Barely lonely runners and very lonely runners: a refined approach to the Lonely Runner Problem

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Abstract. We introduce a sharpened version of the well-known Lonely Runner Conjecture of Wills and Cusick. Given a real number x, let $\|x\|$ denote the distance from x to the nearest integer. For each set of positive integer speeds $v_1, \ldots, v_n$, we define the associated maximum loneliness to be

$$\mathrm{ML}(v_1, \ldots, v_n) = \max_{t \in \mathbb{R}} \min_{1 \leqslant i \leqslant n} \|tv_i\|.$$

The Lonely Runner Conjecture asserts that $\mathrm{ML}(v_1, \ldots, v_n) \geqslant 1/(n+1)$ for all choices of $v_1, \ldots, v_n$. We make the stronger conjecture that for each choice of $v_1, \ldots, v_n$, we have either $\mathrm{ML}(v_1, \ldots, v_n) = s/(ns+1)$ for some $s \in \mathbb{N}$ or $\mathrm{ML}(v_1, \ldots, v_n) \geqslant 1/n$. This view reflects a surprising underlying rigidity of the Lonely Runner Problem. Our main results are: confirming our stronger conjecture for $n \leqslant 3$; and confirming it for $n = 4$ and $n = 6$ in the case where one speed is much faster than the rest.

Mathematics Subject Classifications. 11K60 (primary), 11J13, 11J71, 52C07

1. Introduction

1.1. Background

The Lonely Runner Problem has been a popular research topic ever since it was introduced by Wills [29] and Cusick [15]. Its name comes from the following non-technical formulation. Suppose n runners start at the same point on a circular track of length 1 and begin to run around the track at pairwise distinct constant speeds. We deem a runner "lonely" at a certain time if their distance around the track from every other runner is at least $1/n$. The Lonely Runner Conjecture asserts that regardless of the starting speeds, every runner gets lonely eventually (perhaps at different times for different runners).

Identify the circular track with $\mathbb{R}/\mathbb{Z}$, and consider the frame of reference of a single runner. From this runner's perspective, it doesn't matter in which direction the other runners are going, so we may as well take all of their speeds to be positive. Also, Bohman, Holzman, and Kleitman [6] have shown that it suffices to consider only integer speeds. Given a real number x, let $\|x\|$ denote the distance from x to the nearest integer. For a set of positive integer speeds $v_1, \ldots, v_n$, we define the associated *maximum loneliness* to be

$$\mathrm{ML}(v_1, \ldots, v_n) = \max_{t \in \mathbb{R}} \min_{1 \leqslant i \leqslant n} \|tv_i\|.$$

(The maximum exists because $\min_{1 \leqslant i \leqslant n} \|tv_i\|$ is a continuous periodic function of t.) Then the Lonely Runner Conjecture can be expressed succinctly in terms of this quantity.

Conjecture 1.1 (Lonely Runner Conjecture)**.** For any positive integers $v_1, \ldots, v_n$, we have

$$\mathrm{ML}(v_1, \ldots, v_n) \geqslant \frac{1}{n+1}.$$

If the Lonely Runner Conjecture is true, then the quantity $1/(n+1)$ is the best possible, for there are known equality cases (also called *tight* sets of speeds) with $\mathrm{ML}(v_1, \ldots, v_n) = 1/(n+1)$. One such construction simply sets each $v_i = i$.

The Lonely Runner Conjecture is connected to questions in many fields, such as geometric view-obstruction (e.g., [7, 15]), Diophantine approximation (e.g., [4, 26, 27, 28]), flows in matroids (e.g., [5, 25]), and chromatic numbers of distance graphs (e.g., [21, 22, 30]). The conjecture has received substantial attention in recent decades. The main approach has been to establish the Lonely Runner Conjecture for small values of n; it is now known to hold for $n \leqslant 6$ (see [4] for $n = 2$ and $n = 3$; [16, 5] for $n = 4$; [6, 23] for $n = 5$; [3] for $n = 6$). Another appealing avenue of inquiry has been improving the trivial lower bound $\mathrm{ML}(v_1, \ldots, v_n) \geqslant 1/(2n)$; most recently, Tao [24] showed that $1/(2n)$ can be replaced with

$$\frac{1}{2n} + \frac{c \log n}{n^2 \log\log n}$$

for some constant c and all sufficiently large n. Other investigations into the Lonely Runner Problem include: the work of Goddyn and Wong [18] on tight sets of speeds other than the trivial set $1, \ldots, n$; a series of papers by Chen [8, 9, 10, 11] about an equivalent formulation in terms of simultaneous Diophantine approximation; and Chow and Rimanić's resolution [13] of an analogous problem for function fields. See [20] for more background.

1.2. A new question

In this paper, we introduce a new way to approach the Lonely Runner Problem. A natural but (to our knowledge) hitherto unasked question is:

> **Motivating Question.** If $v_1, \ldots, v_n$ satisfy the Lonely Runner Conjecture but do not form a set of tight speeds, must $\mathrm{ML}(v_1, \ldots, v_n)$ be uniformly bounded away from $1/(n+1)$?

We conjecture that, contrary to what one might expect, this question has an affirmative answer. (Speaking poetically, one might say that lonely runners are always either "barely lonely" or "very lonely".) We in fact offer the following more precise statement about an unexpected rigidity of possible "small" values of $\mathrm{ML}(v_1, \ldots, v_n)$.

Conjecture 1.2 (Loneliness Spectrum Conjecture)**.** For any positive integers $v_1, \ldots, v_n$, we have either

$$\mathrm{ML}(v_1, \ldots, v_n) = \frac{s}{ns+1} \quad \text{for some } s \in \mathbb{N} \quad \text{or} \quad \mathrm{ML}(v_1, \ldots, v_n) \geqslant \frac{1}{n}.$$

Note that this conjecture is strictly stronger than the Lonely Runner Conjecture. In this paper, we will refer to Conjecture 1.1 as the "Lonely Runner Conjecture", and we will refer to Conjecture 1.2 as the "(Loneliness) Spectrum Conjecture". We will often refer to the maximum loneliness amounts of the form $s/(ns+1)$ as the *discrete part* of the (maximum) loneliness spectrum.

We remark that it is natural to focus on maximum loneliness amounts in the interval $[1/(n+1), 1/n)$ because if the Lonely Runner Conjecture is true, then this interval is precisely the "new" regime that is made available with the addition of the n-th runner. That is, heuristically, a set of n speeds must "work together" in some rigid way in order to obtain a maximum loneliness smaller than $1/n$. We mention also that the Spectrum Conjecture provides a satisfying explanation for the appearance of the quantity $1/(n+1)$ in the Lonely Runner Conjecture: the maximum loneliness $1/(n+1)$ is the last element of a highly-structured discrete spectrum.

1.3. Main results and overview of the paper

This paper is based on the author's thesis [20]. It has come to our attention that several of the early results of this paper appear, either explicitly or implicitly, elsewhere in the literature. The methods of proof are different, however, and we include our (strictly speaking, redundant) proofs because they demonstrate a new perspective.

In Section 2, we collect some of the basic tools that we will use throughout the paper. In Section 3, we show that the Spectrum Conjecture is the best possible in the sense that for every n, the entire discrete part of the loneliness spectrum is attained. In Sections 4 and 5, we prove the Loneliness Spectrum Conjecture for $n = 2$ and $n = 3$, respectively. (The $n = 1$ case is trivial.)

Theorem 1.3. *For any positive integers v_1, v_2, we have either*

$$\mathrm{ML}(v_1, v_2) = \frac{s}{2s+1} \quad \textit{for some } s \in \mathbb{N} \quad \textit{or} \quad \mathrm{ML}(v_1, v_2) = \frac{1}{2}.$$

For any positive integers v_1, v_2, v_3, we have either

$$\mathrm{ML}(v_1, v_2, v_3) = \frac{s}{3s+1} \quad \textit{for some } s \in \mathbb{N} \quad \textit{or} \quad \mathrm{ML}(v_1, v_2, v_3) \geqslant \frac{1}{3}.$$

In Section 6, we develop machinery for approaching the Spectrum Conjecture in the regime where one speed is much faster than the rest. We eventually produce an explicit computation involving only the tight sets of $n-1$ speeds which allows us to decide whether or not the Spectrum Conjecture holds for n runners in this regime. Then, using this technique and previous results about tight speed sets for 3 and 5 runners, we establish the Spectrum Conjecture for $n=4$ and $n=6$ when one speed is much faster than the others.

