

TOPPLEABLE PERMUTATIONS, EXCEDANCES AND ACYCLIC ORIENTATIONS

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Abstract. Recall that an excedance of a permutation π is any position i such that $\pi_i > i$. Inspired by the work of Hopkins, McConville and Propp (Elec. J. Comb., 2017) on sorting using toppling, we say that a permutation is toppleable if it gets sorted by a certain sequence of toppling moves. One of our main results is that the number of toppleable permutations on n letters is the same as those for which excedances happen exactly at $\{1, \dots, \lfloor (n-1)/2 \rfloor\}$. Additionally, we show that the above is also the number of acyclic orientations with unique sink (AUSOs) of the complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor + 1}$. We also give a formula for the number of AUSOs of complete multipartite graphs. We conclude with observations on an extremal question of Cameron et al. concerning maximizers of (the number of) acyclic orientations, given a prescribed number of vertices and edges for the graph.

Keywords. Toppleable permutations, acyclic orientations, excedances, collapsed permutations, complete bipartite, complete multipartite, Genocchi numbers

Mathematics Subject Classifications. 05A19, 05A05, 05C30

1. Introduction

The sandpile model on graphs has been extensively studied, ever since its first introduction by Dhar [Dha90] and independently by Björner, Lovasz and Shor [BLS91] (where it is called the chip-firing game). From the mathematical standpoint, this model has influenced subjects as diverse as combinatorics, probability theory, algebraic geometry and combinatorial commutative algebra. See the recent book by Klivans [Kli19] for details.

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In the usual sandpile model, all chips are indistinguishable. An impetus to studying the sandpile model with labeled chips is the recent work of Hopkins, McConville and Propp [HMP16], who studied the model on \mathbb{Z} with chips labeled $1, \dots, n$ initially at the origin. We use the term *toppling* (see formal definition below, in Section 2) to describe this dynamics. For n even, they showed that the resulting configuration is always sorted. But for n odd, they conjectured based on simulations that the probability of being sorted tends to $1/3$ as $n \rightarrow \infty$. This conjecture has motivated a lot of work since then [GHMP18, GHMP19, HP19, KL20b, FK21, KL20a].

In this work, we consider the model on \mathbb{Z} with initial configurations motivated by the analysis of certain special sequences of moves in the original model. We show that the enumerative combinatorics of those initial configurations which get sorted is very rich and related to classical permutation enumerations via the excedance statistic. We also show bijectively that these objects are related to the number of acyclic orientations with a unique sink (AUSOs) of complete bipartite graphs. The relation between acyclic orientations of bipartite graphs and the excedance statistic has also been studied in the context of so-called poly-Bernoulli numbers [BH17]. We then extend our explicit enumeration result on AUSOs of complete bipartite graphs to the N -partite case (for $N \geq 2$), with prescribed sizes for the N parts.

Further motivated by a result in [HMP16], we study a new class of permutations which we call *collapsed permutations*, and show that they are enumerated by the classical Genocchi numbers of the first and second kind. Our final contribution in this work briefly considers a conjecture of Cameron, Glass and Schumacher [CGS14] that Turán graphs maximize the number of acyclic orientations over graphs with given numbers of vertices and edges. We show that when $m \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$, a Turán graph whose parts have size 1 or 2 maximizes the number of AOs over all graphs on n vertices and m edges.

The plan of the rest of the paper is as follows. In Section 2, we motivate and introduce the toppling model on permutations, prove some of its basic properties, and state the main results. In Section 3, we prove one of the main results stated in the previous section. In Section 4, we define the set of collapsed permutations and describe the connections to the Genocchi numbers. In Section 5, we prove the second main result of Section 2. Section 6 is devoted to proving enumeration formulas for AUSOs of complete multipartite graphs. These can also be obtained using the generating function for the chromatic polynomial of complete multipartite graphs given in [Sta99, Solution to Problem 5.6]; see Remark A.4 for more details. Section 7 discusses the extremal question of maximizing the number of acyclic orientations with given numbers of vertices and edges.

2. Toppling on permutations

For $n \geq 2$, we will consider labeled chip configurations on the segment

$$L_n = \{-\lfloor (n+1)/2 \rfloor, \dots, -1, 0, 1, \dots, \lfloor n/2 \rfloor + 1\}.$$

We will think of 0 as the origin. At each position in L_n , we will place a certain number of labeled chips, with a total number of n chips labeled 1 through n . We will consider the discrete dynamical system called *toppling*, defined in [HMP16] (called chip-firing therein), on configurations in L_n as follows:

1. If no position in L_n has two or more chips, stop. Else, go to step 2.
2. Choose a position i uniformly at random among positions occupied by more than one chip.
3. Pick two chips $\alpha < \beta$ uniformly from those at site i .
4. Move α to position $i - 1$ and β to $i + 1$.
5. Go to step 1.

We will be interested in a special class of initial configurations arising from permutations as follows. As usual, let $[n]$ denote $\{1, \dots, n\}$. For $\pi = (\pi_1, \dots, \pi_n) \in S_n$ and an element $r \in [n + 1]$, we will place π_1, \dots, π_n in positions $-\lfloor (n - 1)/2 \rfloor, \dots, -1, 0, 1, \dots, \lfloor n/2 \rfloor$, increase the labels of elements in π greater than or equal to r by 1, and place r at the origin. We will call this initial condition $\pi^{(r)}$. For example, with $r = 2$ and permutations $\rho = (3, 1, 4, 2) \in S_4$ and $\sigma = (2, 5, 1, 3, 4) \in S_5$, we obtain

$$\rho^{(2)} = \begin{array}{cccccc} & & 1 & & & \\ & 4 & 2 & 5 & 3 & \\ -2 & -1 & 0 & 1 & 2 & 3 \end{array}, \quad \sigma^{(2)} = \begin{array}{cccccc} & & & 1 & & \\ & 3 & 6 & 2 & 4 & 5 \\ -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array}. \quad (2.1)$$

For convenience, we will denote configurations in one-line notation similar to that used for permutations, except that when there are multiple chips at a site, we will enclose them in parenthesis. For the above examples, we will write $\rho^{(2)} = (4, (1, 2), 5, 3)$ and $\sigma^{(2)} = (3, 6, (1, 2), 4, 5)$. The vacancies in the first and last position will be understood.

The motivation for studying this model comes from the toppling dynamical system on \mathbb{Z} with $n \geq 2$ chips initially at the origin, studied in detail in [HMP16]. Fix an $r \in [n]$ and consider a special case of topplings of this model where we do not permit r to be picked as one of two chips to be moved. If n is odd, r will end up by itself at the origin and the process will terminate. If n is even, something more interesting happens. At the end of these topplings, r will have a partner at the origin and we will end up with a configuration $\pi^{(r)}$ in L_{n-1} for some $\pi \in S_{n-1}$. If we now allow r to topple, all the chips will be sorted, thanks to [HMP16, Theorem 13]. This tells us that for a large class of permutations π , the final configuration after toppling $\pi^{(r)}$ is sorted. Thus, it is natural to try to understand the set of all permutations for which sorting happens.

We begin with some basic properties of the toppling dynamics in L_n starting with $\pi^{(r)}$.

Proposition 2.1. *Fix $\pi \in S_n$ and $r \in [n + 1]$. The toppling dynamical system on L_n with initial condition $\pi^{(r)}$ satisfies the following properties.*

1. *The final configuration is deterministic: it does not depend on any of the random choices.*
2. *At every step, the configuration lives in L_n . In other words, no chip leaves L_n .*
3. *In the final configuration, there is precisely one chip at every position in L_n , except the origin (resp. position 1) when n is odd (resp. even).*

Proof. We begin with (1) by showing that at each point in time, for any two positions $a < b \in L_n$ each containing two chips, there must be some c with $a < c < b$ and no chips at c . In particular, this implies that no position can ever have more than 2 chips.

The initial configuration $\pi^{(r)}$ satisfies this condition vacuously, as there is only a single position containing 2 chips. Then inducting on the time step t , suppose we toppled position a at time t . Looking at position $a + 1$, it could have contained either 0 or 1 chip before toppling a , by the induction hypothesis. If $a + 1$ contained 1 chip, then it now contains 2 chips. But then for any $b > a$ containing 2 chips, the induction hypothesis gives that there is an empty position between a and b , and it cannot have been $a + 1$, so that empty space separates $a + 1$ from any such b . Likewise, any position to the left of $a + 1$ containing 2 chips is separated from $a + 1$ by the now-empty position a . Otherwise, $a + 1$ did not contain a chip. Then if $a + 1$ was the empty space separating any two positions with 2 chips, the now-empty position a now separates them. The argument for position $a - 1$ is identical.

To complete the proof of (1), note that the randomness due to toppling goes away. Once a site is chosen, nothing happens if it has less than 2 chips, and if it has two chips, the smaller moves left and the larger moves right. Moreover, since any two sites containing 2 chips are far apart, the order of toppling on these sites commutes. Therefore, we have accounted for both sources of randomness.

We will prove (2) inductively. The statement is easily verified for $n = 2, 3$. Suppose $n > 3$. From the proof of (1), we know no position contains more than 2 chips. The only way a chip can leave L_n is if the leftmost or rightmost positions contain more than one chip. Focus on positions $\lfloor n/2 \rfloor$ and $-\lfloor (n-1)/2 \rfloor$. When they contain two chips, they transfer one chip each to the leftmost and rightmost positions in L_n and becomes vacant. Note that we can arrange for both of them to contain two chips simultaneously; for instance, this can be done using the concept of a wave defined in Section 2. Now focus on sites $-\lfloor (n-1)/2 \rfloor, \dots, \lfloor n/2 \rfloor$. The extremal sites are empty and no position contains two or more chips. Up to some trivial relabelling, we obtain a configuration in L_{n-2} . The proof follows now from the induction hypothesis.

The argument for (3) is similar to that of (2). The base cases of $n = 2, 3$ are easily checked. For $n > 3$, once there is a chip at positions $-\lfloor (n+1)/2 \rfloor$ and $\lfloor n/2 + 1 \rfloor$, the adjoining sites are empty and we end up with a configuration in L_{n-2} . \square

By Proposition 2.1(3), the final configuration can be interpreted as a permutation in S_{n+1} and by Proposition 2.1(1), the toppling dynamical system on L_n can be considered as a map $\mathcal{T} : S_n \times [n+1] \rightarrow S_{n+1}$. Let id be the identity (namely sorted) permutation whose size will be clear from the context.

