THE q-ANALOG OF THE MARKOFF INJECTIVITY CONJECTURE OVER THE LANGUAGE OF A BALANCED SEQUENCE

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Abstract. The Markoff injectivity conjecture states that $w \mapsto \mu(w)_{12}$ is injective on the set of Christoffel words where $\mu : \{0,1\}^* \to \mathrm{SL}_2(\mathbb{Z})$ is a certain homomorphism and M_{12} is the entry above the diagonal of a 2×2 matrix M. Recently, Leclere and Morier-Genoud (2021) proposed a q-analog μ_q of μ such that $\mu_q(w)_{12}|_{q=1} = \mu(w)_{12}$ is the Markoff number associated to the Christoffel word w when evaluated at $q = 1$. We show that there exists an order $<_{radix}$ on $\{0,1\}^*$ such that for every balanced sequence $s \in \{0,1\}^{\mathbb{Z}}$ and for all factors u, v in the language of s with $u <_{radix} v$, the difference $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomial of indeterminate q with nonnegative integer coefficients. Therefore, the map $u \mapsto \mu_q(u)_{12}$ is injective over the language of a balanced sequence. The proof uses an equivalence between balanced sequences satisfying some Markoff property and indistinguishable asymptotic pairs.

Keywords. Balance, Markoff spectrum, Sturmian, Christoffel, q-analog

Mathematics Subject Classifications. 11J06, 68R15, 05A30

1. Introduction

A Markoff triple is a positive solution of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{1.1}$$

It was introduced by Markoff [Mar79, Mar80] to describe minima of indefinite real binary quadratic forms. Positive solutions of Equation 1.1 can be computed recursively. If (x, y, z) is a Markoff triple, then $(x, 3xy - z, y)$ and $(y, 3yz - x, z)$ are also Markoff triples, see Figure 1.1. A Markoff triple is called proper as long as x, y and z are pairwise distinct. There are only two

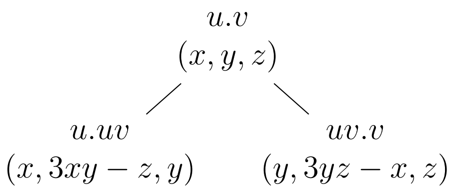

Figure 1.1: Binary tree structure of Christoffel words $u.v$ and Markoff triples (x, y, z).

Markoff triples which are not proper, namely $(1, 1, 1)$ and $(1, 2, 1)$. If (x, y, z) is a proper Markoff triple with $y \geqslant x$ and $y \geqslant z$, then $(x, 3xy - z, y) \neq (y, 3yz - x, z)$ and both $3xy - z$ and $3yz - x$ are greater than y. Hence, the proper Markoff triples naturally label a complete infinite binary tree, see Figure 1.2. All proper Markoff triple have a maximum value. Frobenius [Fro13] asked whether each Markoff number (an element of a Markoff triple) is the maximum of a unique Markoff triple. The question known as the *uniqueness conjecture* is still open. A book was devoted to it and its many equivalent formulations for its 100^{th} anniversary [Aig13].

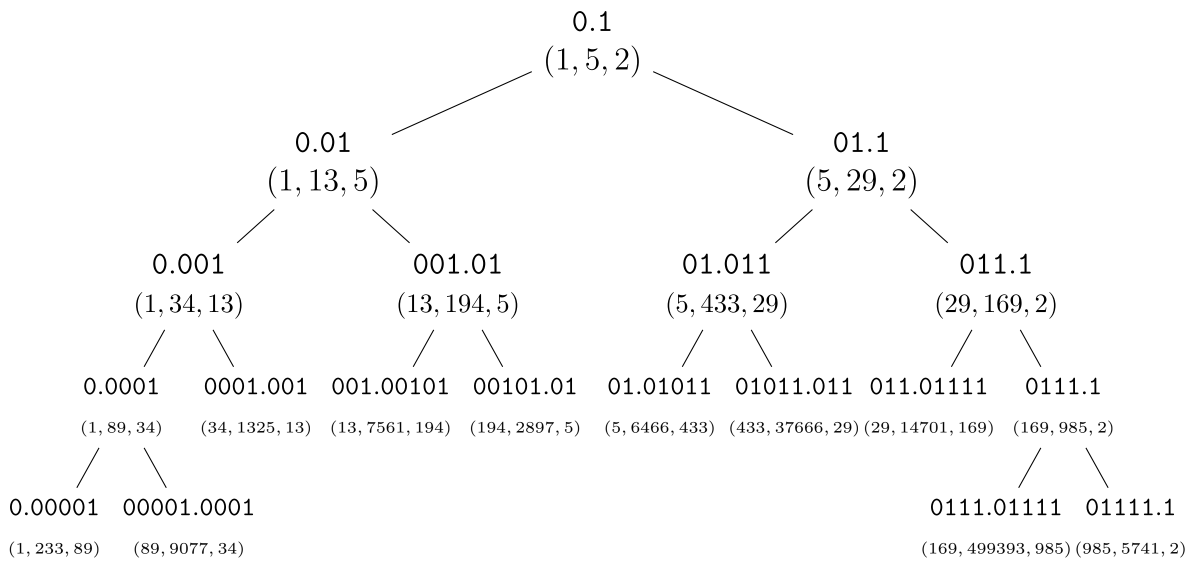

Figure 1.2: Binary tree of proper Christoffel words and proper Markoff triples.

The conjecture can be stated in terms of Christoffel words [Reu19]. Christoffel words are words over the alphabet $\{0, 1\}$ also satisfying a binary tree structure: 0, 1 and 01 are Christoffel words and if $u, v, uv \in \{0, 1\}^*$ are Christoffel words then uuv and uvv are Christoffel words [BLRS09], see Figure 1.1. Note that these are usually named *lower* Christoffel words. It is known that each Markoff number can be expressed in terms of a Christoffel word. More precisely, let μ be the monoid homomorphism $\{0, 1\}^* \to \mathrm{SL}_2(\mathbb{Z})$ defined by

$$\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Each Markoff number is equal to $\mu(w)_{12}$ for some Christoffel word w (see [Reu09]), that is, the element above the diagonal in the matrix $\mu(w)$. Moreover, positive primitive elements of the free group F_2 on two generators are in bijection with Markoff triples [Bom07]. Both results are equivalent since primitive elements of the free group F_2 are in bijection with Christoffel words [KR07].

In other words, the map $w \mapsto \mu(w)_{12}$ from Christoffel words to Markoff numbers is surjective. For example, the Markoff number 194 is associated with the Christoffel word 00101 as it is the entry at position $(1,2)$ in the matrix $\mu(\texttt{00101}) = \left(\begin{smallmatrix} 463 & 194 \\ 284 & 119 \end{smallmatrix}\right)$. Whether this map provides a bijection between Christoffel words and Markoff numbers is equivalent to the uniqueness conjecture. Indeed, the uniqueness conjecture can be expressed in terms of the injectivity of the map $w \mapsto \mu(w)_{12}$ [Reu19, §3.3].

Markoff Injectivity Conjecture. *The map $w \mapsto \mu(w)_{12}$ is injective on the set of Christoffel words.*

The map $w \mapsto \mu(w)_{12}$ is defined over the monoid $\{\texttt{0},\texttt{1}\}^*$ not only on Christoffel words. On this extended domain, Lapointe and Reutenauer showed that $w \mapsto \mu(w)_{12}$ is strictly increasing (thus injective) over the language of factors appearing in a Christoffel word [Lap20, LR21], thus also for all Christoffel words on an infinite path in the binary tree of Christoffel words. The map is not injective on $\{\texttt{0},\texttt{1}\}^*$ as for example, $\mu(\texttt{0011})_{12} = 75 = \mu(\texttt{0101})_{12}$. But Lapointe and Reutenauer conjectured that it is injective on the language of all factors of Christoffel words [LR21, Conjecture 2].

1.1. q-analogs

Markoff numbers and the Markoff injectivity conjecture can be parametrized by introducing a parameter q. Recall that the q-analog of a nonnegative integer n is

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

Recently, the q-analog $\left[\frac{a}{b}\right]_q \in \mathbb{Q}(q)$ of every rational number $\frac{a}{b} \in \mathbb{Q}$ was introduced to be a ratio of polynomials over q defined from the continued fraction expansion of $\frac{a}{b}$ [MGO20]. It also defines naturally the q-analog of all real numbers as an infinite series over the variable q [MGO19]. The approach is based on the following q-deformation of the generators $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\pm\mathrm{Id}$:

$$R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad S_q = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}.$$

Since $\mu(\texttt{0}) = R^2SR$ and $\mu(\texttt{1}) = R^3SR^2SR$, the q-analog of $\mu(\texttt{0})$ and $\mu(\texttt{1})$ are [LMG21]

$$\mu_q(\texttt{0}) = R_q^2 S_q R_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix},$$

$$\mu_q(\texttt{1}) = R_q^3 S_q R_q^2 S_q R_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}.$$

Therefore, this defines a morphism of monoids $\mu_q : \{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}])$.

The q-analog of a nonnegative integer, a rational or a real number α has the property of being equal to α when evaluated at $q = 1$ or more generally when $q \to 1$. Likewise, we may observe that $\mu_q(\mathtt{0})|_{q=1} = \mu(\mathtt{0})$ and $\mu_q(\mathtt{1})|_{q=1} = \mu(\mathtt{1})$. Therefore, we have $\mu_q(w)_{12}|_{q=1} = \mu(w)_{12}$ for all $w \in \{\mathtt{0},\mathtt{1}\}^*$. Thus, if $w \in \{\mathtt{0},\mathtt{1}\}^*$ is a Christoffel word, the polynomial $\mu_q(w)_{12}$ over the variable q is a *q-analog of a Markoff number* satisfying that $\mu_q(w)_{12}$ evaluated at $q = 1$ is a Markoff number, see Figure 1.3. For example,

$$\mu_q(\mathtt{00101})_{12} = 1 + 4q + 10q^2 + 18q^3 + 27q^4 + 33q^5 + 33q^6 + 29q^7 + 21q^8 + 12q^9 + 5q^{10} + q^{11}$$

and

$$\mu_q(\mathtt{00101})_{12}|_{q=1} = 1 + 4 + 10 + 18 + 27 + 33 + 33 + 29 + 21 + 12 + 5 + 1 = 194.$$

A natural question is to understand the structure of the coefficients of $\mu_q(w)_{12}$ whose sum is a Markoff number when w is a Christoffel word.

