Can children represent and compute over mixed sets with the Approximate Number System?

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Abstract

Considerable debate exists over the kinds of numbers the Approximate Number System (ANS) can represent and compute over. Across three experiments (N = 218), we show that children can represent and add large mixed sets (i.e., large collections that include two types of items) with the ANS. In Experiment 1, 5-7-year-olds completed a replication of a large non-symbolic number addition task using an online asynchronous format. In Experiments 2 and 3, 5-7-year-olds completed a variation of that addition task with mixed sets of stimuli either area-controlled or area-correlated and again performed above chance level. Taken together, these findings are a first step in examining whether the ANS can represent all positive rational numbers (i.e., fractions or ratios), as opposed to exclusively integers. Future work can examine whether participants can use ratio representations to complete the task presented here, as opposed to other strategies that are possible in the current experiments. In sum though, our findings suggest that children can represent and compute over large mixed sets of stimuli with the ANS.

Keywords: approximate number system; rational numbers; non-symbolic addition; cognitive development

Introduction

The ability to reason about proportions and probability is fundamental to successfully navigating one's daily life (Denison & Xu, 2014; Kahneman & Tversky, 1973). For example, you may instinctively compare the number of items per person in line to determine which checkout will be quickest or estimate the number of your favorite type of donut available to the number of people in a room to see if you should rush to get one.

In many cases, these judgements arise from the Approximate Number System (ANS), a cognitive system that automatically creates imprecise mental representations of non-symbolic numbers abstracted from the world around us, which can be used to make intuitive judgements, such as comparing quantities or manipulating them arithmetically (e.g., Brannon, 2002; Feigenson et al., 2004; Odic & Starr, 2015). Much prior work has focused on how the ANS represents quantities. While there has been significant debate as to whether the ANS is representing numbers as opposed to other variables that are related to numbers, such as area and density (Cicchini et al, 2016; Clarke & Beck, 2021; Content et al, 2017; DeWind et al, 2015; Gebuis et al. 2016), we (momentarily) put aside that debate and turn to the question of what kinds of numbers the ANS can represent and compute over. As a starting point, any theory that contends that the ANS represents numbers would likely include natural numbers in its representational capacity, such as positive integers. Similarly, most also contend that it cannot represent irrational numbers, such as pi (Ball, 2017; Clarke & Beck, 2021). Conversely, debate exists over whether the ANS can represent rational numbers other than positive integers, such as ratios and fractions (Clarke & Beck, 2021; Gallistel & Gelman, 2000). Some limited evidence supports the hypothesis that the ANS can represent rational numbers. For example, in a study by McCrink & Wynn (2007), 6-month-old infants were habituated to a ratio of dots, before being shown new displays with either the same ratio but different amounts of dots, or a different ratio of dots. Infants were able to discriminate the habituated ratio from a new ratio at above chance level when the ratios differed from each other by a ratio of 2:1 but not 2:3, consistent with the limits on 6-month-old infants' ANS acuity with individual large numbers.

Despite these findings from 6-month-olds, much debate and skepticism still remain as to whether the ANS can represent rational numbers such as ratios (e.g., Leibovich & Henik 2013; Leibovich et al., 2017; Marshall, 2017). For example, Marshall (2017) points to the "denseness" of rational numbers (i.e., between any two rational numbers, there is always a third number), as the reason why the ANS could not represent them. However, the ANS does not need to be able to represent *every* rational number to be able to represent rational numbers. In the same way, the ANS cannot represent every whole number (e.g., one billion), and yet it can still be credited as representing whole numbers (Clark & Beck, 2021).

Previous research has shown that the ANS can enumerate multiple subsets at once (up to two in infancy and childhood, and three in adulthood) and hold them as distinct, simultaneous representations (e.g., 20 blue dots and 10 purple dots are held as two separate ensembles of dots; Feigenson, 2008; Halberda, 2006; Zosh et al., 2011). However, it is unclear whether these representations can be considered in relation to each other (e.g., can 20 blue dots and 10 purple dots be held as one ensemble of dots with a 2:1 ratio), or if

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these representations can be simultaneously manipulated in the same way as a single representation, such as comparing or computing over them. O'Grady & Xu (2020) provide initial support for this possibility by showing that 6-12-year-olds can represent and compare area-controlled mixed sets of stimuli to determine which has a more promising ratio to obtain a certain-colored marble.

