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Crystal structures on FFLV polytopes

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Abstract. In this paper we formulate a conjecture about the crystal structures on Feigin–Fourier–Littelmann–Vinberg (FFLV) polytopes and prove it in small rank examples. In the case of multiples of a fundamental weight this approach recovers the crystal structures defined by Kus. A key step in this approach is the realisation of FFLV polytopes as Minkowski sums of Lusztig polytopes associated to different reduced words.

Keywords. Crystal structure, Lusztig polytope, FFLV polytope

Mathematics Subject Classifications. 05E10, 17B10

1. Introduction

Constructing bases of representations of Lie algebras is one of the central topics in representation theory. For semi-simple Lie algebras and their finite dimensional irreducible representations, various bases (Gelfand–Tsetlin bases, canonical/global crystal bases, standard monomial bases, Poincaré–Birkhoff–Witt-type bases, Mirković–Vilonen bases, bases arising from cluster structures, *etc*) are constructed using quite different methods. Comparing these bases, or more specific, studying base change matrices, is usually a very hard question.

Each of these bases comes with a parametrisation by a polyhedral structure (polyhedral cones, convex polytopes, polyhedral complexes, *etc*). The first step towards studying the base change matrices is to compare the polyhedral structures. For a simple Lie algebra of type $\mathtt{A_n}$ and the finite dimensional irreducible representation $V(\lambda)$ of highest weight λ, there are two PBW-type bases known for $V(\lambda)$.

1. The Feigin–Fourier–Littelmann–Vinberg (FFLV) basis: such a basis is compatible with the PBW filtration on $V(\lambda)$. The FFLV basis is parametrised by lattice points in the FFLV polytope - a lattice polytope having the facet description by Dyck paths.

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2. The canonical basis of Lusztig: for a fixed reduced decomposition of the longest element in the Weyl group, such a basis of $V(\lambda)$ admits a parametrisation by a rational polytope, called Lusztig polytope. The facets of such polytopes are more complicated (see [14, 15] for descriptions using rhombic tilings). They are quite different to FFLV polytopes.

For each of Lusztig polytopes, there exists a unique crystal structure on the set of its lattice points defined using piecewise linear combinatorics in [24]. For FFLV polytopes, such a structure is only known in the case where the highest weight λ is a multiple of a fundamental weight [20].

The first result of this paper is an unexpected relation between these two polytopes: an FFLV polytope can be written as a Minkowski sum of Lusztig polytopes associated to different reduced decompositions (Theorem 2.6). Such a relation between these two polytopes allows us to translate the crystal structure from the Lusztig polytopes to the FFLV polytopes: for multiples of a fundamental weight, we recover the results by Kus in [20]. In small rank examples, when λ is generic, we show that there exist more than one way to implement the crystal structure on the lattice points in the FFLV polytope. We conjecture that when the Lie algebra is of type $\mathtt{A}_\mathtt{n}$, for a generic weight, there exists $n!$ implementations of the crystal structure on the corresponding FFLV polytope.

We will present two different proofs to the aforementioned relation of two polytopes: a representation theoretical proof using essential basis in [8] to compare the lattice points; and a combinatorial proof using rhombic tilings in [14, 15] to compare the facets.

In Section 2, after recalling definitions of the Lusztig polytopes and the FFLV polytopes, we state the main result of the paper (Theorem 2.6). Two proofs of the main theorem are provided in the following two sections: in Section 3 it is proved using representation theory by realising both polytopes as essential polytopes associated to a birational sequence; in Section 4 a combinatorial proof is provided by explicitly writing down the facets of the Lusztig polytopes with the help of rhombic tiling. In the last Section 5 we apply the main result to state the conjecture on the crystal structures on FFLV polytopes, and justify it in the $\mathtt{A}_2$ examples.

2. Polytopes parametrising bases in Lie algebras

2.1. Notations

In this paper we fix $G = \mathrm{SL}_{n+1}(\mathbb{C})$ be the group of $(n+1)\times(n+1)$-complex matrices of determinant 1; its Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ consists of traceless $(n+1)\times(n+1)$-complex matrices.

We fix the triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) consists of strict upper-triangular (resp. strict lower-triangular) matrices and $\mathfrak{h}$ contains the traceless diagonal matrices. Let $U^- \subseteq G$ be the subgroup of unipotent lower-triangular matrices with Lie algebra $\mathfrak{n}^-$. The corresponding universal enveloping algebras will be denoted by $U(\mathfrak{n}^-)$.

Let $n = \dim \mathfrak{h}$ be the rank of $\mathfrak{g}$. The simple roots in $\mathfrak{g}$ will be denoted by $\alpha_1, \ldots, \alpha_n$. Let $\Delta_+ = \{\alpha_{i,j} := \alpha_i + \cdots + \alpha_j \mid 1 \leqslant i \leqslant j \leqslant n\}$ be the set of positive roots in $\mathfrak{g}$ with $N = \#\Delta_+$. For $\beta \in \Delta_+$, we choose a generator f_β of the root space $\mathfrak{g}_{-\beta}$. We fix $U_{-\beta} \subseteq U^-$ to be the unipotent subgroup with Lie algebra $\mathfrak{g}_{-\beta}$. Let $\varpi_1, \ldots, \varpi_n$ be the fundamental weights and

$\Lambda^+ := \sum_{i=1}^n \mathbb{N}\varpi_i$ be the set of dominant integral weights. For $\lambda \in \Lambda^+$, the finite dimensional irreducible representation of $\mathfrak{g}$ associated to λ will be denoted by $V(\lambda)$. We fix a highest weight vector $v_\lambda \in V(\lambda)$.

Let W be the Weyl group of $\mathfrak{g}$ with simple reflections $s_1, \ldots, s_n$, where s_i corresponds to the simple root α_i, and $w_0 \in W$ be the longest element. The length function on W is denoted by ℓ. Let $R(w_0)$ be the set of all reduced decompositions of w_0. An element in $R(w_0)$ will be denoted by either a reduced word $\mathbf{i} = (i_1, \ldots, i_N)$ or a reduced decomposition $\underline{w_0} = s_{i_1} \cdots s_{i_N}$.

We denote by $U_q(\mathfrak{g})$ the quantum group over $\mathbb{C}(q)$ with Chevalley generators E_i, F_i and $K_i^{\pm 1}$ for $1 \leqslant i \leqslant n$; $U_q(\mathfrak{n}^-)$ denotes the $\mathbb{C}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by F_i for $1 \leqslant i \leqslant n$. For $\lambda \in \Lambda^+$, let $V_q(\lambda)$ be the finite dimensional irreducible representation of $U_q(\mathfrak{g})$ of type 1. We fix a highest weight vector $v_\lambda^q \in V_q(\lambda)$. (For readers who are not familiar with quantum groups, we recommend read [16] for details.)

We will consider on $\mathbb{Z}^N$ the following total orderings: for $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{Z}^N$, and $\mathbf{b} = (b_1, \ldots, b_N) \in \mathbb{Z}^N$,

1. opposite lexicographic ordering $>_{\text{oplex}}$: $\mathbf{a} >_{\text{oplex}} \mathbf{b}$ if there exists $1 \leqslant i \leqslant N$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$ and $a_i < b_i$;

2. right opposite lexicographic ordering $>_{\text{roplex}}$: $\mathbf{a} >_{\text{roplex}} \mathbf{b}$ if there exists $1 \leqslant i \leqslant N$ such that $a_N = b_N, \ldots, a_{i+1} = b_{i+1}$ and $a_i < b_i$.

Let $\succ$ be the partial order on $\mathbb{Z}^N$ defined by the intersection of above ordeings: $\mathbf{a} \succ \mathbf{b}$ if both $\mathbf{a} >_{\text{oplex}} \mathbf{b}$ and $\mathbf{a} >_{\text{roplex}} \mathbf{b}$ hold.

We denote $\mathbb{R}^{\Delta_+}$ the set of functions from Δ_+ to $\mathbb{R}$. For such a function $\mathbf{a} \in \mathbb{R}^{\Delta_+}$, we write $a_\beta := \mathbf{a}(\beta)$ for $\beta \in \Delta_+$. Once an enumeration of elements in Δ_+ is fixed, say $\Delta_+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$, we get an identification of $\mathbb{R}^{\Delta_+}$ to $\mathbb{R}^N$ sending a function $\mathbf{a}$ to $(a_{\beta_1}, a_{\beta_2}, \ldots, a_{\beta_N})$.

For two polytopes P and Q in the same vector space $\mathbb{R}^N$, we denote their Minkowski sum by $P + Q$, $P + Q := \{p + q \,|\, p \in P,\, q \in Q\}$.

2.2. Canonical basis and Lusztig polytopes

To a fixed reduced decomposition $\mathbf{i} = (i_1, \ldots, i_N) \in R(w_0)$ we associate an enumeration of positive roots in Δ_+: for $k = 1, \ldots, N$, we set $\beta_k^{\mathbf{i}} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Delta_+$, and $\underline{\beta}^{\mathbf{i}} := (\beta_1^{\mathbf{i}}, \beta_2^{\mathbf{i}}, \cdots, \beta_N^{\mathbf{i}})$.

For $1 \leqslant i \leqslant n$, let $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the Lusztig's automorphism (see [24, Chapter 37] for details, our choice here is $T_i = T''_{i,1}$).