Theorem 1.4. *Let $n=4$ or $n=6$. If $v_1<\cdots<v_n$ are positive integers satisfying $v_n>4v_{n-1}^4$, then we have either*

$$\mathrm{ML}(v_1,\ldots,v_n)=\frac{s}{ns+1}\quad \textit{for some } s\in\mathbb{N}\quad \textit{or}\quad \mathrm{ML}(v_1,\ldots,v_n)\geqslant\frac{1}{n}.$$

Along the way, we raise several questions about the set of all possible maximum loneliness amounts for n runners, and a consequence of this line of inquiry is that the Spectrum Conjecture for n runners immediately implies the Lonely Runner Conjecture for $n-1$ runners as well as for n runners. Finally, in Section 7, we present open questions and promising areas of future inquiry.

Often, the main barrier to proving a long-standing conjecture is that people try to establish the "wrong" version of the statement. This tendency is especially true of induction-type arguments, where having a stronger induction hypothesis can make a proof easier. We hope that the Loneliness Spectrum Conjecture may be the "right" way to approach the Lonely Runner Conjecture.

2. Tools and Preliminary Observations

We begin with a few simple observations that we will use freely later. We always take $v_1,\ldots,v_n$ to be positive integers, with $n\geqslant 2$.

First, note that without loss of generality we can restrict our attention to the case where the speeds $v_1,\ldots,v_n$ do not all share a common factor; indeed, if $\gcd(v_1,\ldots,v_n)=g>1$, then $\mathrm{ML}(v_1,\ldots,v_n)=\mathrm{ML}(v_1/g,\ldots,v_n/g)$, where each v_i/g is an integer. We also lose nothing by taking the speeds to be pairwise distinct.

Second, recall that for fixed $v_1,\ldots,v_n$, we are looking for the real number (time) t that maximizes the function

$$f(t)=\min_{1\leqslant i\leqslant n}\|tv_i\|.$$

Since $f(t+1)=f(t)$, it suffices to consider t in the interval $[0,1)$. In fact, since $f(t)=f(-t)$, we could further restrict our attention to $[0,1/2]$. Moreover, all local maxima of f should occur at times where there are two runners of minimum distance to the origin and these runners are on "different" halves of the circular track, for otherwise we could obtain a larger loneliness by perturbing the time. We make this observation precise in the following simple proposition, which also appears in [17, 19, 22]. (See [20] for more details.)

Proposition 2.1. *Let $v_1,\ldots,v_n$ be positive integers ($n\geqslant 2$) with $\gcd(v_1,\ldots,v_n)=1$. Then every local maximum of the function*

$$f(t)=\min_{1\leqslant i\leqslant n}\|tv_i\|$$

occurs at a time of the form

$$t_0 = \frac{m}{v_i + v_j},$$

where $1 \leqslant i < j \leqslant n$ *and* m *is an integer.*

Of course, the global maximum is among these local maxima for t in the interval $[0, 1)$. This proposition shows that we can determine $\mathrm{ML}(v_1, \ldots, v_n)$ by checking only finitely many times, where this number grows at most cubically with the size of the fastest speed. This observation is particularly useful for performing computational experiments. Another advantage of this perspective is that it gives us a useful way to look at candidate times in "chunks" (according to the pairs i, j) instead of all at once. This way of thinking often reveals underlying structure that is otherwise opaque.

Third, we mention the concept of a *pre-jump*, as used by Bienia, Goddyn, Gvozdjak, Sebő, and Tarsi [5]. Their insight is essentially that if we know the values of $\|t_1 v_i\|$ and $\|t_2 v_i\|$, then we can say something about $\|(t_1 + \alpha t_2)v_i\|$, where α is an integer. This is particularly useful when many of the quantities $\|t_2 v_i\|$ are zero. For instance, if the speeds $v_1, \ldots, v_r$ are all divisible by g and the speeds $v_{r+1}, \ldots, v_n$ are not, then adding multiples of $1/g$ to a time moves the runners with speeds $v_{r+1}, \ldots, v_n$ while fixing the positions of those with speeds $v_1, \ldots, v_r$.

3. Achieving the discrete part of the spectrum

We describe a simple explicit construction that achieves the entire discrete part of the spectrum in the Loneliness Spectrum Conjecture. The motivating idea is that one can obtain small maximum loneliness values with n runners by starting with a tight speed set for $n-1$ runners and choosing the last speed v_n so that $\|t_0 v_n\| = 0$ at every "equality time" t_0 for the first $n-1$ runners. The following result is proven in a different context in [22]; our proof is simpler and more direct.

Theorem 3.1. *For every integer* $n \geqslant 2$ *and every natural number* s*, we have*

$$\mathrm{ML}(1, 2, \ldots, n-1, ns) = \frac{s}{ns+1}.$$

Proof. The $s = 1$ case follows from the known tight case $\mathrm{ML}(1, \ldots, n) = 1/(n+1)$, so we restrict our attention to $s \geqslant 2$. The proof is a straightforward computation with Proposition 2.1. We have to check times with denominators between 1 and $2n-3$ and between $ns+1$ and $ns+n-1$. As before, let $f(t)$ denote the loneliness at time t.

- Consider $t = m/d$, for $1 \leqslant d \leqslant 2n-3$. We know that $f(t) \leqslant 1/n$ because of the speeds $1, \ldots, n-1$. Since $f(t)$ is a nonnegative rational number with denominator at most $2n-3$, we must have either $f(t) = 1/n$ or $f(t) \leqslant 1/(n+1)$. We could have $f(t) = 1/n$ only for $t = m/n$, but in this case $\|t(ns)\| = 0$ gives $f(t) = 0$, so we conclude that $f(t) \leqslant 1/(n+1)$.

- Consider $t = m/(ns+j)$, for $1 \leqslant j \leqslant n-1$. Recall that $f(t) \leqslant 1/n$. So the maximum possible value for $f(t)$ is $s/(ns+j)$, and in fact this value is achieved for $m = s$.

Taking the maximum value of $f(t)$ among these possibilities gives that

$$\mathrm{ML}(1, 2, \ldots, n-1, ns) = \frac{s}{ns+1},$$

as desired. □

We remark that this discrete spectrum can be achieved in much the same way by starting with other known equality cases and adding a new fast runner (see [18, 22]); the case work in the computation becomes more extensive, and sometimes a few of the values for small s are not obtained. We return to this idea in Section 6.

4. Two moving runners

The $n = 2$ case of the Spectrum Conjecture is mentioned in a passing remark of Bienia, Goddyn, Gvozdjak, Sebő, and Tarsi [5]; for the sake of completeness, we provide a proof in the language introduced above.

Theorem 4.1. *Let v_1, v_2 be relatively prime positive integers. If v_1 and v_2 are both odd, then*

$$\mathrm{ML}(v_1, v_2) = \frac{1}{2}.$$

Otherwise,

$$\mathrm{ML}(v_1, v_2) = \frac{s}{2s+1},$$

where $2s + 1 = v_1 + v_2$.

Proof. If v_1 and v_2 are both odd, then $\|(1/2)v_1\| = \|(1/2)v_2\| = 1/2$ gives $\mathrm{ML}(v_1, v_2) = 1/2$, so we restrict our attention to the case where v_1 and v_2 are not both odd. In particular, their sum is odd. By Proposition 2.1, we have to check only times $t = m/(v_1 + v_2)$, where we know that

$$\|tv_1\| = \|tv_2\| = \left\| \frac{mv_1}{v_1 + v_2} \right\|.$$

Note that this quantity is always at most $s/(2s+1)$, where $2s+1 = v_1+v_2$. Since v_1 and v_2 are relatively prime, we also have that v_1 is relatively prime to $v_1 + v_2$. So there exists an integer m such that $mv_1 \equiv s \pmod{2s+1}$, which shows that the loneliness $s/(2s+1)$ is attained. □

5. Three moving runners

The $n = 3$ case of the Spectrum Conjecture is much more delicate than the $n = 2$ case. We later learned that this result is implied by a more precise a theorem of Chen [10] in the context of simultaneous Diophantine approximation. Our method of proof, however, is substantially different from Chen's and appears fit for generalizations in different directions.