Because of this lack of clarity, the full arithmetic capabilities of the ANS are still open for investigation. Previous work has shown that young children, often with little to no formal mathematical training or linguistic understanding of mathematics, can use their ANS to perform computations over their representations, including addition (Barth et al., 2006; McCrink & Wynn, 2004), subtraction (McCrink et al., 2007; McCrink & Wynn, 2009), multiplication (McCrink & Spelke, 2010; Qu et al., 2024), and division (Szkudlarek et al., 2022). For example, one study showed 5- and 6-year-olds a quantity of rectangles before it was occluded, then the amount was divided by either two or four by a "dividing wand" (McCrink & Spelke, 2016). After being presented with a new array of rectangles, children could reliably determine whether the new display had more or less rectangles than the expected quotient based on the first display and the divisor. Currently, research on the ANS's computational abilities has yet to go beyond this single object-type whole number paradigm to more fully assess the representational and computational capabilities of the ANS.

As a starting point to address this gap, we examine whether children can complete arguably the simplest computation (addition) with mixed sets of items (i.e., the building blocks of ratios). Establishing whether children can abstract and add true ratios is a complex question, with many variables to control, and thus this paper presents the first steps of this endeavor. In three experiments with 5- to 7-year-olds we first examined in Experiment 1 whether a classic ANS large number addition paradigm (e.g., Barth et al., 2005; Li et al., 2017) could be run online and asynchronously with this age group. Experiment 2 investigated whether participants could complete a large number addition task when the sets to be added included two types of items intermixed. Finally, Experiment 3 further probed participants' ability to complete mixed-set addition with their ANS by controlling the area and density of the items being added, such that participants could not rely on cues that are confounded with number when completing these addition problems. Based on the design of our stimuli, children could represent either ratios or whole numbers to complete these tasks (see Procedure Section and General Discussion for details). However, this paper adds to our knowledge about children's ANS capacities because even the ability to represent and compute over whole numbers when they are not presented in isolation (i.e., when sets are mixed) is important in itself, as it is a problem humans face in our environments regularly (e.g., people and chairs or tables; animals and available food). But these experiments also lay the foundation for investigating whether children can represent and compute across ratios with their ANS.

Experiment 1

Materials and data for all experiments are available online.

Methods

Participants In all experiments, children were recruited from and virtually tested using the Children Helping Science (CHS; formerly Lookit) platform. Demographic information was not formally collected, but participants recruited through CHS are typically more representative of the United States' general population demographics than lab samples on several measures, including race, socioeconomic status, languages spoken, and parental education (Scott & Schulz, 2017).

We tested 47 five- to seven-year-olds (M_{age}=6.10, 21 female). Data from an additional 19 participants were excluded for not completing the study (*n*=9), failing the practice trial (*n*=5), not paying attention for more than 50% of trials (*n*=3), or attempting to count items for every trial (*n*=2). In this experiment, we aimed to test 20 children per age (in years). We anticipated this rule would ensure a sufficiently large sample to detect effects of age, ratio, and an interaction between them. However, the CHS platform did not allow for age caps to be set per age in years, so the final sample contained fewer 7-year-olds (*n*=5) than 5-year-olds (*n*=21) and 6-year-olds (*n*=21). Participants received a $5 Amazon.com gift card for participating.

Materials and Procedure We devised the procedure for our experiment based on Li et al.'s (2017) paradigm. The stimuli for the experiment were created using PowerPoint slides, and the study was programed and hosted on CHS. Participants completed the study asynchronously in their homes using their personal computers. Prior to beginning the study, children watched an instructional video familiarizing them with the procedures and completed a simple practice trial to make sure they understood the task. The twelve test trials immediately followed the practice trial. In each trial, children watched a set of blue dots move across the screen and behind a box (see Figure 1). Then, a new set of blue dots also moved behind the box. Each set paused on the screen for 1.75 seconds to prevent counting.