For a reduced word $\mathbf{i} \in R(w_0)$ and $m \in \mathbb{N}$, the quantum PBW root vector $F_{\beta_k^{\mathbf{i}}}^{(m)}$ associated to $\beta_k^{\mathbf{i}}$ is defined by:

$$F_{\beta_k^{\mathbf{i}}}^{(m)} := T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(F_{i_k}^{(m)}) \in U_q(\mathfrak{n}^-),$$

where

$$F_i^{(n)} = \frac{F_i^n}{[n]_q!}, \quad [k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}, \text{ and } [k]_q! = [k]_q[k-1]_q \cdots [1]_q.$$

For $\mathbf{m} = (m_1, m_2, \dots, m_N) \in \mathbb{N}^N$, we denote

$$F_{\mathbf{i}}^{(\mathbf{m})} := F_{\beta_1^{\mathbf{i}}}^{(m_1)} F_{\beta_2^{\mathbf{i}}}^{(m_2)} \cdots F_{\beta_N^{\mathbf{i}}}^{(m_N)} \in U_q(\mathfrak{n}^-).$$

According to [24, Corollary 40.2.2], for any $\mathbf{i} \in R(w_0)$, the set $\{F_{\mathbf{i}}^{(\mathbf{m})} \mid \mathbf{m} \in \mathbb{N}^N\}$ forms a vector space basis of $U_q(\mathfrak{n}^-)$.

There exists a bar involution $\bar{\cdot} : U_q(\mathfrak{n}^-) \to U_q(\mathfrak{n}^-)$, which is a $\mathbb{C}$-algebra automorphism uniquely determined by $\overline{q} = q^{-1}$ and $\overline{F_i} = F_i$.

There is a remarkable basis of $U_q(\mathfrak{n}^-)$, whose existence is guaranteed by the following theorem (see [22, 4]).

Theorem 2.1. *Let* $\mathbf{i} \in R(w_0)$.

1. *For any* $\mathbf{m} \in \mathbb{N}^N$, *there exists a unique element* $b_{\mathbf{i}}(\mathbf{m}) \in U_q(\mathfrak{n}^-)$ *satisfying the following properties:*
$$\overline{b_{\mathbf{i}}(\mathbf{m})} = b_{\mathbf{i}}(\mathbf{m}),$$
$$b_{\mathbf{i}}(\mathbf{m}) = F_{\mathbf{i}}^{(\mathbf{m})} + \sum_{\mathbf{n} \prec \mathbf{m}} \lambda_{\mathbf{m}}^{\mathbf{n}} F_{\mathbf{i}}^{(\mathbf{n})}, \ \ \lambda_{\mathbf{m}}^{\mathbf{n}} \in q\mathbb{Z}[q].$$
2. *The map* $b_{\mathbf{i}}$ *sending* $\mathbf{m}$ *to* $b_{\mathbf{i}}(\mathbf{m})$ *gives a bijection between* $\mathbb{N}^N$ *and a basis* $\mathbf{B}$ *of* $U_q(\mathfrak{n}^-)$. *This basis* $\mathbf{B}$ *does not depend on the choice of* $\mathbf{i} \in R(w_0)$.
3. *For* $\lambda \in \Lambda^+$, *we set* $\mathbf{B}(\lambda) := \{b \in \mathbf{B} \mid b \cdot v_\lambda^q \neq 0\}$. *The set* $\{b \cdot v_\lambda^q \mid b \in \mathbf{B}(\lambda)\}$ *is a basis of* $V_q(\lambda)$.

This basis $\mathbf{B}$ is called the *canonical basis* [22] (a.k.a. *global crystal basis* [17]) of $U_q(\mathfrak{n}^-)$, and the map $b_{\mathbf{i}} : \mathbb{N}^N \to \mathbf{B}$ is called the Lusztig parametrisation of the canonical basis corresponding to the reduced decomposition $\mathbf{i}$.

Theorem 2.2 ([2])**.** *For* $\mathbf{i} \in R(w_0)$ *and* $\lambda \in \Lambda^+$, *there exists a rational polytope* $\mathcal{L}_{\mathbf{i}}(\lambda) \subseteq \mathbb{R}^{\Delta_+}$ *satisfying*

$$b_{\mathbf{i}}^{-1}(\mathbf{B}(\lambda)) = \mathcal{L}_{\mathbf{i}}(\lambda)_{\mathbb{Z}},$$

where $\mathcal{L}_{\mathbf{i}}(\lambda)_{\mathbb{Z}} := \mathcal{L}_{\mathbf{i}}(\lambda) \cap \mathbb{Z}^{\Delta_+}$.

The polytope $\mathcal{L}_{\mathbf{i}}(\lambda)$ is called the Lusztig polytope associated to $\mathbf{i}$ and λ. The original definition of Lusztig, Berenstein–Zelevinsky [23, 2] uses piecewise linear combinatorics arising from tropicalisation of positive maps between tori. Later we will present two different approaches to these polytopes.

There are two special reduced decompositions in $R(w_0)$:

$$\text{the lexmin decomposition: } \mathbf{i}_n^{\min} = (1, 2, 1, 3, 2, 1, \dots, n, n-1, \dots, 1),$$

$$\text{the lexmax decomposition: } \mathbf{i}_n^{\max} = (n, n-1, n, n-2, n-1, n, \dots, 1, \dots, n).$$

For a dominant weight $\lambda \in \Lambda^+$, it is well-known that the Lusztig polytope $\mathcal{L}_{\mathbf{i}^{\min}}(\lambda)$ is unimodular equivalent to the Gelfand–Tsetlin polytope (for recent references, see [19, 25]).

Recently, these polytopes are applied to study the branching problem of representations by Molev and Yakimova [26]; the tropical maximal cone of the toric degeneration of the flag variety arising from $\mathcal{L}_{\mathbf{i}^{\min}}(\lambda)$ is determined by Makhlin [25].

2.3. FFLV basis and FFLV polytopes

With a different motivation, the FFLV polytopes [11, 12] appear in the study of bases compatible with the PBW filtration on finite dimensional irreducible representations of a simple Lie algebra. When the Lie algebra is of type A and C, lattice points in these (lattice) polytopes parametrise such a basis in the representation.

We briefly recall the definition and basic properties of these polytopes.

A (type A) Dyck path in Δ_+ is a sequence of positive roots $\mathbf{p} = (\gamma_0, \gamma_1, \dots, \gamma_k)$ for $k \geqslant 0$ satisfying

1. $\gamma_0 = \alpha_i$ and $\gamma_k = \alpha_j$ are simple roots;
2. if $\gamma_r = \alpha_{s,t}$, then γ_{r+1} is either $\alpha_{s+1,t}$ or $\alpha_{s,t+1}$.

For $1 \leqslant i \leqslant j \leqslant n$, we set $\mathbb{P}_n := \bigcup_{1\leqslant i\leqslant j\leqslant n} \mathbb{P}_{i,j}$ where $\mathbb{P}_{i,j}$ is the set of Dyck paths starting from α_i and ending in α_j;

For $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2 + \cdots + \lambda_n\varpi_n \in \Lambda^+$, the polytope $\mathrm{FFLV}_n(\lambda)$ consists of the points $(a_\gamma) \in \mathbb{R}^{\Delta_+}$ satisfying

1. for any $\mathbf{p} \in \mathbb{P}_n$, if $\mathbf{p} \in \mathbb{P}_{i,j}$ then
$$\sum_{\gamma\in\mathbf{p}} a_\gamma \leqslant \lambda_i + \lambda_{i+1} + \cdots + \lambda_j;$$
2. for any $\gamma \in \Delta_+$, $a_\gamma \geqslant 0$.

We denote $\mathrm{FFLV}_n(\lambda)_\mathbb{Z} := \mathrm{FFLV}_n(\lambda) \cap \mathbb{Z}^{\Delta_+}$ to be the set of lattice points in $\mathrm{FFLV}_n(\lambda)$.

For a fixed enumeration $\beta_1, \beta_2, \dots, \beta_N$ of Δ_+ and a lattice point $\mathbf{a} \in \mathrm{FFLV}_n(\lambda)_\mathbb{Z}$, we set

$$f^{\mathbf{a}} := f_{\beta_1}^{a_{\beta_1}} f_{\beta_2}^{a_{\beta_2}} \cdots f_{\beta_N}^{a_{\beta_N}} \in U(\mathfrak{n}^-).$$

Theorem 2.3 ([11]). *The following statements hold:*

1. *The set* $\{f^{\mathbf{a}} \cdot v_\lambda \mid \mathbf{a} \in \mathrm{FFLV}_n(\lambda)_\mathbb{Z}\} \subseteq V(\lambda)$ *forms a basis.*
2. $\mathrm{FFLV}_n(\lambda)$ *is a lattice polytope satisfying the following Minkowski property: for* $\lambda, \mu \in \Lambda^+$,
$$\mathrm{FFLV}_n(\lambda)+\mathrm{FFLV}_n(\mu) = \mathrm{FFLV}_n(\lambda+\mu),\ \mathrm{FFLV}_n(\lambda)_\mathbb{Z}+\mathrm{FFLV}_n(\mu)_\mathbb{Z} = \mathrm{FFLV}_n(\lambda+\mu)_\mathbb{Z}.$$

Remark 2.4. As a consequence of the main result in [13], $\mathrm{FFLV}_n(\lambda)$ is in general not unimodular equivalent to the Lusztig polytope $\mathcal{L}_{\mathbf{i}^{\min}}(\lambda)$ (since it is not unimodular equivalent to the Gelfand–Tsetlin polytope).