Definition 2.2. We say that a permutation π is r -*toppleable* if $\mathcal{T}(\pi, r) = \text{id}$, and we say that π is *toppleable* if π is r -toppleable for all $r \in [n+1]$.

For the examples in (2.1), the final configurations can be seen to be $\mathcal{T}(\rho, 2) = (1, 2, 3, 4, 5)$ and $\mathcal{T}(\sigma, 2) = (1, 2, 3, 4, 6, 5)$. Therefore, ρ is 2-toppleable, but σ is not. It is easy to see that in general, $\mathcal{T}(\text{id}, r) = \text{id}$ for all r and therefore id is always toppleable. The following symmetry property of toppling dynamics can be easily seen by studying what happens during a single toppling move.

Proposition 2.3. *Suppose $n \geq 3$ is odd, $r \in [n + 1]$ and $\pi = (\pi_1, \dots, \pi_n) \in S_n$. Let $\hat{\pi} = (n + 1 - \pi_n, \dots, n + 1 - \pi_1)$. Then the toppling dynamics on $\pi^{(r)}$ is isomorphic to that on $\hat{\pi}^{(n+2-r)}$ via the map which reflects configurations about the origin and interchanges chip i with $n + 2 - i$.*

Since $\widehat{\text{id}} = \text{id}$, π is r -toppleable if and only if $\hat{\pi}$ is $(n + 2 - r)$ -toppleable.

Let $t_r(n)$ be the number of r -toppleable permutations and $t(n)$ be the number of toppleable permutations in S_n . For example, $t_1(3) = t_4(3) = 4$ since there are four 1-toppleable permutations, namely 123, 213, 132 and 231, as well as four 4-toppleable permutations, namely 123, 213, 132 and 312, in S_3 . The common permutations among these turn out also to be 2- and 3-toppleable and hence $t(3) = 3$. Data for $t_r(n)$ for small values of r and n is given in Table 2.1. As expected from Proposition 2.3, $t_i(n) = t_{n+2-i}(n)$ for n odd.

$n \setminus r$	1	2	3	4	5	6	7	8	9
3	4	3	3	4					
4	14	10	7	7	8				
5	46	38	31	31	38	46			
6	230	184	146	115	115	130	146		
7	1066	920	790	675	675	790	920	1066	
8	6902	5836	4916	4126	3451	3451	3842	4264	4718

Table 2.1: The number of r -toppleable permutations, $t_r(n)$, for $3 \leq n \leq 8$.

Recall that an *excedance* of a permutation π is any position i such that $\pi_i > i$. The set of positions at which there are excedances for π is called the *excedance set* of π . For example, the permutations ρ and σ considered in (2.1) have excedance sets $\{1, 3\}$ and $\{1, 2\}$ respectively. There have been a lot of studies of the excedance statistic on permutations. What will be relevant to us is the study of permutations whose excedance set is $\{1, \dots, k\}$ for some k . This was initiated by Ehrenborg and Steingrímsson [ES00], who gave a formula for the number $a_{n,k}$ of such permutations in S_n . The bivariate exponential generating function of $a_{r+s,s}$ is given by the explicit formula [CE10, Theorem 3.1]

$$\sum_{r,s \geq 0} a_{r+s,s} \frac{x^r y^s}{r! s!} = \frac{e^{-x-y}}{(e^{-x} + e^{-y} - 1)^2}. \tag{2.2}$$

Theorem 2.4. *The number of toppleable permutations in S_n satisfies $t(n) = t_{\lfloor n/2 \rfloor + 1}(n) = t_{\lfloor n/2 \rfloor + 2}(n)$ for all n . Furthermore, this number is given by $t(n) = a_{n, \lfloor (n-1)/2 \rfloor}$.*

Using (2.2), de Andrade, Lundberg and Nagle [dLN15, Theorem 1.2] obtained the asymptotic formula,

$$t(n) = \frac{1}{2 \log 2 \sqrt{1 - \log 2} + o(1)} \frac{n!}{(2 \log 2)^n}.$$

These numbers are given by the central diagonal sequence in the triangle [OEI20, Sequence A136126] of the OEIS. Notice that the latter triangle is symmetric and therefore, the answer is unambiguous for even n .

To state our next result, we recall the notion of acyclic orientations. For any simple, undirected graph, an *orientation* is an assignment of arrows to the edges. An *acyclic orientation* (AO) is an orientation in which there is no directed cycle. It is easy to see that every graph has an acyclic orientation and every acyclic orientation has at least one *source* (vertex with no incoming arrows) and one *sink* (vertex with no outgoing arrows). An *acyclic orientation with a unique sink* (AUSO), also known as the *Ursell function* is an acyclic orientation with exactly one sink. Stanley showed that the number of acyclic orientations of any graph (up to sign) is given by the chromatic polynomial of the graph evaluated at -1 [Sta73]. A related result of Greene and Zaslavsky [GZ83] is that the number of acyclic orientations with a unique sink is independent of the sink and equal to (again up to sign) the linear coefficient of the chromatic polynomial.

Our focus here will be on AUSOs of complete bipartite graphs $K_{m,n}$.

Theorem 2.5. *The number of toppleable permutations in S_n , $t(n)$, is the same as the number of acyclic orientations with a fixed unique sink of $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1}$.*

3. Toppleable permutations and excedances

In this section, we will prove Theorem 2.4. To make the presentation cleaner, we state the results separately for odd and even n for the most part. This will avoid the presence of floors and ceilings all over the place. We begin with a monotonicity result.

Theorem 3.1. *Let $\pi \in S_n$ and $m = \lfloor \frac{n}{2} \rfloor$.*

1. *Suppose $2 \leq r \leq m + 1$. Then π is $(r - 1)$ -toppleable if π is r -toppleable.*
2. *Suppose $m + 2 \leq r \leq n$. Then π is $(r + 1)$ -toppleable if π is r -toppleable.*
3. *For $r = m + 1$, π is r -toppleable if and only if π is $(r + 1)$ -toppleable.*

To see that the converse of the first statement is not true, consider the following example with $m = 2$ and $r = 3$. When $\pi = 24135$, $\pi^{(3)} = (2, 5, (1, 3), 4, 6)$ and $\pi^{(2)} = (3, 5, (1, 2), 4, 6)$ couple eventually. But when $\pi = 13452$, $\pi^{(3)} = (1, 4, (3, 5), 6, 2)$ and $\pi^{(2)} = (1, 4, (2, 5), 6, 3)$ never couple and we have $\mathcal{T}(\pi^{(2)}) = 123456$, but $\mathcal{T}(\pi^{(3)}) = 132456$.

To prove Theorem 3.1, we need a lemma:

Lemma 3.2. *Let $\pi \in S_n$ and suppose π is r -toppleable. Then*

1. *for each $1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$, the final move of chip k when toppling $\pi^{(r)}$ is to the left;*
2. *for each $\lfloor \frac{n}{2} \rfloor + 2 \leq k \leq n + 1$, the final move of chip k when toppling $\pi^{(r)}$ is to the right.*
3. *in the final move, chips $\lfloor \frac{n}{2} \rfloor + 1$ and $\lfloor \frac{n}{2} \rfloor + 2$ topple to their correct positions.*

Proof. To prove (1), we use induction on k . First, since $\pi^{(r)}$ topples to the identity, chip 1 ends in the leftmost position of L_n . Then, Proposition 2.1(2) ensures that the final move of 1 into this position must be to the left, since otherwise the chip would have to lie outside L_n directly before its final move.

Now, using the induction hypothesis, we have that the final move of chip $k - 1$ is to the left, say into position $p - 1$. This leaves an empty space in position p . Moreover, after this final move, no chip may land on position $p - 1$. We know, since π is r -toppleable and $k \leq \lfloor \frac{n}{2} \rfloor + 1$, that position p must eventually hold chip k . And since chip k cannot land on position $p - 1$, it must make its final move to the left from position $p + 1$.

(2) is proved similarly. For (3), recall Proposition 2.1(3) gives that the final configuration contains no chips at position 0 (resp. 1) if n is odd (resp. even). Since $\pi^{(r)}$ topples to the identity, we have $\lfloor \frac{n}{2} \rfloor + 1$ directly left of this empty position, and $\lfloor \frac{n}{2} \rfloor + 2$ directly to the right. It is clear that the only way to arrive at this configuration is for the final topple to have occurred at position 0 (resp. 1) containing these two chips. \square

Proof of Theorem 3.1. Our strategy for the proof is as follows. Suppose $\pi \in S_n$ is r -toppleable. Then, the only difference between $\pi^{(r)}$ and $\pi^{(r-1)}$ is that the positions of $r - 1$ and r are interchanged. By definition, r is positioned at the origin of L_n in $\pi^{(r)}$ and let j be the position of $r - 1$ in $\pi^{(r)}$. If $j = 0$, then $\pi^{(r)} = \pi^{(r-1)}$ and the result trivially holds. If not, there are two possibilities. Either $j > 0$ or $j < 0$. We might need different arguments in both cases.

At each step of the toppling procedure, $\pi^{(r)}$ and $\pi^{(r-1)}$ continue to differ only in their positions of $r - 1$ and r . This will be the case until we reach the point when $r - 1$ and r are at the same position. At this point, the toppling procedure is coupled and the final result is identity. The only problem with this argument is that we could have reached the final result without ever being coupled.

Now we assume that $2 \leq r \leq m + 1$. For the proof of statement (1), we will track the positions of $r - 1$ and r as time evolves only in $\pi^{(r)}$. If we can show that there is a time when $r - 1$ and r are at the same site, then we are done since $\pi^{(r)}$ and $\pi^{(r-1)}$ will be coupled.

If $j > 0$, that means $r - 1$ is to the right of r in $\pi^{(r)}$. But we know that eventually $r - 1$ will end up to the left of r , since π is r -toppleable. Therefore, there will necessarily be a time when $r - 1$ and r are at the same site.