Children were then shown images of two new sets of blue dots: the correct set had the same number of dots as the summed total in the box, and the comparison set did not. Children were instructed to click on the set that had the same total number of blue dots as the box. The number of dots in the two sets presented for children to choose between differed from each other by a ratio of 1:2, 2:3, or 3:4, with 3:4 being one of the most difficult ratios children in this age range can distinguish (Barth et al., 2005; Halberda & Feigenson, 2008). Trials were counterbalanced, such that the incorrect answer had the larger number of dots for half of the trials for each ratio, and the correct answer was presented on the left side of the box for half of the trials and on the right side for the other half. The order that the twelve trials appeared in was randomized in blocks of three, such that participants would complete one trial for each ratio within every block of trials.

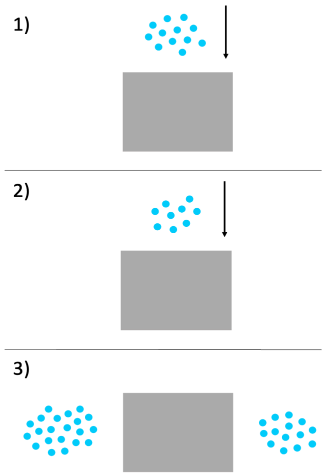

Figure 1: Schematic of the procedure for Experiment 1

Results

A Generalized Estimating Equations (GEE) binary logistic regression with ratio as a within-subject factor and age (in months) centered around the mean as a covariate revealed a significant main effect of ratio on accuracy, *Wald* X^2=21.19, p<.001, as well as a significant main effect of age, *Wald* X^2=10.59, p=.001 (see Figure 2). These main effects were qualified by a significant interaction of ratio by age, *Wald* X^2=6.13, p=.047.

Bonferroni-corrected pairwise comparisons revealed that participants performed poorer on the more difficult 3:4 ratio trials compared to the 1:2 (p<.001) and 2:3 ratio trials (p=.028), while performance did not significantly differ between 1:2 and 2:3 ratio trials (p=.707). Further analyses probing the interaction revealed that this pattern held true for six-year-olds such that they performed significantly poorer on the 3:4 ratio trials compared to the 1:2 (p<.001) and 2:3 ratio trials (p=.032) and their performance did not significantly differ on the 1:2 and 2:3 ratio trials (p=.706). However, performance did not significantly differ between trial types for five- (all p's>.352) and seven-year-olds (all p's>.497). Additionally, five-year-olds' performance was significantly lower than six- (p=.010) and seven-year-olds' (p=.005), while performance did not significantly differ between six- and seven-year-olds (p=.283), likely due to the loss of power from the smaller sample size of seven-year-olds.

Binomial tests were conducted by age group to determine whether children's performance differed from chance level. The results successfully replicated the findings from Li et al. (2017), such that 5-year-olds, M=67.5%, $t(20)$=4.87, p<.001, 6-year olds, M=77.4%, $t(20)$=10.23, p<.001, and 7-year-olds, M=86.7%, $t(4)$=5.88, p=.004, performed reliably above chance level on the task. Additionally, a Chi-Square Goodness-of-Fit analysis showed that the observed distribution of results by age group did not significantly differ from the original distribution of results found by Li et al. (2017), X^2=0.47, p=.792.

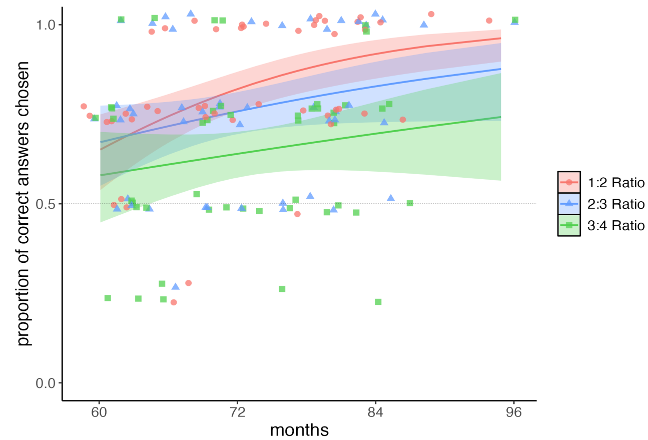

Figure 2: Results for Experiment 1. Colored bands show 95% confidence intervals; points are jittered to avoid overplotting; the dashed line represents chance (50%).