The main idea of our proof is that we use a pre-jump to handle the case where two speeds share a large common factor, after which we can control the remaining cases more precisely. We require the following technical lemma, which says that if we consider all times with denominator the sum of two fixed speeds, then we should get a loneliness of at least $1/3$, up to a rounding error, unless the third runner always "stays" at 0. We remark that the condition $\gcd(v_1, v_2) \leqslant 2$ is not necessary in the statement of the lemma; we omit the case $\gcd(v_1, v_2) > 2$ (which is easily handled with a pre-jump) simply because we do not need it for the main result of this section.

Lemma 5.1. *Let v_1, v_2, v_3 be positive integers with $\gcd(v_1, v_2, v_3) = 1$, and suppose that $\gcd(v_1, v_2) \leqslant 2$. Let*

$$r = \left\lfloor \frac{v_1 + v_2}{3} \right\rfloor,$$

and let L denote the maximum loneliness that is achieved at a time of the form

$$t = \frac{m}{v_1 + v_2}$$

($m \in \mathbb{Z}$). Then we have the following dichotomy:

- *If v_3 is a multiple of $v_1 + v_2$, then $L = 0$.*
- *If v_3 is not a multiple of $v_1 + v_2$, then $L \geqslant r/(v_1 + v_2)$.*

Proof. The first statement is trivial. For second statement, fix v_1, v_2, v_3 with v_3 not a multiple of $v_1 + v_2$. We condition on $\gcd(v_1, v_2)$, which must be 1 or 2 by assumption. Recall that at every $t = m/(v_1 + v_2)$, we have $\|tv_1\| = \|tv_2\|$.

First, suppose $\gcd(v_1, v_2) = 1$. Then there is an integer u such that $uv_1 \equiv 1 \pmod{v_1 + v_2}$. Write $m \equiv \ell u \pmod{v_1 + v_2}$, so that ℓ ranges over the residues modulo $v_1 + v_2$ as m does so, and consider the times

$$t = \frac{\ell u}{v_1 + v_2}.$$

Then for all $r \leqslant \ell \leqslant v_1 + v_2 - r$, we have that $\|tv_1\|$ (equivalently, $\|tv_2\|$) is at least $r/(v_1 + v_2)$. We now claim that

$$\|tv_3\| \geqslant \frac{r}{v_1 + v_2}$$

for some ℓ in this range. In other words, some element of

$$ruv_3, (r+1)uv_3, \ldots, (v_1 + v_2 - r)uv_3$$

leaves a residue between r and $v_1 + v_2 - r$ modulo $v_1 + v_2$.

How this comes about depends on the residue of uv_3 modulo $v_1 + v_2$. We may take this residue to be between 1 and $(v_1 + v_2)/2$ since otherwise we can replace v_3 with $c(v_1 + v_2) - v_3$ for some large integer c that makes this quantity positive. We handle the various possibilities separately:

- Suppose $uv_3 \equiv 1 \pmod{v_1 + v_2}$. Then $ruv_3 \equiv r \pmod{v_1 + v_2}$, as desired.

- Suppose $uv_3 \equiv j \pmod{v_1+v_2}$, for some $2 \leqslant j \leqslant v_1+v_2-2r+1$. The upper bound on j tells us that if $\ell j < k(v_1+v_2)+r$ for some integer k, then $(\ell+1)j \leqslant (k+1)(v_1+v_2)-r$. In other words, incrementing ℓ cannot make ℓj "skip" over all of the residue classes (strictly) between r and v_1+v_2-r. So it remains only to show that the ℓj's are not all contained in an interval of the form

$$k(v_1+v_2)-r+1, \dots, k(v_1+v_2)+r-1$$

for any integer k. The difference between the largest and smallest elements of this interval is $2r-2$. At the same time, the lower bound on j gives

$$(v_1+v_2-r)j - rj \geqslant 2(v_1+v_2-2r) \geqslant 2r > 2r-2,$$

so the ℓj's cannot all be contained in such a short interval. We conclude that some ℓj leaves a residue between r and v_1+v_2-r, as desired.

- Suppose $uv_3 \equiv j \pmod{v_1+v_2}$, for some $v_1+v_2-2r+2 \leqslant j < (v_1+v_2)/2$. Recall that for $r \leqslant \ell \leqslant v_1+v_2-r$, we are done if the quantity ℓj ever leaves a residue between r and v_1+v_2-r. In particular, this possibility obtains if there is any $r \leqslant \ell \leqslant v_1+v_2-r-1$ such that the residue of ℓj is between $r-j$ and v_1+v_2-r-j, for then the residue of $(\ell+1)j$ is between r and v_1+v_2-r. So it suffices to show that the residues of ℓj, for $r \leqslant \ell \leqslant v_1+v_2-r-1$, cannot be confined to the intervals

$$I_1 = \{v_1+v_2-r-j+1, v_1+v_2-r-j+2, \dots, r-1\}$$

and

$$I_2 = \{v_1+v_2-r+1, v_1+v_2-r+2, \dots, v_1+v_2+r-j-1\}.$$

Note that the difference between the largest and smallest elements of I_1 is

$$(r-1)-(v_1+v_2-r-j+1) < \frac{1}{6}(v_1+v_2)-2 < j,$$

where we used the upper bound on j. Similarly, the difference between the largest and smallest elements of I_2 is

$$(r-j-1)-(-r+1) = 2r-j-2 < j.$$

These bounds imply that consecutive residues of ℓj and $(\ell+1)j$ cannot both lie in a single one of these two intervals, so we have to worry about only the possibility in which ℓj alternately lies in I_1 and I_2 as ℓ grows from r to v_1+v_2-r-1. If this were the case, we would have at least

$$\left\lfloor \frac{(v_1+v_2-r-1)-r}{2} \right\rfloor \geqslant \frac{1}{6}(v_1+v_2)-1$$

values of ℓ (increasing in increments of 2) with ℓj leaving a residue in I_1. Note that as ℓ increases by 2, the residue of ℓj increases by

$$h = v_1+v_2-2j,$$

which is nonzero by the condition on j. So the difference between the largest and smallest of these residues is at least $\frac{1}{6}(v_1+v_2)-2$, but then it is impossible to fit this entire arithmetic progression into I_1. So we conclude that in fact some ℓj leaves a residue between r and v_1+v_2-r, as desired.

- Suppose $uv_3 \equiv (v_1+v_2)/2 \pmod{v_1+v_2}$. Then either ruv_3 or $(r+1)uv_3$ leaves a residue of $(v_1+v_2)/2$ modulo v_1+v_2, and this is certainly between r and v_1+v_2-r.

This concludes the argument for the $\gcd(v_1,v_2)=1$ case.

Second, suppose $\gcd(v_1,v_2)=2$. Then there is an integer u such that $uv_1 \equiv 2 \pmod{v_1+v_2}$. Note that v_3 is odd due to our gcd restrictions. Now, write even m as $m \equiv \ell u \pmod{v_1+v_2}$, so that ℓ ranges over the residues $1,2,\dots,(v_1+v_2)/2$ modulo v_1+v_2 as m ranges over the even residues modulo v_1+v_2, and consider times $t=(\ell u)/(v_1+v_2)$. Then for all $r/2 \leqslant \ell \leqslant (v_1+v_2-r)/2$, we have that $\|tv_1\|$ (equivalently, $\|tv_2\|$) is at least $r/(v_1+v_2)$. We now claim that either

$$\|tv_3\| \geqslant \frac{r}{v_1+v_2} \quad \text{or} \quad \|tv_3\| \leqslant \frac{1}{2}-\frac{r}{v_1+v_2}$$

for some ℓ in this range. The second possibility is sufficient to establish the desired result because at the time $t+1/2$ (which is still of the form $m/(v_1+v_2)$, where m now might be odd), we have

$$\left\|\left(t+\frac{1}{2}\right)v_1\right\| = \|tv_1\|, \quad \left\|\left(t+\frac{1}{2}\right)v_2\right\| = \|tv_2\|, \text{ and } \left\|\left(t+\frac{1}{2}\right)v_3\right\| = \frac{1}{2}-\|tv_3\|.$$

(We can think of this manipulation as a pre-jump with the times t and $1/2$.) So our claim is that some element of

$$\left(\frac{r}{2}\right)uv_3, \left(\frac{r}{2}+1\right)uv_3, \dots, \left(\frac{v_1+v_2-r}{2}\right)uv_3$$

leaves a residue between r and v_1+v_2-r or between $r-(v_1+v_2)/2$ and $(v_1+v_2)/2-r$ modulo v_1+v_2.