2.4. Main result: statement

We start with defining some special reduced decomposition in $R(w_0)$.

For a fixed $1 \leqslant k \leqslant n$, we define

$$\underline{w}_{k,n} := (s_k s_{k-1} \cdots s_1)(s_{k+1} s_k \cdots s_2) \cdots (s_n s_{n-1} \cdots s_{n-k+1}),$$

where $w_{k,n}$ is the corresponding element in the Weyl group. We set furthermore

$$\underline{w}_{n-(k-1)+[k-1]} := (s_n s_{n-1} \cdots s_{n-k+2})(s_n s_{n-1} \cdots s_{n-k+3}) \cdots (s_n s_{n-1}) s_n,$$

and

$$\underline{w}_{[n-k]} = (s_1 s_2 \cdots s_{n-k})(s_1 s_2 \cdots s_{n-k-1}) \cdots (s_1 s_2) s_1.$$

Then $\underline{w}^{-1}_{n-(k-1)+[k-1]}$ is a shift of $\mathbf{i}^{\max}_{k-1}$ and $\underline{w}^{-1}_{[n-k]}$ coincides with $\mathbf{i}^{\min}_{n-k}$, therefore

$$\ell(\underline{w}_{k,n}) = (n-k+1)k, \quad \ell(\underline{w}_{n-(k-1)+[k-1]}) = \frac{k(k-1)}{2}, \quad \ell(\underline{w}_{[n-k]}) = \frac{(n-k+1)(n-k)}{2}.$$

Lemma 2.5. *For any* $k = 1, \ldots, n$, $\underline{w}_{k,n} \cdot \underline{w}_{n-(k-1)+[k-1]} \cdot \underline{w}_{[n-k]} \in R(w_0)$.

Proof. The wiring diagram corresponding to this decomposition has the following form: the first k wires go parallel to the NE direction until the wire labeled k touches the “roof”, and the wires $k+1, \ldots, n+1$ go parallel to the SE direction until the wire labeled $k+1$ touches the “floor”. Then the first k wires go to the east and each two of them cross follow the lexmax reduced decomposition of the longest permutation of $\mathfrak{S}_{k-1}$ in the alphabet $\{1, \ldots, k\}$, and the wires $k+1, \ldots, n+1$ got to the east and each two of them cross follow the lexmin reduced decomposition of the inverse permutations of $\mathfrak{S}_{n-k}$ in the alphabet $\{k+1, \ldots, n+1\}$.

We see that the intersection of the first k wires with the wires labeled by $k+1, \ldots, n+1$ form a rectangular being rotated around the corner corresponding to the intersection of the k and $k+1$ wires.

An example with $n = 5$, $k = 3$ is illustrated in Figure 2.1 □

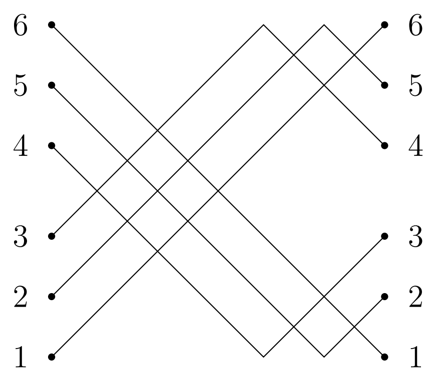

Figure 2.1: Wiring diagram for $n = 5$ and $k = 3$.

We will denote $\mathbf{i}^k$ the concatenation of those reduced words in the above lemma: it is a reduced decomposition of w_0. For instance, when $n = 3$ we have:

$$\mathbf{i}^1 = (1,2,3,1,2,1),\ \ \mathbf{i}^2 = (2,1,3,2,3,1),\ \ \mathbf{i}^3 = (3,2,1,3,2,3).$$

Theorem 2.6. *For $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2 + \cdots + \lambda_n\varpi_n \in \Lambda^+$, as polytopes in $\mathbb{R}^{\Delta_+}$,*

$$\mathrm{FFLV}_n(\lambda) = \mathcal{L}_{\mathbf{i}^1}(\lambda_1\varpi_1) + \mathcal{L}_{\mathbf{i}^2}(\lambda_2\varpi_2) + \cdots + \mathcal{L}_{\mathbf{i}^n}(\lambda_n\varpi_n).$$

Moreover, on the level of lattice points,

$$\mathrm{FFLV}_n(\lambda)_{\mathbb{Z}} = \mathcal{L}_{\mathbf{i}^1}(\lambda_1\varpi_1)_{\mathbb{Z}} + \mathcal{L}_{\mathbf{i}^2}(\lambda_2\varpi_2)_{\mathbb{Z}} + \cdots + \mathcal{L}_{\mathbf{i}^n}(\lambda_n\varpi_n)_{\mathbb{Z}}.$$

Remark 2.7. When $\lambda = r\varpi_k$, Theorem 2.6 explains why in [7] we can get FFLV polytopes for multiples of fundamental weights from a particular chart of positive Grassmannians, although there is no connection known between FFLV bases and total positivity.

In the next two sections, we will present two different proofs of this theorem:

1. The first representation-theoretic proof based on realising both polytopes as Newton–Okounkov bodies. We will apply the representation-theoretic interpretation (essential monomials) of the lattice points in these Newton–Okounkov bodies given in [8].

2. The second convex-geometric proof relies on an explicit description to the defining inequalities of the Lusztig polytope $\mathcal{L}_{\mathbf{i}^k}(r\varpi_k)$ arising from an interplay of the crystal structure and the cluster structure [14].

3. Algebraic proof

3.1. Birational sequences

Let $S = (\beta_1, \ldots, \beta_N)$ with $\beta_i \in \Delta_+$ be a sequence of positive roots (repetitions allowed). It is called a *birational sequence*, if the multiplication map

$$U_{-\beta_1} \times U_{-\beta_2} \times \cdots \times U_{-\beta_N} \to U^-,$$

$$(u_1, u_2, \ldots, u_N) \mapsto u_1 u_2 \ldots u_N$$

is birational.

Let $>$ be a fixed total ordering on $\mathbb{N}^N$. We will associate to a birational sequence $S = (\beta_1, \ldots, \beta_N)$ and this total ordering a semigroup and a cone. By fixing these data one defines a filtration on $U(\mathfrak{n}^-)$ by setting for $\mathbf{m} \in \mathbb{N}^N$,

$$U(\mathfrak{n}^-)_{\leqslant \mathbf{m}} := \mathrm{span}\{f^{\mathbf{a}} := f_{\beta_1}^{a_1} \cdots f_{\beta_N}^{a_N} \mid \mathbf{a} \leqslant \mathbf{m}\};$$

and similarly we have $U(\mathfrak{n}^-)_{<\mathbf{m}} \subseteq U(\mathfrak{n}^-)_{\leqslant \mathbf{m}}$. These filtrations on $U(\mathfrak{n}^-)$ induce filtrations on $V(\lambda)$ by requiring for $\mathbf{m} \in \mathbb{N}^N$

$$V(\lambda)_{\leqslant \mathbf{m}} := U(\mathfrak{n}^-)_{\leqslant \mathbf{m}} \cdot v_\lambda,\ \ V(\lambda)_{<\mathbf{m}} := U(\mathfrak{n}^-)_{<\mathbf{m}} \cdot v_\lambda.$$

Definition 3.1 ([8])**.** An element $\mathbf{m} \in \mathbb{N}^N$ is called an *essential exponent* with respect to $(S, >)$ if

$$\dim \left(V(\lambda)_{\leqslant \mathbf{m}}/V(\lambda)_{<\mathbf{m}}\right) = 1.$$

The set of essential exponents in $V(\lambda)$ will be denoted by $\mathrm{es}_\lambda(S, >)$. We set

$$\Gamma_\lambda(S, >) := \bigcup_{k \in \mathbb{N}} \{k\} \times \mathrm{es}_{k\lambda}(S, >) \subseteq \mathbb{N} \times \mathbb{N}^N,$$

and define the *global essential monoid*

$$\Gamma(S, >) := \bigcup_{\lambda \in \Lambda^+} \{\lambda\} \times \mathrm{es}_\lambda(S, >) \subseteq \Lambda^+ \times \mathbb{N}^N.$$

By taking the function Ψ in [8], Proposition 1 to be the zero function, we have:

Proposition 3.2. *With the induced structures from $\mathbb{N} \times \mathbb{N}^N$ and $\Lambda^+ \times \mathbb{N}^N$, the sets $\Gamma_\lambda(S, >)$ and $\Gamma(S, >)$ are monoids.*

3.2. Realisation of polytopes

We set $\Lambda := \mathbb{Z}\varpi_1 + \cdots + \mathbb{Z}\varpi_n$ and $\Lambda_\mathbb{R} := \Lambda \otimes_\mathbb{Z} \mathbb{R}$. Let

$$C_\lambda(S, >) \subseteq \mathbb{R} \times \mathbb{R}^N \text{ and } C(S, >) \subseteq \Lambda_\mathbb{R} \times \mathbb{R}^N$$

be the cones generated by the sets $\Gamma_\lambda(S, >)$ and $\Gamma(S, >)$, respectively. By cutting the cone $C_\lambda(S, >)$ we obtain a convex body

$$\Delta_\lambda(S, >) := (\{1\} \times \mathbb{R}^N) \cap C_\lambda(S, >),$$

called the Newton–Okounkov body associated to $(S, >)$ and the weight λ. It is shown in [8] that $\Delta_\lambda(S, >)$ coincides with the Newton–Okounkov body associated to a valuation.