Discussion

Experiment 1 found that five- to seven-year-olds could successfully complete large-number addition with their ANS when the two choice sets differed from each other by a ratio of 1:2, 2:3, or 3:4. This provides further evidence that children can add two arrays of dots using their ANS, which in turn strengthens the argument that the ANS represents numbers, since numeric representations are essential to being able to complete arithmetic calculations, such as addition (Barth et al., 2005; Barth et al., 2006; Li et al., 2017).

As well, this replication demonstrates that experiments studying the ANS can successfully be run virtually and asynchronously using the CHS platform and produce robust results comparable to those ran in-person within a lab setting. With this confirmation, we began investigating whether children can use their ANS to add mixed sets. To do so, we created stimuli with two item-types rather than a single type per set, such that children could represent the ratio[1] of items in the set, as well as the total number of each item.

Experiment 2

Methods

Participants 64 five- to seven-year-olds participated in the study on CHS (M_{age} = 6.24, 32 female). An additional 15 participants were excluded for not completing the study (n=6), failing the practice trial (n=5), or attempting to count for more than 50% of trials (n=4). The final sample again contained fewer 7-year-olds (n=14) than 5-year-olds (n=25) and 6-year-olds (n=25). Participants received one draw entry for a $50 Amazon.com gift card as compensation for participating.

[1] We often refer to our mixed sets as "ratios" for ease of expression in Experiments 2 and 3, but we do not know that children are representing ratios per se, and not something else, like the total number of each individual set of items.

Materials and Procedure The general procedure was the same as Experiment 1 with different stimuli entering the box (see Figure 3). In each trial, two mixed sets of apples and oranges moved across the screen and into a box. While each pair of sets had different ratios of apples to oranges, the total number of fruits in each set were always equal (e.g., set one 5:10=15, set two 9:6=15). This was done to allow for the ratios to be added, without children needing to shift the weighting of each ratio based on the number of stimuli in each set. Participants were then shown two new sets of fruit. Both new sets had the same total number of fruits, such that participants could not rely on the total number of fruits to arrive at the correct answer. The correct set had the same ratio of apples and oranges as the box, while the comparison set had an incorrect ratio.

Trials were subdivided into three trial types based on the difference between the correct set and the comparison set: type 1 trials were reciprocal ratios (i.e., the correct ratio and the reverse of the correct ratio; 3:5 vs 5:3), such that participants needed to associate the ratio with the corresponding items rather than only track the absolute ratio; for type 2 trials, both choice options had the same majority to a differing degree (e.g., both options had more apples than oranges), such that participants could not rely on an imprecise adding strategy (e.g., the sum has more apples than oranges); and in type 3 trials, the correct answer always had an added ratio of 1:1 with varying comparisons, such that the ratio of ratios between the correct and comparison sets could be manipulated to be closer together and thus more difficult to differentiate.

Trials were further divided into three alphabetical classification subsets to indicate the difficulty of the ratios, calculated using the Weber Fraction of the ratio of ratios, where (a) trials were the easiest and (c) trials were the most difficult (formulas, calculations, and exact trials are provided in the supplement). This led to a total of nine test trials in a 3 (type: reciprocal, same-majority, 1:1) x 3 (difficulty: easier, moderate, harder) design, with the ratio of ratios ranging from 1:3 to 3:4, and Weber's Fractions ranging from 1.48 to 0.67.