As before, we divide cases according to the residue of uv_3, where we can take this residue to be between 1 and $(v_1+v_2)/2$. Because the arguments are essentially the same as what we presented above in the $\gcd(v_1,v_2)=1$ case, we provide only sketches.

- Suppose $uv_3 \equiv j \pmod{v_1+v_2}$, for some $1 \leqslant j \leqslant v_1+v_2-2r+1$. The upper bound on j tells us that incrementing ℓ cannot make ℓj "skip" over either of the two forbidden intervals of residues, so we have to worry about only the cases where the ℓj's are either all contained in

$$\frac{v_1+v_2}{2}-r+1, \frac{v_1+v_2}{2}-r+2, \dots, r-1$$

or all contained in

$$v_1+v_2-r+1, v_1+v_2-r+2, \dots, \frac{v_1+v_2}{2}+r-1.$$

But neither of these intervals is long enough to contain the entire arithmetic progression of ℓj's.

- Suppose $uv_3 \equiv j \pmod{v_1+v_2}$, for some $v_1+v_2-2r+2 \leqslant j < (v_1+v_2)/2$. The argument then goes roughly as in the third bullet above, except that we now have $h \geqslant 2$ since v_1+v_2 is even.

- Suppose $uv_3 \equiv (v_1+v_2)/2 \pmod{v_1+v_2}$. Then ruv_3 leaves a residue of 0 or $(v_1+v_2)/2$ modulo v_1+v_2, either of which is sufficient.

This concludes the argument for the $\gcd(v_1, v_2) = 2$ case. □

Now, Lemma 5.1 will handle most of the "difficult" sets of speeds in the $n=3$ case of the Spectrum Conjecture.

Theorem 5.2. *Let v_1, v_2, v_3 be positive integers with $\gcd(v_1, v_2, v_3) = 1$. Then we have either*

$$\mathrm{ML}(v_1, v_2, v_3) = \frac{s}{3s+1} \quad \textit{for some } s \in \mathbb{N} \quad \textit{or} \quad \mathrm{ML}(v_1, v_2, v_3) \geqslant \frac{1}{3}.$$

Proof. First of all, suppose some two of the speeds have a common factor of at least 3, say, $\gcd(v_1, v_2) = g \geqslant 3$. Note that $\gcd(g, v_3) = 1$. By Theorem 4.1, there exists a time t such that both $\|tv_1\|$ and $\|tv_2\|$ are at least $1/3$. By the Pigeonhole Principle, there is an integer h such that

$$\left\| \left(t + \frac{h}{g}\right) v_3 \right\| \geqslant \frac{1}{2} - \frac{1}{2g} \geqslant \frac{1}{3}.$$

We also know that $\|(t+h/g)v_1\| = \|tv_1\|$ and $\|(t+h/g)v_2\| = \|tv_2\|$, so the loneliness at time $t+h/g$ is at least $1/3$. (We are using a pre-jump with the times t and $1/g$.) We conclude that $\mathrm{ML}(v_1, v_2, v_3) \geqslant 1/3$. Henceforth, we restrict our attention to the case where no two speeds have a common factor greater than 2.

Next, suppose no speed is a multiple of 3. Then the loneliness at time $t = 1/3$ is at least $1/3$, and we are done. So we can restrict our attention to the case where exactly one speed is a multiple of 3.

For each $1 \leqslant i < j \leqslant 3$, let

$$r_{i,j} = \left\lfloor \frac{v_i + v_j}{3} \right\rfloor,$$

and let $L_{i,j}$ denote the maximum loneliness that is achieved at a time of the form $t = m/(v_i+v_j)$. Let k be the remaining element of $\{1,2,3\}$. Lemma 5.1 provides the following dichotomy for each pair i, j:

- If v_k is a multiple of $v_i + v_j$, then $L_{i,j} = 0$.

- If v_k is not a multiple of $v_i + v_j$, then $L_{i,j} \geqslant r_{i,j}/(v_i+v_j)$.

Recall that $\mathrm{ML}(v_1, v_2, v_3)$ is the maximum of the three values $L_{i,j}$. If any $L_{i,j} \geqslant 1/3$, then we are done, so we restrict our attention to the case where this does not occur. In particular, the second case of the dichotomy collapses to

$$L_{i,j} = \frac{r_{i,j}}{v_i + v_j}$$

if $v_i + v_j$ is not a multiple of 3, and the second case becomes completely disallowed if $v_i + v_j$ is a multiple of 3.

Suppose the second possibility of the dichotomy obtains for each pair i, j, i.e., each $L_{i,j} = r_{i,j}/(v_i + v_j)$, where $v_i + v_j$ is not a multiple of 3. Thus, the residues of v_1, v_2, v_3 modulo 3 are either $1, 1, 0$ or $2, 2, 0$. In the first scenario, write $v_1 = 3a + 1$, $v_2 = 3b + 1$, and $v_3 = 3c$, where $a < b$. Then we have

$$L_{1,2} = \frac{a+b}{3a+3b+2}, \quad L_{1,3} = \frac{a+c}{3a+3c+1}, \quad \text{and} \quad L_{2,3} = \frac{b+c}{3b+3c+1}.$$

Direct comparison shows that $L_{2,3}$ is the largest of these three quantities, so

$$\mathrm{ML}(v_1, v_2, v_3) = \frac{b+c}{3b+3c+1},$$

as desired.

In the second scenario, write $v_1 = 3a + 2$, $v_2 = 3b + 2$, and $v_3 = 3c$, where $a < b$. Then we have

$$L_{1,2} = \frac{a+b+1}{3a+3b+4}, \quad L_{1,3} = \frac{a+c}{3a+3c+2}, \quad \text{and} \quad L_{2,3} = \frac{b+c}{3b+3c+2}.$$

We are done if $L_{1,2}$ is the largest of these three quantities. Assume (for contradiction) this does not occur; since $L_{2,3} > L_{1,3}$, we must have $L_{2,3} > L_{1,2}$. Direct computation gives the inequality

$$c > 2a + b + 2.$$

Now, consider the time $t = 1/3 - 1/(9c)$. We compute that the fractional parts of tv_1 and tv_2 are, respectively,

$$\frac{2}{3} - \frac{3a+2}{9c} \quad \text{and} \quad \frac{2}{3} - \frac{3b+2}{9c},$$

whence both $\|tv_1\|$ and $\|tv_2\|$ are greater than $1/3$. Also,

$$\|tv_3\| = 0 + \frac{3c}{9c} = \frac{1}{3}.$$

This alltogether implies that $\mathrm{ML}(v_1, v_2, v_3) \geqslant 1/3$, contrary to our assumption. So $L_{1,2}$ must be the largest, as desired. This exhausts the cases in which the second possibility of the dichotomy obtains for each pair i, j.

It remains to treat the case in which the first possibility of the dichotomy obtains for some pair, say, $i = 1$, $j = 2$. Then we have

$$L_{1,2} = 0, \quad L_{1,3} = \frac{r_{1,3}}{v_1 + v_3}, \quad \text{and} \quad L_{2,3} = \frac{r_{2,3}}{v_2 + v_3}.$$

If both $v_1 + v_3$ and $v_2 + v_3$ are equivalent to 1 modulo 3, then we are done. Similarly, if $v_1 + v_3 \equiv 1 \pmod 3$ and $v_2 + v_3 \equiv 2 \pmod 3$, then $L_{1,3} > L_{2,3}$ by direct computation, and we are also done.

