Large hypergraphs without tight cycles

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Abstract. An r-uniform tight cycle of length $\ell > r$ is a hypergraph with vertices $v_1, \dots, v_\ell$ and edges $\{v_i, v_{i+1}, \dots, v_{i+r-1}\}$ (for all i), with the indices taken modulo ℓ. It was shown by Sudakov and Tomon that for each fixed $r \geqslant 3$, an r-uniform hypergraph on n vertices which does not contain a tight cycle of any length has at most $n^{r-1+o(1)}$ hyperedges, but the best known construction (with the largest number of edges) only gives $\Omega(n^{r-1})$ edges. In this note we prove that, for each fixed $r \geqslant 3$, there are r-uniform hypergraphs with $\Omega(n^{r-1} \log n / \log\log n)$ edges which contain no tight cycles, showing that the $o(1)$ term in the exponent of the upper bound is necessary.

Mathematics Subject Classifications. 05C65, 05C38

1. Introduction

A well-known basic fact about graphs states that a graph on n vertices containing no cycle of any length has at most $n-1$ edges, with this upper bound being tight. To find generalisations of this result (and other results concerning cycles) for r-uniform hypergraphs with $r \geqslant 3$, we need a corresponding notion of cycles in hypergraphs. There are several types of hypergraph cycles for which Turán-type problems have been widely studied, including Berge cycles and loose cycles [1, 2, 3, 4, 6, 7]. In this note we will consider tight cycles, for which it appears to be rather difficult to obtain extremal results.

Given positive integers $r \geqslant 2$ and $\ell > r$, an r-uniform tight cycle of length ℓ is a hypergraph with vertices $v_1, \dots, v_\ell$ and edges $\{v_i, v_{i+1}, \dots, v_{i+r-1}\}$ for $i = 1, \dots, \ell$, with the indices taken modulo ℓ. Observe that for $r = 2$ a tight cycle of length ℓ is just a cycle of length ℓ in the usual sense. Let $f_r(n)$ denote the maximal number of edges that an r-uniform hypergraph on n vertices can have if it has no subgraph isomorphic to a tight cycle of any length. So $f_2(n) = n-1$. It is easy to see that the hypergraph obtained by taking all edges containing a certain point is tight-cycle-free, giving a lower bound $f_r(n) \geqslant \binom{n-1}{r-1}$. Sós and independently Verstraëte (see [10])

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raised the problem of estimating $f_r(n)$, and asked whether the lower bound $\binom{n-1}{r-1}$ is tight. This question was answered in the negative by Huang and Ma [5], who showed that for $r \geqslant 3$ there exists $c_r > 0$ such that if n is sufficiently large then $f_r(n) \geqslant (1 + c_r)\binom{n-1}{r-1}$. Very recently, Sudakov and Tomon [9] showed that $f_r(n) \leqslant n^{r-1+o(1)}$ for each fixed r, and commented that it is widely believed that the correct order of magnitude is $\Theta(n^{r-1})$. The main result of this paper is the following theorem, which disproves this conjecture.

Theorem 1.1. *For each fixed $r \geqslant 3$ we have $f_r(n) = \Omega(n^{r-1} \log n / \log \log n)$. In particular, $f_r(n)/n^{r-1} \to \infty$ as $n \to \infty$.*

The upper bound of Sudakov and Tomon [9] is $n^{r-1}e^{c_r\sqrt{\log n}}$, although they remark that it might be possible to use their approach to get an upper bound of $n^{r-1}(\log n)^{O(1)}$.

Concerning tight cycles of a given length, we mention the following interesting problem of Conlon (see [8]), which remains open.

Question 1.2 (Conlon)**.** Given $r \geqslant 3$, does there exist some $c = c(r)$ constant such that whenever $\ell > r$ and ℓ is divisible by r then any r-uniform hypergraph on n vertices which does not contain a tight cycle of length ℓ has at most $O(n^{r-1+c/\ell})$ edges?

Note that we need the assumption that ℓ is divisible by r, otherwise a complete r-uniform r-partite hypergraph has no tight cycle of length ℓ and has $\Theta(n^r)$ edges.

2. Proof of our result

The key observation for our construction is the following lemma.

Lemma 2.1. *Assume that n, k, t are positive integers and $G_1, \ldots, G_t$ are edge-disjoint subgraphs of $K_{n,n}$ such that no G_i contains a cycle of length at most $2k$. Assume furthermore that $kt \leqslant n$. Then there is a tight-cycle-free 3-partite 3-uniform hypergraph on at most $3n$ vertices having $k \sum_{i=1}^{t} |E(G_i)|$ hyperedges.*

Proof. Let the two vertex classes of $K_{n,n}$ be X and Y, and let $Z = [t] \times [k]$. (As usual, $[m]$ denotes $\{1, \ldots, m\}$.) Our 3-uniform hypergraph has vertex classes X, Y, Z and hyperedges

$$\{\{x, y, z\} : x \in X, y \in Y, z \in Z, z = (i, s) \text{ for some } i \in [t] \text{ and } s \in [k], \text{ and } \{x, y\} \in E(G_i)\}.$$

In other words, for each G_i we add k new vertices (denoted (i, s) for $s = 1, \ldots, k$), and we replace each edge of G_i by the k hyperedges obtained by adding one of the new vertices corresponding to G_i to the edge.