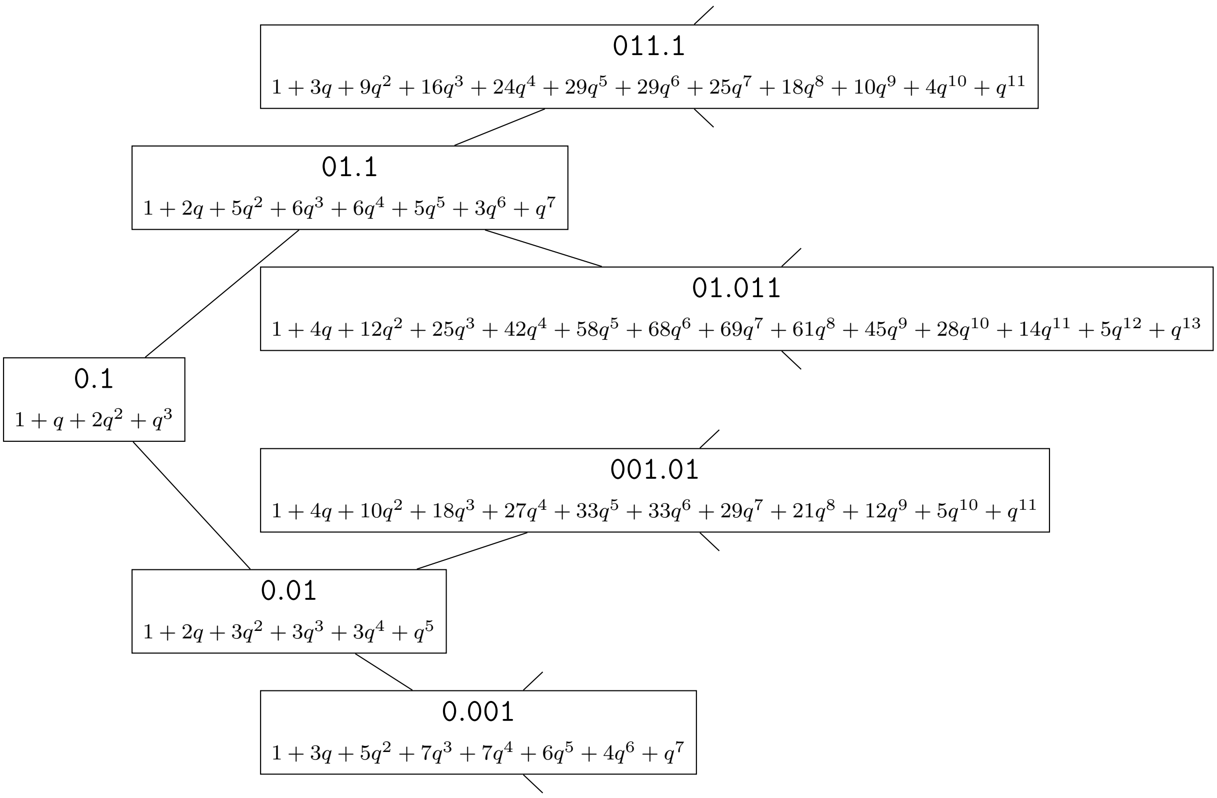

Figure 1.3: Binary tree of proper Christoffel words w and proper q-Markoff numbers $\mu_q(w)_{12}$. The sequence of polynomials associated to words $w \in \{\mathtt{00}^*\mathtt{1}\}$ is a subsequence of a sequence indexed in the Online Encyclopedia of Integer Sequences at `http://oeis.org/A123245`.

Question 1.1. Is there a combinatorial interpretation for the degree and coefficients of the q-Markoff number $\mu_q(w)_{12} \in \mathbb{Z}[q]$ associated to a Christoffel word $w \in \{\mathtt{0},\mathtt{1}\}^*$?

Also, a natural extension of the Markoff injectivity conjecture to the q-analog of Markoff numbers is the following.

Conjecture 1.2 (q-analog of the Markoff Injectivity Conjecture)**.** The map $\{0,1\}^* \to \mathbb{Z}[q]$ defined by $w \mapsto \mu_q(w)_{12}$ is injective over the set of Christoffel words.

Markoff Injectivity Conjecture implies Conjecture 1.2, since if $\mu_q(u)_{12}|_{q=1} \neq \mu_q(v)_{12}|_{q=1}$, then $\mu_q(u)_{12} \neq \mu_q(v)_{12}$. However, Conjecture 1.2 is weaker than the classical Markoff Injectivity Conjecture since two different polynomials may take the same value at $q = 1$. So a priori Conjecture 1.2 may be "easier" than the classical conjecture.

As mentioned above, the Markoff Injectivity Conjecture was extended to the language of factors of all Christoffel words [LR21, Conjecture 2]. This language is equal to the language of all balanced sequences over a binary alphabet. Balanced sequences include biinfinite periodic ${}^\infty w^\infty$ repetitions of a Christoffel word, Sturmian sequences which are aperiodic and more (ultimately periodic biinfinite words which are not purely periodic, called skew by Morse and Hedlund [MH40]). See its definition in Section 2. We extend the q-analog of the Markoff Injectivity Conjecture to the language of all balanced sequences $\mathcal{B} = \{s \in \{0,1\}^{\mathbb{Z}} \colon s \text{ is balanced}\}$.

Conjecture 1.3. The map $\{0,1\}^* \to \mathbb{Z}[q]$ defined by $w \mapsto \mu_q(w)_{12}$ is injective over the language $\mathcal{L}(\mathcal{B}) = \bigcup_{s\in\mathcal{B}} \mathcal{L}(s)$ of all balanced sequences.

Conjecture 1.3 implies Conjecture 1.2 since the set of Christoffel words is a subset of $\mathcal{L}(\mathcal{B})$.

1.2. Main results

In this article, we propose a result which progresses in the direction of Conjecture 1.3. More precisely, we prove that the map $w \mapsto \mu_q(w)_{12}$ is strictly increasing with respect to the radix order on the language of a fixed balanced sequence.

Recall that the radix order is defined for every $u, v \in \{0,1\}^*$ as

$$u <_{radix} v \qquad \text{if} \qquad \begin{cases} |u| < |v| & \text{or} \\ |u| = |v| & \text{and} \quad u <_{lex} v. \end{cases}$$

Also we define a partial order $\prec$ on $\mathbb{Z}[q]$ as

$$f \prec g \quad \text{if and only if} \quad f \neq g \text{ and } g - f \in \mathbb{Z}_{\geqslant 0}[q].$$

Theorem A. *Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence and $u, v \in \mathcal{L}(s)$ be two factors in the language of s. If $u <_{radix} v$, then $\mu_q(u)_{12} \prec \mu_q(v)_{12}$, i.e., $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomial of indeterminate q with nonnegative integer coefficients.*

Theorem A is illustrated in Table 6.1 in the appendix which shows the values of $\mu(w)_{12}$ and $\mu_q(w)_{12}$ for the small factors in the Fibonacci word, the most well-known balanced sequence. We observe that the coefficients of the polynomials over q are increasing from one row to the next.

As a consequence, we prove Conjecture 1.3 when restricted to the language of a given balanced sequence.

Corollary 1. *Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence. The map $u \mapsto \mu_q(u)_{12}$ is injective over the language $\mathcal{L}(s)$.*

Remark that Corollary 1 can also be deduced from Corollary 1 of [LR21] since two polynomials evaluated at $q = 1$ are distinct implies that the polynomials are distinct. We also state a corollary of Theorem A when evaluating polynomials at $q = \gamma$ for all positive real numbers $\gamma > 0$ improving Corollary 1 from [LR21].

Corollary 2. *Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence. For every $\gamma > 0$, the map $\{0,1\}^* \to \mathbb{R}$ defined by $w \mapsto \mu_q(w)_{12}|_{q=\gamma}$ is strictly increasing and injective over the language $\mathcal{L}(s)$ with respect to the radix order $<_{radix}$.*

To each word $w \in \{0,1\}^*$ corresponds a function $\gamma \mapsto \mu_q(w)_{12}|_{q=\gamma}$. The graph of these functions is shown on the interval $0 < \gamma < 100$ with a logarithmic y-scale in Figure 1.4 for every short factors w in Fibonacci word.

The proof of Theorem A follows the same idea as the proof that the map $\{0,1\}^* \to \mathbb{Z}_{\geqslant 0} : w \mapsto \mu(w)_{12}$ is strictly increasing (for the radix order) over the factors of a Christoffel word [Lap20, LR21]. When listing the conjugates of a Christoffel word in lexicographic order, only a flip of two letters happens between consecutive conjugates [BR06, Corollary 5.1]. This observation was done in the context of the Burrows–Wheeler transform [MRS03]. Recall that Burrows–Wheeler transform of a finite word w is obtained from w by first listing the conjugates of w in lexicographic order and then concatenating the final letters of the conjugates in this order, see [FMMB07]. When listing lexicographically the $n+1$ factors of a balanced language, at most two letters are changed from one word to the next (see Lemma 4.1). This allows to prove Theorem A for the language of a balanced biinfinite sequence.

This article is structured as follows. Section 2 gathers many equivalent characterizations of balanced sequences including properties introduced by Markoff [Mar79, Mar82]. It is then related to indistinguishable asymptotic pairs [BLS21] which we state as Theorem B since it provides a link between an old notion of Markoff with a recent one. Indistinguishable asymptotic pairs naturally provides two compact representations of the language of length n of a balanced sequence as the factors appearing in two words of length $2n$, see Corollary 2.6. This is used to show that only small local changes appear between a factor and the next factor according to the radix order over the language of a balanced sequence. In Section 3, we show that the map $w \mapsto \mu_q(w)_{12}$ is increasing over the listed small local changes. The proof of Theorem A is done in Section 4. In Section 5, we conclude with few examples illustrating that Theorem A can unfortunately not be extended to the language of all balanced sequences using the radix order.