So it remains only to treat the case where both $v_1 + v_3$ and $v_2 + v_3$ are equivalent to 2 modulo 3. In particular, v_1 and v_2 leave the same residue modulo 3, so this residue is not 0. This in turn implies that v_3 is a multiple of 3 (since exactly one of the speeds is a multiple of 3). Then v_1 and v_2 leave a remainder of 2 modulo 3. At the time $t = 1/3 - 1/(3v_3)$, we obtain a loneliness of $1/3$ (as above), which contradicts our earlier assumption. This completes the proof. □

A close inspection of the previous two proofs reveals that a maximum loneliness of $1/4$ is obtained only for the speeds $1, 2, 3$.

Corollary 5.3. *The only tight set of speeds for $n = 3$ (up to scaling) is $1, 2, 3$.*

Proof (sketch). If the largest $L_{i,j}$ is $L_{1,3} = 1/4$, then we must have $v_1+v_3 = 4$, which implies that (without loss of generality) $v_1 = 1$ and $v_3 = 3$. Direct computation shows that $\mathrm{ML}(1, v_2, 3) > 1/4$ whenever $v_2 \geqslant 4$: consider times $5/12 < t < 7/12$, where both $\|t\|$ and $\|3t\|$ are greater than $1/4$; if in this interval the v_2 runner traverses a distance greater than $1/2$, then we find a time with loneliness larger than $1/4$. □

In principle, one could reproduce the program of this section for 4 or more moving runners: if many of the speeds share a large common factor, then the induction hypothesis together with a pre-jump gives the desired result; otherwise, there are only finitely many cases to consider in establishing an analog of Lemma 5.1, after which ad hoc arguments could take care of the remaining sporadic cases. Given the difficulty and length of the proof for $n = 3$, however, this program is probably infeasible for $n \geqslant 4$, as least for a non-computer-assisted proof.

6. One very fast runner

6.1. An asymptotic version of the Spectrum Conjecture

It is natural to try to use induction for the Lonely Runner Problem. One appealing strategy is the following: given speeds $v_1, \ldots, v_n$, use an induction hypothesis to obtain a lower bound for $\mathrm{ML}(v_1, \ldots, v_{n-1})$, with this loneliness achieved at some time t_0, then modify t_0 in order to obtain a time t_1 where every $\|t_1 v_i\|$ (now for $1 \leqslant i \leqslant n$) remains large. The following innocuous proposition demonstrates how this approach could play out if one runner is much faster than the rest.

Proposition 6.1. *Let $v_1 < \cdots < v_{n-1}$ be positive integers ($n \geqslant 2$) with $\mathrm{ML}(v_1, \ldots, v_{n-1}) \geqslant L$, and fix some $0 < \varepsilon < L$. Then we have that*

$$\mathrm{ML}(v_1, \ldots, v_n) \geqslant L - \varepsilon$$

whenever

$$v_n \geqslant \left(\frac{L-\varepsilon}{\varepsilon}\right) v_{n-1}.$$

Proof. Choose a time t_0 such that $\|t_0 v_i\| \geqslant L$ for all $1 \leqslant i \leqslant n-1$. We know that in a time interval of length ε/v_{n-1}, each such runner traverses a distance of at most ε. Let

$$I = \left[t_0 - \frac{\varepsilon}{v_{n-1}}, t_0 + \frac{\varepsilon}{v_{n-1}}\right]$$

be the closed interval of all times at most ε/v_{n-1} away from t_0. Consequently, for every $t \in I$, we have $\|t v_i\| \geqslant L - \varepsilon$ for $1 \leqslant i \leqslant n-1$. In any interval of length $(2\varepsilon)/v_{n-1}$, the runner with speed v_n traverses a distance of

$$\frac{2 v_n \varepsilon}{v_{n-1}} \geqslant 2(L-\varepsilon),$$

which in particular implies that there is some $t \in I$ with $\|tv_n\| \geqslant L - \varepsilon$. □

This observation motivates giving special attention to the case where one speed is significantly larger than the rest. More precisely, we examine the following weak ("asymptotic") version of the Loneliness Spectrum Conjecture.

Conjecture 6.2. For every integer $n \geqslant 4$, there exists a function $f_n : \mathbb{N} \to \mathbb{N}$ such that the following holds: for any positive integers $v_1 < \cdots < v_n$ with $v_n > f_n(v_{n-1})$, we have either

$$\mathrm{ML}(v_1, \ldots, v_n) = s/(ns + 1) \quad \text{for some } s \in \mathbb{N} \quad \text{or} \quad \mathrm{ML}(v_1, \ldots, v_n) \geqslant 1/n.$$

(Note that we do not require the speeds to lack a common factor.) In this section and the next two sections, we develop an explicit way to determine whether or not this conjecture holds for n runners based on the tight speed sets for $n - 1$ runners.

The conjecture immediately splits into two cases, depending on whether or not $\mathrm{ML}(v_1, \ldots, v_{n-1})$ is strictly larger than $1/n$. For the case $\mathrm{ML}(v_1, \ldots, v_{n-1}) > 1/n$, we quickly obtain an affirmative answer to Conjecture 6.2 with $f_n(v)$ quadratic in v (and independent of n). A statement of a similar flavor appears in [2].

Lemma 6.3. *Let $v_1 < \cdots < v_{n-1}$ be positive integers ($n \geqslant 3$) with*

$$\mathrm{ML}(v_1, \ldots, v_{n-1}) = L > \frac{1}{n}.$$

Then we have

$$ML(v_1, \ldots, v_n) \geqslant \frac{1}{n}$$

whenever

$$v_n \geqslant v_{n-1}(2v_{n-1} - 1).$$

Proof. By Proposition 2.1, we know that

$$L = \frac{m}{v_i + v_j}$$

for some integer m and some $1 \leqslant i < j \leqslant n - 1$. In particular, the inequality

$$nm > v_i + v_j$$

in the integers implies that

$$nm \geqslant v_i + v_j + 1.$$

We then compute

$$L \geqslant \frac{v_i + v_j + 1}{n(v_i + v_j)} = \frac{1}{n} + \frac{1}{n(v_i + v_j)} \geqslant \frac{1}{n} + \frac{1}{n(2v_{n-1} - 1)}.$$

Applying Proposition 6.1 with $\varepsilon = \frac{1}{n(2v_{n-1}-1)}$ and $L' = 1/n + \varepsilon$ shows that

$$\mathrm{ML}(v_1, \ldots, v_n) \geqslant \frac{1}{n},$$

as desired. □

We remark that if we assume that the Spectrum Conjecture holds for $n-1$ runners, then we can take $f_n(v)$ to be linear in v rather than quadratic in v (at the cost of a dependence on n). This improvement comes from the assumption that non-tight sets of $n-1$ speeds have maximum loneliness uniformly bounded away from $1/n$.

6.2. Accumulation points

We temporarily take a step back and consider the big picture. For each positive integer n, define the set

$$\mathcal{S}(n) = \{\mathrm{ML}(v_1, \dots, v_n) : v_1, \dots, v_n \text{ positive integers}\}$$

to consist of all maximum loneliness amounts achieved by sets of n runners. It is immediate that $\mathcal{S}(n) \subset \mathcal{S}(n+1)$ and that $\mathcal{S}(n) \subset (0, 1/2]$. The Lonely Runner Conjecture asserts that $\mathcal{S}(n) \subset [1/(n+1), 1/2]$; the Loneliness Spectrum Conjecture, its refinement, asserts that $\mathcal{S}(n)$ is the union of $\{s/(ns+1) : s \in \mathbb{N}\}$ and a subset of $[1/n, 1/2]$.

We define a real number A to be an *accumulation point* for n runners if the set $\mathcal{S}(n)$ contains elements that are arbitrarily close to A. On a more fine-grained view, we define A to be a *lower accumulation point* for n runners if $\mathcal{S}(n)$ contains a sequence of elements approaching A from below; we define A to be an *upper accumulation point* for n runners if $\mathcal{S}(n)$ contains a sequence of elements approaching A from above. For instance, Theorem 3.1 shows that $1/n$ is a lower accumulation point for n runners.

