

# A NOTE ON SATURATION FOR $k$ -WISE INTERSECTING FAMILIES

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**Abstract.** A family  $\mathcal{F}$  of subsets of  $\{1, \dots, n\}$  is called  $k$ -wise intersecting if any  $k$  members of  $\mathcal{F}$  have non-empty intersection, and it is called maximal  $k$ -wise intersecting if no family strictly containing  $\mathcal{F}$  satisfies this condition. We show that for each  $k \geq 2$  there is a maximal  $k$ -wise intersecting family of size  $O(2^{n/(k-1)})$ . Up to a constant factor, this matches the best known lower bound, and answers an old question of Erdős and Kleitman, recently studied by Hendrey, Lund, Tompkins, and Tran.

**Keywords.** Intersecting family, saturation, set system

**Mathematics Subject Classifications.** 05D05

## 1. Introduction

Given positive integers  $k \geq 2$  and  $n$ , we say that a family  $\mathcal{F}$  of subsets of  $[n] = \{1, 2, \dots, n\}$  is  *$k$ -wise intersecting* if whenever  $X_1, \dots, X_k \in \mathcal{F}$  then  $X_1 \cap \dots \cap X_k \neq \emptyset$ . It is well known (and easy to see) that any (2-wise) intersecting family over  $[n]$  has size at most  $2^{n-1}$ , so the largest possible size of a  $k$ -wise intersecting family is clearly  $2^{n-1}$  for all  $k$ . This is achieved, for example, by taking  $\mathcal{F} = \{A \in \mathcal{P}([n]) : 1 \in A\}$ , where  $\mathcal{P}(X)$  denotes the set of subsets of  $X$ .

However, the corresponding saturation problem of finding the smallest possible size of a maximal  $k$ -wise intersecting family is more interesting for  $k \geq 3$ . (A family  $\mathcal{F}$  of subsets of  $[n]$  is *maximal  $k$ -wise intersecting* if it is  $k$ -wise intersecting but no family  $\mathcal{F}'$  over  $[n]$  strictly containing  $\mathcal{F}$  is  $k$ -wise intersecting.) This problem was originally raised by Erdős and Kleitman [EK74] in 1974. Very recently, Hendrey, Lund, Tompkins, and Tran [HLTT21] studied this problem for  $k = 3$ . They determined the smallest possible size of a maximal 3-wise intersecting family exactly when  $n$  is sufficiently large and even. For general  $k$ , they showed that the smallest possible size  $f_k(n)$  of a maximal  $k$ -wise intersecting family satisfies

$$c_k \cdot 2^{n/(k-1)} \leq f_k(n) \leq d_k \cdot 2^{n/\lceil k/2 \rceil}$$

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(for some constants  $c_k, d_k > 0$ ). They asked about closing the exponential gap between the lower and upper bounds.

In this note we prove the following result, which shows that the lower bound gives the right order of magnitude.

**Theorem 1.1.** *For each  $k \geq 3$  there exists some constant  $C_k$  such that for all  $n$  there is a maximal  $k$ -wise intersecting family over  $[n]$  of size at most  $C_k \cdot 2^{n/(k-1)}$ .*

In the case when  $n \geq 2(k-1)$  is a multiple of  $k-1$ , the exact value of our upper bound will be  $2^{n/(k-1)+k-3}(k-1) - (2^{k-1} - 1)(k-2)$ . In the special case  $k = 3$  this upper bound is  $2^{n/2+1} - 3$ , which is tight for  $n$  sufficiently large (as shown by Hendrey, Lund, Tompkins, and Tran [HLTT21]), but for  $k \geq 4$  the construction has more complicated structure than for  $k = 3$ . In fact, in [HLTT21] it was shown that for  $k = 3$  (and  $n$  large and even), the unique maximal 3-wise intersecting families of smallest possible size are given by  $\{A^c : A \in (\mathcal{P}(X) \setminus \{X\}) \cup (\mathcal{P}(Y) \setminus \{Y\})\}$  for some partition  $X \cup Y$  of  $[n]$  into two equal parts. This was proved by first obtaining a stability result stating that for any ‘small’ maximal 3-wise intersecting family  $\mathcal{F}$ , the family  $\bar{\mathcal{F}} = \{A^c : A \in \mathcal{F}\}$  must be ‘close’ to the union of two cubes  $\mathcal{P}(X) \cup \mathcal{P}(Y)$  (with  $X, Y$  as above).

However, the following result shows that, for  $k \geq 4$ , directly generalising this approach cannot work, and it is necessary to have more complicated structure.