We provide some examples of this construction. A reduced word $\mathbf{i} = (i_1, \ldots, i_N) \in R(w_0)$ gives a birational sequence

$$P_\mathbf{i} = (\beta_1^\mathbf{i}, \ldots, \beta_N^\mathbf{i}).$$

We fix the right opposite lexicographic ordering $>_{\mathrm{roplex}}$ on $\mathbb{Z}^N$. The following theorem identifies the Newton–Okounkov bodies associated to $(P_\mathbf{i}, >_{\mathrm{roplex}})$ with the known polytopes.

By fixing the enumeration $\beta_1^\mathbf{i}, \ldots, \beta_N^\mathbf{i}$ of Δ_+, we identify $\mathbb{R}^{\Delta_+}$ with $\mathbb{R}^N$.

Theorem 3.3 ([8], Theorem 2)**.** *We have $\Delta_\lambda(P_\mathbf{i}, >_{\mathrm{roplex}}) = \mathcal{L}_\mathbf{i}(\lambda)$.*

3.3. Proof of Theorem

We start with showing some properties of the reduced word $\mathbf{i}^k$. Let $(\beta_1^{\mathbf{i}^k}, \beta_2^{\mathbf{i}^k}, \ldots, \beta_N^{\mathbf{i}^k})$ be the enumeration of positive roots associated to $\mathbf{i}^k$.

For a positive root $\alpha_{i,j} \in \Delta_+$, we set $\mathrm{ht}(\alpha_{i,j}) = j - i + 1$ to be the height of the root.

Lemma 3.4. *The following statements hold:*

1. *We have* $\{\beta_1^{\mathbf{i}^k}, \ldots, \beta_{k(n-k+1)}^{\mathbf{i}^k}\} = \{\alpha_{i,j} \mid 1 \leqslant i \leqslant k \leqslant j \leqslant n\}$.
2. *We have* $\{\beta_{k(n-k+1)+1}^{\mathbf{i}^k}, \ldots, \beta_N^{\mathbf{i}^k}\} = \{\alpha_{i,j} \mid k \notin [i,j]\}$.
3. *We denote* $\beta_s^{\mathbf{i}^k} = \alpha_{p_1,q_1}$ *and* $\beta_t^{\mathbf{i}^k} = \alpha_{p_2,q_2}$. *If* $s < t \leqslant k$, *then either* $q_1 > q_2$ *or* $q_1 = q_2$, $p_2 < p_1$.

Proof. It suffices to notice that by definition, $(\beta_1^{\mathbf{i}^k}, \beta_2^{\mathbf{i}^k}, \ldots, \beta_{k(n-k+1)}^{\mathbf{i}^k})$ is

$$(\alpha_{k,k}, \alpha_{k-1,k}, \ldots, \alpha_{1,k}, \alpha_{k,k+1}, \ldots, \alpha_{1,k+1}, \ldots, \alpha_{k,n}, \ldots, \alpha_{1,n}). \qquad \square$$

We first consider the case of a fundamental weight.

Proposition 3.5. *We have* $\mathrm{FFLV}_n(\varpi_k) = \mathcal{L}_{\mathbf{i}^k}(\varpi_k)$.

We need some preparations for the proof:

Lemma 3.6. *For* $\mathbf{a} \in \mathcal{L}_{\mathbf{i}^k}(\varpi_k)$, *if* $\ell > k(n-k+1)$, *then* $a_{\beta_\ell^{\mathbf{i}^k}} = 0$.

Proof. This holds by the definition of the filtration on $V(\varpi_k)$. According to Lemma 3.4 (2), if $\ell > k(n-k+1)$ then $f_{\beta_\ell^{\mathbf{i}^k}}$ acts by zero on the highest weight vector v_{ϖ_k}. $\square$

We recall the description of the lattice points in the FFLV polytope in [10, 9].

As representations of $\mathfrak{g}$, we have $V(\varpi_k) \cong \Lambda^k \mathbb{C}^{n+1}$. We fix the standard basis $\{e_1, \ldots, e_{n+1}\}$ of $\mathbb{C}^{n+1}$. Then $V(\varpi_k)$ has a linear basis

$$\{e_{j_1} \wedge \cdots \wedge e_{j_k} \mid 1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n+1\},$$

where $e_1 \wedge \cdots \wedge e_k$ is a highest weight vector. We fix the index s satisfying

$$1 \leqslant j_1 < \cdots < j_s \leqslant k < j_{s+1} < \cdots < j_k \leqslant n+1,$$

and set $\{p_1, \ldots, p_{k-s}\} = \{1, \ldots, k\} \setminus \{j_1, \ldots, j_s\}$ with the ordering $p_1 < \cdots < p_{k-s}$.

For $\mathbf{a} \in \mathbb{N}^{\Delta_+}$, we will denote $a_\ell^k := a_{\beta_\ell^{\mathbf{i}^k}}$. If $\mathbf{a}$ satisfies for any $\ell > k(n-k+1)$, $a_\ell^k = 0$, we will write the monomial

$$f^{\mathbf{a}} := f_{\beta_1^{\mathbf{i}^k}}^{a_1^k} f_{\beta_2^{\mathbf{i}^k}}^{a_2^k} \cdots f_{\beta_k^{\mathbf{i}^k}}^{a_k^k}.$$

Note that this monomial does not depend on the order of the root vectors f_β it contains. We will write $f_{i,j} := f_{\alpha_{i,j}}$ for short.

A monomial $f^{\mathbf{a}}$ satisfies

$$f^{\mathbf{a}} \cdot e_1 \wedge \cdots \wedge e_k \text{ is proportional to } e_{j_1} \wedge \cdots \wedge e_{j_k}$$

if and only if it has the form

$$f_{p_{\sigma(1)}, j_k - 1} f_{p_{\sigma(2)}, j_{k-1} - 1} \cdots f_{p_{\sigma(k-s)}, j_{s+1} - 1} \tag{3.1}$$

for some $\sigma \in \mathfrak{S}_{k-s}$.

The point in $\mathrm{FFLV}_n(\varpi_k)$ corresponding to the basis $e_{j_1} \wedge \cdots \wedge e_{j_k}$ is given by the function $\mathbf{p}_{j_1,\ldots,j_k}$ defined by:

$$\mathbf{p}_{j_1,\ldots,j_k}(\alpha_{i,j}) := \begin{cases} 1, & \text{if } (i,j) = (p_r, j_{k-r+1} - 1) \text{ for some } r \in [1, k-s]; \\ 0, & \text{otherwise.} \end{cases}$$

It corresponds to the case $\sigma = \mathrm{id}$ in (3.1).

Remark 3.7. Such a function $\mathbf{p}_{j_1,\ldots,j_k}$ corresponds to corners of a path from $\alpha_{1,k}$ to $\alpha_{k,n}$ in the rectangular consisting of roots in Lemma 3.4 (1).

Proposition 3.8 ([10])**.** *We have* $\mathrm{FFLV}_n(\varpi_k)_{\mathbb{Z}} = \{\mathbf{p}_{j_1,\ldots,j_k} \mid 1 \leqslant j_1 < \cdots < j_k \leqslant n+1\}$.

We turn to the proof of Proposition 3.5.

Proof of Proposition 3.5. We first show that $\mathrm{FFLV}_n(\varpi_k)_{\mathbb{Z}} = \mathcal{L}_{\mathbf{i}^k}(\varpi_k)_{\mathbb{Z}}$. Putting together the discussions above, it suffices to show that for any $1 \leqslant j_1 < \cdots < j_k \leqslant n+1$, $\mathbf{p}_{j_1,\ldots,j_k} \in \mathcal{L}_{\mathbf{i}^k}(\varpi_k)$. According to Theorem 3.3, we show that $\mathbf{p}_{j_1,\ldots,j_k} \in \Delta_{\varpi_k}(P_{\mathbf{i}^k}, >_{\mathrm{roplex}})$.

By Definition 3.1 and Lemma 3.6, this amount to determine under the right opposite lexicographic ordering, which monomial in (3.1) is minimal. According to Lemma 3.4 (3), we opt to choose the root vectors $f_{i,j}$ where the second index is large, and the first index is small. In (3.1), the second index satisfies $j_k - 1 > j_{k-1} - 1 > \cdots > j_{s+1} - 1$: it suffices to choose σ such that $\sigma(1) < \cdots < \sigma(k-s)$, that is to say, $\sigma = \mathrm{id}$.

It remains to show that the polytopes are the same. Since $\mathrm{FFLV}_n(\varpi_k)$ is a lattice polytope, $\mathrm{FFLV}_n(\varpi_k) \subseteq \mathcal{L}_{\mathbf{i}^k}(\varpi_k)$. It is clear that they have the same dimension. Since both of them are Newton–Okounkov bodies, their volumes can be computed as the (normalized) leading coefficient of the polynomial function $m \mapsto \dim V(m\varpi_k)$. By Weyl dimension formula, they share the same volume, implying the equality. □

Combining Proposition 3.5 and the Minkowski property in Theorem 2.3 proves Theorem 2.6 for multiples of fundamental weights.

Corollary 3.9. *For any* $r \geqslant 1$, $\mathrm{FFLV}_n(r\varpi_k) = \mathcal{L}_{\mathbf{i}^k}(r\varpi_k)$.

