/min" and followed by a minus sign. He immediately recast the rate of incoming salt as "0.3(2)," also omitting the units of measurement. Then he proceeded to the part following the minus sign that was "also a function of time." He knew that "the amount of fluid that is flowing out is 2." What he thus wanted to figure out was how the amount of salt that was leaving depended upon the concentration of salt solution in the flask. The amount of salt in the tank "is always gonna be x over 10." At this point he had in effect written out the complete differential equation. After a lengthy pause (of 12 seconds) he came to the conclusion "that this might be the right idea, really, 'cause this says that the rate at which the amount of salt changes depends upon how fast you add it." Despite having put the complete differential equation on paper, S had not yet fully conceived of it as the appropriate mathematical model. There was no doubt for him that the rate of incoming salt was the determined constant (Episode IX). It was also clear that "the rate at which it flows out is the same at which it enters, so that's gonna be 2." But "what about the changes due to the increasing concentration?" "How [did he] know what the concentration of salt is?" He admitted, he didn't know. At this point another insight rescued him. Checking the units of measurement for the right- and left-hand side of his equation (Episode X) he was just delighted to find that "this has the units, this has the right units!" He restated the equation in a concise form as $\dot{x} = 0.6 - 0.2x$.

Now he had to solve the equation. He anticipated a little bit of effort and challenge because he would have to "integrate or something" and he would "end up with some kinda exponential function" (Episode XI). He reformulated the equation in Leibnizian notation and proceeded to a separation of variables. But before taking the solution any further, that is, proceeding to integrate, he wanted to "think" and "see" once more whether he liked "the way this is going" (Episode XII). He was "comfortable" with the "constant time derivative which is because there is a constant source." The question was "how do you deal with a changing rate of outflow?" He could "see" that it was changing because "salt concentration is rising and therefore more is flowing out." There was no salt in the flask "to start with" and in the "extreme future you would reach an equilibrium situation where all of the original water had been replaced and therefore the concentration inside the tank would be 0.3 kilograms per liter and at that point you would have 3 kilograms in there." "Unfortunately the problem didn't ask about the extremes," that is, the initial amount and the amount at equilibrium; it asked for "the middle." An answer to this question could be found by solving the differential equation; therefore, he thought, "it's worth doing."

He proceeded with the solution by employing a substitution procedure and after several steps arrived at the following intermediary result $0.6 - 0.2x = e^{-0.2t}$ (Episode XIII). He solved for the dependent variable x and checked his solution by evaluating the equation at t=0. It turned out that "x is not equal to 0", thus the initial condition according to which initially pure water was in the flask was not met. Clearly something was "not right" (Episode XIV). He diagnosed the problem: "Uh oh, I think I see that's because I didn't do the right integration." Nevertheless he recognized "elements of truth" because in the "long-term limit" he would "get the right answer, " he would get the 3 kilograms. He had to "figure out how to do the integrals," thus handling the "initial and final conditions correctly." He started over after his substitution procedure noting definite integrals to be evaluated between 0 and x and 0 and t, respectively (Episode XV). Now he was convinced he was on the right track: "All right, so I know I have some good stuff going on here, I just didn't get enough of it into my equation." Evaluating the definite integrals he arrived at $x(t) = 3(1 - e^{-0.2t})$. This was a result he "liked," because "as t approaches infinity x approaches 3" and at "t equals 0, x is equal to 0" (Episode XVI) . Everything was ready for "the question we want to answer," namely, "what is x of 5?" (Episode XVII). He used a pocket calculator to determine the numerical solution (Episode XVIII) and declared "1.896 kilograms" to be his answer to the problem.

Agency and Disciplinarity

At the outset, it is worth noting that the problem was seriously demanding for the expert participants--while this is a prerequisite for rich verbal protocols, it was somewhat surprising at first, until the effort necessary to initially represent the problem became apparent. Participants T and U, both retreated to approaches utilizing discrete time intervals when faced with difficulty on a conceptual level. However, their approximation procedures were very different in character. Participant S employed a genuine modeling approach. I will concentrate here on differences in their representations of change as such, particularly on the agency implicated by these representations.