In this section, we make progress towards determining the lower and upper accumulation points for n runners. As a starting point, the classification in Theorem 4.1 immediately answers both questions for $n = 2$: the set of lower accumulation points for 2 runners is precisely $\mathcal{S}(1) = \{1/2\}$, and there are no upper accumulation points for 2 runners.

We now establish a refinement of Proposition 6.1 in the case where we have additional information about the factors of v_n.

Lemma 6.4. *Let $v_1 < \cdots < v_{n-1}$ be positive integers ($n \geqslant 2$) with*

$$\mathrm{ML}(v_1, \dots, v_{n-1}) = L.$$

Let $t_1, \dots, t_r$ be the times in the interval $[0, 1)$ at which $\|tv_i\| \geqslant L$ for all $1 \leqslant i \leqslant n-1$. For each t_j, let ρ_j be the largest index such that $t_j v_{\rho_j}$ has remainder L modulo 1, and let λ_j be the largest index such that $t_j v_{\lambda_j}$ has remainder $1 - L$ modulo 1. Finally, define

$$\mu = \min_{1 \leqslant j \leqslant r} \{\rho_j, \lambda_j\},$$

where the minimum runs over all values of j. Then

$$\mathrm{ML}(v_1, \dots, v_n) = \frac{v_n L}{v_n + v_\mu}$$

whenever v_n is a sufficiently large integer multiple of the lcm of the denominators of the t_j's.

Let us state a "physical" interpretation of what is expressed above in symbols: at each equality time t_j, we identify the fastest runner whose position in $\mathbb{R}/\mathbb{Z}$ is L and the fastest runner whose position is $1-L$, then we let v_μ be the slowest of these runners, across all the t_j's. We also remark that carefully carrying all of the bounds through the proof shows that $v_n > 4v_{n-1}^3 v_1$ is sufficiently large; we make no attempts to optimize this quantity.

Proof. The divisibility condition on v_n ensures that at every "equality time" t_j, we have $\|t_j v_n\| = 0$, whence we conclude that

$$\mathrm{ML}(v_1, \ldots, v_n) < L$$

strictly. We now record a few properties of the real function

$$e(t) = \min_{1 \leqslant i \leqslant n-1} \|tv_i\|.$$

Note that $e(t)$ is continuous and piecewise linear and the slope of each linear segment has absolute value between v_1 and v_{n-1}. Moreover, $e(t)$ has a local minimum at t_0 if and only if $e(t_0) = 0$.

Fix some sufficiently small $\varepsilon > 0$ (say, $\varepsilon \leqslant 1/(4v_{n-1}^3)$). The above observations tell us that if $e(t) \geqslant L - \varepsilon$, then t is a distance at most ε/v_1 from some t' with $e(t') = L$. In particular, if we want to identify all possible times t (modulo 1) with $e(t) \geqslant L - \varepsilon$, then it suffices to examine the closed ε/v_1-neighborhoods of the t_j's. Moreover, since ε is sufficiently small, we have

$$e(t) = \left\|tv_{\rho_j}\right\| \quad \text{for all } t_j - \frac{\varepsilon}{v_1} \leqslant t \leqslant t_j$$

and

$$e(t) = \left\|tv_{\lambda_j}\right\| \quad \text{for all } t_j \leqslant t \leqslant t_j + \frac{\varepsilon}{v_1}.$$

This characterization allows us to compute the maximum loneliness explicitly when we include the runner with speed v_n.

Fix some t_j, and recall that $\|t_j v_n\| = 0$. Write $t = t_j + \delta$. As δ is increased from 0, the quantity $\|tv_n\|$ increases from 0 at a rate of v_n and the quantity $\left\|tv_{\lambda_j}\right\|$ decreases from L at a rate of v_{λ_j}. We obtain equality $\|tv_n\| = \left\|tv_{\lambda_j}\right\|$ when $\delta = \delta_0 = L/(v_n + v_{\lambda_j})$, at which point

$$\|tv_n\| = \left\|tv_{\lambda_j}\right\| = \frac{Lv_n}{v_n + v_{\lambda_j}}.$$

By choosing v_n sufficiently large (say, $v_n \geqslant (Lv_1)/\varepsilon$), we guarantee that $\delta_0 \leqslant \varepsilon/v_1$. Moreover, since the quantity $\left\|tv_{\lambda_j}\right\|$ is monotonically decreasing as δ increases, we conclude that

$$\frac{Lv_n}{v_n + v_{\lambda_j}}$$

is the largest loneliness amount achieved for the speeds $v_1, \ldots, v_n$ for times $t_j \leqslant t \leqslant t_j + \varepsilon/v_1$.

The same reasoning shows that the largest loneliness amount achieved for the speeds $v_1, \ldots, v_n$ for times $t_j - \varepsilon/v_1 \leqslant t \leqslant t_j$ is

$$\frac{Lv_n}{v_n + v_{\rho_j}}.$$

Taking the maximum of all such quantities over j gives that

$$\mathrm{ML}(v_1, \ldots, v_n) = \frac{v_n L}{v_n + v_\mu},$$

as desired. □

This lemma allows us to deduce that every element of $\mathcal{S}(n-1)$ is a lower accumulation point for n runners.

Theorem 6.5. *For every $n \geqslant 2$, the set of lower accumulation points for n runners contains $\mathcal{S}(n-1)$.*

Proof. Fix some $L \in \mathcal{S}(n-1)$. Then there exist positive integers $v_1, \ldots, v_{n-1}$ such that $\mathrm{ML}(v_1, \ldots, v_{n-1}) = L$. By Lemma 6.4, there is an increasing sequence of values for v_n such that the quantity $\mathrm{ML}(v_1, \ldots, v_n)$ approaches L from below. □

In the absence of any straightforward constructions for upper accumulation points or other lower accumulation points, it is natural to suspect that all accumulation points for n runners arise through the "mechanism" of Theorem 6.5, and we present the following pair of questions.

Question 6.6. For $n \geqslant 2$, is the set of lower accumulation points for n runners always precisely $\mathcal{S}(n-1)$?

Question 6.7. For $n \geqslant 2$, are there any upper accumulation points for n runners, or are all accumulation points only lower accumulation points?

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An immediate corollary to Theorem 6.5 is that Loneliness Spectrum Conjecture (as well as its weakened version, Conjecture 6.2) for n runners implies the Lonely Runner Conjecture for $n-1$ runners. We contrast this implication with the situation for the Lonely Runner Conjecture alone, where (to our knowledge), the statement for n runners does not directly imply the statement for $n-1$ runners. We record this observation in the following corollary.

Corollary 6.8. *Fix some. $n \geqslant 2$. The Loneliness Spectrum Conjecture for n runners implies Conjecture 6.2 for n runners, which in turn implies the Lonely Runner Conjecture for $n-1$ runners.*

6.3. Towards Conjecture 6.2

We now present a generalization of Lemma 6.4 to the scenario in which v_n has a fixed residue with respect to certain moduli; the idea is substantively the same as the idea of Lemma 6.4, but the result is messier to state, so we introduce notation and an informal description before giving the precise statement and proof sketch.

As in Lemma 6.4, we fix positive integers $v_1, \dots, v_{n-1}$ with

$$\mathrm{ML}(v_1, \dots, v_{n-1}) = L,$$

and we let $t_1, \dots, t_r$ be the "equality times" in $[0, 1)$, i.e., the times for which $\|tv_i\| \geqslant L$ for all $1 \leqslant i \leqslant n-1$. Let D be the least common multiple of the denominators appearing in the reduced-fraction representations of the t_j's. Fix some integer

$$-\frac{D}{2} < Q \leqslant \frac{D}{2}$$

such that $\|t_jQ\| < L$ for all $1 \leqslant j \leqslant r$. (Call such a value of Q *admissible*. We will look at values of v_n that are equivalent to Q modulo D. Lemma 6.4 is the special case $Q = 0$.) For each j, let u_j denote the real number in $(-1/2, 1/2]$ that is equivalent to t_jQ modulo 1. As in Lemma 6.4, for each t_j we let ρ_j be the largest index such that $t_jv_{\rho_j}$ has remainder L modulo 1, and we let λ_j be the largest index such that $t_jv_{\lambda_j}$ has remainder $1 - L$ modulo 1. Now, instead of taking a minimum over the ρ_j's and λ_j's, we take the following "weighted minimum": let μ be the ρ_j or λ_j that minimizes the quantity

$$v_{\rho_j}(L - u_j) \quad \text{or} \quad v_{\lambda_j}(L + u_j),$$

respectively. We break ties between ρ_j's in favor of larger u_j (and arbitrarily beyond that point), and we break ties between λ_j's in favor of smaller u_j (and arbitrarily beyond that point). We break ties between $\rho_{j'}$ and $\lambda_{j''}$ in favor of larger u_j (and arbitrarily beyond that point). So, keeping track of which j our μ "came from" and whether it came from a ρ or from a λ, we write either $\mu = \rho_k$ or $\mu = \lambda_k$. We can finally state the lemma.