If $j < 0$, then we perform induction on the difference ℓ in the locations of $r - 1$ and r , assuming that r is not a singleton and no positions between r and $r - 1$ are vacant. If $\ell = 0$, then we are done as argued before. Suppose that $\ell \geq 1$, and for induction suppose that if r is $\ell - 1$ positions to the right of $r - 1$ with r not a singleton and no vacancies between them, then they will couple.

Now, for distance ℓ , the initial situation is

$$\dots \quad r - 1 \quad \underbrace{\dots}_{\ell - 1} \quad \begin{matrix} r \\ a \end{matrix} \quad b \quad \dots,$$

where $a, b \in [n + 1]$. We topple the site containing a and r , then there are now two sub-cases.

(1) If $r < a$, then we land in either

$$\dots \quad \begin{matrix} r \\ r - 1 \end{matrix} \quad _ \quad \begin{matrix} a \\ b \end{matrix} \quad \dots$$

if $\ell = 1$, or in

$$\dots \quad r-1 \quad \underbrace{\dots}_{\ell-2} \quad x \quad - \quad \begin{matrix} r & a \\ & b \end{matrix} \quad \dots$$

if $\ell > 1$ and x is the chip immediately to the left of a initially. In the former case, $r-1$ and r are already at the same site, and hence coupled. In the latter, we are in a similar situation as what we started with, but ℓ has reduced by 1. Therefore, we are done by the induction assumption.

(2) Suppose $a < r$ and hence $a < r-1$. Then, in the first step, we arrive in

$$\dots \quad r-1 \quad \underbrace{\dots}_{\ell-2} \quad \begin{matrix} a & r \\ & x - b \end{matrix} \quad \dots,$$

where again x is the chip immediately to the left of a initially, and $x = r-1$ if $\ell = 1$. Note in particular that Lemma 3.2 ensures that when r moves to the right, that is not its final move. Therefore, there must eventually be some b (maybe after toppling some chips to the right of r) which lands in the same position as r as shown above.

Now, we only perform topplings on the left half until a particle, say y , lands on the same site as $r-1$. Ignoring all the other particles, we then have

$$\dots \quad z \quad r-1 \quad - \quad \underbrace{\dots}_{\ell-1} \quad \begin{matrix} y & r \\ & b \end{matrix} \quad \dots$$

The key observation is that $y < r-1$. Therefore, at the next stage, we will arrive at

$$\dots \quad z \quad - \quad r-1 \quad \underbrace{\dots}_{\ell-1} \quad \begin{matrix} y & r \\ & b \end{matrix} \quad \dots$$

Therefore, we are back in the same situation as before with ℓ sites in between, except that both $r-1$ and r are shifted to the right. Now, if $b > r$, we are back in Item (1) and we are done by induction. If not, we repeat this argument and end up again with ℓ sites between $r-1$ and r . Since r will have to move to the left eventually (again, by Lemma 3.2), we will arrive in the situation with Item (1) and the result is proved by induction.

The proof of statement (2) is similar—it follows directly by symmetry in the case that n is odd, and by a similar argument to the above in the even case. The proof of (3) follows from Lemma 3.2(3): chips $r = \lfloor \frac{n}{2} \rfloor + 1$ and $r+1$ must couple in the final move. \square

We now move towards a structural characterization for toppleable permutations. To do so, we will find it useful to define the notion of a *wave* for a fixed sequence of topplings; recall that the sequence of topplings does not matter by Proposition 2.1(1).

Let $\pi \in S_n$. Then, we denote the tuple counting the number of chips at each site of L_n in $\pi^{(r)}$ by $p^{(r)} = (-, 1, \dots, 1, \hat{2}, 1, \dots, 1, -)$, where we have marked the origin with a hat. Equivalently, this is the corresponding unlabeled configuration of $\pi^{(r)}$. Let us consider what happens to $p^{(r)}$ after the first few topplings:

$$\begin{aligned} p^{(r)} &\rightarrow (-, 1, \dots, 1, 1, 2, \hat{2}, 2, 1, 1, \dots, 1, -) \\ &\rightarrow (-, 1, \dots, 1, 2, -, \hat{2}, -, 2, 1, \dots, 1, -) \\ &\rightarrow (-, 1, \dots, 2, -, 1, \hat{2}, 1, -, 2, \dots, 1, -). \end{aligned}$$

At this point, we leave the origin unchanged and start to topple the vertices with 2 chips both on the left and right simultaneously, until we reach the end. We then arrive at the configuration $(1, -, 1, \dots, 1, \hat{2}, 1, \dots, 1, -, 1)$. Now, the extremal points cannot be modified by any further topplings and are fixed. We call this sequence of topplings the *first wave*. This consists of n individual topplings. Similarly, the *second wave* will be initiated by toppling the origin in a similar way, and we will end up with $(1, 1, -, 1, \dots, 1, \hat{2}, 1, \dots, 1, -, 1, 1)$. We continue this way. If n is odd, then we see that after $(n + 1)/2$ waves, the configuration will freeze leaving the origin empty. If n is even, then after $n/2 - 1$ moves, we end up with $(1, \dots, 1, -, \hat{2}, 1, -, 1, \dots, 1)$. We then declare the $(n/2)$ 'th wave to be the one that topples at the origin and site 1, freezing the configuration leaving site 1 empty.

We make two elementary observations about these moves. First, every chip between the two vacant sites topples at least once in every wave. Second, if $\pi \in S_{2m+1}$ is r -toppleable, then for $1 \leq i \leq m + 1$, i and $2m + 2 - i$ get fixed in their correct positions at the end of the i 'th wave. For example, the result of waves on the $\rho^{(2)}$ and $\sigma^{(2)}$ from (2.1) are as follows:

$$\begin{array}{cccccccccccc} \rho^{(2)} & \xrightarrow[\text{wave}]{\text{first}} & & & 2 & & & \xrightarrow[\text{wave}]{\text{second}} & 1 & 2 & 3 & - & 4 & 5, \\ & & 1 & - & 4 & 3 & - & 5 & & & & & & \\ \sigma^{(2)} & \xrightarrow[\text{wave}]{\text{first}} & & & 2 & & & & & & & & & \\ & & 1 & - & 3 & 6 & 4 & - & 5 & & & & & \\ & & & & & & & & & & & & & \\ & \xrightarrow[\text{wave}]{\text{second}} & & & 3 & & & \xrightarrow[\text{wave}]{\text{third}} & 1 & 2 & 3 & - & 4 & 6 & 5. \\ & & 1 & 2 & - & 4 & - & 6 & 5 & & & & & & \end{array}$$

Lemma 3.3. *If $\pi \in S_{2m+1}$ is toppleable, then 1 is in position at most $m + 1$ in π . Conversely, if 1 (resp n) is in position at most $m + 1$ (at least $m + 1$) in π , then 1 (resp $n + 1$) is in the first (resp. last) position in $\mathcal{T}(\pi^{(m+1)})$.*

If $\pi \in S_{2m}$ is toppleable, then 1 is in position at most m in π . Conversely, if 1 (resp. $2m$) is in position at most m (resp. at least m) in π , then 1 (resp. $2m + 1$) is in the first (resp. last) position in $\mathcal{T}(\pi^{(m+1)})$.

Proof. By Theorem 3.1, it suffices to consider $\pi^{(\lfloor n/2 \rfloor + 1)}$. Suppose 1 is to the right of the origin in $\pi^{(m+1)}$. Then, in the first wave, 1 will move exactly one position to the left (since it is smallest) at the end of the first wave. Therefore, 1 is not frozen in its correct position, which is the extreme left. So π cannot be toppleable.

Conversely, suppose 1 is in a position on or to the left of the origin in $\pi^{(m+1)}$. Then it gets a partner at some point during the first wave. After that time, it keeps moving left for all future

times until the first wave ends and gets placed at the extreme left, its correct position. A similar argument works for even n , completing the proof. \square

We are now in a position to characterize toppleable permutations. This characterization involves bounds on the difference between values and positions and is in the spirit of so-called *Vesztergombi permutations* [Ves74, LV78], where these differences have global bounds.

Theorem 3.4. *A permutation $\pi \in S_{2m+1}$ is $(m+1)$ -toppleable if and only if*

$$\begin{aligned} \pi_i &\leq m+i, & 1 \leq i \leq m, \\ \pi_i &\geq i-m, & m+1 \leq i \leq 2m+1. \end{aligned}$$

Equivalently,

$$\begin{aligned} \pi_i^{-1} &\in \{1, \dots, m+i\}, & 1 \leq i \leq m+1, \\ \pi_i^{-1} &\in \{i-m, \dots, 2m+1\}, & m+2 \leq i \leq 2m+1. \end{aligned}$$

A permutation $\pi \in S_{2m}$ is $(m+1)$ -toppleable if and only if

$$\begin{aligned} \pi_i &\leq m+i, & 1 \leq i \leq m, \\ \pi_i &\geq i-m+1, & m+1 \leq i \leq 2m. \end{aligned}$$

Equivalently,

$$\begin{aligned} \pi_i^{-1} &\in \{1, \dots, m+i-1\}, & 1 \leq i \leq m, \\ \pi_i^{-1} &\in \{i-m, \dots, 2m\}, & m+1 \leq i \leq 2m. \end{aligned}$$

Proof. We will prove the forward implication first. Suppose for some $1 \leq i \leq m$, i is in a position greater than $i+m$ in π . Therefore, it is to the right of the origin in $\pi^{(m+1)}$. If $i=1$, we are done by Lemma 3.3. If not, consider the situation after $i-1$ waves. After each wave, i either moves left by exactly one position or moves to the right by an arbitrary number of positions. Therefore, at the end of the $(i-1)$ 'th wave, i is strictly to the right of the origin. Therefore, arguing exactly as in Lemma 3.3, i cannot be in its correct position at the end of the i 'th wave. So, π cannot be $(m+1)$ -toppleable. The case when $i \geq m+2$ is in a position to the left of $i-m$ in π is done by symmetry.

For the converse, suppose $i \in \{\pi_1, \dots, \pi_{m+i}\}$ for $1 \leq i \leq m+1$. As i lies left of position $m+i+1$ in π , then in starting configuration $\pi^{(m+1)}$, i lies to the left of position i in L_n . Denote the position of i in L_n after the j 'th wave by $p_j(i)$. We will show for every j : after the j 'th wave, we have $i=1, \dots, j$ fixed in their correct positions, and for each $i=j+1, \dots, m+1$, $p_j(i) < i-j$ (Notice this means that every $i \leq m+1$ is fixed correctly after the $(m+1)$ 'th wave).