Applying again the Minkowski property in Theorem 2.3 terminates the proof of Theorem 2.6.

4. Geometric proof

To simplify notations, we set $m = n + 1$ in this section.

4.1. Rhombic tiling

The inequalities defining type A Lusztig polytopes can be described using rhombic tilings and Reineke vectors. We briefly recall these constructions following [6, 14, 15].

First draw a $2m$-gon C_{2m} on the plane and fix a vertex v_0 of C_{2m}. One labels the edges of C_{2m} clockwise starting from v_0 by $1, 2, \ldots, m$ until a vertex v_1; these edges are called left boundary. Then continue labelling the edges starting from v_1 by $1, 2, \ldots, m$ and call them the right boundary.

We fix a reduced decomposition $\mathbf{i} = (i_1, \ldots, i_N) \in R(w_0)$ and an enumeration of positive roots $\Delta_+ = \{\beta^{\mathbf{i}}_1, \ldots, \beta^{\mathbf{i}}_N\}$ where $\beta^{\mathbf{i}}_k = \alpha_{s_k,t_k}$.

We start from α_{s_1,t_1}: this is a simple root hence $t_1 = s_1$; we complete the edges on the left boundary labeled by s_1, s_1+1 to a rhombus inside of C_{2m}. The opposite edges in this rhombus will be labelled by the same number. This gives us a new connected set of edges labeled by $1, \ldots, m$.

We move to this new set of edges and consider the second positive root α_{s_2,t_2}. In this new set of edges, edges labelled by s_2 and t_2+1 are neighbours, we complete them into a rhombus inside of C_{2m}, label the opposite edges by the same number and switch to this new set of edges. According to [6], when this procedure is applied consecutively to $\alpha_{s_1,t_1}, \ldots, \alpha_{s_N,t_N}$, we obtain a rhombic tiling $\mathcal{T}$ of the $2m$-gon C_{2m}. Every edge in the tiling is labeled by a number in $[m]$.

A set of edges is called *connected*, if there exists one and only one path between any two vertices. A connected set of edges in $\mathcal{T}$ is called a *border*, if it contains exactly one edge with each label.

Each tile T in $\mathcal{T}$ has exactly two edge labels: if these edge labels are $1 \leqslant s \neq t \leqslant m$, we will denote the tile by $T = [s, t]$.

A sequence $\gamma = (\gamma_i)_{1 \leqslant i \leqslant r}$ of tiles in $\mathcal{T}$ is termed *neighbour sequence*, if for any $1 \leqslant i \leqslant r-1$, the tiles γ_i and γ_{i+1} share an edge.

For $1 \leqslant t \leqslant m$, the t-strip $\mathcal{S}^t$ is defined to be the neighbour sequence $\gamma = (\gamma_i)_{1 \leqslant i \leqslant m-1}$ such that for any $1 \leqslant i \leqslant m-1$, one of the edges of γ_i is labeled by t and one edge labeled by t in γ_1 is on the left boundary. We will denote $\mathcal{S}^t_k := \gamma_k$.

Example 4.1. The rhombic tiling associated to the reduced word $\mathbf{i}^{\min}_n$ (resp. $\mathbf{i}^{\max}_n$) is called a *standard* (resp. *anti-standard*) tiling.

For any $s \in [2m]$ we define a partial order $\preceq_s$ on the tiles in $\mathcal{T}$ in the following way: we label the boundary of C_{2m} starting from v_0 by $b_1, \ldots, b_{2m}$. Let B_1 be the border consisting of edges $b_{m+s+1}, \ldots, b_{2m+s}$ where the indices are understood modulo $2m$.

Denote $\mathcal{T}^s_1$ to be the set of tiles in $\mathcal{T}$ intersecting B_1 in two edges. We move to a new border B_2 obtained from B_1 by: for every tile in $\mathcal{T}^s_1$, replace the two edges intersecting B_1 by the other

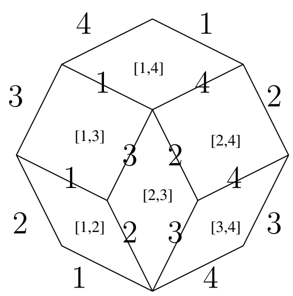

Figure 4.1: rhombic tiling for $\mathbf{i}^{\min}_3$.

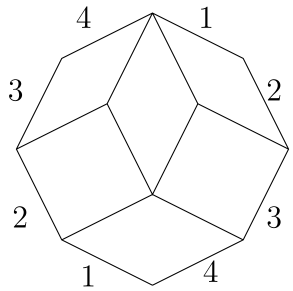

Figure 4.2: rhombic tiling for $\mathbf{i}_3^{\max}$.

two edges. Then denote $\mathcal{T}_2^s$ to be the set of tiles in $\mathcal{T} \setminus \mathcal{T}_1^s$ intersecting B_2 with two edges and repeat the above procedure.

Eventually one obtain a partition of $\mathcal{T}$ into a disjoint union of $\mathcal{T}_1^s$, $\mathcal{T}_2^s, \ldots$.

For two tiles $T, T' \in \mathcal{T}$ such that $T \in \mathcal{T}_t^s$ and $T' \in \mathcal{T}_r^s$, we will denote $T \preceq_s T'$ if $t \leqslant r$.

For $s \in [2m]$ a neighbour sequence $\gamma = (\gamma_1, \ldots, \gamma_m)$ in $\mathcal{T}$ is called *s-ascending*, if for any $1 \leqslant i \leqslant m-1$, $\gamma_i \prec_s \gamma_{i+1}$.

An s-ascending neighbour sequence $(\gamma_1, \ldots, \gamma_p)$ in $\mathcal{T}$ is called an *s-crossing*, if $\gamma_1 = \mathcal{S}_1^s$ and $\gamma_p = \mathcal{S}_1^{s+1}$.

To an s-ascending neighbour sequence $(\gamma_1, \ldots, \gamma_p)$ in $\mathcal{T}$ we associate a *strip sequence* $(s_1, \ldots, s_r)$ in the following way: there exists $1 \leqslant j_1 < \cdots < j_r < j_{r+1} = p$ such that $\gamma_1, \ldots, \gamma_{j_1} \in \mathcal{S}^{s_1}$, $\gamma_{j_1+1}, \ldots, \gamma_{j_2} \in \mathcal{S}^{s_2}, \ldots, \gamma_{j_r+1}, \ldots, \gamma_p \in \mathcal{S}^{s_r}$.

An s-crossing is uniquely determined by its strip sequence.

Let Γ_s denote the set of s-crossings in $\mathcal{T}$. We denote $\mathcal{W}_s$ the s-crossings given by the strip sequence $(s, s+1)$. Such an s-crossing exists, and will be called an *s-comb*.

4.2. Dual Reineke vectors and H-description of Lusztig polytopes

We introduce a dual version of the constructions above. They give the potential facets of Lusztig polytopes.

The set Γ_s^* of dual s-crossings consists of $(m+s)$-ascending neighbour sequences $(\gamma_1, \ldots, \gamma_p)$ at $\gamma_1 = \mathcal{S}_n^s$ and ending at $\gamma_p = \mathcal{S}_n^{s+1}$. One can similarly define the strip sequence of a dual s-crossing. A dual s-crossing is called a dual s-comb, if its strip sequence is $(s, s+1)$.

A dual s-crossing $\gamma = (\gamma_1, \ldots, \gamma_p) \in \Gamma_s$ is called a *dual Reineke s-crossing*, if for any $\gamma_i = [a, b]$ such that $\gamma_{i-1}, \gamma_i, \gamma_{i+1}$ lie in the same strip $\mathcal{S}^a$,

- if $b \leqslant s$ then $a > b$;
- if $b \geqslant s+1$ then $a < b$.

Let $\mathcal{R}_s^* \subseteq \Gamma_s^*$ denote the set of dual Reineke s-crossings.

Let $\gamma = (\gamma_1, \ldots, \gamma_p) \in \Gamma_s$ with strip sequence $(s_1 = s, \ldots, s_q = s+1)$, we define

$$(\mathsf{r}(\gamma))_T := \begin{cases} \operatorname{sgn}(s_{i+1} - s_i), & \text{if } T = [s_i, s_{i+1}] \text{ for some } 1 \leqslant i \leqslant q-1; \\ 0, & \text{otherwise;} \end{cases}$$

$$\varepsilon_s(T) := \begin{cases} 1, & \text{if } T = [a,b] \in \mathcal{T} \text{ and } a \leqslant s < s+1 \leqslant b; \\ -1 & \text{else;} \end{cases}$$

and for $\mathbf{x} \in \mathbb{R}^{\mathcal{T}}$,

$$\mathsf{s}(\gamma)(\mathbf{x}) = \sum_{T\in\Gamma,\ \varepsilon_s(T)=1} x_T - \sum_{\substack{T\in\gamma,\ \varepsilon_s(T)=-1,\\ (\mathsf{r}(\gamma))_T=0}} x_T.$$

Theorem 4.2 ([15])**.** *For* $\lambda = \lambda_1\varpi_1 + \cdots + \lambda_n\varpi_n \in \Lambda^+$,

$$\mathcal{L}_{\mathbf{i}}(\lambda) = \left\{\mathbf{x} \in \mathbb{R}^{\Delta_+}_{\geqslant 0} \mid \textit{for any } s \in [m],\ \gamma \in \mathcal{R}_s^*,\ \mathsf{s}(\gamma)(\mathbf{x}) \leqslant \lambda_s\right\}.$$

4.3. Proof of Theorem

The goal of this subsection is to give a second proof to Theorem 2.6.