Lemma 6.9. *Let $v_1, \dots, v_{n-1}$ be positive integers ($n \geqslant 2$) with*

$$\mathrm{ML}(v_1, \dots, v_{n-1}) = L.$$

Define $t_1, \dots, t_r$ and D as above, and fix some admissible integer

$$-\frac{D}{2} < Q \leqslant \frac{D}{2}.$$

Now define the u_j's, ρ_j's, and λ_j's as above, along with the resulting $\mu = \rho_k$ or $\mu = \lambda_k$. Then

$$\mathrm{ML}(v_1, \dots, v_n) = \begin{cases} L - \frac{v_{\rho_k}}{v_{\rho_k}+v_n}(L - u_k), & \text{if } \mu = \rho_k \\ L - \frac{v_{\lambda_k}}{v_{\lambda_k}+v_n}(L + u_k), & \text{if } \mu = \lambda_k \end{cases}$$

whenever v_n is a sufficiently large integer that is equivalent to Q modulo D.

Proof (sketch). The choice of Q guarantees that $\|t_j v_n\| = \|t_j Q\| < L$ for every t_j, which in turn implies that

$$\mathrm{ML}(v_1, \dots, v_n) < L.$$

Moreover, note that at the time t_j, the runner with speed v_n is at the position u_j (which lies strictly between $-L$ and L). The argument from the proof of Lemma 6.4 shows that if $\|tv_i\| \geqslant L - \varepsilon$ for all $1 \leqslant i \leqslant n-1$, then t is within ε/v_1 of some t_j, so we restrict our attention to these neighborhoods. For times slightly larger than t_j, the greatest loneliness achieved by speeds $v_1, \dots, v_n$ is precisely

$$L - \frac{v_{\lambda_j}}{v_{\lambda_j} + v_n}(L + u_j),$$

and for times slightly smaller than t_j, the greatest loneliness achieved is

$$L - \frac{v_{\rho_j}}{v_{\rho_j} + v_n}(L - u_j).$$

Taking a minimum over all such expressions (for sufficiently large n) gives the desired result. (In other words, there is one expression that "wins out" for all sufficiently large n.) □

We can now give a necessary and sufficient condition to determine whether or not Conjecture 6.2 holds for speeds $v_1, \dots, v_n$, where we fix $v_1, \dots, v_{n-1}$.

Theorem 6.10. *Let $v_1 < \cdots < v_{n-1}$ be positive integers ($n \geqslant 2$) with $\mathrm{ML}(v_1, \dots, v_{n-1}) = L$. Then the following are equivalent:*

(I) For every sufficiently large integer v_n, we have either

$$\mathrm{ML}(v_1, \dots, v_n) = \frac{s}{ns+1} \quad \textit{for some } s \in \mathbb{N} \quad \textit{or} \quad \mathrm{ML}(v_1, \dots, v_n) \geqslant \frac{1}{n}.$$

(II) One of the following holds:

(a) $L > 1/n$.

(b) $L = 1/n$; and (in the notation of Lemma 6.9) for each admissible residue Q we have, when $\mu = \rho_k$ (respectively, $\mu = \lambda_k$), both the equality

$$Q = -nv_{\rho_k} u_k \quad \textit{(respectively, } Q = nv_{\lambda_k} u_k\textit{)}$$

and the property that

$$\frac{D}{nv_{\rho_k}(1 - nu_k)} \quad \textit{(respectively, } \frac{D}{nv_{\lambda_k}(1 + nu_k)}\textit{)}$$

is an integer.

Proof. It is clear from Lemma 6.9 that if $L > 1/n$, then $\mathrm{ML}(v_1, \ldots, v_n) > 1/n$ for all sufficiently large v_n; the reverse implication follows from Lemma 6.4. It is also clear from Lemma 6.4 that if $L < 1/n$, then (I) does not hold. So it remains to consider the case where $L = 1/n$. For each integer $-D/2 < Q \leqslant D/2$, we consider sufficiently large values of n with (fixed) residue Q modulo D. If there is any t_j with $\|t_j Q\| \geqslant 1/n$, then the speeds $v_1, \ldots, v_n$ achieve a loneliness of $1/n$ at that time, whence we conclude that $\mathrm{ML}(v_1, \ldots, v_n) = 1/n$. So, as in Lemma 6.9 we restrict our attention to admissible values of Q.

Fix some such Q, and write $v_n = mD + Q$. Suppose that in Lemma 6.9, we have $\mu = \rho_k$. Then for sufficiently large m (i.e., sufficiently large v_n), we have

$$\mathrm{ML}(v_1, \ldots, v_n) = \frac{1}{n} - \frac{v_{\rho_k}}{v_{\rho_k} + v_n}\left(\frac{1}{n} - u_k\right).$$

Suppose this quantity equals $s/(ns+1)$ for some $s \in \mathbb{N}$, i.e.,

$$\frac{v_{\rho_k}}{v_{\rho_k} + v_n}\left(\frac{1}{n} - u_k\right) = \frac{1}{n(ns+1)}.$$

Substituting for v_n and rearranging gives

$$\frac{mD + Q + v_{\rho_k}}{v_{\rho_k}(1 - nu_k)} = ns + 1.$$

Each (sufficiently large) m can have a corresponding s only if incrementing m increments the left-hand side by an integer multiple of n, i.e.,

$$\frac{D}{nv_{\rho_k}(1 - nu_k)}$$

is an integer. Moreover, we then see that we also require

$$\frac{Q + v_{\rho_k}}{v_{\rho_k}(1 - nu_k)} = 1,$$

or, equivalently,

$$Q = -nv_{\rho_k}u_k.$$

These are precisely the two conditions of (II.b). The same argument for $\mu = \lambda_k$ gives the analogous pair of conditions in the statement of the lemma. Since these steps are reversible, we see that (II.b) also implies (I). Finally, to achieve a uniform bound (over choices of Q) on how large v_n must be, we simply take the maximum of the bounds obtained for the various Q's. □

This theorem reduces Conjecture 6.2 for n runners to an explicit computation once we know all of the tight speed sets for $n-1$ runners; moreover, this computation is finite if there are only finitely many tight sets of speeds (up to scaling). Recall from Corollary 6.8 that if the Lonely Runner Conjecture does not hold for $n-1$ runners, then Conjecture 6.2 does not hold for n runners.

Theorem 6.11. *Fix some $n \geqslant 4$. Suppose the Lonely Runner Conjecture holds for $n-1$ runners and we know all of the (finitely many) tight speed sets for $n-1$ runners. Then Conjecture 6.2 holds for n runners if and only if every such tight set of speeds $v_1 < \cdots < v_{n-1}$ with $\gcd(v_1, \ldots, v_{n-1}) = 1$ satisfies the conditions in (II.b) of Theorem 6.10 (each of which can be checked with an explicit finite computation).*

Proof. First, recall that Conjecture 6.2 asks for a function $f_n : \mathbb{N} \to \mathbb{N}$ such that the set of speeds $\mathrm{ML}(v_1, \ldots, v_n)$ has certain properties whenever $v_n > f_n(v_{n-1})$; this is *uniform* bound on v_n in terms of v_{n-1}. Since there are only finitely many sets of positive speeds $v_1 < \cdots < v_{n-1}$ for each value of v_{n-1}, however, it suffices to consider "sufficiently large" v_n for each set $v_1, \ldots, v_{n-1}$ separately and then take $f_n(v_{n-1})$ to be the maximum of the bounds obtained.