We will prove this by double induction, first on j : the base case is simply the 0'th wave, which is satisfied by our assumptions. Then supposing our statement holds for wave $j-1$, consider wave j . Notice first that for $i=j$, we have $p_{j-1}(j) \leq 0$ by the induction hypothesis. Since j is the smallest non-fixed chip, then by the same reasoning as Lemma 3.3, in the j 'th wave, j

topples left until it is fixed into its correct position (for $j = 1$, this is exactly Lemma 3.3). Thus, after wave j , chips $1, \dots, j$ are correctly fixed.

Now, for chips $i = j + 1, \dots, m + 1$, we will use induction on i to argue that during the j 'th wave, i never moves to the right of position $i - j$, and must end strictly left of position $i - j$ (i.e. $p_j(i) < i - j$ as desired). The base case is $i = j$, and simply follows from our previous argument about $i = j$. Now, supposing the statement holds for $j, \dots, i - 1$, recall from our (outer) induction hypothesis that $p_{j-1}(i) \leq i - j$. We consider two cases:

1. If $p_{j-1}(i) < 0$, then in wave j it can move at most one space to the right, so it will fall in a desired position (left of $i - j$).
2. If $p_{j-1}(i) \geq 0$, then it may move to the right multiple times. However, it either never reaches position $i - j$, in which case it is in a desired position, or it reaches (or starts at) position $i - j$. But upon reaching this position, it must topple left, as the (inner) induction hypothesis gives that no smaller chip can reach that position. Thus, in either case, it ends at a position left of $i - j$, so $p_j(i) < i - j$ as desired.

When $\pi_i^{-1} \in \{i - m, \dots, 2m + 1\}$ for $m + 2 \leq i \leq 2m + 1$, an identical argument shows that each of these chips lies in its correct position after $m + 1$ waves. The argument for even n is mostly identical and omitted. \square

We now prove a bijective correspondence relating $(m + 1)$ -toppleable permutations to permutations with the correct excedance set,

Lemma 3.5. *Permutations $\pi \in S_{2m+1}$ such that $\pi_i \leq m + i$ for $1 \leq i \leq m$ and $\pi_i \geq i - m$ for $m + 1 \leq i \leq 2m + 1$ are in bijection with permutations in S_{2m+1} whose excedance set is $\{1, \dots, m\}$.*

Permutations $\pi \in S_{2m}$ such that $\pi_i \leq m + i$ for $1 \leq i \leq m$ and $\pi_i \geq i - m + 1$ for $m + 1 \leq i \leq 2m$ are in bijection with permutations in S_{2m} whose excedance set is $\{1, \dots, m - 1\}$.

Proof. We consider the odd case first. Suppose $\pi \in S_{2m+1}$ satisfies the above conditions. Then we define $\sigma \in S_{2m+1}$ as

$$\sigma_i = \begin{cases} 2m + 2 - \pi_{m+1-i} & 1 \leq i \leq m, \\ 2m + 2 - \pi_{3m+2-i} & m + 1 \leq i \leq 2m + 1. \end{cases}$$

Then we claim that σ has excedance set $\{1, \dots, m\}$. First, suppose σ has the desired excedance set. By definition, $\sigma_i > i$ for $1 \leq i \leq m$. Thus, $\pi_{m+1-i} < 2m + 2 - i$, or equivalently, $\pi_i < m + i + 1$. Also, $\sigma_i \leq i$ for $m + 1 \leq i \leq 2m + 1$. Therefore, $\pi_{3m+2-i} \geq 2m + 2 - i$ for this range of i . Equivalently, $\pi_i \geq i - m$. Thus, π satisfies the above conditions.

Conversely, suppose π satisfies the above conditions. There are three cases to consider. First, consider the entries i such that $1 \leq i \leq m + 1$. Then $i \in \{\pi_1, \dots, \pi_{m+i}\}$. Thus, $2m + 2 - i \in \{\sigma_1, \dots, \sigma_m\} \cup \{\sigma_{2m+2-i}, \dots, \sigma_{2m+1}\}$, where $m + 1 \leq 2m + 2 - i$. Now, if i belongs to the set $\{\pi_1, \dots, \pi_m\}$, then $2m + 2 - i \in \{\sigma_1, \dots, \sigma_m\}$ and we get an excedance in $\{1, \dots, m\}$. If not, $2m + 2 - i \in \{\sigma_{2m+2-i}, \dots, \sigma_{2m+1}\}$, and we do not get an excedance in $\{2m + 2 - i, \dots, 2m + 1\}$.

Second, consider the entries i such that $m + 2 \leq i \leq 2m$. Then $i \in \{\pi_{i-m}, \dots, \pi_{2m+1}\}$. Therefore, $2m + 2 - i \in \{\sigma_1, \dots, \sigma_{2m+1-i}\} \cup \{\sigma_{m+1}, \dots, \sigma_{2m+1}\}$, where $2 \leq 2m + 2 - i \leq m$. Now, if i belongs to the set $\{\pi_{i-m}, \dots, \pi_m\}$, then $2m + 2 - i \in \{\sigma_1, \dots, \sigma_{2m+1-i}\}$ and we get an excedance in $\{1, \dots, 2m + 1 - i\} \subset \{1, \dots, m\}$. If not, $2m + 2 - i \in \{\sigma_{m+1}, \dots, \sigma_{2m+1}\}$, and we do not get an excedance in $\{m + 1, \dots, 2m + 1\}$.

Lastly, the entry $2m + 1 \in \{\pi_{m+1}, \dots, \pi_{2m+1}\}$. Thus, $1 \in \{\sigma_{m+1}, \dots, \sigma_{2m+1}\}$ and the value 1 can never contribute as an excedance. Thus, we do not get an excedance in $\{m + 1, \dots, 2m + 1\}$. Therefore, every chip which lands in positions $\{1, \dots, m\}$ of σ gives an excedance and which lands outside it does not, proving that σ has excedance set exactly $\{1, \dots, m\}$.

For the even case, suppose $\pi \in S_{2m}$ satisfies the conditions above. Then we define $\sigma \in S_{2m}$ as

$$\sigma_i = \begin{cases} 2m + 1 - \pi_{m-i} & 1 \leq i \leq m - 1, \\ 2m + 1 - \pi_{3m-i} & m \leq i \leq 2m. \end{cases}$$

The overall strategy of proof is similar to the odd case. \square

Example 3.6. As an illustration of Lemma 3.5 in the odd case, let $\pi = 31524 \in S_5$ which satisfies $\pi_i \leq 2 + i$ for $1 \leq i \leq 2$ and $\pi_i \geq i - 2$ for $3 \leq i \leq 5$. Then $\sigma = 53241$, which has excedance set $\{1, 2\}$.

For the even case, let $\pi = 216435 \in S_6$ which satisfies $\pi_i \leq 3 + i$ for $1 \leq i \leq 3$ and $\pi_i \geq i - 2$ for $4 \leq i \leq 6$. Then $\sigma = 652431$, which has excedance set $\{1, 2\}$.

We are now in a position to prove our main result.

Proof of Theorem 2.4. By Theorem 3.1, we see that $\pi \in S_n$ is toppleable if it is $(\lfloor n/2 \rfloor + 1)$ -toppleable. According to Theorem 3.4, $\pi_i \leq \lfloor n/2 \rfloor + i$ for $1 \leq i \leq \lfloor n/2 \rfloor$ and $\pi_i \geq i - \lfloor n/2 \rfloor + 1$ for $\lfloor n/2 \rfloor + 1 \leq i \leq n$. Now, Lemma 3.5 proves that the number of such permutations is $a_{n, \lfloor (n-1)/2 \rfloor}$ bijectively, completing the proof. \square

4. Collapsed permutations

As explained before, Hopkins, McConville and Propp considered toppling on \mathbb{Z} starting with n chips at the origin. One of their main results [HMP16, Theorem 13] is that when n is even, the final configuration is always id, the identity permutation. Along the way, they prove bounds on the possible positions of chip k at every step of the toppling process. Specifically, they show [HMP16, Lemma 12] that the position of chip k lies between $-\lfloor (n + 1 - k)/2 \rfloor$ and $\lfloor k/2 \rfloor$. When n is odd (resp. even), $n = 2m + 1$ (resp. $n = 2m$), the final configuration will contain single chips in all positions (resp. all positions but the origin) $-m$ through m . We now apply this condition to count possible permutations arising from this condition switching notation to considering permutations as bijections from the set $[n]$ to itself.

Definition 4.1. We say that a permutation $\pi \in S_n$ is collapsed if

$$\pi_k^{-1} \geq \begin{cases} \lfloor k/2 \rfloor & n \text{ odd,} \\ 1 + \lfloor k/2 \rfloor & n \text{ even} \end{cases} \quad \text{and} \quad \pi_k^{-1} \leq \lfloor n/2 \rfloor + \lfloor k/2 \rfloor.$$

Let G_n be the subset of collapsed permutations in S_n .

For example, $G_3 = \{123, 132, 213\}$ and $G_4 = \{1234, 1324\}$. Since, for n even, the only permutation that appears as a result of toppling is id, counting the cardinality of G_n in that case is not directly relevant to the toppling problem.

To state our results, we recall a well-known combinatorial triangle. The *Seidel triangle* is the triangular sequence $S_{n,k}$ for $n \geq 1$ given by

$$\begin{aligned}
 S_{1,1} &= 1, \\
 S_{n,k} &= 0, \quad k < 2 \text{ or } (n+3)/2 < k, \\
 S_{2n,k} &= \sum_{i \geq k} S_{2n-1,i}, \\
 S_{2n+1,k} &= \sum_{i \leq k} S_{2n,i}.
 \end{aligned}$$

The first few rows of the Seidel triangle are given in Table 4.1.

$n \backslash k$	2	3	4	5	6
1	1				
2	1				
3	1	1			
4	2	1			
5	2	3	3		
6	8	6	3		
7	8	14	17	17	
8	56	48	34	17	
9	56	104	138	155	155
10	608	552	448	310	155

Table 4.1: Rows 1 through 10 of the Seidel triangle.