We start from considering the case $\lambda = \varpi_k$ for $1 \leqslant k < m = n+1$ and the rhombic tiling associated to $\mathbf{i}^k$. Such a tiling can be obtained by gluing together the following three parts:

- a rectangular tableau of size $k \times (m-k)$ slightly rotated counterclockwise around its SW-corner;
- the east border of the rectangle is glued with the left border of the anti-standard tiling for SL_k, where the top vertex in the anti-standard tiling is glued together with the NE-corner vertex of the rectangle; we denote the tiles in the anti-standard tiling by $\mathcal{T}'$;
- the south border of the rectangular tableau is glued with the left border of the standard tiling for SL_{m-k}, where the bottom vertex in the standard tiling is glued together with the SW-corner vertex of the rectangle; we denote the tiles in the standard tiling by $\mathcal{T}''$.

An example for $m = 7$ and $k = 3$ is illustrated in Figure 4.3.

In the following we will denote the tile $[s,t]$ by $T_{[s,t]}$; for $\mathbf{x} \in \mathbb{R}^{\mathcal{T}}$, the value assigned to the tile $[s,t]$ is $x_{s,t}$.

Lemma 4.3. *For* $\mathbf{x} \in \mathcal{L}_{\mathbf{i}^k}(r\varpi_k)$*, if* $a \leqslant k$ *and* $k+1 \leqslant b$ *do not hold simultaneously, then* $x_{a,b} = 0$.

Proof. The tiles in the anti-standard (resp. standard) part $\mathcal{T}'$ (resp. $\mathcal{T}''$) have the form $T_{[a,b]}$ where $a, b < k$ (resp. $a, b \geqslant k+1$).

We look at the set Γ_s for $s < k$. First notice that the dual s-crossings will not go outside of the tiles in $\mathcal{T}'$, and any tile in $\mathcal{T}'$ is contained in some dual Reineke s-crossing for $s < k$. By Theorem 4.2, $\lambda_s = 0$ implies that for any $a \leqslant b < k$, $x_{a,b} = 0$.

A similar argument shows that for any $k+1 \leqslant a \leqslant b$, $x_{a,b} = 0$. ☐

We consider the dual k-crossings Γ_k^*. The dual k-comb is the union of the k-strip $\mathcal{S}^k$ and the $(k+1)$-strip $\mathcal{S}^{k+1}$; it turns from the k-strip to the $(k+1)$-strip at the tile $T_{[k,k]}$.

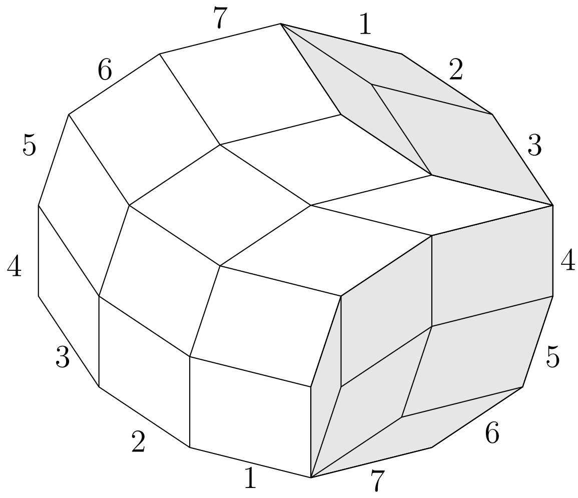

Figure 4.3: Rhombic tiling for $m = 7$ associated to $\mathbf{i}^3$.

Lemma 4.4. *The following statements hold:*

1. *Each dual k-crossing is contained in the dual k-comb.*
2. *We have $\Gamma_k^* = \mathcal{R}_k^*$.*

Proof. 1. This follows from the dual version of the poset structure on Γ_k^*, where the dual k-comb is the maximal element [14, Section 2.4].

2. Let $\gamma_i = [a, b]$ be contained in a k-crossing such that $\gamma_{i-1}, \gamma_{i+1}$ are both in the a-strip. First assume that $b \leqslant k$, such a b-strip is horizontal, to cross it the a-strip must be vertical, hence $a > b$. Similarly if $b \geqslant k+1$, such a b-strip is vertial, hence $a < b$.

□

We consider a dual k-crossing in Γ_k^*: such a crossing starts from the right boundary at the tile having a boundary labeled by k. Before going to the rectangular tableau, it goes along the tiles $T_{[1,k]}, \ldots, T_{[r,k]}$ for some $r \leqslant k-1$. Then the dual k-crossing goes into the rectangular tableau at one of the tiles $T_{[1,m]}, \ldots, T_{[k,m]}$.

Inside the rectangular tableau, when the crossing reaches a tile $T_{[p,q]}$, the next tile in the crossing can only be $T_{[p-1,q]}$ or $T_{[p,q-1]}$. The crossing goes until it reaches the 1-strip, say, tiles $T_{[1,k+1]}, \ldots, T_{[1,m]}$. After that the crossing goes into the standard tiling, along the $(k+1)$-strip until it reaches the tile having a boundary labeled by $k+1$.

As a conclusion, inside the rectangular tableau, a dual k-crossing gives a Dyck path, which is saturated in the sense that there is no such a Dyck path in the rectangular tableau containing it.

It remains to consider $\mathsf{s}(\gamma)$ for $\gamma \in \mathcal{R}_k^*$. Every tile T in the rectangular tableau gets $\varepsilon_k(T) = 1$. We do not need to care about $\varepsilon_k(T)$ for T being outside of the rectangular tableau since the corresponding coordinates are always zero by Lemma 4.3. Therefore the Lusztig polytope $\mathcal{L}_{\mathbf{i}^k}(r\varpi_k)$ admits the following description:

- for any $r \leqslant s$ such that $k \notin [r, s]$, $x_{r,s} = 0$;
- for any Dyck path starting from the right border of the rectangular tableau and ending up with the bottom of the tableau, the sum of the coordinates associated to the tiles is less or equal to r.

These are nothing but the defining inequalities of $\mathrm{FFLV}(r\varpi_k)$. The proof is complete.

5. Crystal structure

As an application to the main result, we propose a conjecture on crystal structures on FFLV parametrisations and examine it in small rank examples.

5.1. Crystal structure on Lusztig polytopes

Recall (see Lemma 3.4) that for the reduced decomposition $\mathbf{i}^k$, the corresponding enumeration of positive roots begins with $k \times (n-k+1)$ roots $\alpha_{i,j}$, $1 \leqslant i \leqslant k \leqslant j \leqslant n$.

For $1 \leqslant a \leqslant n$, let f_a denote the Kashiwara operator corresponding to a. We will denote $f_{a,k}$ the Kashiwara operator for $\mathcal{L}_{\mathbf{i}^k}(r\varpi_k)$ to emphasise its connection to the fundamental weight ϖ_k.

The crystal structure on lattice points of the Lusztig polytope $\mathcal{L}_{\mathbf{i}^k}(r\varpi_k)$ is defined by the set of Reineke vectors in [14, Section 4].

Precisely, for a lattice point $\mathbf{x} \in \mathcal{L}_{\mathbf{i}^k}(r\varpi_k)$, the point $f_{a,k}(\mathbf{x})$ takes one of the following forms:

1. when $1 \leqslant a < k$, there exists $k \leqslant j \leqslant n$ such that $f_{a,k}(\mathbf{x}) = \mathbf{x} - \delta_{a+1,j} + \delta_{a,j}$;
2. when $k < a \leqslant n$, there exists $1 \leqslant i \leqslant k$ such that $f_{a,k}(\mathbf{x}) = \mathbf{x} - \delta_{i,a-1} + \delta_{i,a}$;
3. when $a = k$, $f_{a,k}(\mathbf{x}) = \mathbf{x} + \delta_{k,k}$;

where $\delta_{i,j}$ is the function in $\mathbb{R}^{\Delta_+}$ taking value 1 on $\alpha_{i,j}$ and 0 on the other positive roots.

As a consequence of Theorem 2.6, on $\mathrm{FFLV}_n(r\varpi_k)$ there exists a crystal structure. Such a structure coincides with the one defined explicitly by Kus in [20]. The main results in [20, Section 3] then follow from the crystal structures on Lusztig polytopes described above.

5.2. Crystal structures on FFLV polytopes

For $\lambda \in \Lambda^+$, we define an edge-colored directed graph structure on the set $\mathrm{FFLV}_n(\lambda)_{\mathbb{Z}}$ with colors $\{1, 2, \ldots, n\}$. For a point $\mathbf{z} \in \mathrm{FFLV}_n(\lambda)_{\mathbb{Z}}$ and $1 \leqslant a \leqslant n$, for $k = 1, \ldots, n$, there exists an edge colored by a from $\mathbf{z}$ to $f_{a,k}(\mathbf{z})$ if $f_{a,k}(\mathbf{z})$ is a lattice points in $\mathrm{FFLV}_n(\lambda)_{\mathbb{Z}}$ (see Section 5.1 for the definition of $f_{a,k}(\mathbf{z})$).

We denote this colored directed graph by $\mathrm{PB}_n(\lambda)$. Below is an example of $\mathrm{PB}_3(\varpi_1 + \varpi_2)$, where the color 1 (resp. 2) is displayed by red (resp. blue). The coordinates e_1, e_{12}, e_2 stand for the functions in $\mathbb{R}^{\Delta_+}$ corresponding to $\alpha_1, \alpha_1 + \alpha_2, \alpha_2$, respectively.