**Lemma 1.2.** *Let  $k \geq 4$ , let  $X_1 \cup \dots \cup X_{k-1}$  be a partition of  $[n]$  with each  $X_i$  having size  $n/(k-1) + O(1)$ , and let  $Q = \mathcal{P}(X_1) \cup \dots \cup \mathcal{P}(X_{k-1})$ . If  $|\mathcal{F} \setminus Q| = o(2^{n/(k-1)})$  and  $n$  is large enough, then  $\bar{\mathcal{F}} = \{A^c : A \in \mathcal{F}\}$  cannot be maximal  $k$ -wise intersecting.*

We mention that many other saturation problems have already been studied in the context of set systems and intersection properties. For example, several authors gave bounds for the smallest possible size  $m(r)$  of a set system which is maximal among (2-wise) intersecting families  $\mathcal{F} \subseteq \binom{[n]}{r}$  consisting of sets of size  $r$  – see, for example, [DDFL85, Blo87, BFK89]. A linear lower bound follows from a result of Erdős and Lovász [EL73], and Dow, Drake, Füredi, and Larson [DDFL85] showed that in fact  $m(r) \geq 3r$  for  $r \geq 4$ . Blokhuis [Blo87] proved a polynomial upper bound of  $m(r) \leq r^5$ , and for certain values of  $r$  quadratic upper bounds are also known – see, e.g., [Blo87, BFK89]. Finding the order of magnitude of  $m(r)$  is still an open problem. See the introduction and the references in [HLTT21] for other related saturation problems.

## 2. The construction

We now prove Theorem 1.1 by describing a family of size  $O(2^{n/(k-1)})$  and showing that it is maximal  $k$ -wise intersecting over  $[n]$ . For simplicity, we will work with complements, using the observation that  $\bar{\mathcal{G}} = \{X^c : X \in \mathcal{G}\}$  is  $k$ -wise intersecting if and only if no  $k$  elements of  $\mathcal{G}$  have union  $[n]$ .

Fix some  $k \geq 3$  and  $n \geq 2(k-1)$  integers. Partition  $[n]$  into  $k-1$  sets  $A_1, \dots, A_{k-1}$  which are as close as possible in size, and pick ‘special’ elements  $a_i \in A_i$  for each  $i$ . Consider the

following families. (All indices will be understood mod  $k - 1$ , so, for example,  $a_0 = a_{k-1}$ .)

$$\begin{aligned} \mathcal{F}_1(i) &= \mathcal{P}(A_i) \setminus \{A_i\} \\ \mathcal{F}_2(i) &= \{X \cup Y : X \subseteq A_i, Y \subseteq \{a_1, a_2, \dots, a_{k-1}\} \setminus \{a_{i-1}, a_i\}, X \neq A_i \setminus \{a_i\}, X \neq A_i\} \\ \mathcal{F} &= \bigcup_{i=1}^{k-1} (\mathcal{F}_1(i) \cup \mathcal{F}_2(i)). \end{aligned}$$

We will show that  $\{A^c : A \in \mathcal{F}\}$  is maximal  $k$ -wise intersecting. Note that if  $k = 3$  then  $\mathcal{F}_2(i) \subseteq \mathcal{F}_1(i)$  for each  $i = 1, 2$ , so  $\mathcal{F}$  is simply  $\mathcal{F}_1(1) \cup \mathcal{F}_1(2) = (\mathcal{P}(A_1) \cup \mathcal{P}(A_2)) \setminus \{A_1, A_2\}$ . This was shown to be (up to isomorphism) the unique minimal-sized construction when  $k = 3$  and  $n$  is even and sufficiently large by Hendrey, Lund, Tompkins, and Tran [HLTT21]. Furthermore, note that

$$|\mathcal{F}| = 2^{n/(k-1)+k-3}(k-1) - (2^{k-1} - 1)(k-2)$$

if  $k - 1$  divides  $n$ . Indeed, the number of subsets of  $\{a_1, \dots, a_{k-1}\}$  appearing in  $\mathcal{F}$  is  $2^{k-1} - 1$ , the number of sets which are not subsets of  $\{a_1, \dots, a_{k-1}\}$  but appear in some  $\mathcal{F}_2(i)$  is  $(k-1) \cdot (2^{n/(k-1)} - 4) \cdot 2^{k-3}$ , and finally, the only elements of  $\mathcal{F}$  we have not yet counted are the  $(k-1)$  sets  $A_i \setminus \{a_i\}$  for  $i = 1, \dots, k-1$ . Summing these contributions gives the formula above.