2. Balanced sequences

2.1. Definition and example

Let $s \in \Sigma^{\mathbb{Z}}$ be a sequence over a finite set Σ. The language of s is $\mathcal{L}(s) = \{s_k s_{k+1} \cdots s_{k+n-1} \mid k \in \mathbb{Z}, n \geqslant 0\} \subset \Sigma^*$ is the set of subwords (or factors) occurring in s. The language of subwords of length $n \in \mathbb{Z}_{\geqslant 0}$ is $\mathcal{L}_n(s) = \mathcal{L}(s) \cap \Sigma^n$. The *reversal* of a finite word $w = (w_i)_{1 \leqslant i \leqslant n}$

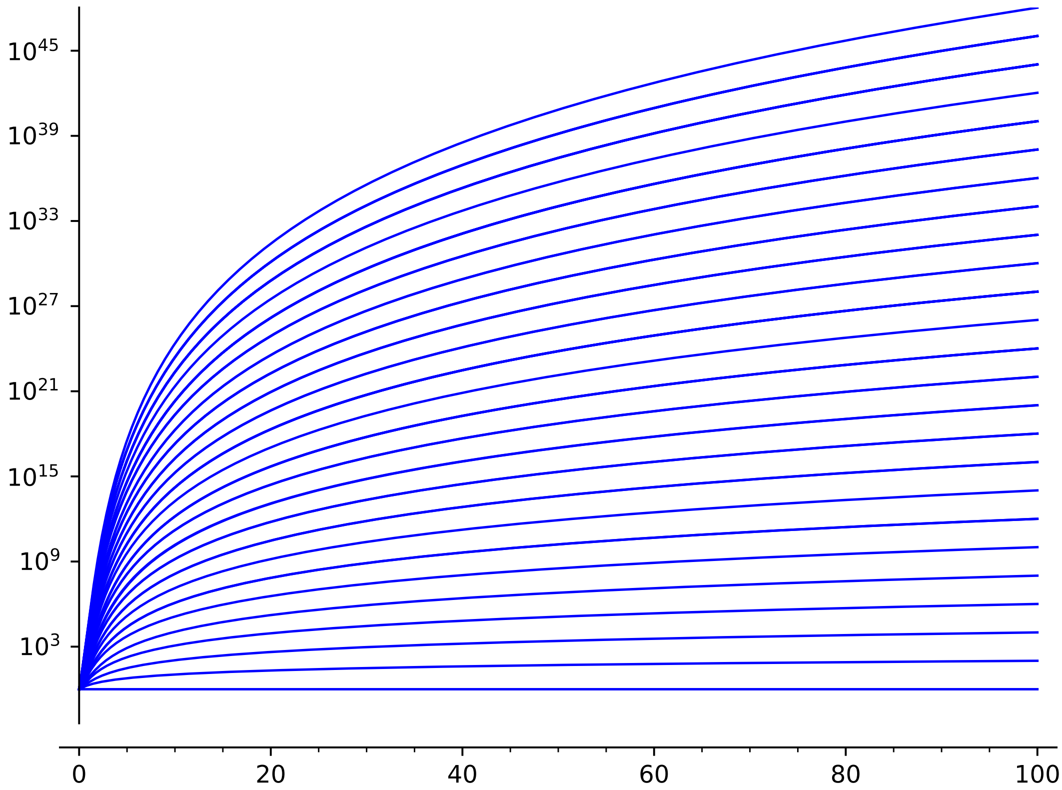

Figure 1.4: The graph of the curves $(\gamma, \mu_q(w)_{12}|_{q=\gamma})$ for $0 \leqslant \gamma \leqslant 100$ for all 55 factors w of length $|w| < 10$ in the Fibonacci word. The 55 polynomials $\mu_q(w)_{12}$ shown in the appendix are ranging from degree 0 to 24 explaining why we see 25 curves in the figure instead of 55. A consequence of Theorem A is that the 55 curves do not intersect when $\gamma > 0$. The difference between two polynomials of the same degree is very small and can not be distinguished (the y axis uses a logarithmic scale). For example, there are four factors of length 9 whose images under $w \mapsto \mu_q(w)_{12}$ are polynomials of degree 23 whose pairwise difference is a polynomial of degree 20, 18 or 14, see Table 6.1.

is $\widetilde{w} = (w_{n+1-i})_{1 \leqslant i \leqslant n}$. Similarly, the *reversal* of a right infinite sequence $s = (s_i)_{i \in \mathbb{Z}_{\geqslant 0}}$ is the left infinite sequence $\widetilde{s} = (s_{-i})_{i \in \mathbb{Z}_{\leqslant 0}}$.

Definition 2.1. A sequence $s \in \Sigma^{\mathbb{Z}}$ is *balanced* if for every positive integer n, for every $u, v \in \mathcal{L}_n(s)$ and every letter $a \in \Sigma$, the number of a's occurring in u and v differ by at most 1.

For example, the right-infinite Fibonacci word

$$F = \mathtt{0100101001001010010 1} \ldots \in \Sigma^{\mathbb{Z}_{\geqslant 0}}$$

over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$ is such that both

$$\widetilde{F} \cdot \mathtt{01} \cdot F = \ldots \mathtt{1010010100100101001 0} \cdot \mathtt{01} \cdot \mathtt{0100101001001010010 1} \ldots$$

and

$$\widetilde{F} \cdot \mathtt{10} \cdot F = \ldots \mathtt{1010010100100101001 0} \cdot \mathtt{10} \cdot \mathtt{0100101001001010010 1} \ldots$$

are balanced sequences. This observation is illustrated for factors of length up to six in the following table.

n	$\mathcal{L}_n(\widetilde{F}\mathtt{01}F)$	number of $\mathtt{0}$'s	number of $\mathtt{1}$'s
0	$\{\varepsilon\}$	0	0
1	$\{\mathtt{0},\mathtt{1}\}$	0 or 1	0 or 1
2	$\{\mathtt{00},\mathtt{01},\mathtt{10}\}$	1 or 2	0 or 1
3	$\{\mathtt{001},\mathtt{010},\mathtt{100},\mathtt{101}\}$	1 or 2	1 or 2
4	$\{\mathtt{0010},\mathtt{0100},\mathtt{0101},\mathtt{1001},\mathtt{1010}\}$	2 or 3	1 or 2
5	$\{\mathtt{00100},\mathtt{00101},\mathtt{01001},\mathtt{01010},\mathtt{10010},\mathtt{10100}\}$	3 or 4	1 or 2
6	$\{\mathtt{001001},\mathtt{001010},\mathtt{010010},\mathtt{010100},\mathtt{100100},\mathtt{100101},\mathtt{101001}\}$	3 or 4	2 or 3

From the table, we confirm that the number of $\mathtt{0}$'s and the number of $\mathtt{1}$'s occurring in two factors of the same length differ by at most 1.

2.2. The Markoff property

It is worth recalling that balanced sequences appeared in the work of Markoff himself [Mar79, Mar82] under an equivalent condition, called Markoff property (M) in [Reu06].

Definition 2.2. [Reu06] We say that a biinfinite word $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ satisfies the *Markoff property* if for any factorization $s = uxyv$, where $\{x, y\} = \{\mathtt{0},\mathtt{1}\}$, one has

- either $\widetilde{u} = v$,
- or there is a factorization $u = u'ym$, $v = \widetilde{m}xv'$.

The Markoff property is related to the Markoff spectrum. Let $U = (a_i)_{i\in\mathbb{Z}}$ be a biinfinite sequence such that a_i are positive integers. For $i \in \mathbb{Z}$, let

$$\lambda_i(U) = a_i + [0; a_{i+1}, a_{i+2}, \dots] + [0; a_{i-1}, a_{i-2}, \dots].$$

The *Markoff supremum of* U is

$$M(U) = \sup_{i\in\mathbb{Z}} \lambda_i(U).$$

Two results of Markoff can be stated in terms of Christoffel words and balanced sequences as follows where σ is the substitution from $\{\mathtt{0},\mathtt{1}\}^*$ to $\{1,2\}^*$ defined by $\mathtt{0} \mapsto 11$ and $\mathtt{1} \mapsto 22$. It provides an equivalence between sequences satisfying the Markoff property and sequences of positive integers such that the Markoff supremum is at most 3. The equivalence between sequences satisfying the Markoff property and balanced sequences was not proved by Markoff himself: it was stated without proof in [CF89] and a proof was provided in [Reu06].

Theorem 2.3 (Markoff)**.** [Reu06, Theorem 3.1 and 7.1] *Let $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a biinfinite word. The following conditions are equivalent:*

- *s satisfies the Markoff property,*

- *s is balanced,*
- $M(\sigma(s)) \leqslant 3$.

The Markoff supremum of a purely periodic balanced sequence can be computed from the Markoff number associated to the Christoffel word which is a period of the sequence.

Theorem 2.4 (Markoff)**.** [Reu19, Theorem 6.2.1] *Let w be some lower Christoffel word associated with Markoff number $m = \mu(w)_{12}$. Let s be the biinfinite sequence ${}^\infty\sigma(w)^\infty$. Then $M(s) = \sqrt{9 - \frac{4}{m^2}}$.*

2.3. Mechanical sequences

It is known that right-infinite aperiodic balanced sequences correspond to mechanical sequences [MH40] which are binary encodings of irrational rotations, see the chapters [Fog02, Chapter 6], [Lot02, Chapter 2] and [AS03, Chapter 9]. A biinfinite balanced sequence can also be periodic and in this case expressed in terms of Christoffel words. To be more precise, let $\alpha \in [0, 1]$ and $\rho \in \mathbb{R}$ and consider the *lower* and *upper mechanical sequences* $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ with *slope* α and *intercept* ρ given respectively by

$$\begin{array}{rccl} s_{\alpha,\rho}: & \mathbb{Z} & \to & \{0, 1\} \\ & n & \mapsto & \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor \end{array}$$

and

$$\begin{array}{rccl} s'_{\alpha,\rho}: & \mathbb{Z} & \to & \{0, 1\} \\ & n & \mapsto & \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil. \end{array}$$

When α is rational, the sequences $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are periodic and their period corresponds to a Christoffel word [BLRS09]. When α is irrational, then $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are not periodic. It is clear that if $\rho - \rho'$ is an integer, then $s_{\alpha,\rho} = s_{\alpha,\rho'}$ and $s'_{\alpha,\rho} = s'_{\alpha,\rho'}$. Thus we may always assume $0 \leqslant \rho < 1$. Moreover, if $\mathbb{Z} \cap \alpha\mathbb{Z} + \rho = \varnothing$ then $s_{\alpha,\rho} = s'_{\alpha,\rho}$.

2.4. Four classes of balanced sequences

Biinfinite balanced sequences can be split into four different types of sequences. Reutenauer proposed the following refinement of the Markoff property [Reu06] which was restated in [GLS08] as follows. If a biinfinite sequence $u \in \{0, 1\}^{\mathbb{Z}}$ satisfies the Markoff property, then it falls into exactly one of the following classes:

(M_1) u cannot be written as $u = \widetilde{p}xyp$ where $\{x, y\} = \{0, 1\}$ and the lengths of the Christoffel words occurring in u are bounded;

(M_2) u cannot be written as $u = \widetilde{p}xyp$ where $\{x, y\} = \{0, 1\}$ and the lengths of the Christoffel words occurring in u are unbounded;

(M_3) u has a unique factorization $u = \widetilde{p}xyp$ where $\{x, y\} = \{0, 1\}$;

(M_4) u has at least two factorizations $u = \widetilde{p}xyp$ where $\{x, y\} = \{0, 1\}$.

Morse and Hedlund gave a classification of balanced biinfinite sequences into three classes (periodic, Sturmian, skew) [MH40]. Since the Sturmian case naturally splits into two, Reutenauer proposed the following four classes $(MH_i)_{i\in\{1,2,3,4\}}$ and proved their equivalence with the (M_i).