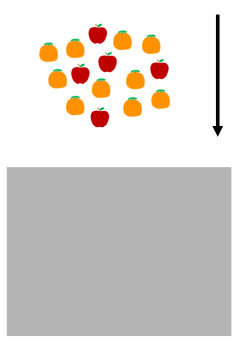

Figure 3: Example mixed set of stimuli for Experiment 2

Results

A GEE binary logistic regression with trial type and difficulty as within-subject factors and age (in months) centered around the mean as a covariate revealed a significant main effect of

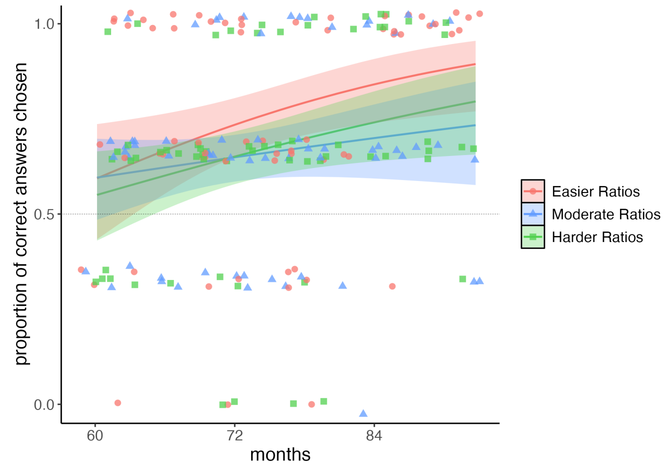

Figure 4: Results for Experiment 2. Colored bands show 95% confidence intervals; points are jittered to avoid overplotting; the dashed line represents chance (50%).

age on accuracy, *Wald* X^2=9.85, *p*=.002 (see Figure 4). There was not a significant main effect of trial difficulty, *Wald* X^2=5.57, *p*=.062, or trial type, *Wald* X^2=3.11, *p*=.211, nor any significant interactions (all *p*'s>.264). Bonferroni-corrected pairwise comparisons revealed that seven-year-olds performed significantly better than five- (*p*=.002) and six-year-olds (*p*=.002). However, performance did not significantly differ between five- and six-year-olds (*p*=.998).

Binomial tests examining whether performance differed from chance found that children performed at above chance level across all trials, *M*=0.70, *p*<.001. Tests conducted by age group found that 5-year-olds, *M*=0.67, *p*<.001, 6-year-olds, *M*=0.65, *p*=.004, and 7-year-olds, *M*=0.83, *p*<.001 all performed reliably above chance level on the task. To examine performance on each trial individually, binomial tests performed for each trial found that children's performance was significantly above chance level for all trials (all *M*'s>0.63, all *p*'s<.044), except for trial 2b (*M*=0.59, *p*=.175).

Discussion

Five- to seven-year-old children were able to add together mixed sets of items and select the summed total from two choices at above chance level. As such, Experiment 2 provides preliminary evidence that children can represent and compute over mixed sets of stimuli with their ANS.

However, the stimuli used thus far was not controlled for visual confounds, so it is possible that participants could be using the average diameter, cumulative area, convex hull, or density of dots rather than the number or ratio of dots to succeed at the task. To address this common critique of ANS confounds and to provide a replication of the general finding in Experiment 2 (e.g., Gebuis et al., 2016; Leibovich & Henik, 2013; Leibovich et al., 2017), Experiment 3 contrasted children's performance on mixed sets addition with consistently sized stimuli versus area- and density-controlled stimuli.

Experiment 3

Methods

Participants 107 five- to seven-year-olds participated in the study on CHS (M_{age} = 6.49, 62 females). Participants were randomly assigned to an area condition, such that the stimuli for all trials consistently had either same-sized dots (n=50, M_{age} = 6.55, 31 females) or density- and area-controlled dots (n=57, M_{age} = 6.44, 31 females) moving into the box. An additional 31 participants were excluded for not completing the study (n=10), failing both practice trials (n=16), not paying attention for more than 50% of trials (n=3), or attempting to count for more than 50% of the trials (n = 2). Participants received one draw entry for a $50 Amazon.com gift card for participating.

Materials and Procedure The procedure was the same as Experiments 1 and 2, with different stimuli entering the box (see Figure 5). In each trial, two mixed sets of purple and blue dots moved across the screen and into the box. Children were randomly assigned to either the same-sized dot condition, where the dots that moved into the box were all the same size regardless of the set's ratio, or the area-controlled dot condition, where the purple and blue dots in a set occupied the same total area on the screen and had the same total color density, thus dot size changed depending on the ratio and number of dots (i.e., a larger number of dots equals smaller dots). Additionally, each pair of sets that entered the box were controlled for equal area and density. Regardless of the condition, all dots in the two final choice sets were the same size to prevent children from simply tracking the size of the dots to identify the correct answer. Participants completed two counterbalanced practice trials: one with same-sized dots, and one with area controlled (AC) dots. The trial type classification system from Experiment 2 was implemented, but trial difficulty was reduced to two alphabetic subsets, such that (a) trials were easier and (b) trials were more difficult, leading to six test trials per area condition in a 3 (type: reciprocal, same-majority, 1:1) x 2 (difficulty: easier, harder) x 2 (area condition: same-sized, controlled) design.