**Claim 2.1.** *The family  $\mathcal{F}$  contains no  $k$  elements having union  $[n]$ .*

*Proof.* Suppose we have sets  $Y_1, \dots, Y_k \in \mathcal{F}$  satisfying  $Y_1 \cup \dots \cup Y_k = [n]$ . Then clearly at least one of them must come from some  $\mathcal{F}_2(i)$ , we may assume  $Y_1 \in \mathcal{F}_2(1)$ . If there is some  $j \neq 1$  and  $i \neq 1$  such that  $Y_j \in \mathcal{F}_2(i)$ , then for each  $t$  we can pick  $b_t \in A_t \setminus \{a_t\}$  such that  $b_t \notin Y_1 \cup Y_j$ . Then no element of  $\mathcal{F}$  contains more than one  $b_t$ , but  $\{b_1, \dots, b_{k-1}\} \subseteq \bigcup_{\ell \neq 1, j} Y_\ell$ , giving a contradiction. On the other hand, if there is no such pair  $(i, j)$ , then  $\{Y_\ell : \ell \neq 1\}$  must contain at least one non-empty set from  $\mathcal{F}_1(t)$  (to cover  $A_t \setminus \{a_t\}$ ) for  $t = 2, \dots, k-2$ , at least two different sets from  $\mathcal{F}_1(k-1)$  (to cover  $A_{t-1}$ ), and at least one set having an element in  $A_1$ , again giving a contradiction since these  $k$  sets must all be different.  $\square$

**Claim 2.2.** *For any  $X \in \mathcal{P}([n]) \setminus \mathcal{F}$  there exist  $X_1, \dots, X_{k-1} \in \mathcal{F}$  such that  $X_1 \cup X_2 \cup \dots \cup X_{k-1} \cup X = [n]$ .*

*Proof.* We first consider the following five cases, then check that each choice of  $X$  belongs to at least one of these cases.

- **Case 1:**  $X \cap A_i \neq \emptyset$  for all  $i$ . Then let  $X_i = A_i \setminus X$ , so  $X_i \in \mathcal{F}_1(i)$  and the  $X_i$  satisfy the conditions.
- **Case 2:** There is some  $i$  such that  $X \cap A_i$  contains an element  $b_i$  with  $b_i \neq a_i$  and  $X \cap A_{i-1} \neq \emptyset$ . We may assume that  $i = 1$ . Then let  $X_1 = (A_1 \setminus \{b_1\}) \cup \{a_2, a_3, \dots, a_{k-2}\}$  (so  $X_1 \in \mathcal{F}_2(1)$ ), let  $X_j = A_j \setminus \{a_j\}$  for  $j = 2, \dots, k-2$  (so  $X_j \in \mathcal{F}_1(j)$ ) and let  $X_{k-1} = A_{k-1} \setminus X$  (so  $X_{k-1} \in \mathcal{F}_1(k-1)$ ). These clearly satisfy the conditions.

- Case 3: There exist  $i, j, b_i, b_j$  such that  $i \neq j, b_i \in X \cap (A_i \setminus \{a_i\}), b_j \in X \cap (A_j \setminus \{a_j\})$ . Then let  $X_i = (A_i \setminus X) \cup \{a_\ell : \ell \neq i, i-1\}$ ,  $X_j = (A_j \setminus X) \cup \{a_\ell : \ell \neq j, j-1\}$ , and  $X_\ell = A_\ell \setminus \{a_\ell\}$  for  $\ell \neq i, j$ . Then  $X_i \in \mathcal{F}_2(i), X_j \in \mathcal{F}_2(j)$ , and  $X_\ell \in \mathcal{F}_1(\ell)$  for  $\ell \neq i, j$ , so it is easy to see that the conditions are satisfied.
- Case 4:  $X \supseteq (A_i \setminus \{a_i\}) \cup \{a_j\}$  for some  $i, j$  with  $j \neq i$ . Then let  $X_i = \{a_t : t \neq j\}$  (so  $X_i \in \mathcal{F}_2(j+1)$ ), and let  $X_\ell = A_\ell \setminus \{a_\ell\}$  for  $\ell \neq i$  (so  $X_\ell \in \mathcal{F}_1(\ell)$ ). It is easy to see that the conditions are satisfied.
- Case 5:  $X = A_i$  for some  $i$ . Then let  $X_i = \{a_t : t \neq i\}$  (so  $X_i \in \mathcal{F}_2(i+1)$ ) and  $X_\ell = A_\ell \setminus \{a_\ell\}$  for  $\ell \neq i$  (so  $X_\ell \in \mathcal{F}_1(\ell)$ ). Then the conditions are again satisfied.