The (unsigned) *Genocchi numbers of the first kind* g_{2n} , $n \geq 1$ are the numbers on the right-most diagonal of the Seidel triangle and they count the number of permutations in S_{2n-3} whose excedance set is $\{1, 3, \dots, 2n-5\}$. The exponential generating function of g_{2n} is given by

$$\sum_{n \geq 0} g_{2n} \frac{x^{2n}}{(2n)!} = x \tan\left(\frac{x}{2}\right).$$

The first few numbers of this sequence are [OEI20, A110501]

$$\{g_{2n}\}_{n=1}^8 = \{1, 1, 3, 17, 155, 2073, 38227, 929569\}.$$

Theorem 4.2. *The cardinality of G_{2n+1} is g_{2n+4} .*

Proof. Suppose $\pi \in G_{2n+1}$. Then one can check that the definition of G_n in Definition 4.1 is equivalent to saying $\pi_i \leq 2i$ and $\pi_{n+1+i} \geq 2i$ for $1 \leq i \leq n$. Define a map $f : G_{2n+1} \rightarrow S_{2n+1}$ which send $\pi \mapsto \sigma = (\sigma_1, \dots, \sigma_{2n+1})$ such that $\sigma_{2i} = \pi_i$ and $\sigma_{2i-1} = \pi_{n+1+i}$ for $1 \leq i \leq n$, and $\sigma_{2n+1} = \pi_{n+1}$. By the previous sentence, it is immediate that the excedance set of σ is $\{1, 3, \dots, 2n-3, 2n-1\}$. Since f is injective, one can check that the inverse map is well-defined and that $f^{-1}(\sigma) \in G_{2n+1}$ for $\sigma \in S_{2n+1}$ with excedance set $\{1, 3, \dots, 2n-3, 2n-1\}$ in a similar way. \square

As an example, the bijection for $n = 1$ is illustrated below:

G_3	$\pi \in S_3$ with excedance set $\{1\}$
132	213
123	312
213	321

The *median Genocchi numbers* or the *Genocchi numbers of the second kind*, denoted H_{2n+1} are given by the leftmost diagonal of the Seidel triangle (see Table 4.1). They count among other things, ordered pairs $((a_1, \dots, a_{n-1}), (b_1, \dots, b_{n-1})) \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$ such that $0 \leq a_k \leq k$ and $1 \leq b_k \leq k$ for all k and $\{a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1}\} = [n-1]$ [Het19]. The first few numbers of this sequence are [OEI20, A005439]

$$\{H_{2n-1}\}_{n=1}^8 = \{1, 2, 8, 56, 608, 9440, 198272, 5410688\}.$$

No particularly simple formula or generating function seems to be known for H_{2n+1} . In terms of the Genocchi numbers of the first kind, we have [HZ99]

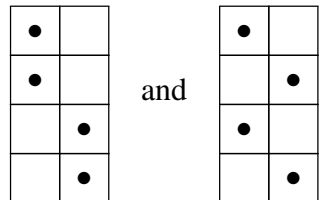
$$H_{2n+1} = \sum_{i=0}^n g_{2n-2i} \binom{n}{2i+1}.$$

Although it is not clear either from the above definition or the formula, it is true that H_{2n-1} is divisible by 2^{n-1} . The numbers $h_n = H_{2n+1}/2^n$ are called the *normalized median Genocchi numbers*. The first few numbers of this sequence are [OEI20, A000366]

$$\{h_n\}_{n=0}^7 = \{1, 1, 2, 7, 38, 295, 3098, 42271\}.$$

A classical combinatorial interpretation for these are the Dellac configurations, which we now define.

A *Dellac configuration* of order n is a $2n \times n$ array containing $2n$ points, such that every row has a point, every column has two points, and the points in column j lie between rows j and $n+j$, both inclusive, $1 \leq j \leq n$. For example, the two Dellac configurations of order 2 are



Dellac's result [Del00] is that the number of Dellac configurations of order n is h_n .

Theorem 4.3. *The cardinality of G_{2n} is given by H_{2n-1} .*

Proof. Suppose $\pi \in G_{2n}$. Directly from the definition of G_n in Definition 4.1, we see that both $2i$ and $2i + 1$ have to lie in positions between $i + 1$ and $i + n$, both inclusive, for $1 \leq i \leq n - 1$. Therefore, interchanging $2i$ and $2i + 1$ will also give a permutation in G_{2n} , and this can be done independently for each i . Thus, $\#G_{2n}$ is divisible by 2^{n-1} .

Now, we focus on $\pi \in G_{2n}$ such that $\pi_{2i}^{-1} < \pi_{2i+1}^{-1}$ for $1 \leq i \leq n - 1$, i.e. those for which $2i$ precedes $2i + 1$ in one-line notation. We will now construct a bijection between such permutations and Dellac configurations of order $n - 1$. Again, from the definition of G_n in Definition 4.1, we have that $\pi_1 = 1, 2 \leq \pi_i \leq 2i - 1$ for $1 \leq i \leq n, 2(i - n) \leq \pi_i \leq 2n - 1$ for $n + 1 \leq i \leq 2n - 1$, and $\pi_{2n} = 2n$. Since the first and last entries are forced, we focus on $(\pi_2, \dots, \pi_{2n-1})$. Construct a configuration C of points on an $(2n - 2) \times (n - 1)$ array as follows: for $2 \leq i \leq 2n - 1$, place a point in position $(i - 1, \lfloor \pi_i/2 \rfloor)$. We claim that C is a Dellac configuration. First of all, it is obvious that there is one point in every row and two points in every column. Now, if $\lfloor \pi_i/2 \rfloor = j, \pi_i$ is either $2j$ and $2j + 1$ and in both cases, $j \leq i - 1 \leq j + n - 1$ (from the first paragraph of the proof). This is precisely the condition for C to be a Dellac configuration of order $n - 1$. For example, the permutation $(1, 2, 4, 3, 6, 5, 7, 8)$ is in bijection with the Dellac configuration

•		
	•	
•		
		•
	•	
		•

Conversely, given a Dellac configuration C of order $n - 1$, label the points in column j as $2j$ and $2j + 1$ from top to bottom for all $1 \leq j \leq n - 1$. Now, read the labels in each row from top to bottom to form the tuple $\pi = (\pi_2, \dots, \pi_{2n-1})$. By construction, this is a permutation of the set $\{2, \dots, 2n - 1\}$. Because of the Dellac constraint, $2j$ and $2j + 1$ lie between positions j and $n - 1 + j$ in π for each j . Prepending 1 and appending $2n$ to π therefore gives a permutation in G_{2n} , completing the proof. □

Remark 4.4. Even though several combinatorial interpretations are known for the median Genocchi numbers H_{2n-1} , it seems like G_{2n} is one of the few for which the divisibility by 2^{n-1} is manifest.

5. AUSOs of complete bipartite graphs and excedances

Recall the definition of excedance, acyclic orientation (AO) and acyclic unique sink orientation (AUSO) from Section 2. We denote the set of AUSOs of a graph G for which the unique sink is always the fixed vertex s as $\mathcal{A}(G, s)$. Recall also from Section 2 that the number of AUSOs with fixed sink is independent of the choice of sink. We therefore pay no mind to the actual choice

of vertex s in the following sections. In this section, we will prove Theorem 2.5 bijectively. The discussion will be made simpler by considering *topological sorts* of the vertices of an acyclic orientation.

Definition 5.1. Given a directed acyclic graph G on vertex set $[n]$, a *topological sort* of G is a total ordering τ on the vertices such that for every directed edge $e = (u, v)$, we have $\tau(u) < \tau(v)$. We will often say that τ is a topological sort of the *undirected* graph G , if there is an AO on G with topological sort τ .

Example 5.2. Notice that a given directed acyclic graph can have multiple topological sorts. For example,

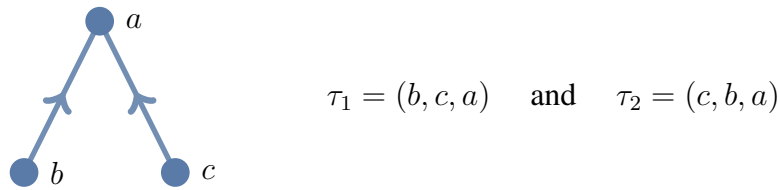


Figure 5.1: Directed acyclic graph with two different topological sorts τ_1 and τ_2 .

In a directed acyclic graph, we call two vertices *incomparable* if there is no directed path from one vertex to the other. Likewise, in a topological sort of an undirected graph G , two vertices are incomparable if they are incomparable in the AO induced by the sort. A topological sort of G in which every pair of vertices that are incomparable appear in increasing numerical order is called a *canonical topological sort* of G .

Lemma 5.3. *If K is a complete multipartite graph on vertex set $[n]$, then there is a bijection between AOs and canonical topological sorts of K .*

Proof. It is immediate that any canonical topological sort of the vertices of any graph G uniquely induces an AO on G .

Conversely, given an AO \mathcal{O} of K , we claim it has a canonical preimage. Two vertices are incomparable in \mathcal{O} if and only if they:

- (i) are in the same part of the partition of the vertices induced by K , and
- (ii) have identical in/out-neighborhoods.

Otherwise, there would be a directed path from one to the other. Thus, incomparability is an equivalence relation, and any topological sort of \mathcal{O} must order the equivalence classes with respect to one another. Any reordering of the vertices within an equivalence class is still a sort of \mathcal{O} , and so simply ordering each class in increasing order gives a canonical topological sort that induces AO \mathcal{O} on K . \square

For $0 \leq m \leq n - 1$, denote by $\mathcal{E}(n, m)$ the set of permutations in S_n with excedance set exactly $\{1, \dots, m\}$. That is, denoting the excedance set by $\text{ex}(\sigma)$,

$$\mathcal{E}(n, m) = \{\sigma \in S_n \mid \text{ex}(\sigma) = \{1, \dots, m\}\}.$$

While discussing the complete bipartite graph $K_{m,n}$, we refer to the ‘Left’ vertex set containing m vertices as $L = \{1, \dots, m\}$, and likewise the ‘Right’ vertex set containing n vertices as $R = \{m + 1, \dots, m + n\}$.