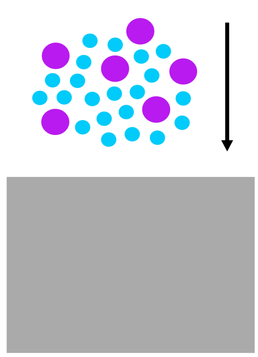

Figure 5: Example area- and density-controlled mixed set of stimuli for Experiment 3

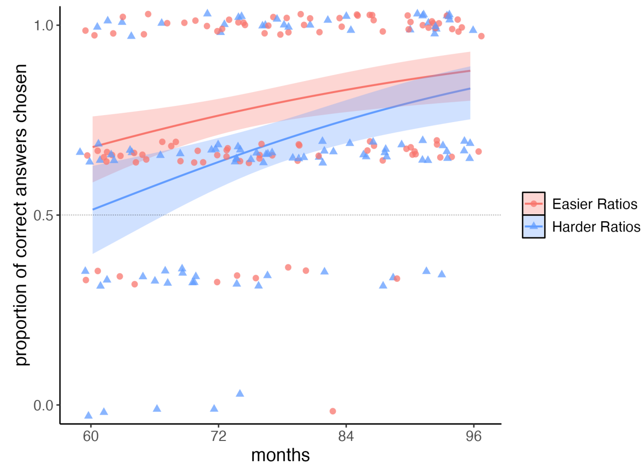

Figure 6: Results for Experiment 3. Colored bands show 95% confidence intervals; points are jittered to avoid overplotting; the dashed line represents chance (50%).

Results

A GEE model with trial type and difficulty as within-subjects factors, area condition as a between-subjects factor, and mean-centered age as a continuous covariate revealed significant main effects of age, *Wald* X^2=31.89, p<.001, and trial difficulty, *Wald* X^2=5.87, p=.015 (see Figure 6). Bonferroni-corrected pairwise comparisons revealed that five-year-olds performed significantly poorer than six- (p=.033) and seven-year-olds (p<.001), while six- and seven-year-olds' performance did not significantly differ (p=.333). As well, participants performed significantly better on the easier (a) trials than the harder (b) trials (p=.001). While there was a significant main effect of trial type, *Wald* X^2=7.40, p=.025, Bonferroni-adjusted post hoc tests showed that performance did not significantly differ when comparing each trial type, Type 1 vs 2, p=.245; Type 1 vs 3, p=.999; Type 2 vs 3, p=.089. There was not a significant main effect of area condition, *Wald* X^2=0.52, p=.471, nor any significant interactions (all p's>.070).

Binomial tests examining whether performance differed from chance found that children performed at above chance level across all trials, M=0.73, p<.001, and for each trial individually, all M's>0.63, all p's<.008. Additional tests by age group found that 5-year-olds, M=0.63, p<.001, 6-year-olds, M=0.74, p<.001, and 7-year-olds, M=0.81, p<.001 all performed reliably above chance level on the task.

Discussion

Five- to seven-year-old children were able to add together mixed sets of dots and select the summed total from two choices at above chance level, replicating the pattern of findings from Experiment 2. Critically, there was not a significant difference in children's performance on trials using same-sized dots versus density- and area-controlled dots. This provides further support that 5- to 7-year-old children can complete mixed sets addition with their ANS and are not relying on a confounding feature of the stimuli to succeed at the task, such as dot area or density. Taken together, the results from Experiments 2 and 3 lay the groundwork for future studies using mixed sets addition to investigate whether the ANS can represent and compute over rational numbers.