We now check that any  $X \in \mathcal{P}([n]) \setminus \mathcal{F}$  belongs to at least one of these cases. If  $X \subseteq \{a_1, \dots, a_{k-1}\}$  then in fact  $X = \{a_1, \dots, a_{k-1}\}$  and we are in Case 1. Otherwise there is some  $i$  and some  $b_i \in A_i$  such that  $b_i \in X$  and  $b_i \neq a_i$ . If  $X \cap A_{i-1} \neq \emptyset$  then we are in Case 2. Otherwise, if  $X$  is not a subset of  $A_i \cup \{a_\ell : \ell \neq i, i-1\}$  then we are in Case 3. Finally, if  $X$  is a subset of  $A_i \cup \{a_\ell : \ell \neq i, i-1\}$ , then we are in Case 4 or Case 5 since  $X \notin \mathcal{F}_1(i) \cup \mathcal{F}_2(i)$ .  $\square$

*Proof of Theorem 1.1.* By Claims 2.1 and 2.2, the family  $\bar{\mathcal{F}} = \{X^c : X \in \mathcal{F}\}$  is maximal  $k$ -wise intersecting for all  $k \geq 3$  and  $n \geq 2(k-1)$ . However,  $\mathcal{F}_1(i)$  and  $\mathcal{F}_2(i)$  both have size  $O_k(2^{n/(k-1)})$  for each  $i$ , so  $|\mathcal{F}| = O_k(2^{n/(k-1)})$ . The result follows.  $\square$

### 3. Non-existence of certain types of maximal families

Finally, we prove Lemma 1.2 stating that there can be no construction for  $k \geq 4$  which is close to the union of  $k-1$  cubes.

*Proof of Lemma 1.2.* Suppose that  $\mathcal{F}$  is as in the statement of the lemma, and  $\bar{\mathcal{F}}$  is maximal  $k$ -wise intersecting. Then at least one  $X_i$  does not appear in  $\mathcal{F}$ , we may assume  $X_{k-1} \notin \mathcal{F}$ . Let  $\mathcal{G}$  be the family of sets  $S$  over  $[n]$  satisfying the following conditions.

- The set  $S$  cannot be written as  $S_1 \cup \dots \cup S_{k-1}$  such that  $S_i \in \mathcal{F}$  for all  $i$  and  $S_1 \in \mathcal{F} \setminus Q$ .
- For all  $S' \in \mathcal{F} \setminus Q$  and  $i \leq k-2$ , we have  $S' \cap X_i \neq S \cap X_i$ .
- For all  $i$ ,  $S \cap X_i$  is non-empty.
- We have  $S^c \cap (X_1 \cup \dots \cup X_{k-2}) \notin \mathcal{F}$ .

It is easy to see that each of these conditions fails for only  $o(2^n)$  subsets of  $[n]$ , so  $\mathcal{G}$  is non-empty. Pick any  $S \in \mathcal{G}$ , and let  $T = S^c \cap (X_1 \cup \dots \cup X_{k-2})$ . Then  $T \notin \mathcal{F}$ , so (by maximality) there are  $T_1, \dots, T_{k-1} \in \mathcal{F}$  such that  $T \cup T_1 \cup T_2 \cup \dots \cup T_{k-1} = [n]$ . Furthermore, by maximality,  $\mathcal{F}$  must be a down-set, so we may assume that in fact  $T^c = T_1 \cup \dots \cup T_{k-1}$ . So  $S \cup X_{k-1} = T_1 \cup \dots \cup T_{k-1}$ . Write  $S_i = T_i \setminus (X_{k-1} \setminus S)$ , then  $S = S_1 \cup \dots \cup S_{k-1}$  and  $S_i \in \mathcal{F}$  for all  $i$  (as  $\mathcal{F}$  is a down-set). So we must have  $S_1, \dots, S_{k-1} \in Q \cap \mathcal{F}$ . Since  $S \cap X_i$  is non-empty for all  $i$ , we may assume that  $S_i = S \cap X_i$  for all  $i$ . Then, for  $i \leq k-2$ ,  $T_i \cap X_i = S \cap X_i$ , so  $T_i \notin \mathcal{F} \setminus Q$ . Hence  $T_i \in Q$  and  $T_i = S_i$ . But then  $T_{k-1} = X_{k-1}$ , giving a contradiction.  $\square$

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