Denote by $\mathcal{R}(m, n)$ the set of AOs of $K_{m,n}$ in which there are no sinks in L (sinks only allowed in R). By simply adding a vertex to L to get $K_{m+1,n}$ and demanding this vertex be a sink, we maintain the acyclic property, and the added vertex becomes the unique sink. Removing the vertex gives the inverse, and the resulting bijection shows

$$|\mathcal{R}(m, n)| = |\mathcal{A}(K_{m+1,n}, s)|.$$

We will give a bijection between AUSOs $\mathcal{A}(K_{m+1,n}, s)$ and permutations with a fixed excedance set $\mathcal{E}(m + n, m)$. We use the discussion above to write the bijection in terms of the cycle decomposition of permutations in $\mathcal{E}(m + n, m)$ and orientations in $\mathcal{R}(m, n)$.

Let $f: \mathcal{E}(m + n, m) \rightarrow \mathcal{R}(m, n)$ be defined as follows. For $\sigma \in \mathcal{E}(m + n, m)$, construct $f(\sigma)$ by:

1. Write each cycle in the cycle decomposition of σ such that its least element appears first.
2. Order the cycles relative to each other from right to left by least element (disjoint cycles commute).
3. Remove the parentheses around the cycle decomposition as written in step (2), and consider the resulting sequence of vertices as a topological sort inducing an orientation on $K_{m,n}$.

Example 5.4. Suppose $m = 4, n = 5$, and we are given the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 9 & 6 & 7 & 5 & 2 & 4 & 8 & 1 \end{pmatrix} = (13629)(47)(5)(8).$$

It is easy to verify that $\sigma \in \mathcal{E}(9, 4)$. To construct $f(\sigma)$ notice that each cycle is already written according to step 1. Step 2 yields the decomposition $(\underline{8})(\underline{5})(\underline{47})(\underline{13629})$. Finally, step 3 gives the topological sort 8, 5, 4, 7, 1, 3, 6, 2, 9 of $K_{4,5}$ with the resulting orientation being acyclic and containing no sink in $L = \{1, 2, 3, 4\}$. We can check this by seeing that the last vertex in the topological sort is in R , so no vertex in L can be a sink.

Lemma 5.5. *The function f is well-defined.*

Proof. Consider any permutation σ . Since $f(\sigma)$ is an orientation determined by a topological sort, it must be acyclic. To check that f indeed gives an AO with no sink in L , write σ according to step 2 of the description of f as $\sigma = \sigma_1 \cdots \sigma_k$. The rightmost vertex v in the topological sort

$f(\sigma)$ is the final term in the cycle $\sigma_k = (\underline{1}, \dots, v)$. In particular, $\sigma(v) = 1$ and since 1 is the least element, $\sigma(v) \leq v$. Therefore $v \notin \text{ex}(\sigma)$, so

$$v \in (L \cup R) \setminus \text{ex}(\sigma) = \{m + 1, \dots, m + n\} = R.$$

Since v is the rightmost term of the topological sort, it is a sink. Since it is in R , it has all vertices in L as neighbors, and so no vertex in L can be a sink. \square

Proof of Theorem 2.5. We now prove that f is a bijection by constructing its inverse. Given an orientation $\mathcal{O} \in \mathcal{R}(m, n)$, construct $f^{-1}(\mathcal{O})$ as follows:

Since \mathcal{O} is acyclic, it has a unique canonical topological sort. For each equivalence class, from Lemma 5.3, of incomparable vertices in R , reverse its order. This defines a topological sort v_1, \dots, v_{m+n} , and there is a unique way to insert parentheses into the sequence to make it into a cycle decomposition written as specified in step 2 of the description of f . In particular, the rightmost cycle contains 1 and all elements to its right, the next cycle contains the next smallest unused element and all unused elements to its right, etc. Thus we have a permutation $\sigma = f^{-1}(\mathcal{O})$.

Observe our ordering of incomparable vertices and the fact that all elements of L are less than those of R implies that in the topological sort, $v_j < v_{j+1}$ if and only if $v_j \in L$ for all j . To see that σ has excedance set L as desired, it remains to check that for each cycle $(v_i \dots v_{i+k})$, we have $\sigma(v_j) > v_j$ if and only if $v_j \in L$ for $j = i, \dots, i + k$.

- For $j = i, \dots, i + k - 1$, this follows immediately from the observation that $v_j < v_{j+1}$ if and only if $v_j \in L$.
- For $j = i + k$, our choice of how to insert parentheses ensures that $\sigma(v_{i+k}) = v_i \leq v_{i+k}$ and also that the first element of the next cycle satisfies $v_{i+k} > v_{i+k+1}$. Therefore, the same observation gives $v_{i+k} \in R$. \square

Example 5.6. As an example of f^{-1} , we reverse the example we did while defining f (recall that $m = 4, n = 5$). We start with an orientation which has canonical topological sort 5, 8, 4, 7, 1, 3, 6, 2, 9 (notice that there is no sink in L) as in Figure 5.2.

Notice that in this orientation, vertices 5, 8 $\in R$ are incomparable, as well as vertices 1, 3 $\in L$. So, we reverse only the those incomparable vertices in R to get topological sort: 8, 5, 4, 7, 1, 3, 6, 2, 9. Now, we insert parentheses first left of 1, then left of 4, then 5, then 8 to obtain $f^{-1}(\mathcal{O}) = (8)(5)(47)(13629)$. Notice that for this permutation, $f^{-1}(f(\sigma)) = \sigma$, as we expected.

6. Enumeration of AOs and AUSOs of complete bipartite graphs

In this section, we give formulas for the number of acyclic orientations and acyclic orientations with unique sink of complete bipartite graphs. In Appendix A, we extend these techniques to give formulas for the number of AOs and AUSOs of all complete multipartite graphs.

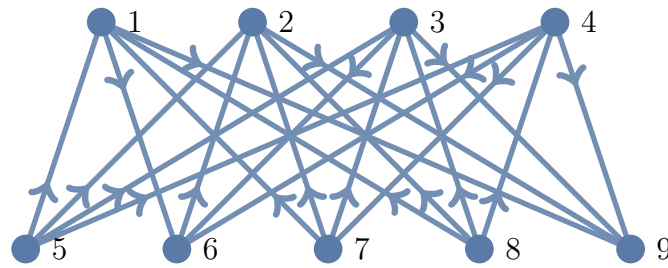


Figure 5.2: The AO \mathcal{O} on $K_{4,5}$. The topological sort specified in the description of f^{-1} is $(8, 5, 4, 7, 1, 3, 6, 2, 9)$, and this produces the permutation $(8)(5)(47)(13629)$.

We know from [ES00], that

$$\begin{aligned} |\mathcal{E}(m+n, m)| &= \sum_{i=1}^{m+1} (-1)^{m+1-i} \cdot i! \cdot i^{n-1} \cdot \left\{ \begin{matrix} m+1 \\ i \end{matrix} \right\} \\ &= \sum_{i=1}^n (-1)^{n-i} \cdot i! \cdot i^m \cdot \left\{ \begin{matrix} n \\ i \end{matrix} \right\} \\ &= n!n^m - \sum_{j=1}^{n-1} (-1)^{j-1} \cdot (n-j)! \cdot (n-j)^m \cdot \left\{ \begin{matrix} n \\ n-j \end{matrix} \right\}, \end{aligned}$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the Stirling number of the second kind, the second equality follows from the symmetry $\mathcal{E}(m+n, m) = \mathcal{E}(m+n, n-1)$ [ES00] (which itself isn't immediately obvious), and the third follows by pulling out the largest term of the sum, then reversing the order of summation.

Recall from Section 5 following Lemma 5.3 that if we let $\mathcal{R}(m, n)$ be the set of AOs of $K_{m,n}$ in which there are no sinks in L (the left vertex set), then there is a simple bijection showing $|\mathcal{R}(m, n)| = |\mathcal{A}(K_{m+1,n}, s)|$.

Theorem 6.1.

$$|\mathcal{R}(m, n)| = n!n^m - \sum_{j=1}^{n-1} (-1)^{j-1} \cdot (n-j)! \cdot (n-j)^m \cdot \left\{ \begin{matrix} n \\ n-j \end{matrix} \right\}.$$

Proof. Consider the graph $K'_{m,n}$, the complete bipartite graph with m labeled vertices $L = \{1, \dots, m\}$, and n unlabeled vertices $R = \{r, \dots, r\}$.

By Lemma 5.3, Counting the number of AOs on $K'_{m,n}$ such that there are no sinks in L can be done by counting canonical topological sorts of the vertices such that the final vertex in the sort is from R . To do this, first write out all n vertices from R . Then, for each $v \in L$, v can be placed in any of the n spaces between these vertices (not counting the final space, since we don't allow v to be a sink).

Once such a space is chosen, v 's ordering with respect to the other vertices of L in the same space is uniquely determined, since we are counting canonical sorts. Thus, we have n^m such AOs of $K'_{m,n}$.

$$_r_r_r_r \quad \longrightarrow \quad r13r2r4r$$

Now, we want to relabel the vertices of R , and count the resulting number of canonical topological sorts for $K_{m,n}$. For a given canonical topological sort of $K'_{m,n}$, there are $n!$ ways to relabel R , but some may result in a non-canonical sort.