Theorem 2.5. [Reu06, Theorem 6.1] *Let $u \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence. For every $i \in \{1,2,3,4\}$, u satisfies (M_i) if and only if u satisfies (MH_i) where*

(MH_1) *u is a purely periodic word ${}^\infty w^\infty$ for some Christoffel word w,*

(MH_2) *u is a generic aperiodic Sturmian word, i.e., $u = s_{\alpha,\rho} = s'_{\alpha,\rho}$ for some $\alpha \in [0,1]\setminus\mathbb{Q}$ and $\rho \in \mathbb{R}$ such that $\mathbb{Z}\cap\alpha\mathbb{Z}+\rho = \varnothing$.*

(MH_3) *u is a characteristic aperiodic Sturmian word, i.e., $u = s_{\alpha,\rho}$ or $u = s'_{\alpha,\rho}$ for some $\alpha \in [0,1]\setminus\mathbb{Q}$ and $\rho \in \mathbb{R}$ such that $\mathbb{Z}\cap\alpha\mathbb{Z}+\rho \neq \varnothing$.*

(MH_4) *u is an ultimately periodic word but not purely periodic, i.e., $u = \cdots xxyxx\cdots$ or $u = \cdots(ymx)(ymx)(ymy)(xmy)(xmy)\cdots$ where $\{x,y\} = \{0,1\}$ and $0m1$ is a Christoffel word.*

2.5. Indistinguishable asymptotic pairs

In this section, we give equivalent conditions for balanced sequences satisfying cases (M_3) or (M_4). Cases (M_3) and (M_4) can be expressed in terms of limits of mechanical words toward an irrational or rational slope from above or from below which were shown to be equivalent to sequences that belong to an indistinguishable asymptotic pair [BLS21].

Concretely, given a finite set Σ, we consider the space of sequences $\Sigma^{\mathbb{Z}} = \{s\colon \mathbb{Z}\to\Sigma\}$ endowed with the prodiscrete topology and the *shift* $\sigma\colon \Sigma^{\mathbb{Z}}\to\Sigma^{\mathbb{Z}}$ where

$$(\sigma(s))_m = s_{m+1} \qquad \text{for every } m\in\mathbb{Z} \text{ and } s\in\Sigma^{\mathbb{Z}}.$$

The shift on $\Sigma^{\mathbb{Z}}$ is invertible and extends to a shift action $\mathbb{Z}\overset{\sigma}{\curvearrowright}\Sigma^{\mathbb{Z}}$. In this setting, two sequences $s,t\in\Sigma^{\mathbb{Z}}$ are *asymptotic* if s and t differ in finitely many positions of $\mathbb{Z}$. The finite set $F = \{n\in\mathbb{Z} : s_n \neq t_n\}$ is called the *difference set* of (s,t).

Given two asymptotic sequences $s,t\in\Sigma^{\mathbb{Z}}$, we may compare the number of occurrences of a fixed pattern. A *pattern* is a function $p\colon S\to\Sigma$ where S, called *support*, is a finite subset of $\mathbb{Z}$. The *occurrences* of a pattern $p\in\Sigma^S$ in a sequence $s\in\Sigma^{\mathbb{Z}}$ is the set $\operatorname{occ}_p(s) := \{n\in\mathbb{Z}\colon \sigma^n(s)|_S = p\}$. Observe that when $s,t\in\Sigma^{\mathbb{Z}}$ are asymptotic sequences, the difference $\operatorname{occ}_p(s)\setminus\operatorname{occ}_p(t)$ is finite because the occurrences of p are the same outside the difference set. We say that (s,t) is an *indistinguishable asymptotic pair* if s and t are asymptotic and

$$\#\left(\operatorname{occ}_p(s)\setminus\operatorname{occ}_p(t)\right) = \#\left(\operatorname{occ}_p(t)\setminus\operatorname{occ}_p(s)\right)$$

for every finite support $S\subset\mathbb{Z}$ and every pattern $p\in\Sigma^S$. Extending the results proved in [BLS21] about indistinguishable asymptotic pairs, we may prove equivalent conditions for balanced sequences satisfying Markoff property (M_3) or (M_4). In the statement, we denote the position of the origin of a biinfinite sequence $s = \cdots s_{-2}s_{-1}.s_0s_1s_2\cdots\in\Sigma^{\mathbb{Z}}$ with a dot (.) between positions -1 and 0.

Theorem B. *Let $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ and $n_0 \in \mathbb{Z}$. The following are equivalent conditions describing balanced sequences satisfying Markoff property (M_3) or (M_4):*

(1) the sequence s has a factorization $\sigma^{n_0}s = \widetilde{p}x.yp$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$;

(2) there exists a sequence $(\alpha_k)_{k\in\mathbb{Z}_{\geqslant 0}}$ with $\alpha_k \in [0,1]\setminus\mathbb{Q}$ such that $\sigma^{n_0}s = \lim_{k\to\infty} s_{\alpha_k,0}$ or $\sigma^{n_0}s = \lim_{k\to\infty} s'_{\alpha_k,0}$;

(3) there exists a sequence $t \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ such that (s,t) is an indistinguishable asymptotic pair with difference set $\{n_0-1,n_0\}$.

Proof. (1) $\implies$ (2). It is sufficient to prove it for $n_0 = 0$. We suppose that s has a factorization $s = \widetilde{p}x.yp$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$. The symmetry $n \mapsto -n-1$ keeps the sequence s invariant except at $\{-1,0\}$. In other words, $s(n) = s(-n-1)$ for every $n \in \mathbb{Z}\setminus\{-1,0\}$ and $\{s(-1),s(0)\} = \{\mathtt{0},\mathtt{1}\}$.

Suppose that s satisfies case (M_3). From Theorem 2.5, s also satisfies case (MH_3), that is, there exists an irrational number for some $\alpha \in [0,1]\setminus\mathbb{Q}$ and $\rho \in \mathbb{R}$ such that $s = s_{\alpha,\rho}$ or $s = s'_{\alpha,\rho}$ with $\mathbb{Z}\cap\alpha\mathbb{Z}+\rho \neq \varnothing$. Since $s(n) = s(-n-1)$ for every $n \in \mathbb{Z}\setminus\{-1,0\}$, we must have $\rho = 0$. Because α is irrational, for every $n \in \mathbb{Z}\setminus\{-1,0\}$ and every sequence $(\alpha_k)_{k\in\mathbb{Z}_{\geqslant 0}}$ of irrational numbers α_k such that $\lim_{k\to\infty}\alpha_k = \alpha$, we have

$$s_{\alpha,0}(n) = \lfloor \alpha(n+1)\rfloor - \lfloor \alpha n\rfloor = \lim_{k\to\infty} \lfloor \alpha_k(n+1)\rfloor - \lfloor \alpha_k n\rfloor = \lim_{k\to\infty} s_{\alpha_k,0}(n),$$
$$s'_{\alpha,0}(n) = \lceil \alpha(n+1)\rceil - \lceil \alpha n\rceil = \lim_{k\to\infty} \lceil \alpha_k(n+1)\rceil - \lceil \alpha_k n\rceil = \lim_{k\to\infty} s'_{\alpha_k,0}(n).$$

Also, since each α_k is irrational, we have $s_{\alpha,0}(0)s_{\alpha,0}(1) = \mathtt{10} = \lim_{k\to\infty} s_{\alpha_k,0}(0)s_{\alpha_k,0}(1)$ and $s'_{\alpha,0}(0)s'_{\alpha,0}(1) = \mathtt{01} = \lim_{k\to\infty} s'_{\alpha_k,0}(0)s'_{\alpha_k,0}(1)$. We conclude that $s = \lim_{k\to\infty} s_{\alpha_k,0}$ or $s = \lim_{k\to\infty} s'_{\alpha_k,0}$.

Suppose that s satisfies case (M_4). From Theorem 2.5, s also satisfies case (MH_4), that is, s is an ultimately periodic word but not purely periodic, i.e., $s = \cdots xxyxx\cdots$ or $s = \cdots(ymx)(ymx)(ymy)(xmy)(xmy)\cdots$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$ and $\mathtt{0}m\mathtt{1}$ is a Christoffel word. From [BLS21, Lemma 4.2], there exists $a,b \in \mathbb{Z}_{\geqslant 0}$ coprime integers such that

$$\widetilde{p}\mathtt{1}.\mathtt{0}p = \lim_{\alpha\to\frac{a}{a+b}^+} s_{\alpha,0} \quad\text{and}\quad \widetilde{p}\mathtt{0}.\mathtt{1}p = \lim_{\alpha\to\frac{a}{a+b}^+} s'_{\alpha,0} \quad \text{(limit from above)}$$

or

$$\widetilde{p}\mathtt{1}.\mathtt{0}p = \lim_{\alpha\to\frac{a}{a+b}^-} s_{\alpha,0} \quad\text{and}\quad \widetilde{p}\mathtt{0}.\mathtt{1}p = \lim_{\alpha\to\frac{a}{a+b}^-} s'_{\alpha,0} \quad \text{(limit from below).}$$

(2) $\implies$ (1). If $\lim_{k\to\infty}\alpha_k \in \mathbb{Q}$, then from [BLS21, Lemma 4.2], we directly have that $\sigma^{n_0}s$ has a factorization $\sigma^{n_0}s = \widetilde{p}x.yp$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$. If $\lim_{k\to\infty}\alpha_k = \alpha \in [0,1]\setminus\mathbb{Q}$, then $\lim_{k\to\infty} s_{\alpha_k,0} = s_{\alpha,0}$ and $\lim_{k\to\infty} s'_{\alpha_k,0} = s'_{\alpha,0}$. Both are symmetric satisfying $s_{\alpha,0}(n) = s_{\alpha,0}(-n-1)$ and $s'_{\alpha,0}(n) = s'_{\alpha,0}(-n-1)$ for every $n \in \mathbb{Z}\setminus\{-1,0\}$ and $\{s_{\alpha,0}(-1),s_{\alpha,0}(0)\} = \{s'_{\alpha,0}(-1),s'_{\alpha,0}(0)\} = \{\mathtt{0},\mathtt{1}\}$.

(2) $\iff$ (3). It was proved in [BLS21, Theorem B] that (2) holds if and only if there exists a sequence $t' \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ such that $(\sigma^{n_0}s,t')$ is an indistinguishable asymptotic pair with difference set $\{-1,0\}$. This holds if and only if $(s,\sigma^{-n_0}t')$ is an indistinguishable asymptotic pair with difference set $\{n_0-1,n_0\}$ since the shift preserves indistinguishable asymptotic pairs [BLS21, Proposition 2.5]. It concludes the proof if we let $t = \sigma^{-n_0}t'$. □

2.6. Language of a balanced sequence

Balanced sequences have other equivalent definitions, for example, in terms of factor complexity [CH73]. A balanced sequence satisfies Markoff properties (M_2), (M_3) or (M_4) if and only if it has complexity $n+1$, see [Lot02, Theorem 2.1.13] stated for right-infinite sequences.

The language of a balanced sequence of complexity $n+1$ can be compactly represented in two ways as described in the following result. It shows that there exist two words of length $2n$ that contains the language of factors of length $n \geqslant 1$ occurring in a balanced sequence. The proof follows easily from the notion of indistinguishable asymptotic pairs [BLS21].