We need to show that our hypergraph contains no tight cycles. Since our hypergraph is 3-partite, it is easy to see that any tight cycle is of the form $x_1y_1z_1x_2y_2z_2 \ldots x_\ell y_\ell z_\ell$ (for some $\ell \geqslant 2$ positive integer) with $x_j \in X, y_j \in Y, z_j \in Z$ for all j. Assume that $z_1 = (i, s_1)$. Then $\{x_1, y_1\}, \{y_1, x_2\}, \{x_2, y_2\} \in E(G_i)$. But $\{x_2, y_2\} \in E(G_i)$ implies that z_2 must be of the form (i, s_2) for some s_2. Repeating this argument, we deduce that there are $s_j \in [k]$ such that $z_j = (i, s_j)$ for all j, and $x_jy_j, y_jx_{j+1} \in E(G_i)$ for all j (with the indices taken mod ℓ). Hence $x_1y_1x_2y_2 \ldots x_\ell y_\ell$ is a cycle in G_i, giving $\ell > k$. But the vertices $z_j = (i, s_j)$ $(j = 1, \ldots, \ell)$ must all be distinct, and there are k possible values for the second coordinate, giving $\ell \leqslant k$. We get a contradiction, giving the result. □

We mention that Lemma 2.1 can be generalised to give $(r + r')$-uniform tight-cycle-free hypergraphs if we have edge-disjoint r-uniform hypergraphs $G_1, \ldots, G_t$ not containing tight cycles of length at most rk and edge-disjoint r'-uniform hypergraphs $H_1, \ldots, H_t$ not containing tight cycles of length more than $r'k$. Indeed, we can take all edges $e \cup f$ with $e \in E(G_i), f \in E(H_i)$ for some i. (Then Lemma 2.1 may be viewed as the special case $r = 2, r' = 1$.)

Lemma 2.2. *There exists $\alpha > 0$ such that whenever $k \leqslant \alpha \log n / \log\log n$ then we can find edge-disjoint subgraphs $G_1, \ldots, G_t$ of $K_{n,n}$ with $t = \lfloor n/k \rfloor$ such that no G_i contains a cycle of length at most $2k$, and $\sum_{i=1}^{t} |E(G_i)| = (1 - o(1))n^2$.*

Proof. It is well-known (and can be proved by a standard probabilistic argument) that there are constants $\beta, c > 0$ such that if n is sufficiently large and $k \leqslant \beta \log n$ then there exists a subgraph H of $K_{n,n}$ which has no cycle of length at most $2k$ and has $|E(H)| \geqslant n^{1+c/k}$. We randomly and independently pick copies $H_1, \ldots, H_t$ of H in $K_{n,n}$. Let $G_1 = H_1$ and $E(G_i) = E(H_i) \setminus \bigcup_{j=1}^{i-1} E(H_j)$ for $i \geqslant 2$. Then certainly the G_i are edge-disjoint and no G_i contains a cycle of length at most $2k$. Furthermore, the probability that a given edge is not contained in any H_i is

$$\begin{aligned}(1 - |E(H)|/n^2)^t &\leqslant \exp\left(-|E(H)|t/n^2\right)\\ &\leqslant \exp\left(-n^{1+c/k}\lfloor n/k \rfloor/n^2\right) = \exp\left(-n^{c/k}/k(1 + o(1))\right).\end{aligned}$$

This is $o(1)$ as long as $k \leqslant \alpha \log n / \log\log n$ for some constant $\alpha > 0$. Therefore the expected value of $\left|\bigcup_{i=1}^{t} E(H_i)\right|$ is $(1 - o(1))n^2$. Since $\sum_{i=1}^{t} |E(G_i)| = \left|\bigcup_{i=1}^{t} E(H_i)\right|$, the result follows. □

Proof of Theorem 1.1. First consider the case $r = 3$. Lemma 2.2 and Lemma 2.1 together show that if $k \leqslant \alpha \log n / \log\log n$ then there is a tight-cycle-free 3-partite 3-uniform hypergraph on $3n$ vertices with $(1 - o(1))kn^2$ edges. This shows $f_3(n) = \Omega(n^2 \log n / \log\log n)$, as claimed.

For $r \geqslant 4$, observe that $f_r(2n) \geqslant f_{r-1}(n)n$. Indeed, if H is an $(r-1)$-uniform tight-cycle-free hypergraph on n vertices, then we can construct a tight-cycle-free r-uniform hypergraph H' on $2n$ vertices with $n|E(H)|$ edges as follows. The vertex set of H' is the disjoint union of $[n]$ and the vertex set $V(H)$ of H, and the edges are $e \cup \{i\}$ with $e \in E(H)$ and $i \in [n]$. Then any tight cycle in H' must be of the form $v_1v_2 \ldots v_{\ell r}$ with $v_i \in V(H)$ if i is not a multiple of r and $v_i \in [n]$ if i is a multiple of r. But then we get a tight cycle $v_1v_2 \ldots v_{r-1}v_{r+1}v_{r+2} \ldots v_{2r-1}v_{2r+1} \ldots v_{\ell r-1}$ in H by removing each vertex from $[n]$ from this cycle. This is a contradiction, so H' contains no tight cycles. The result follows. □

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