Given any sort \mathcal{O} for $K'_{m,n}$, we may refer to the i 'th occurrence of r as r_i , and for a particular relabeling ℓ , the corresponding label is denoted $\ell(r_i)$. Also, let $r_i \sim r_{i+1}$ denote that they are incomparable (i.e. no vertex from L was placed in the space between them). Then, we may count the non-canonical sort/relabeling pairs as the union of sets:

$$A_i = \{(\mathcal{O}, \ell) \mid r_i \sim r_{i+1} \text{ and } \ell(r_i) > \ell(r_{i+1})\}$$

for each $i = 1, \dots, n-1$ (it is clear that every non-canonical sort/ relabeling pair must be in at least one such set). Then, inclusion-exclusion gives a way to count all non-canonical sort/relabeling pairs:

$$\left| \bigcup_{i=1}^{n-1} A_i \right| = \sum_{\emptyset \neq J \subseteq [n-1]} (-1)^{|J|-1} \left| \bigcap_{i \in J} A_i \right| = \sum_{j=1}^{n-1} (-1)^{j-1} \sum_{\substack{J \subseteq [n-1] \\ |J|=j}} \left| \bigcap_{i \in J} A_i \right|.$$

For fixed J , $\left| \bigcap_{i \in J} A_i \right|$ is the number of sort/label pairs such that for each $i \in J$, $r_i \sim r_{i+1}$, and within each group of incomparable vertices from R , the relabeling is in reverse order (by definition of A_i). This corresponds uniquely with an ordered partition of $\{\ell(r_1), \dots, \ell(r_n)\}$. In particular, for a fixed j , we would like to count the ways to partition $\{\ell(r_1), \dots, \ell(r_n)\}$ into $n-j$ non-empty parts, and order the parts with respect to one another. This is precisely a surjection from $[n]$ to $[n-j]$, so is counted by $(n-j)! \left\{ \begin{smallmatrix} n \\ n-j \end{smallmatrix} \right\}$ [Sta12]. Then, after choosing a J of size j and the labels for the r_i , we must complete the orientation by choosing the placement of vertices from L . Since we need $r_i \sim r_{i+1}$ for each $i \in J$, there are j spaces disallowed for placement of vertices from L , so there are $(n-j)^m$ choices of for each J . In total, we get that

$$\sum_{\substack{J \subseteq [n-1] \\ |J|=j}} \left| \bigcap_{i \in J} A_i \right| = (n-j)^m (n-j)! \left\{ \begin{smallmatrix} n \\ n-j \end{smallmatrix} \right\}$$

and therefore

$$\left| \bigcup_{i=1}^{n-1} A_i \right| = \sum_{j=1}^{n-1} (-1)^{j-1} (n-j)^m (n-j)! \left\{ \begin{smallmatrix} n \\ n-j \end{smallmatrix} \right\}.$$

Finally, as these are the non-canonical sort/labelling pairs, we subtract from the total number to get the canonical ones

$$\begin{aligned} |\mathcal{R}(m, n)| &= n! n^m - \left| \bigcup_{i=1}^{n-1} A_i \right| \\ &= n! n^m - \sum_{j=1}^{n-1} (-1)^{j-1} \cdot (n-j)! \cdot (n-j)^m \cdot \left\{ \begin{smallmatrix} n \\ n-j \end{smallmatrix} \right\}, \end{aligned}$$

completing the proof. □

7. Turán graphs and Extremality

Recall that the *Turán graph* $T(n, r)$ is a graph on n vertices, where the vertex set is partitioned into r subsets of sizes as close to each other as possible, and edges connecting vertices if and only if they belong to different subsets. To be precise, it is the complete multipartite graph of the form $K_{\lfloor n/r \rfloor, \lfloor n/r \rfloor, \dots, \lfloor n/r \rfloor, \lfloor n/r \rfloor}$ with r parts.

Let $u_{n,r} = |\mathcal{A}(T(n, r), s)|$ denote the number of acyclic orientations with unique sink for the Turán graph $T(n, r)$. Given a sequence $\{a_n\}$, define the difference operator δ by $\delta(\{a_n\}) = \{a_n - a_{n-1}\}$.

Theorem 7.1. *Fix an integer $k \geq 0$. Then $u_{r+k,r}$ is given by the $(r + k)$ 'th entry of the sequence $\delta^k\{(r + k - 1)!\}$ provided that $r \geq k$. In particular, for such r , $u_{r+k,r} = u_{r+k,r+1} - u_{r+k-1,r}$.*

Some data for the number of acyclic orientations with fixed sink is given in Table 7.1. Clearly, the rightmost column is the sequence $\{(n - 1)!\}$ because $T(n, n)$ is the complete graph K_n and there are $(n - 1)!$ such acyclic orientations. The column corresponding to $r = 2$ is covered by the discussion on bipartite graphs in Section 5.

$n \setminus r$	1	2	3	4	5	6	7
1	<u>1</u>						
2	<u>0</u>	<u>1</u>					
3	0	<u>1</u>	<u>2</u>				
4	0	<u>3</u>	<u>4</u>	<u>6</u>			
5	0	7	<u>14</u>	<u>18</u>	<u>24</u>		
6	0	31	<u>64</u>	<u>78</u>	<u>96</u>	<u>120</u>	
7	0	115	284	<u>426</u>	<u>504</u>	<u>600</u>	<u>720</u>

Table 7.1: The number of acyclic orientations with fixed sink of $T(n, r)$ for small values of n and r . The underlined numbers fall under the purview of Theorem 7.1.

To prove Theorem 7.1, we will appeal to a basic result about AUSOs. For a graph G containing an edge e , let $G \setminus e$ denote the graph G with the edge e removed and G/e denote the graph G with the edge e contracted. For a bijective proof of the following lemma, see Lemma 2.2 in [GS00].

Lemma 7.2. *For any graph G , we have that*

$$|\mathcal{A}(G, s)| = |\mathcal{A}(G \setminus e, s)| + |\mathcal{A}(G/e, s)|.$$

Proof of Theorem 7.1. For $r \geq k$, the graph $T_{r+k,r}$ has r parts with the first k parts having two vertices each and the remainder having one. Now, consider the graph G obtained by adding the edge e joining the two vertices in the k 'th part. The key observations are that $G = T_{r+k,r+1}$ and that $G/e = T_{r+k-1,r}$. By Lemma 7.2, we find that $u_{r+k,r} = u_{r+k,r+1} - u_{r+k-1,r}$. By appealing to the case of $k = 0$, for which the result is $(r - 1)!$, we obtain the proof.

Another way to prove Theorem 7.1 is starting with Theorem A.5 for the case $n_1 = 1$, $n_2 = \dots = n_k = 2$ and $n_{k+1} = \dots = n_r = 1$. Then the sum simplifies to (recalling for (j_2, \dots, j_k) our notation $|j| = \sum_{i=2}^k j_i$)

$$u_{r+k,r} = \sum_{(j_2, \dots, j_k) \in [2]^{k-1}} (-1)^{|j|} (|j| + r - k) (|j| + r - k)!. \quad (7.1)$$

Now, using (7.1), we can rewrite by summing over the j_k variable to obtain

$$\begin{aligned} u_{r+k,r} &= \sum_{(j'_2, \dots, j'_{k-1}) \in [2]^{k-2}} (-1)^{|j'|} \left[\begin{aligned} &- (|j'| + r - k + 1) (|j'| + r - k + 1)! \\ &+ (|j'| + r - k + 2) (|j'| + r - k + 2)! \end{aligned} \right] \\ &= -u_{r+k-1,r} + u_{r+k,r+1}, \end{aligned}$$

proving the result. \square

7.1. Extremal Graphs for AOs

We conclude this contribution by addressing the following extremal problem on the number of acyclic orientations: given a fixed number n of vertices, and m of edges, which graph(s) maximize the number of AOs?

It is conjectured by Cameron, Glass, and Schumacher [CGS14] that Turán graphs with two parts maximize the number of AOs over graphs with the same number of vertices and edges. We prove here that Turán graphs with parts of size at most 2 are also maximizers.

Lemma 7.3. *Let G be any graph containing an edge $e = (a, b)$ such that*

$$N(a) \setminus \{b\} \supseteq N(b) \setminus \{a\},$$

where N denotes the set of neighboring vertices. Then, for any vertex c such that the edge $e' = (c, b)$ is not in G , $|\mathcal{A}(G)| \leq |\mathcal{A}(G \setminus e + e')|$.

Proof. Let $G' = G \setminus e + e'$. We know by the deletion-contraction recurrence for AOs that $|\mathcal{A}(G)| = |\mathcal{A}(G \setminus e)| + |\mathcal{A}(G/e)|$ (equivalently for G' and e'). But notice that $G \setminus e = G' \setminus e'$, so we just need show that $|\mathcal{A}(G/e)| \leq |\mathcal{A}(G'/e')|$.

First, $G/e \cong G \setminus \{b\}$, since the new vertex ab resulting from contracting $e = (a, b)$ has neighborhood

$$N(ab) = (N(a) \setminus \{b\}) \cup (N(b) \setminus \{a\}) = N(a) \setminus \{b\},$$

while the rest of G is unchanged. On the other hand, G'/e' has new vertex bc resulting from contracting $e' = (c, b)$ with the neighborhood

$$N(bc) = (N(b) \setminus \{c\}) \cup (N(c) \setminus \{b\}) \supseteq N(c) \setminus \{b\}.$$

Therefore, $G'/e' \supseteq G \setminus \{b\}$, as the neighborhood of bc is no smaller than that of c . This gives the desired

$$G/e \subseteq G'/e' \implies |\mathcal{A}(G/e)| \leq |\mathcal{A}(G'/e')| \implies |\mathcal{A}(G)| \leq |\mathcal{A}(G')|. \quad \square$$

Theorem 7.4. For $m \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$, the graph whose complement is a matching maximizes the number of AOs over all graphs on the same number of vertices and edges.

Proof. Lemma 7.3 can be interpreted in the complement: if there is an edge $e = (c, b)$ in \overline{G} , and a vertex a with $N_{\overline{G}}(a) \subseteq N_{\overline{G}}(b)$, then we can replace e with $e' = (a, b)$ in the complement without decreasing the number of AOs in G .

In particular, if a is an isolated vertex, we can always “slide” any edge to a . If there are at most $\lfloor \frac{n}{2} \rfloor$ edges in the complement and it is not a matching, then it has an isolated vertex. So for any graph whose complement is not a matching, there is a series of edge slides which do not decrease the number of AOs, eventually resulting in a graph whose complement is a matching. \square

Remark 7.5. The complement of a matching is a Turán graph with parts of sizes 1 or 2, so Theorem 7.4 proves the claim.

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A. Enumerating AOs of Complete Multipartite Graphs

Here, we consider AOs of complete multipartite graphs K_{n_1, \dots, n_N} . We refer to the vertex sets in the separate parts as V_1, \dots, V_N . To simplify notation, we set $|k| = \sum_{i=2}^N k_i$ for any $k = (k_2, \dots, k_N) \in \mathbb{N}^{N-1}$.