Corollary 2.6. *Let $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a balanced sequence having at least one factorization $s = \widetilde{p}xyp$ where $\{x,y\} = \{\mathtt{0},\mathtt{1}\}$. Let $n \geqslant 1$ and w be the prefix of p of length $n-1$. The two words $\widetilde{w}\mathtt{01}w$ and $\widetilde{w}\mathtt{10}w$ of length $2n$ contain the $n+1$ factors of s. More precisely, $\mathcal{L}_n(s) = \mathcal{L}_n(\widetilde{w}\mathtt{01}w) = \mathcal{L}_n(\widetilde{w}\mathtt{10}w)$.*

Proof. Let $n_0 \in \mathbb{Z}$ be such that $\sigma^{n_0}s = \widetilde{p}x.yp$. We may assume that $n_0 = 0$ and $\sigma^{n_0}s = s$ since shifting the sequence s preserves its language. From Theorem B, there exists a sequence $t \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ such that (s,t) is an indistinguishable asymptotic pair with difference set $\{n_0-1,n_0\} = \{-1,0\}$. In particular, s and t are equal outside of the difference set, i.e., $s|_{\mathbb{Z}\setminus\{-1,0\}} = t|_{\mathbb{Z}\setminus\{-1,0\}}$, and different on the difference set. Since the alphabet is binary, we must have $t_{-1}t_0 = yx$. Therefore the sequence t satisfies $t = \widetilde{p}y.xp$. Replacing $\{x,y\}$ by $\{\mathtt{0},\mathtt{1}\}$, we have that the indistinguishable asymptotic pair is of the form $\{s,t\} = \{\widetilde{p}\mathtt{0}.\mathtt{1}p, \widetilde{p}\mathtt{1}.\mathtt{0}p\}$. Thus, for every prefix w of length $n-1$ of p, we have $\{s_{-n}s_{-n+1}\cdots s_{n-1}, t_{-n}t_{-n+1}\cdots t_{n-1}\} = \{\widetilde{w}\mathtt{01}w, \widetilde{w}\mathtt{10}w\}$. From [BLS21, Corollary 3.6], we have

$$\begin{aligned}\mathcal{L}_n(s) &= \mathcal{L}_n(s_{-n}s_{-n+1}\cdots s_{n-1}) = \mathcal{L}_n(t_{-n}t_{-n+1}\cdots t_{n-1})\\ &= \mathcal{L}_n(\widetilde{w}\mathtt{01}w) = \mathcal{L}_n(\widetilde{w}\mathtt{10}w).\end{aligned}$$

□

For example, the following two words of length 16 contain one occurrence of each factor of length 8 occurring in the Fibonacci word:

$$\mathtt{1010010.01.0100101}$$
$$\mathtt{1010010.10.0100101}$$

We use Corollary 2.6 in Lemma 4.1 to show the existence of a bijection $f : \mathcal{L}_n(s) \to \mathcal{L}_n(s)$ which is a cyclic permutation and having the property, except for two words in $\mathcal{L}_n(s)$, of flipping a $\mathtt{01}$ into a $\mathtt{10}$ from u to $f(u)$. This property is well-known in the context of the Burrows–Wheeler transform of a Christoffel word, see [MRS03, BR06] or more recently [Reu19, Theorem 15.2.4].

Small local changes from a factor to the next (in radix order) can also be seen in the language of a balanced sequence. For example, we list below the factors of length 8 in the Fibonacci word as well as the greatest (for the radix order) factor of length 7 and the smallest factor of length 9:

$$
\begin{array}{lclcl}
 & & \underline{\mathtt{1}}\mathtt{010010} & = & \underline{\mathtt{1}}m \\
\mathtt{0010}\underline{\mathtt{01}}\mathtt{01} & = & \mathtt{0010}\underline{\mathtt{01}}\mathtt{01} & = & \mathtt{0}m\mathtt{1} \\
\mathtt{0}\underline{\mathtt{01}}\mathtt{01001} & = & \mathtt{0}\underline{\mathtt{01}}\mathtt{01001} & & \\
\mathtt{010010}\underline{\mathtt{01}} & = & \mathtt{010010}\underline{\mathtt{01}} & & \\
\mathtt{010}\underline{\mathtt{01}}\mathtt{010} & = & \mathtt{010}\underline{\mathtt{01}}\mathtt{010} & = & \widetilde{u}\mathtt{01}v \\
\underline{\mathtt{01}}\mathtt{010010} & = & \underline{\mathtt{01}}\mathtt{010010} & = & \widetilde{u}\mathtt{10}v \\
\mathtt{10010}\underline{\mathtt{01}}\mathtt{0} & = & \mathtt{10010}\underline{\mathtt{01}}\mathtt{0} & & \\
\mathtt{10}\underline{\mathtt{01}}\mathtt{0100} & = & \mathtt{10}\underline{\mathtt{01}}\mathtt{0100} & & \\
\mathtt{10100100} & = & \mathtt{1010010}\underline{\mathtt{0}} & = & \mathtt{1}m\underline{\mathtt{0}} \\
 & & \underline{\mathtt{1}}\mathtt{0100101} & = & \underline{\mathtt{1}}m\mathtt{1} \\
 & & \mathtt{001001010} & = & \mathtt{0}m\mathtt{10}
\end{array}
$$

In the left column, the 8 cyclic conjugates of the Christoffel word $\mathtt{00100101}$ are listed in lexicographic order. It illustrates what happens in the context of the Burrows–Wheeler transform: in each word (except the last), the factor $\mathtt{01}$ that is changed into a $\mathtt{10}$ is underlined. In the middle column, the 9 factors of length 8 in the language of the Fibonacci word. There is also the lexicographically largest factor of length 7 and the lexicographically smallest factor of length 9. As for the conjugates of a Christoffel word, we observe that at most two letters change from a word to the next. The type of changes summarized in the right column can be verified on the factors of the Fibonacci word of length up to 9 ordered in radix order in Table 6.1 in the appendix.

This observation proved in Lemma 4.1 is a key point in the proof of Theorem A. More precisely, we observe that the small local changes are of the following forms:

$$\widetilde{u}\mathtt{01}v \mapsto \widetilde{u}\mathtt{10}v, \qquad w\mathtt{0} \mapsto w\mathtt{1}, \qquad \mathtt{1}w \mapsto \mathtt{0}w\mathtt{0}, \qquad \mathtt{1}w \mapsto \mathtt{0}w\mathtt{1} \tag{2.1}$$

where $u, v, w \in \{\mathtt{0}, \mathtt{1}\}^*$ and u is a prefix of v or vice versa.

3. Increasing over small local changes

In this section, we prove that the map $w \mapsto \mu_q(w)_{12}$ is increasing over the small local changes listed in Equation (2.1). Recall that we use the partial order $\prec$ on $\mathbb{Z}[q]$ is defined as

$$f \prec g \quad \text{if and only if} \quad f \neq g \text{ and } g - f \in \mathbb{Z}_{\geqslant 0}[q].$$

More precisely, we prove the following two propositions.

Proposition 3.1. *For every $w \in \{\mathtt{0}, \mathtt{1}\}^*$,*

$$\mu_q(w\mathtt{0})_{12} \prec \mu_q(w\mathtt{1})_{12} \quad \textit{and} \quad \mu_q(\mathtt{1}w)_{12} \prec \mu_q(\mathtt{0}w\mathtt{0})_{12} \prec \mu_q(\mathtt{0}w\mathtt{1})_{12}.$$

Proposition 3.2. *Let $u, v \in \{\mathtt{0}, \mathtt{1}\}^*$ such that u is a prefix of v or vice versa. Then*

$$\mu_q(\widetilde{u}\mathtt{01}v)_{12} \prec \mu_q(\widetilde{u}\mathtt{10}v)_{12}.$$

The proof of each proposition is preceded by a lemma. The proof of the two lemmas is the q-analog of the proofs made in [Lap20, LR21] over the integer entries. In particular, the next lemma extends Lemma 2 from [LR21].

Lemma 3.3. *Let $w \in \{0,1\}^*$ and polynomials $m, n, o, p \in \mathbb{Z}[q]$ such that $\mu_q(w) = \left(\begin{smallmatrix} m & n \\ o & p \end{smallmatrix}\right)$. Then m, p and*

$$qm - q^2 n + o, \tag{3.1}$$
$$(q+q^2)m - (q^2+q^3+q^4)n + o - qp, \tag{3.2}$$

are nonzero polynomials with nonnegative coefficients. Moreover, o and n are nonzero polynomials with nonnegative coefficients except if w is empty in which case $o = n = 0$.

Proof. The proof is done by induction on the length of w. If w is the empty word, then $m = p = 1$ and $n = o = 0$. Therefore,

$$\begin{aligned} &qm - q^2 n + o = q \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}, \\ &(q+q^2)m - (q^2+q^3+q^4)n + o - qp = (q+q^2) - q = q^2 \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}. \end{aligned}$$

Let $w \in \{0,1\}^*$ such that $\mu_q(w) = \left(\begin{smallmatrix} m & n \\ o & p \end{smallmatrix}\right)$ for some polynomials $m, n, o, p \in \mathbb{Z}_{\geqslant 0}[q]$. Assume by induction that m, p, (3.1) and (3.2) are nonzero polynomials with nonnegative coefficients.

Let $w' \in \{0,1\}^*$ be a nonempty word $w' = w0$ or $w' = w1$. We have $\mu_q(w') = \left(\begin{smallmatrix} m' & n' \\ o' & p' \end{smallmatrix}\right)$ for some polynomials $m', n', o', p' \in \mathbb{Z}[q]$. If $w' = w0$, then

$$\mu(w') = \begin{pmatrix} m' & n' \\ o' & p' \end{pmatrix} = \begin{pmatrix} m & n \\ o & p \end{pmatrix} \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix} = \begin{pmatrix} (q+q^2)m + qn & m+n \\ (q+q^2)o + qp & o+p \end{pmatrix}. \tag{3.3}$$

We observe that m', n', o' and p' are nonzero polynomials with nonnegative coefficients. Also, we have

$$\begin{aligned} qm' - q^2 n' + o' &= q\left((q+q^2)m + qn\right) - q^2(m+n) + \left((q+q^2)o + qp\right) \\ &= q\left(q^2 m + (q+1)o + p\right) \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}, \end{aligned}$$

since $m, p \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}$ and $o \in \mathbb{Z}_{\geqslant 0}[q]$. Moreover, using the induction hypothesis, we have

$$\begin{aligned} (q+q^2)m' - (q^2+q^3+q^4)n' + o' - qp' &= (q+q^2)\left((q+q^2)m + qn\right) \\ &\quad - (q^2+q^3+q^4)(m+n) \\ &\quad + \left((q+q^2)o + qp\right) - q(o+p) \\ &= q^2\left(qm - q^2 n + o\right) \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\} \end{aligned}$$

by Equation (3.1).