An edge-colored directed graph with vertices $\mathrm{FFLV}_n(\lambda)_{\mathbb{Z}}$ is called an *FFLV-crystal graph*, if

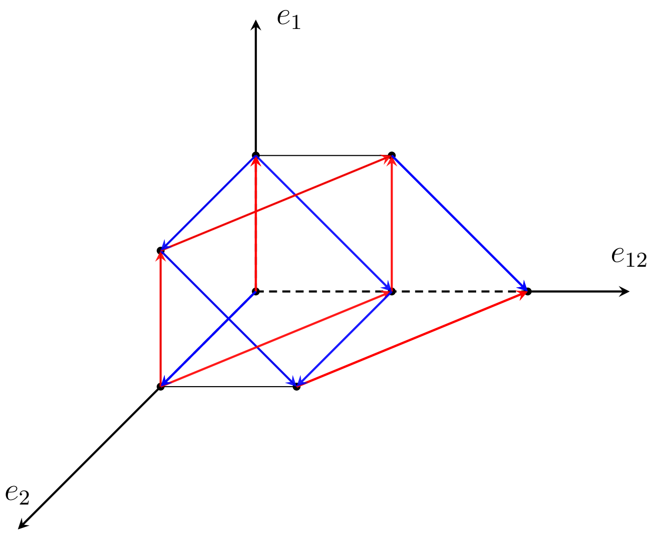

Figure 5.1: The directed graph $\mathrm{PB}_3(\varpi_1 + \varpi_2)$.

- at each vertex, for a fixed color, it has at most one of the emanating edges in $\mathrm{PB}_n(\lambda)$ having this color;
- the Stembridge axioms in [27] (or equivalently, local conditions (A1)–(A4) in [5]) for any pair of colors a and b with $|a - b| = 1$ are satisfied.

Note that $\mathrm{PB}_n(\lambda)$ needs not to be an FFLV-crystal graph. For example, for $\mathrm{PB}_3(\varpi_1 + \varpi_2)$ (see Figure 5.1), the node e_2 has two outgoing red edges, node $e_1 + e_{12}$ has two ingoing red edges, node e_1 has two ingoing blue edges and $e_2 + e_{12}$ has two ingoing blue edges. In order to get an FFLV-crystal graph, we have to delete some of edges. One can check that the only possibilities to delete extra edges of $\mathrm{PB}_3(\varpi_1 + \varpi_2)$ and get an FFLV-crystal graph are depicted in Figure 5.4 and Figure 5.8. Thus, in such a case, we get two different FFLV-crystal graphs.

Conjecture 5.1. When λ is regular, there exist $n!$ FFLV-crystal graphs on $\mathrm{FFLV}_n(\lambda)_{\mathbb{Z}}$. Each of crystal graphs is defined by fixing an element $\sigma \in \mathfrak{S}_n$.

To be precise, this choice of $\sigma \in \mathfrak{S}_n$ defines a total ordering on the fundamental weights by setting

$$\varpi_{\sigma(1)} > \varpi_{\sigma(2)} > \cdots > \varpi_{\sigma(n)}.$$

Such a choice of the total ordering means that there exists an iterative process such that at each vertex $\mathbf{z} \in \mathrm{FFLV}_n(\lambda)_{\mathbb{Z}}$, if there are several, allowed due to this process, edges in $\mathrm{PB}_n(\lambda)$ emanating from this vertex to points $f_{a,k_1}(\mathbf{z}), \ldots, f_{a,k_r}(\mathbf{z})$, we choose the edge towards $f_{a,k}(\mathbf{z})$ where k corresponds to the maximal element in $\{\varpi_{k_1}, \ldots, \varpi_{k_r}\}$ with respect to the fixed total ordering. Our refined conjecture affirms that after making such choices for all vertices in $\mathrm{FFLV}_n(\lambda)_{\mathbb{Z}}$, there exists a unique FFLV-crystal graph.

5.3. Small rank cases

We study this conjecture in the case of SL_3. In this case $\mathbf{i}^1 = (1,2,1)$ and $\mathbf{i}^2 = (2,1,2)$. Let $\lambda = a\varpi_1 + b\varpi_2 \in \Lambda^+$. We will describe two crystal graphs on $\mathrm{FFLV}_2(\lambda)_{\mathbb{Z}}$.

First we choose $\sigma = \mathrm{id} \in \mathfrak{S}_2$, it corresponds to a total ordering $\varpi_1 > \varpi_2$.

Let $B^{>}(a,b)$ be the edge-colored directed graph on lattice points of $\mathrm{FFLV}_2(a\varpi_1 + b\varpi_2)_{\mathbb{Z}}$ such that its monochromatic paths are defined in the following way:

1. For the color 1, we take the "sky" paths depicted in Figure 5.2 and their translations by vectors $(-k,k,0)$, $k = 1,\dots$ (precisely we takes the parts of translated paths which belong to $\mathrm{FFLV}_2(\lambda)$).

2. For the color 2, we take the "ground" paths depicted in Figure 5.3 and their translations by vectors $(k,0,k)$, $k = 1,\dots$ (precisely we takes the parts of translated paths which belong to $\mathrm{FFLV}_2(\lambda)$).

Proposition 5.2. *The edge-colored graph $B^{>}(a,b)$ is an FFLV-crystal graph.*

The corresponding FFLV-crystals graphs $B^{>}(1,1)$ and $B^{>}(2,2)$ are depicted in Figures 5.4 and 5.5. One can verify immediately that they are indeed FFLV-crystals graphs.

Proof of Proposition 5.4. For the proof we take the crystal graph $K(a,b)$ constructed in [5, Theorem 3.1] which satisfies (A1)-(A4) and establish a crystal bijection $\kappa : K(a,b) \to B^{>}(a,b)$.

Let H be the linear hyperplane in $\mathbb{R}^{\Delta_+}$ having normal vector $(1,0,-1)$ and denote by $C(a,b)$ the set of lattice points in the intersection

$$\mathrm{FFLV}_2(a\varpi_1 + b\varpi_2) \cap H.$$

Without loss of generality we assume that $b \geqslant a$ (the other case can be treated similarly).

The set $C(a,b)$ is the set of critical points (for the definition see [5]) in $B^{>}(a,b)$, which has cardinality $(a+1)(b+1)$: it is constituted from the lattice points of union the rectangular of size $a \times (b-a)$ and the half of the rectangular of size $2a \times a$, along the common edge of length a.

The crystal $K(a,b)$ has the same amount of critical points ([5, Corollary 3.2]).

By the construction of $K(a,b)$ in the beginning of [5, Section 3], the critical points in $K(a,b)$ can be identified with the integral points in the rectangular $[0,a] \times [0,b]$. The map κ sends the corners of the rectangle to the following corners in $C(a,b)$: $(0,0,0)$, $(0,a,0)$, $(0,a+b,0)$, and $(a,b-a,a)$ (precisely, $\kappa^{-1}(0,a+b,0)$ is opposite to $\kappa^{-1}(0,0,0)$ and $\kappa^{-1}(a,b-a,a)$ is opposite to $\kappa^{-1}(0,a,0)$).

We first define the image of the $(b+1)$-copies (labeled by $0,1,\dots,b$) of $K(a,0)$ under the map κ. The image of the 0-th copy of $K(a,0)$ is the subgraph of $B^{>}(a,b)$ bounded by the path from $(0,0,0)$ to $(a,0,0)$ of color 1, the path from $(a,0,0)$ to $(0,a,0)$ of color 2, and the critical points on the segment from $(0,0,0)$ to $(0,a,0)$. The image of the m-th copy, where $1 \leqslant m \leqslant b$, is obtained as follows. We denote by π_m the path in the set of "sky" crystal paths emanating from $(0,0,m)$. Consider the part of π_m between its critical point and its endpoint: such a 1-color path, denoted by π_m^+, has length a. To each vertex in π_m^+, we engraft the part of the path

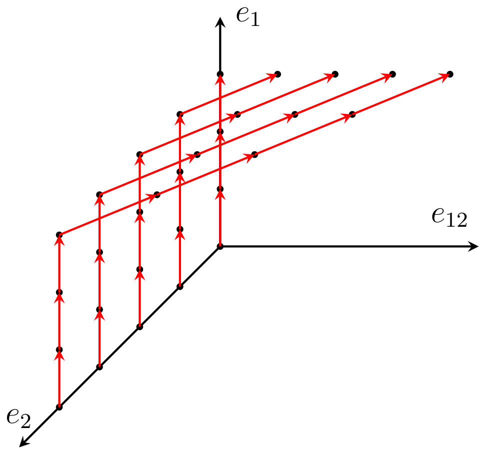

Figure 5.2: “Sky” crystal paths of color 1, $a = 3$ and $b = 4$.