General Discussion

We investigated whether children can compute over large mixed sets using their ANS. Our first experiment demonstrated that ANS experiments can successfully be ran asynchronously online and produce robust results comparable to experiments ran in-lab. Experiment 2 provided initial evidence that 5- to 7-year-old children can compute over mixed sets of stimuli. Our third experiment then further supported these findings by contrasting children's performance on trials with same-sized stimuli to trials with area- and density-controlled stimuli. In this experiment, children's performance did not significantly differ based on the stimuli, and their performance was significantly above chance level for both trial types.

Together, these findings suggest that children can compute over mixed sets using their ANS and may be representing ratios to do so. It is important to consider, however, that children could have succeeded on this task by only representing and computing across whole numbers. Children could have used three possible strategies to succeed at this task: first, they could track and add only one colour of dots (e.g., 5 blue + 10 blue = 15 blue); second, they could simultaneously track and add each colour of dots separately (e.g., 5 blue + 10 blue = 15 blue and 10 purple + 5 purple = 15 purple); finally, they could be adding together the ratio of each set (e.g., 1:2 blue to purple + 2:1 blue to purple = 1:1). All three strategies would enable children to identify the correct answer, but only the third strategy involves computing across ratios. Thus, an important future direction will be to eliminate participants' ability to rely on the total number of each color of stimuli in the final sets in a way that still enables participants to identify the correct set by the ratio.

In an ongoing experiment, we are manipulating the mixed-sets addition paradigm such that participants will be unable to use the total number of each-colored stimuli to choose the correct answer and can only use the ratio on the final selection. To do so, we will introduce a cup that scoops a sample of stimuli from the box (after each mixed set has entered the box) and ask participants to identify the set that matches the cup, which will contain the same ratio of stimuli as the box but a smaller quantity of each type. This can determine whether children are abstracting the ratio and selecting it accordingly in tasks like Experiments 2 and 3 here to choose the correct final ratio. However, additional work will be needed to examine whether children are computing over ratios in this task (i.e., adding ratios). For this, a paradigm that adds particular ratios of items but in different absolute quantities (such that the ratios must be weighted) would be required.

We are also working towards more systematic manipulations of difficulty and trial type. Note that in the current experiments, we had multiple levels of difficulty, which were generally based on the Weber's fractions and ratio of ratios between the correct and comparison choice sets (i.e., closer ratios are of course harder to distinguish; formulas and calculations are provided in the supplement). We always kept these ratios within the known capacities of children's acuity for discriminating ratios at these ages (i.e., on single-number addition tasks, 5-7-year-old's acuity tends to be constrained to a maximum ratio of 3:4, so all ratios used in our experiments were equal to or further apart than 3:4; Barth et al., 2005; Halberda & Feigenson, 2008). Future work can examine children's limits in these kinds of tasks much more comprehensively, by varying both the input ratios (i.e., the two sets of items going into the box) and the correct versus comparison choice ratios (the ratio of ratios between the correct response and the alternative). We also classified trials in the current experiments as one of three types based on the difference between the correct and comparison choice ratios: reciprocal ratios (e.g., 3:5, 5:3), same-majority pairs (e.g., 3:5, 2:7), and one-to-one comparisons (e.g., 3:5, 1:1). However, additional trial types and ratio manipulations warrant systematic investigation in future experiments.

Understanding the types of numbers that the ANS can and cannot represent and compute across may reveal critical developmental milestones in children's acquisition of informal and formal arithmetic. By identifying which aspects of numerical cognition are rooted in approximate processing and which require more precise symbolic manipulation, this understanding of the ANS could inform educational developments aimed at bridging the gap between intuitive number sense and more complex arithmetic skills. Further, longitudinal research has shown that the ANS may play a significant role in children's later mathematical skills (Bonny & Lourenco, 2013; Halberda et al., 2008; Liang et al., 2022; Malone et al., 2021). For example, children's ANS acuity at 12 months predicted their symbolic arithmetic skills at 4 years, even after controlling for their general intelligence and perceptual skills (Decarli et al., 2023). Given the importance of mathematical skills in both children's and adults' lives, understanding how these skills develop is crucial, and the ANS could be key to this understanding.

Acknowledgments

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