If $w' = w1$, then

$$\begin{aligned} \mu(w') = \begin{pmatrix} m' & n' \\ o' & p' \end{pmatrix} &= \begin{pmatrix} m & n \\ o & p \end{pmatrix} \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (q+2q^2+q^3+q^4)m + (q+q^2)n & (1+q)m+n \\ (q+2q^2+q^3+q^4)o + (q+q^2)p & (1+q)o+p \end{pmatrix} \end{aligned}$$

We observe that m', n', o' and p' are nonzero polynomials with nonnegative coefficients. Also we have that both

$$\begin{aligned} qm' - q^2n' + o' &= q\left(\left(q+2q^2+q^3+q^4\right)m + \left(q+q^2\right)n\right) - q^2\left((1+q)m+n\right) \\ &\quad + \left(\left(q+2q^2+q^3+q^4\right)o + \left(q+q^2\right)p\right) \\ &= q\left((q^2+q^3+q^4)m + q^2n + (1+2q+q^2+q^3)o + (1+q)p\right) \end{aligned}$$

and

$$\begin{aligned} &(q+q^2)m' - (q^2+q^3+q^4)n' \\ &\qquad +o' - qp' = (q+q^2)\left(\left(q+2q^2+q^3+q^4\right)m + \left(q+q^2\right)n\right) \\ &\qquad\qquad - (q^2+q^3+q^4)\left((1+q)m+n\right) \\ &\qquad\qquad + \left(\left(q+2q^2+q^3+q^4\right)o + \left(q+q^2\right)p\right) \\ &\qquad\qquad - q\left((1+q)o+p\right) \\ &\qquad = q^2((q+q^2+q^3+q^4)m + qn + (1+q+q^2)o + p) \end{aligned}$$

belong to $\mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}$ since $m, p \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}$ and $n, o \in \mathbb{Z}_{\geqslant 0}[q]$. □

Proof of Proposition 3.1. Let $w \in \{\mathtt{0}, \mathtt{1}\}^*$ such that $\mu_q(w) = \left(\begin{smallmatrix} m & n \\ o & p \end{smallmatrix}\right)$ where $m, n, o, p \in \mathbb{Z}_{\geqslant 0}[q]$ from Lemma 3.3. Firstly, we have

$$\begin{aligned} \mu_q(w\mathtt{1}) - \mu_q(w\mathtt{0}) &= \mu_q(w)\left[\mu_q(\mathtt{1}) - \mu_q(\mathtt{0})\right] \\ &= \begin{pmatrix} m & n \\ o & p \end{pmatrix} \left[\begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix} - \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} m & n \\ o & p \end{pmatrix} \begin{pmatrix} q^2+q^3+q^4 & q \\ q^2 & 0 \end{pmatrix} \end{aligned}$$

We compute the entry $(1, 2)$ of the above matrix and we obtain

$$\mu_q(w\mathtt{1})_{12} - \mu_q(w\mathtt{0})_{12} = mq \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}$$

since $m \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}$.

Secondly, we have

$$\begin{aligned} \mu_q(\mathtt{0}w\mathtt{0}) - \mu_q(\mathtt{1}w) &= \mu_q(\mathtt{0})\left(\begin{smallmatrix} m & n \\ o & p \end{smallmatrix}\right)\mu_q(\mathtt{0}) - \mu_q(\mathtt{1})\left(\begin{smallmatrix} m & n \\ o & p \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} q+q^2 & 1 \\ q & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} m & n \\ o & p \end{smallmatrix}\right)\left(\begin{smallmatrix} q+q^2 & 1 \\ q & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} m & n \\ o & p \end{smallmatrix}\right). \end{aligned}$$

We compute the entry $(1, 2)$ of the above matrix and from Lemma 3.3, we obtain

$$\begin{aligned} \mu_q(\mathtt{0}w\mathtt{0})_{12} - \mu_q(\mathtt{1}w)_{12} &= \left(q+q^2\right)m + \left(q+q^2\right)n + o + p \\ &\quad - \left(q+2\,q^2+q^3+q^4\right)n - (1+q)p \\ &= (q+q^2)m - (q^2+q^3+q^4)n + o - qp \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\}. \end{aligned}$$

From the first part of the proof, we also have $\mu_q(\mathtt{0}w\mathtt{0})_{12} - \mu_q(\mathtt{0}w\mathtt{1})_{12}$ is a nonzero polynomial with nonnegative coefficients. □

Let

$$D_q = \mu_q(\mathtt{10}) - \mu_q(\mathtt{01}) = \begin{pmatrix} 0 & q+q^4 \\ -q^2-q^5 & 0 \end{pmatrix}.$$

The matrix D_q represent the flip between two consecutive factors in the language of a balanced language. Hence, properties of this matrix are used to prove our main result. In the next lemma, we use the following notation. If $u \in \{\mathtt{0},\mathtt{1}\}^*$ and $a \in \{\mathtt{0},\mathtt{1}\}$, then $|u|_a$ is the number of occurrences of the letter a in u.

Lemma 3.4. *Let $u \in \{\mathtt{0},\mathtt{1}\}^*$. Then*

$$\mu_q(\widetilde{u}\mathtt{10}u) - \mu_q(\widetilde{u}\mathtt{01}u) = q^n D_q = \det(\mu_q(u))D_q$$

where $n = 2|u|_{\mathtt{0}} + 4|u|_{\mathtt{1}}$.

Proof. It follows from the following two equalities

$$\begin{aligned}\mu_q(\mathtt{0})D_q\mu_q(\mathtt{0}) &= \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix}\begin{pmatrix} 0 & q+q^4 \\ -q^2-q^5 & 0 \end{pmatrix}\begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix}\\ &= \begin{pmatrix} 0 & q^3+q^6 \\ -q^4-q^7 & 0 \end{pmatrix} = q^2D_q = \det(\mu_q(\mathtt{0}))D_q\end{aligned}$$

and

$$\begin{aligned}\mu_q(\mathtt{1})D_q\mu_q(\mathtt{1}) &= \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}\begin{pmatrix} 0 & q+q^4 \\ -q^2-q^5 & 0 \end{pmatrix}\\ &\quad\begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}\\ &= \begin{pmatrix} 0 & q^5+q^8 \\ -q^6-q^9 & 0 \end{pmatrix} = q^4D_q = \det(\mu_q(\mathtt{1}))D_q.\end{aligned}$$

□

Proof of Proposition 3.2. There are three cases: $u = v$, v is longer than u or u is longer than v. First assume $u = v$. We use the identity $\mu_q(\widetilde{u})D_q\mu_q(u) = q^nD_q$ for some $n > 0$ from Lemma 3.4. We have

$$\begin{aligned}\mu_q(\widetilde{u}\mathtt{10}v)_{12} - \mu_q(\widetilde{u}\mathtt{01}v)_{12} &= \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)\mu_q(\widetilde{u})[\mu_q(\mathtt{10}) - \mu_q(\mathtt{01})]\mu_q(u)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\\ &= \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)\cdot q^nD_q\cdot\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = q^n\left(q^4+q\right) \in \mathbb{Z}_{\geqslant 0}[q]\setminus\{0\}.\end{aligned}$$

Assume $|v| > |u|$ and let $s \in \{\mathtt{0},\mathtt{1}\}^+$ such that $v = us$. We compute

$$\begin{aligned}\mu_q(\widetilde{u}\mathtt{10}v)_{12} - \mu_q(\widetilde{u}\mathtt{01}v)_{12} &= \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)\mu_q(\widetilde{u})[\mu_q(\mathtt{10}) - \mu_q(\mathtt{01})]\mu_q(u)\mu_q(s)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\\ &= \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)\cdot q^nD_q\cdot\mu_q(s)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\\ &= q^n\left(\begin{smallmatrix}0 & q^4+q\end{smallmatrix}\right)\mu_q(s)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \in \mathbb{Z}_{\geqslant 0}[q]\setminus\{0\},\end{aligned}$$

since s is non-empty and from Lemma 3.3 the entries of $\mu_q(s)$ are nonzero polynomials with nonnegative coefficients.

Assume $|u| > |v|$ and let $s \in \{0,1\}^+$ such that $u = vs$. We obtain

$$\begin{aligned}\mu_q(\widetilde{u}10v)_{12} - \mu_q(\widetilde{u}01v)_{12} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mu_q(\widetilde{s})\mu_q(\widetilde{v}) \left[\mu_q(10) - \mu_q(01)\right] \mu_q(v) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mu_q(\widetilde{s}) \cdot q^n D_q \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= q^n \begin{pmatrix} 1 & 0 \end{pmatrix} \mu_q(\widetilde{s}) \begin{pmatrix} q^4+q \\ 0 \end{pmatrix} \in \mathbb{Z}_{\geqslant 0}[q] \setminus \{0\},\end{aligned}$$

since s is non-empty and from Lemma 3.3 the entries of $\mu_q(\widetilde{s})$ are nonzero polynomials with nonnegative coefficients. □

4. Proof of Theorem A

The following lemma can be seen as a extension to biinfinite balanced sequences of complexity $n+1$ of Theorem 15.2.4 from [Reu19] stated for the n conjugates of a Christoffel word of length n.

Lemma 4.1. *Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence having at least one factorization $s = \widetilde{p}xyp$ where $\{x,y\} = \{0,1\}$. Let $n \geqslant 1$ and $u_0, \ldots, u_n$ be the $n+1$ factors of length n of s such that*

$$u_0 <_{lex} \cdots <_{lex} u_n.$$

If w is the prefix of length $n-1$ of p, we have

- *$u_0 = 0w$ and $u_n = 1w$,*
- *there exists $i \in \{0, \ldots, n-1\}$ such that $u_i = \widetilde{w}0$ and $u_{i+1} = \widetilde{w}1$,*
- *for all $j \in \{0, \ldots, n-1\} \setminus \{i\}$, there exist prefixes x, y of w such that $u_j = \widetilde{x}01y$ and $u_{j+1} = \widetilde{x}10y$.*

Proof. Let $f : \mathcal{L}_n(s) \to \mathcal{L}_n(s)$ be the map such that $f(u)$ is the factor appearing in $\widetilde{w}10w$ at the same position as the occurrence of u in $\widetilde{w}01w$. From Corollary 2.6, f is a bijection.