Suppose we have verified the conditions in (II.b) of Theorem 6.10 for every tight set of speeds $v_1 < \cdots < v_{n-1}$ with $\gcd(v_1, \ldots, v_{n-1}) = 1$. Then we claim that (I) is also satisfied for every set of tight speeds. Indeed, let $v_1' < \cdots < v_{n-1}'$ be positive integers with $\mathrm{ML}(v_1', \ldots, v_{n-1}') = 1/n$. Then let $g = \gcd(v_1', \ldots, v_{n-1}')$, and write $v_i' = g v_i$ for $1 \leqslant i \leqslant n-1$, where we know that (II.b) is satisfied for the speeds $v_1, \ldots, v_{n-1}$. If $g = 1$, then we are done, so consider $g \geqslant 2$. The equality times for $v_1', \ldots, v_{n-1}'$ are precisely the times of the form

$$\frac{t_j + h}{g},$$

where t_j is an equality time for $v_1, \ldots, v_{n-1}$ and h is an integer. Thus, we have $D' = gD$, where D' (respecticely, D) is the lcm of the equality times in $[0,1)$ for the speeds $v_1', \ldots, v_{n-1}'$ (respectively, $v_1, \ldots, v_{n-1}$). We now consider various admissible residues Q' (modulo D'). Each admissible Q' must be a multiple of g: otherwise, we could add time increments of $1/g$ (pre-jump) to find an equality time t_j' for $v_1', \ldots, v_{n-1}'$ at which

$$\left\| t_j' Q' \right\| \geqslant \frac{1}{2} - \frac{1}{2g} \geqslant \frac{1}{4} \geqslant \frac{1}{n}.$$

So Q' is a multiple of g, and any v_n' that is equivalent to Q' modulo D' can be written as $v_n' = g v_n$. But then

$$\mathrm{ML}(v_1', \ldots, v_n') = \mathrm{ML}(v_1, \ldots, v_n),$$

and we know that the quantity on the right-hand side satisfies condition (I) of Theorem 6.10. So we conclude that it suffices to check tight instances for $n-1$ runners where the speeds do not all share a common factor. □

Another point of interest of this theorem is that it provides a potential way to refute the Spectrum Conjecture.

We now apply Theorem 6.10 to the tight set of speeds $1, \ldots, n-1$ and then use Theorem 6.11 to resolve Conjecture 6.2 for 4 moving runners. The computation is straightforward, and we remark that it would be interesting to carry out these computations for the other tight sets of speeds discussed in Goddyn and Wong [18].

Proposition 6.12. *Consider the tight set of speeds $v_1 = 1, v_2 = 2, \ldots, v_{n-1} = n-1$ ($n \geqslant 4$). This set of speeds satisfies the conditions of (II.b) in Theorem 6.10.*

Proof. The equality times for $v_1, \ldots, v_{n-1}$ are precisely the times of the form

$$\frac{m}{n},$$

where m is relatively prime to n, so $D = n$. The only admissible value of Q is 0 because otherwise we would have $\|tQ\| \geqslant 1/n$ at the equality time $t = 1/n$. Since $Q = 0$, we are in fact in the setting of Lemma 6.4, so we simply have to find the smallest ρ_j or λ_j. We get $\rho_1 = 1$ at the time $t_1 = 1/n$, and this is the smallest possible. Since $Q = 0$ implies that $u_1 = 0$, the first condition of (II.b) is immediately satisfied. For the second condition, it suffices to observe that

$$\frac{D}{nv_{\rho_1}(1 - nu_1)} = \frac{n}{n(1)(1-0)} = 1$$

is an integer. □

Recall from Corollary 5.3 that the only tight speed set for 3 runners is, up to scaling, $1, 2, 3$. It then follows from the preceding discussion that Conjecture 6.2 holds for $n = 4$ (and the function f_4 can be taken to be quartic in v_{n-1} since we stayed in the Lemma 6.4 "special subcase" of Lemma 6.9).

Corollary 6.13. *There exists a function $f_4 : \mathbb{N} \to \mathbb{N}$ such that for any positive integers $v_1 < v_2 < v_3 < v_4$ with $v_4 > f_4(v_3)$, we have either*

$$\mathrm{ML}(v_1, v_2, v_3, v_4) = \frac{s}{4s+1} \quad \textit{for some } s \in \mathbb{N} \quad \textit{or} \quad \mathrm{ML}(v_1, v_2, v_3, v_4) \geqslant \frac{1}{4}.$$

We can carry out the same program for $n = 6$ by making use of Bohman, Holzman, and Kleitman's determination [6] of all of the tight sets of speeds for 5 moving runners.

Theorem 6.14 (Bohman, Holzman, and Kleitman [6])**.** *The Lonely Runner Conjecture holds for $n = 5$. Moreover, if $v_1, \ldots, v_5$ are positive integers with $\gcd(v_1, \ldots, v_5) = 1$ and $\mathrm{ML}(v_1, \ldots, v_5) = 1/6$, then $v_1, \ldots, v_5$ are (in some order) either $1, 2, 3, 4, 5$, or $1, 3, 4, 5, 9$.*

Corollary 6.15. *There exists a function $f_6 : \mathbb{N} \to \mathbb{N}$ such that for any positive integers $v_1 < \cdots < v_6$ with $v_6 > f_6(v_5)$, we have either*

$$\mathrm{ML}(v_1, \ldots, v_6) = \frac{s}{6s+1} \quad \textit{for some } s \in \mathbb{N} \quad \textit{or} \quad \mathrm{ML}(v_1, \ldots, v_6) \geqslant \frac{1}{6}.$$

Proof. The set of tight speeds $1, 2, 3, 4, 5$ is handled by Proposition 6.12. For $1, 3, 4, 5, 9$, the only equality times in $[0, 1)$ are $1/6$ and $5/6$. So $D = 6$, and it is easy to check that only $Q = 0$ is admissible. As in the proof of Proposition 6.12, we find ourselves in the setting of Lemma 6.4, where we get the "best possible" value $v_{\rho_1} = 1$ at the time $t_1 = 1/n$, and the conditions of (II.b) are satisfied in the same way. □

It is curious that the literature seems not to contain a characterization of all tight sets of speeds with 4 moving runners. It is widely known (see, e.g., [18]) that $1, 3, 4, 7$ is a tight speed set in addition to the trivial $1, 2, 3, 4$. The reader may easily verify that the conditions of (II.b) are satisfied for $1, 3, 4, 7$. It appears likely that there are no other tight sets of speeds (up to scaling) (see also [1]), in which case we would also be able to confirm Conjecture 6.2 for $n = 5$.

7. Concluding remarks

We gather here a number of questions, problems, and ideas that could be fruitful starting points for future research on the loneliness spectrum.

- Prove the Spectrum Conjecture for $n = 4$. Is it feasible to extend the techniques that we used for the $n = 3$ case?
- Determine whether or not $1, 2, 3, 4$ and $1, 3, 4, 7$ are (up to scaling) the only tight speed sets for 4 runners.
- Determine the set of accumulation points for n runners. Are there any upper accumulation points?
- In all of our applications of Theorem 6.10, only $Q = 0$ is admissible. Is this the case for all tight speed sets?
- Check the conditions in (II.b) of Theorem 6.10 for the families of tight speed sets in Goddyn and Wong [18].
- To what extent does discrete behavior persist in values of $\mathcal{S}(n)$ that are slightly larger than $1/n$? (See the preliminary results in [11].)
- Tao [24] has proven the Lonely Runner Conjecture for the case where all of the speeds are at most $1.2n$. It is easy to see that the Lonely Runner Conjecture is equivalent to the Spectrum Conjecture in the regime where all speeds are at most $1.5n$; it would be desirable to establish the Spectrum Conjecture for positive integer speeds up to βn, for some $\beta > 1.5$.
- There has also been interest in a "shifted" variant of the Lonely Runner Problem in which one allows the runners to start at different positions on the track. Cslovjecsek, Malikiosis, Naszódi, and Schymura [14] have recently shown that the analog of the Spectrum Conjecture fails in this new setting, and they have proposed a slightly weaker version (allowing more discrete values). Many of the questions addressed in the present paper could also be investigated for the shifted Lonely Runner Problem.

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