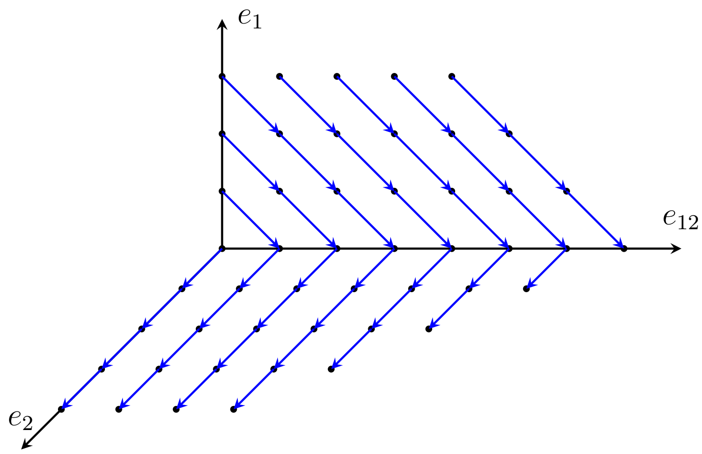

Figure 5.3: “Ground” crystal paths of color 2, $a = 3$ and $b = 4$.

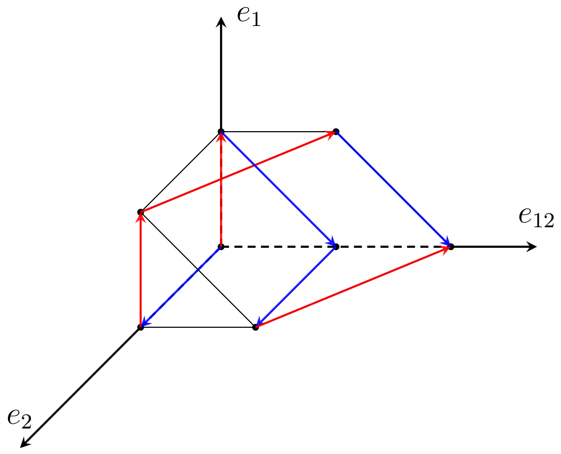

Figure 5.4: The crystal $B^{>}(1,1)$ with respect to $\varpi_1 > \varpi_2$.

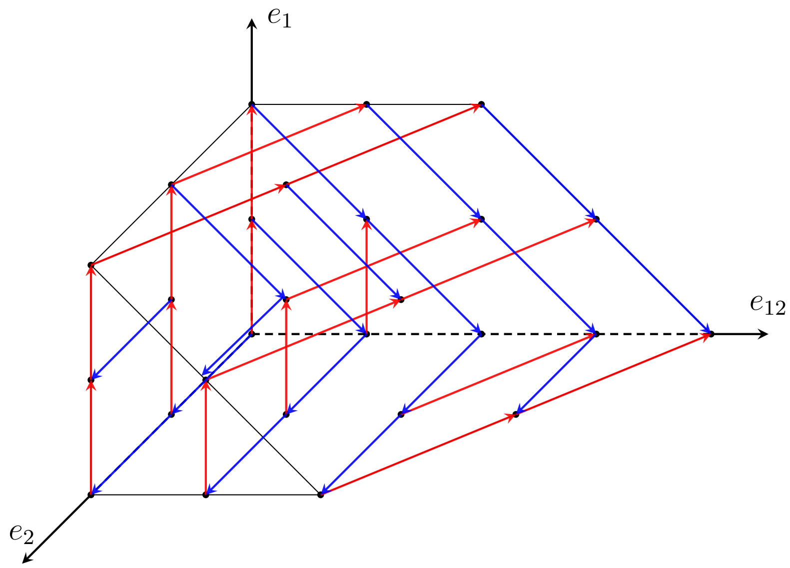

Figure 5.5: The crystal $B^{>}(2,2)$ with respect to $\varpi_1 > \varpi_2$.

of color 2 emanating from this vertex until its critical point. The subgraph of $B^{>}(a,b)$ with this set of vertices at these parts of paths is the image of the m-th copy of $K(a,0)$ under κ.

It is easy to see from this construction that such defined subgraphs are isomorphic to $K(a,0)$ and they cover the set of critical points properly without overlapping.

We define the images of the $(a+1)$-copies (labeled by $0,1,\dots,a$) of $K(0,b)$ under the map κ. For the image of the 0-th copy of $K(0,b)$, we take the part of the path π_b from its beginning to the critical point $(a,b-a,a)$; to each vertex in this path, we consider the path of color 2 terminating at this vertex and take the part of such a path between its critical point and the endpoint. We finally take the subgraph of $B^{>}(a,b)$ with vertices on all such defined paths. This is the image of the 0-th copy of $K(0,b)$.

For the image of the k-th copy, where $1\leqslant k\leqslant a$, we consider the translation of the 1-colored path π_b by the vector $(-k,k,0)$ and take from it the part starting from its beginning to its critical point: such a path has length equal to b. To each vertex on such a path, we engraft the part of the 2-colored path terminating at this vertex from its critical point. The sub-crystal with such defined set of vertices is the image of the k-th copy $K(0,b)$. Since the 2-colored paths appearing in the construction are translations of those in the "ground" crystal paths, we get that such a sub-crystal is isomorphic to $K(0,b)$. Note that these images of copies $K(0,b)$ cover the set of critical points properly without overlapping.

According to [5, Theorem 3.1], such defined map κ is a crystal bijection, the proof terminates. □

Remark 5.3. We can consider a kind of $B(\infty)$ crystal of the above form by sending a and b to $+\infty$. Then we will get the crystal graph on lattice points of the positive orthant, which has the monochromatic paths of color 1 being vertical rays $\mathbf{z}+\mathbb{R}e_1$, and the monochromatic path of color 2 being translations of the "ground" paths with $a=b=+\infty$. Unfortunately, we can not embed any of FFLV-crystals to such kind of $B(\infty)$.

The case $\sigma=(1,2)\in\mathfrak{S}_2$ corresponding to $\varpi_2>\varpi_1$. It can be treated similarly. Let $B^{<}(a,b)$ be an edge-colored graph on lattice points in $\mathrm{FFLV}_2(a\varpi_1+b\varpi_2)_{\mathbb{Z}}$ such that its monochromatic paths are defined in the following way:

1. For the color 1, we take the "ground" paths depicted in Figure 5.6 and their translations by the vector $(k,0,k)$, $k=1,\dots$ (precisely we takes the parts of translated paths which belong to $\mathrm{FFLV}_2(a,b)$).
2. For the color 2, we take the "wall" paths depicted in Figure 5.7 and their translations by the vector $(0,k,-k)$, $k=1,\dots$ (precisely we takes the parts of translated paths which belong to $\mathrm{FFLV}_2(a,b)$).

Proposition 5.4. *The edge-colored graph $B^{<}(a,b)$ is an FFLV-crystal graph.*

Similarly to the proof of Proposition 5.4, one can establish a crystal bijection $\kappa':K(a,b)\to B^{<}(a,b)$. We leave details to the reader.

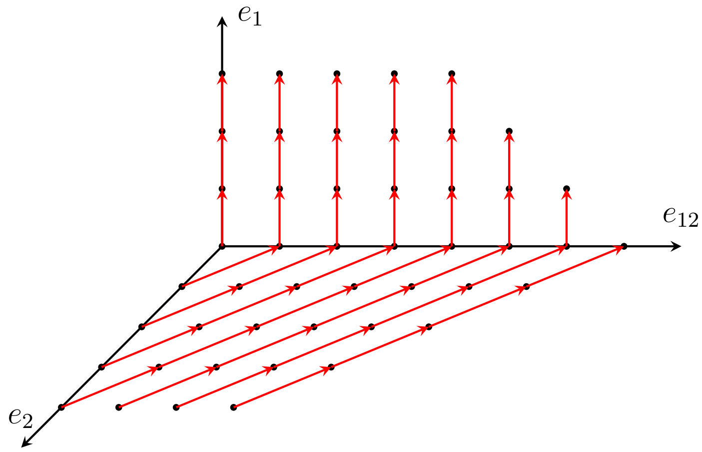

Figure 5.6: "Ground" crystal paths of color 1, $a = 3$ and $b = 4$.

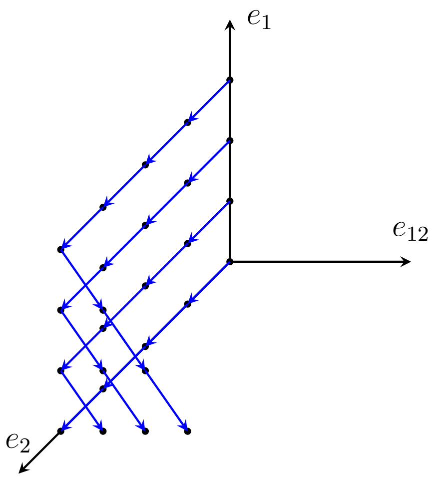

Figure 5.7: "Wall" crystal paths of color 2, $a = 3$ and $b = 4$.

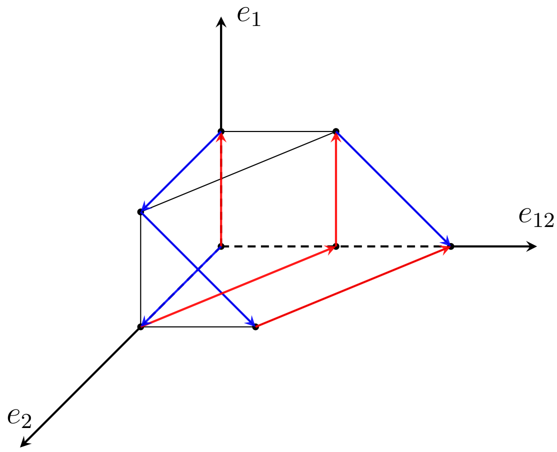

Figure 5.8: The crystal $B^{<}(1,1)$ with respect to $\varpi_2 > \varpi_1$.

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