From the definition of f, if $u \neq 1w$ then $u <_{lex} f(u)$. Thus f is a cyclic permutation (if f had at least 2 cycles, there would exist two distinct words u such that $u \not<_{lex} f(u)$). Thus there exists a minimal word u_0 for the lexicographic order such that

$$u_0 <_{lex} f(u_0) <_{lex} f^2(u_0) <_{lex} \cdots <_{lex} f^n(u_0) \not<_{lex} f^{n+1}(u_0) = u_0$$

It also implies that the maximal factor for the lexicographic order is $f^n(u_0) = 1w$ and the minimal one is $u_0 = f(f^n(u_0)) = f(1w) = 0w$. This ends the proof if we let $u_i = f^i(u_0)$ for all $i \in \{1, \ldots, n\}$. □

We may now show that the q-analog of the Markoff injectivity conjecture holds over the language of a balanced sequence satisfying property (M_3) or (M_4).

Proposition 4.2. *Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence having at least one factorization $s = \widetilde{p}xyp$ where $\{x,y\} = \{0,1\}$. Let $u, v \in \mathcal{L}(s)$ be two factors in the language of s. If $u <_{radix} v$, then $\mu_q(u)_{12} \prec \mu_q(v)_{12}$.*

Proof. Let $(x_i)_{0\leqslant i\leqslant n}$ be a maximal chain for the radix order such that

$$u = x_0 <_{radix} x_1 <_{radix} \cdots <_{radix} x_n = v.$$

Let $i \in \{0, \ldots, n-1\}$. Since the chain is maximal, we have $|x_{i+1}| = |x_i|$ or $|x_{i+1}| = |x_i| + 1$. First assume that $|x_i| = |x_{i+1}|$. Since $x_i <_{radix} x_{i+1}$, we have $x_i <_{lex} x_{i+1}$. From Lemma 4.1 and Proposition 3.2, we conclude that $\mu_q(x_i)_{12} \prec \mu_q(x_{i+1})_{12}$.

Now assume that $|x_{i+1}| = |x_i| + 1$. Since the chain is maximal, then x_i is the lexicographically maximal factor of its length and x_{i+1} is the lexicographically minimal factor of its length. From Lemma 4.1, the prefix w of length $|x_i| - 1$ of p is such that $x_i = \mathtt{1}w$ and $x_{i+1} \in \{\mathtt{0}w\mathtt{0}, \mathtt{0}w\mathtt{1}\}$. From Proposition 3.1, we obtain that $\mu_q(x_i)_{12} \prec \mu_q(x_{i+1})_{12}$. □

We may now prove the main result and its corollaries.

Proof of Theorem A. Let $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a balanced sequence over the alphabet $\Sigma = \{\mathtt{0},\mathtt{1}\}$. The sequence s is in one of the four cases: (M_1), (M_2), (M_3) or (M_4).

If s satisfies case (M_1), then from Theorem 2.5, s is a purely periodic sequence ${}^{\infty}w^{\infty}$ for some Christoffel word w. From Theorem 2.5, there exists an eventually periodic sequence s' satisfying (MH_4) and (M_4) such that $\mathcal{L}(s) \subset \mathcal{L}(s')$. If s satisfies case (M_2), then there exists a characteristic Sturmian sequence s' of the same slope satisfying case (M_3) such that $\mathcal{L}(s) = \mathcal{L}(s')$. If s already satisfies case (M_3) or (M_4), then let $s' = s$.

In summary, s' is a balanced sequence satisfying property (M_3) or (M_4) such that $\mathcal{L}(s) \subset \mathcal{L}(s')$. Thus $s' \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ is a balanced sequence having at least one factorization $s' = \widetilde{p}xyp$ where $\{x, y\} = \{\mathtt{0},\mathtt{1}\}$. From Proposition 4.2, if $u, v \in \mathcal{L}(s')$ such that $u <_{radix} v$, then $\mu_q(u)_{12} \prec \mu_q(v)_{12}$. □

Proof of Corollary 1. Let $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ be a balanced sequence. Let $u, v \in \mathcal{L}(s)$ such that $u \neq v$. Without loss of generality, we may assume that $u <_{radix} v$. From Theorem A, $\mu_q(u)_{12} \prec \mu_q(v)_{12}$. In particular, $\mu_q(u)_{12} \neq \mu_q(v)_{12}$. Therefore, the map $u \mapsto \mu_q(u)_{12}$ is injective over the language $\mathcal{L}(s)$. □

Proof of Corollary 2. Let $u, v \in \mathcal{L}(s)$ be two factors in the language of a balanced sequence $s \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$ such that $u <_{radix} v$. From Theorem A, $\mu_q(u)_{12} \prec \mu_q(v)_{12}$. Thus $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomial with nonnegative coefficients. In particular, for every $\gamma > 0$, $(\mu_q(v)_{12} - \mu_q(u)_{12})|_{q=\gamma} > 0$ or equivalently $\mu_q(u)_{12}|_{q=\gamma} < \mu_q(v)_{12}|_{q=\gamma}$. In particular, $\mu_q(u)_{12}|_{q=\gamma} \neq \mu_q(v)_{12}|_{q=\gamma}$. Thus the map $\{\mathtt{0},\mathtt{1}\}^* \to \mathbb{R}$ defined by $w \mapsto \mu_q(w)_{12}|_{q=\gamma}$ is strictly increasing and injective over $\mathcal{L}(s)$ for every $\gamma > 0$. □

5. Conclusion

As we have shown in Theorem A, the map $w \mapsto \mu_q(w)_{12}$ is increasing over the language of a balanced sequence. But this is not true for the language of all balanced sequences. Thus the Markoff injectivity conjecture can not be extended to the injectivity of the map $w \mapsto \mu_q(w)_{12}|_{q=\gamma}$ for all $\gamma > 0$. We provide few counterexamples below.

Observe that 011 $<_{radix}$ 100 but

$$\mu_q(\texttt{100})_{12} = 1 + 3q + 4q^2 + 4q^3 + 4q^4 + 2q^5 + q^6,$$
$$\mu_q(\texttt{011})_{12} = 1 + 2q + 5q^2 + 6q^3 + 6q^4 + 5q^5 + 3q^6 + q^7,$$

and

$$\mu_q(\texttt{100})_{12} - \mu_q(\texttt{011})_{12} = q - q^2 - 2q^3 - 2q^4 - 3q^5 - 2q^6 - q^7$$

has negative coefficients.

Another example is 011101 $<_{radix}$ 101011 but

$$\mu_q(\texttt{101011})_{12} - \mu_q(\texttt{011101})_{12} = q + 3q^2 + 7q^3 + 12q^4 + 17q^5 + 20q^6 + 21q^7 + 19q^8 + 14q^9 + 9q^{10} + 4q^{11} + q^{12} - q^{13} - q^{14}$$

which has negative coefficients.

The words 100, 101011 and 011101 are not Christoffel words, but even in the case of Christoffel words, there are counterexamples. For example,

$$\mu_q(\texttt{00001})_{12} = 1 + 4q + 8q^2 + 13q^3 + 16q^4 + 17q^5 + 14q^6 + 10q^7 + 5q^8 + q^9,$$
$$\mu_q(\texttt{0111})_{12} = 1 + 3q + 9q^2 + 16q^3 + 24q^4 + 29q^5 + 29q^6 + 25q^7 + 18q^8 + 10q^9 + 4q^{10} + q^{11},$$

and their difference

$$\mu_q(\texttt{00001})_{12} - \mu_q(\texttt{0111})_{12} = q - q^2 - 3q^3 - 8q^4 - 12q^5 - 15q^6 - 15q^7 - 13q^8 - 9q^9 - 4q^{10} - q^{11}$$

has negative coefficients. Another counterexample is given by the Christoffel words $\texttt{0}^{12}\texttt{1}$ and $\texttt{01}^7$.

The above counterexamples show that Corollary 2 does not hold for $\gamma > 0$ over the language of all balanced sequences. Thus a proof of Markoff Injectivity Conjecture needs to use the hypothesis that polynomials are evaluated only at $q = 1$, perhaps extending the approach used in [BRS09].

Finally, we observe that the map $w \mapsto \mu_q(w)_{12}$ from $\{0, 1\}^*$ to polynomials in the indeterminate q is not injective, as for example:

$$\mu_q(\texttt{000111})_{12} = 1 + 5q + 16q^2 + 38q^3 + 70q^4 + 109q^5 + 145q^6 + 168q^7 + 171q^8 + 152q^9 + 118q^{10} + 79q^{11} + 44q^{12} + 19q^{13} + 6q^{14} + q^{15} = \mu_q(\texttt{011001})_{12}.$$

6. Appendix: values of $\mu_q(w)_{12}$ over the language of the Fibonacci word

The following table gathers the values of $\mu(w)_{12}$ and $\mu_q(w)_{12}$ over the language of the Fibonacci word for factors of length up to 9 sorted in radix order. We observe that the coefficients of the polynomials are increasing from one row to the next. The graph of the 55 polynomials listed in the table is shown in Figure 1.4. The difference between two polynomials of the same degree can't be seen in the figure as it is relatively very small.

w	$\mu(w)_{12}$	$\mu_q(w)_{12}$
ε	0	0
0	1	1
1	2	$1+q$
00	3	$1+q+q^2$
01	5	$1+q+2q^2+q^3$
10	7	$1+2q+2q^2+q^3+q^4$
001	13	$1+2q+3q^2+3q^3+3q^4+q^5$
010	17	$1+2q+4q^2+4q^3+3q^4+2q^5+q^6$
100	19	$1+3q+4q^2+4q^3+4q^4+2q^5+q^6$
101	31	$1+3q+5q^2+6q^3+7q^4+5q^5+3q^6+q^7$
0010	44	$1+3q+6q^2+8q^3+9q^4+8q^5+5q^6+3q^7+q^8$
0100	46	$1+3q+6q^2+9q^3+9q^4+8q^5+6q^6+3q^7+q^8$
0101	75	$1+3q+7q^2+11q^3+14q^4+14q^5+12q^6+8q^7+4q^8+q^9$
1001	81	$1+4q+8q^2+12q^3+15q^4+15q^5+13q^6+8q^7+4q^8+q^9$
1010	105	$1+4q+9q^2+14q^3+18q^4+19q^5+17q^6+12q^7+7q^8+3q^9+q^{10}$
00100	119	$1+4q+9q^2+15q^3+20q^4+22q^5+19q^6+15q^7+9q^8+4q^9+q^{10}$
00101	194	$1+4q+10q^2+18q^3+27q^4+33q^5+33q^6+29q^7+21q^8+12q^9+5q^{10}+q^{11}$
01001	196	$1+4q+10q^2+19q^3+27q^4+33q^5+34q^6+29q^7+21q^8+12q^9+5q^{10}+q^{11}$
01010	254	$1+4q+11q^2+21q^3+32q^4+40q^5+42q^6+39q^7+30q^8+19q^9+10q^{10}+4q^{11}+q^{12}$
10010	274	$1+5q+13q^2+24q^3+35q^4+43q^5+46q^6+41q^7+31q^8+20q^9+10q^{10}+4q^{11}+q^{12}$
10100	284	$1+5q+13q^2+24q^3+36q^4+45q^5+47q^6+43q^7+33q^8+21q^9+11q^{10}+4q^{11}+q^{12}$
001001	507	$1+5q+14q^2+29q^3+48q^4+67q^5+79q^6+81q^7+71q^8+54q^9+34q^{10}+17q^{11}+6q^{12}+q^{13}$
001010	657	$1+5q+15q^2+32q^3+55q^4+79q^5+96q^6+102q^7+94q^8+76q^9+52q^{10}+30q^{11}+14q^{12}+5q^{13}+q^{14}$
010010	663	$1+5q+15q^2+33q^3+56q^4+80q^5+97q^6+103q^7+95q^8+76q^9+52q^{10}+30q^{11}+14q^{12}+5q^{13}+q^{14}$
010100	687	$1+5q+15q^2+33q^3+57q^4+82q^5+100q^6+107q^7+99q^8+80q^9+55q^{10}+32q^{11}+15q^{12}+5q^{13}+q^{14}$
100100	741	$1+6q+18q^2+38q^3+64q^4+90q^5+109q^6+115q^7+105q^8+84q^9+57q^{10}+33q^{11}+15q^{12}+5q^{13}+q^{14}$
100101	1208	$1+6q+19q^2+43q^3+78q^4+119q^5+156q^6+178q^7+179q^8+158q^9+121q^{10}+80q^{11}+44q^{12}+19q^{13}+6q^{14}+q^{15}$
101001	1210	$1+6q+19q^2+43q^3+78q^4+119q^5+156q^6+179q^7+179q^8+158q^9+122q^{10}+80q^{11}+44q^{12}+19q^{13}+6q^{14}+q^{15}$
0010010	1715	$1+6q+20q^2+48q^3+91q^4+145q^5+198q^6+237q^7+249q^8+233q^9+192q^{10}+138q^{11}+86q^{12}+45q^{13}+19q^{14}+6q^{15}+q^{16}$