Participant T built his representation of the problem by means of finite time intervals with the intent to develop eventually an "instantaneous picture." In the limiting case this approach could consider infinitely small intervals, or differentials; as such his approach was Leibnizian. By contrast, the representation that participant U employed can be characterized as Newtonian. Newton's and Leibniz' calculi were concerned with variable quantities, but they conceived them very differently (Bos, 1980). With Leibniz a variable quantity ranged over a sequence of infinitely close values. With Newton's fluxionary calculus variable quantities were conceived as changing in time, and thus as dynamic (see Freyd, 1987). In Newton's own words (cited from his *Tractatus de quadratura curvarum* of 1704 as translated and reprinted in Struik, 1969): "I consider mathematical quantities in this place not as consisting of very small parts; but as described by a continued motion. Lines are described, and thereby generated not by the apposition of parts, but by the continued motion of points; superficies [surfaces] by the motion of lines; solids by the motion of superficies; angles by rotation of sides; portions of time by a continual flux: and so in other quantities." Fluxions, the fundamental concept in Newton's calculus, are the velocities or rates of change of these variable quantities. In Leibniz' calculus, on the other hand, differentials were fundamental (Bos, 1974). A differential is the infinitely small difference between successive values in a sequence of infinitely close values, that is, the variable quantity.

T *chose* time intervals, first of 1 minute, then of 30 seconds and finally (at least hypothetically) of 15 seconds in an attempt to "refine until nothing," until the differences--read time intervals--had vanished. In this sense choice was fundamental to his representation of change. Similarly, the choice of infinitely close values of a variable quantity was at the heart of Leibniz' calculus. Bos (1993) has emphasized that Leibniz did not conceive of differentials by means of a local limit process (like the derivative). Leibniz' limit taking was global. With respect to a curve, for instance, this meant that the curve remained composed of the sides of a polygon even after extrapolation to the infinite case of an infinitangular polygon. T's plan to "refine until nothing" might therefore not have been Leibnizian in intent, however, his realization in terms of *fixed* increments was.

U *transformed* the change in salt into the continuous motion of a point creating a graph in time. This transformation was the result of her knowledge about process characteristics (the steady state) and a particular way of constructing an asymptotic graph in a Cartesian coordinate system. Her approximation method finally paralleled her construction of this graph: First she determined how much was added, then she made sure that she was "a little low," that is, that she subtracted about the right amount, then she used these points as reference points for the extrapolation of the corresponding "motion." Similarly, Newton conceived of mathematical quantities as motion of geometrical objects.

Mathematical representations of change, like those employed by participants T and U, include representation of the problem solver's own agency (Kurz & Tweney, 1998). Agency is here understood as the task assigned to the problem solver by the representation. The agency entailed in participant T's Leibnizian representation can be characterized as *choice*, the agency in the case of participant U's Newtonian representation as *transformation*. In the sense that the problem solver "knows" the representation he or she is building and employing, he or she has "knowledge" about the entailed agency. As Gooding has pointed out (1992), agency is a matter of skill.

What kind of agency was implicated by participant S's modeling approach? Participant S's protocol was largely concerned with finding a match between a mathematical model and an understanding of the physical process. This is in line with the fact that he had formulated the appropriate differential equation long before he recognized the equation as the appropriate mathematical model. In working out this match he made the physical process "observable" and "manipulable" in his mind's eye. This mental "simulation" of the physical process "observed" a physical process in time, from "no salt in there to start with" to an "extreme future" in which "you would reach an equilibrium." S's model, mathematical and physical, *described* a physical process changing in real-time. In some sense description was observation in this case; this is parallel to what Nersessian (1992) has pointed out for thought experiments. The agency of *observation and manipulation*, the unity of which Tweney (1992) and Gooding (1990) have emphasized in their accounts of Michael Faraday's experimental investigations, occurred at the interface of S's understanding of the physical process and of his mathematical model. It could be argued that the differential equation *is* the physical process model. This is the case in a strict sense; however, the identification of the understanding of the physical process with the mathematical model stood at the *end* of S's solution process; this in fact was his primary achievement.

The wide spread application of calculus in the sciences seems to suggest that the calculus does not recognize disciplinary boundaries. The representational practice of calculus, however, has always recognized boundaries. In the 18th century, when Newtonian calculus was predominant on the British Isles and Leibnizian calculus on the Continent, the English Channel was such a boundary. As much as participants T, U and S's approaches are typical of particular academic disciplines or fields, the calculus still seems to acknowledge boundaries--nowadays, those that map our academic specializations.

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