Lemma A.1. *Consider the complete N -partite graph K'_{n_1, \dots, n_N} with the vertices in each of the n_2, \dots, n_N -sets unlabeled within their vertex set. Then*

$$|\mathcal{A}(K'_{n_1, \dots, n_N})| = (1 + |n|)^{n_1} \cdot \binom{|n|}{n_2, \dots, n_N}.$$

Proof. The proof idea is a generalization of Theorem 6.1. We refer to the unlabeled vertex sets as V_2, \dots, V_N , and the labeled n_1 -set as V_1 . We will count ways to construct an AO of K'_{n_1, \dots, n_N} by counting canonical topological sorts of K'_{n_1, \dots, n_N} . First the edges disjoint from V_1 will be oriented, then all edges containing a vertex from V_1 will be oriented.

To orient edges disjoint from V_1 , we pick the relative order in the topological sort of vertices in $V_2 \cup \dots \cup V_N$. Since these vertices are unlabeled (within their respective vertex sets), we will collectively refer to any vertex in V_i with the label v^i . We can order them by writing a sequence containing n_2 copies of v^2 , n_3 copies of v^3 , and so on until n_N copies of v^N . The number of such sequences is given by the multinomial coefficient

$$\binom{|n|}{n_2, \dots, n_N}.$$

Next, given any such sequence S , to pick the orientations for the edges containing vertices in V_1 , we simply pick where to insert each vertex $a_1, \dots, a_{n_1} \in V_1$ into S to create a canonical topological sort. S is of length $|n|$, so there are $1 + |n|$ “spaces” in which we can place a_1, \dots, a_{n_1} (including the spaces on the far left and far right). Notice that if two a_i, a_j get placed in the same space, then they are incomparable—their order within that space is determined since we are counting canonical sorts. So, we simply choose a space for each vertex in V_1 , giving $(1 + |n|)^{n_1}$ choices.

The equivalence between canonical sorts and AOs of K_{n_1, \dots, n_N} given in Lemma 5.3 completes the proof. □

Remark A.2. Notice that we could easily alter this proof to count AOs of K'_{n_1, \dots, n_N} such that there are no sinks in vertex set V_1 . The only modification needed is to disallow placement of vertices from V_1 in the rightmost space of the topological sort. This means that there are only $|n|$ choices for each $a \in V_1$, which gives the total number of such AOs as

$$|n|^{n_1} \cdot \binom{|n|}{n_2, \dots, n_N}.$$

This will allow us to easily count the number of AUSOs of multipartite graphs as well.

Theorem A.3. $|\mathcal{A}(K_{n_1, \dots, n_N})|$, the number of AOs of the complete N -partite graph K_{n_1, \dots, n_N} is given by

$$\sum_{(k_2, \dots, k_N) \in \mathcal{K}} (-1)^{|n| - |k|} (1 + |k|)^{n_1} |k|! \prod_{i=2}^N \left\{ \begin{matrix} n_i \\ k_i \end{matrix} \right\},$$

where $\mathcal{K} = [n_2] \times [n_3] \times \dots \times [n_N]$.

Proof. First, we write an equivalent formula by reversing the order of summation along each $[n_i]$. Let $\mathcal{J} = \{0, \dots, n_2 - 1\} \times \dots \times \{0, \dots, n_N - 1\}$, and make the substitutions $j_i = n_i - k_i$. We will instead show that $|\mathcal{A}(K_{n_1, \dots, n_N})|$ is given by

$$\sum_{(j_2, \dots, j_N) \in \mathcal{J}} (-1)^{|j|} \cdot (1 + |n| - |j|)^{n_1} (|n| - |j|)! \cdot \prod_{i=2}^N \left\{ \begin{matrix} n_i \\ n_i - j_i \end{matrix} \right\}.$$

Lemma A.1 gives a way to count $|\mathcal{A}(K'_{n_1, \dots, n_N})|$, and we may consider all *relabelings* of vertices in sets V_2, \dots, V_N . There are $\prod_{i=2}^N n_i!$ such relabelings, and this gives a multiset \mathcal{A} of AOs of K_{n_1, \dots, n_N} of size

$$|\mathcal{A}| = |\mathcal{A}(K'_{n_1, \dots, n_N})| \cdot \prod_{i=2}^N n_i! = (1 + |n|)^{n_1} \cdot \binom{|n|}{n_2, \dots, n_N} \cdot \prod_{i=2}^N n_i!$$

If we denote the set of all relabelings as \mathcal{L} , then there is an obvious correspondence between \mathcal{A} and $\mathcal{A}(K'_{n_1, \dots, n_N}) \times \mathcal{L}$. It is also clear that eliminating duplicates in the multiset would exactly yield $\mathcal{A}(K_{n_1, \dots, n_N})$. In particular, using the equivalence of Lemma 5.3, some of the topological sorts resulting from relabeling are non-canonical. We will count how many are non-canonical in a similar manner to Theorem 6.1.

For any canonical topological sort (an orientation) \mathcal{O} of K'_{n_1, \dots, n_N} , we can refer to the k 'th occurrence of v^i , a vertex from the unlabeled set V_i , as v_k^i . Then, if ℓ is a relabeling, we denote the label of this vertex as $\ell(v_k^i)$. Let $v_k^i \sim v_{k+1}^i$ denote that these two unlabeled vertices (from the same vertex set) are adjacent in the sort. That is to say, they are incomparable. Then, a non-canonical relabeling of \mathcal{O} must have some $v_k^i \sim v_{k+1}^i$ with $\ell(v_k^i) > \ell(v_{k+1}^i)$.

Thus, for each vertex set V_2, \dots, V_N and each vertex $v_k^i \in V_i$ consider the set of non-canonical orientations / labeling pairs:

$$\mathcal{B}_{V_i, k} = \{(\mathcal{O}, \ell) \in \mathcal{A}(K'_{n_1, \dots, n_N}) \times \mathcal{L} \mid v_k^i \sim v_{k+1}^i, \ell(v_k^i) > \ell(v_{k+1}^i)\}.$$

The union of these sets is the set of all non-canonical orientations / labeling pairs in \mathcal{A} . We will count this with inclusion-exclusion. Let

$$\mathcal{I} = \bigcup_{i=2}^N \{(V_i, k) \mid 1 \leq k < n_i\}.$$

Then the principle of inclusion exclusion gives that the number of non-canonical pairs is

$$\sum_{\emptyset \neq J \subseteq \mathcal{I}} (-1)^{|J|-1} \left| \bigcap_{(V,k) \in J} \mathcal{B}_{V,k} \right|.$$

Analogously to Theorem 6.1, we first consider all J such that j_2 pairs in V_2 are incomparable, j_3 pairs in V_3 are incomparable, \dots , j_N pairs in V_N are incomparable for some fixed j_2, \dots, j_N . We can look at each vertex set independently and use Stirling numbers of the second kind to count ordered partitions of the labels to see that in total, the number of ways to choose j_i indices in vertex set V_i and relabel to get something in $\bigcap_{(V,k) \in J} \mathcal{B}_{V,k}$ is

$$\prod_{i=2}^N (n_i - j_i)! \left\{ \begin{matrix} n_i \\ n_i - j_i \end{matrix} \right\}.$$

Then, for each such J , regardless of the indices and labelling, the number of topological sorts can be counted identically to Lemma A.1 by first ordering the groups of incomparable vertices in $V_2 \cup \dots \cup V_N$, then selecting a space for each $a_1, \dots, a_{n_1} \in V_1$. There are

$$(1 + |n| - |j|)^{n_1} \cdot \binom{|n| - |j|}{n_2 - j_2, \dots, n_N - j_N}$$

ways. Therefore, if for $J \subseteq \mathcal{I}$, we denote $J_i = J \cap \{(V_i, k) \mid 1 \leq k < n_i\}$, then for any fixed j_2, \dots, j_N ,

$$\sum_{\substack{J \subseteq \mathcal{I} \\ \forall i, |J_i| = j_i}} \left| \bigcap_{(V,k) \in J} \mathcal{B}_{V,k} \right| = (1 + |n| - |j|)^{n_1} (|n| - |j|)! \prod_{i=2}^N \left\{ \begin{matrix} n_i \\ n_i - j_i \end{matrix} \right\}.$$

This lets us rewrite the inclusion-exclusion as

$$\sum_{(j_2, \dots, j_N) \in \mathcal{J}'} (-1)^{|j|-1} (1 + |n| - |j|)^{n_1} (|n| - |j|)! \prod_{i=2}^N \left\{ \begin{matrix} n_i \\ n_i - j_i \end{matrix} \right\},$$

where $\mathcal{J}' = \mathcal{J} \setminus \{(0, \dots, 0)\}$. Subtracting this from $|\mathcal{A}|$, we get the desired result. □

Remark A.4. Richard Stanley kindly suggested to us that an alternative approach to this enumeration problem might be to use the generating function for the chromatic polynomial $\chi(q)$ of

complete multipartite graphs K_{n_1, \dots, n_k} for fixed $k \geq 1$ [Sta99, Solution to Problem 5.6]. Differentiating this expression with respect to q and setting $q = 0$ to get (up to sign) the number of AOs with a fixed sink gives the recurrence

$$|\mathcal{A}(K_{n_1, \dots, n_N})| + \sum_{\ell=0}^{n_1-1} (-1)^{n_1-\ell} \binom{n_1}{\ell} |\mathcal{A}(K_{\ell, n_2, \dots, n_N})| + \dots + \sum_{\ell=0}^{n_N-1} (-1)^{n_N-\ell} \binom{n_N}{\ell} |\mathcal{A}(K_{n_1, \dots, n_{N-1}, \ell})| = 0.$$

Theorem A.5. *The number of AUSOs with fixed sink of $K_{n_1+1, n_2, \dots, n_N}$ is given by*

$$|\mathcal{A}(K_{n_1+1, n_2, \dots, n_N}, s)| = \sum_{(k_2, \dots, k_N) \in \mathcal{K}} (-1)^{|n|-|k|} \cdot |k|^{n_1} |k|! \prod_{i=2}^N \begin{Bmatrix} n_i \\ k_i \end{Bmatrix}.$$

Proof. The proof is identical by just counting the number of AOs of K_{n_1, \dots, n_N} which have no sinks in vertex set V_1 , and using the remark after Lemma A.1. \square