0010100	1777	$1+6q+20q^2+48q^3+92q^4+148q^5+203q^6+244q^7+259q^8+243q^9+201q^{10}+146q^{11}+91q^{12}+48q^{13}+20q^{14}+6q^{15}+q^{16}$
0100100	1793	$1+6q+20q^2+49q^3+94q^4+150q^5+206q^6+247q^7+261q^8+245q^9+202q^{10}+146q^{11}+91q^{12}+48q^{13}+20q^{14}+6q^{15}+q^{16}$
0100101	2923	$1+6q+21q^2+54q^3+110q^4+188q^5+276q^6+356q^7+405q^8+411q^9+371q^{10}+296q^{11}+207q^{12}+125q^{13}+63q^{14}+25q^{15}+7q^{16}+q^{17}$
0101001	2927	$1+6q+21q^2+54q^3+110q^4+188q^5+276q^6+356q^7+406q^8+412q^9+371q^{10}+297q^{11}+208q^{12}+125q^{13}+63q^{14}+25q^{15}+7q^{16}+q^{17}$
1001001	3157	$1+7q+25q^2+63q^3+126q^4+211q^5+306q^6+390q^7+439q^8+441q^9+394q^{10}+312q^{11}+216q^{12}+129q^{13}+64q^{14}+25q^{15}+7q^{16}+q^{17}$
1001010	4091	$1+7q+26q^2+68q^3+140q^4+241q^5+358q^6+467q^7+542q^8+562q^9+521q^{10}+433q^{11}+319q^{12}+207q^{13}+116q^{14}+55q^{15}+21q^{16}+6q^{17}+q^{18}$
1010010	4093	$1+7q+26q^2+68q^3+140q^4+241q^5+358q^6+468q^7+542q^8+562q^9+522q^{10}+433q^{11}+319q^{12}+207q^{13}+116q^{14}+55q^{15}+21q^{16}+6q^{17}+q^{18}$
00100101	7561	$1+7q+27q^2+75q^3+166q^4+309q^5+496q^6+701q^7+881q^8+994q^9+1008q^{10}+920q^{11}+753q^{12}+548q^{13}+351q^{14}+194q^{15}+89q^{16}+32q^{17}+8q^{18}+q^{19}$
00101001	7571	$1+7q+27q^2+75q^3+166q^4+309q^5+496q^6+701q^7+882q^8+995q^9+1010q^{10}+922q^{11}+754q^{12}+550q^{13}+352q^{14}+194q^{15}+89q^{16}+32q^{17}+8q^{18}+q^{19}$
01001001	7639	$1+7q+27q^2+76q^3+169q^4+314q^5+504q^6+711q^7+893q^8+1006q^9+1018q^{10}+928q^{11}+758q^{12}+551q^{13}+352q^{14}+194q^{15}+89q^{16}+32q^{17}+8q^{18}+q^{19}$
01001010	9899	$1+7q+28q^2+81q^3+185q^4+353q^5+579q^6+836q^7+1075q^8+1242q^9+1296q^{10}+1222q^{11}+1040q^{12}+797q^{13}+545q^{14}+329q^{15}+172q^{16}+76q^{17}+27q^{18}+7q^{19}+q^{20}$
01010010	9901	$1+7q+28q^2+81q^3+185q^4+353q^5+579q^6+836q^7+1075q^8+1243q^9+1296q^{10}+1222q^{11}+1041q^{12}+797q^{13}+545q^{14}+329q^{15}+172q^{16}+76q^{17}+27q^{18}+7q^{19}+q^{20}$
10010010	10679	$1+8q+33q^2+95q^3+214q^4+401q^5+649q^6+926q^7+1178q^8+1348q^9+1393q^{10}+1303q^{11}+1100q^{12}+836q^{13}+567q^{14}+339q^{15}+176q^{16}+77q^{17}+27q^{18}+7q^{19}+q^{20}$
10010100	11065	$1+8q+33q^2+95q^3+215q^4+406q^5+661q^6+947q^7+1211q^8+1393q^9+1446q^{10}+1358q^{11}+1152q^{12}+879q^{13}+598q^{14}+359q^{15}+186q^{16}+81q^{17}+28q^{18}+7q^{19}+q^{20}$
10100100	11069	$1+8q+33q^2+95q^3+215q^4+406q^5+661q^6+948q^7+1212q^8+1393q^9+1447q^{10}+1359q^{11}+1152q^{12}+879q^{13}+598q^{14}+359q^{15}+186q^{16}+81q^{17}+28q^{18}+7q^{19}+q^{20}$

10100101	18045	$1+8q+34q^2+102q^3+242q^4+481q^5+826q^6+1251q^7+1692q^8+2063q^9+2278q^{10}+2284q^{11}+2078q^{12}+1712q^{13}+1270q^{14}+841q^{15}+490q^{16}+246q^{17}+103q^{18}+34q^{19}+8q^{20}+q^{21}$
001001010	25606	$1+8q+35q^2+109q^3+268q^4+551q^5+977q^6+1527q^7+2132q^8+2686q^9+3071q^{10}+3198q^{11}+3037q^{12}+2626q^{13}+2063q^{14}+1464q^{15}+930q^{16}+522q^{17}+254q^{18}+104q^{19}+34q^{20}+8q^{21}+q^{22}$
001010010	25610	$1+8q+35q^2+109q^3+268q^4+551q^5+977q^6+1527q^7+2132q^8+2686q^9+3072q^{10}+3199q^{11}+3037q^{12}+2627q^{13}+2064q^{14}+1464q^{15}+930q^{16}+522q^{17}+254q^{18}+104q^{19}+34q^{20}+8q^{21}+q^{22}$
010010010	25840	$1+8q+35q^2+110q^3+272q^4+560q^5+993q^6+1550q^7+2162q^8+2720q^9+3105q^{10}+3228q^{11}+3060q^{12}+2642q^{13}+2072q^{14}+1468q^{15}+931q^{16}+522q^{17}+254q^{18}+104q^{19}+34q^{20}+8q^{21}+q^{22}$
010010100	26774	$1+8q+35q^2+110q^3+273q^4+565q^5+1007q^6+1580q^7+2214q^8+2797q^9+3208q^{10}+3349q^{11}+3187q^{12}+2763q^{13}+2175q^{14}+1546q^{15}+983q^{16}+552q^{17}+268q^{18}+109q^{19}+35q^{20}+8q^{21}+q^{22}$
010100100	26776	$1+8q+35q^2+110q^3+273q^4+565q^5+1007q^6+1580q^7+2214q^8+2798q^9+3208q^{10}+3349q^{11}+3188q^{12}+2763q^{13}+2175q^{14}+1546q^{15}+983q^{16}+552q^{17}+268q^{18}+109q^{19}+35q^{20}+8q^{21}+q^{22}$
010100101	43651	$1+8q+36q^2+117q^3+302q^4+653q^5+1219q^6+2008q^7+2958q^8+3937q^9+4763q^{10}+5261q^{11}+5315q^{12}+4910q^{13}+4141q^{14}+3176q^{15}+2200q^{16}+1363q^{17}+744q^{18}+350q^{19}+137q^{20}+42q^{21}+9q^{22}+q^{23}$
100100101	47081	$1+9q+42q^2+137q^3+351q^4+750q^5+1384q^6+2254q^7+3286q^8+4331q^9+5194q^{10}+5690q^{11}+5702q^{12}+5229q^{13}+4378q^{14}+3333q^{15}+2292q^{16}+1409q^{17}+763q^{18}+356q^{19}+138q^{20}+42q^{21}+9q^{22}+q^{23}$
100101001	47143	$1+9q+42q^2+137q^3+351q^4+750q^5+1384q^6+2254q^7+3287q^8+4334q^9+5199q^{10}+5697q^{11}+5712q^{12}+5239q^{13}+4387q^{14}+3341q^{15}+2297q^{16}+1412q^{17}+764q^{18}+356q^{19}+138q^{20}+42q^{21}+9q^{22}+q^{23}$
101001001	47159	$1+9q+42q^2+137q^3+351q^4+750q^5+1384q^6+2255q^7+3289q^8+4336q^9+5202q^{10}+5700q^{11}+5714q^{12}+5241q^{13}+4388q^{14}+3341q^{15}+2297q^{16}+1412q^{17}+764q^{18}+356q^{19}+138q^{20}+42q^{21}+9q^{22}+q^{23}$
101001010	61111	$1+9q+43q^2+144q^3+378q^4+826q^5+1556q^6+2585q^7+3844q^8+5171q^9+6336q^{10}+7105q^{11}+7310q^{12}+6905q^{13}+5985q^{14}+4749q^{15}+3434q^{16}+2249q^{17}+1321q^{18}+687q^{19}+310q^{20}+118q^{21}+36q^{22}+8q^{23}+q^{24}$

Table 6.1: The values of $\mu(w)_{12}$ and $\mu_q(w)_{12}$ for the factors of the Fibonacci word of length up to 9 ordered in radix order. Factors of the Fibonacci word that are Christoffel words and Markoff numbers are underlined. Christoffel words form a sparse subset of the language of the Fibonacci word.

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