

TRIANGULATIONS, ORDER POLYTOPES, AND GENERALIZED SNAKE POSETS

Matias von Bell^{*1}, Benjamin Braun^{†2}, Derek Hanely^{‡3}, Khrystyna Serhiyenko^{§4},
Julianne Vega⁵, Andrés R. Vindas-Meléndez^{¶6}, and Martha Yip^{||7}

¹*Institute of Geometry, Graz University of Technology, Austria*
matias.vonbell@gmail.com

^{2,4,7}*Department of Mathematics, University of Kentucky, U.S.A.*
benjamin.braun@uky.edu, *khrystyna.serhiyenko@uky.edu*, *martha.yip@uky.edu*

³*Department of Mathematics, Penn State Behrend, U.S.A.*
derek.hanely@psu.edu

⁵*Maret School, Whashington D.C., U.S.A.*
jvega@maret.org

⁶*Department of Mathematics, University of California, Berkeley, U.S.A.*
andres.vindas@berkeley.edu

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Abstract. This work regards the order polytopes arising from the class of generalized snake posets and their posets of meet-irreducible elements. Among generalized snake posets of the same rank, we characterize those whose order polytopes have minimal and maximal volume. We give a combinatorial characterization of the circuits in related order polytopes and then conclude that all of their triangulations are unimodular. For a generalized snake word, we count the number of flips for the canonical triangulation of these order polytopes. We determine that the flip graph of the order polytope of the poset whose lattice of upper order ideals comes from a ladder is the Cayley graph of a symmetric group. Lastly, we introduce an operation on triangulations called twists and prove that twists preserve regular triangulations.

Keywords. Order polytopes, triangulations, flow polytopes, circuits

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1. Introduction

In 1986, Richard Stanley [Sta86] introduced two geometric objects associated to a finite partially ordered set, or *poset*, known as the order polytope and the chain polytope. Since then, the study of order polytopes has been an active area of research, e.g., geometric and algebraic properties [DR16, HKT20, HLL⁺19, HM16, HMOS15], connections between flow polytopes and order polytopes [LMSD19, MMS19], and lattice-point enumeration [CFS17, LT19].

One of Stanley’s fundamental observations is that the arrangement given by all hyperplanes of the form $x_i = x_j$ for $i \neq j$ induces a regular unimodular triangulation of the order polytope for any poset. This triangulation is known as the *canonical triangulation* of an order polytope, see Subsection 2.2. It is well-known that the set of all regular triangulations of a polytope correspond to the vertices of its secondary polytope, and that these triangulations are connected via flips; definitions are given in Subsection 2.1 and further information can be found in [DLRS10]. Various triangulations of order polytopes have been constructed or considered, often for special classes of posets. See, for example, Santos, Stump, and Welker for products of chains [SSW17], Féray and Reiner for non-unimodular triangulations related to graph-associahedra [FR12], Reiner and Welker for graded posets [RW05], Bränden and Solus for s-lecture hall order polytopes [BS20], disjoint unions of chains [DLRS10, Section 6.2], and others. However, the general space of regular triangulations of an order polytope, i.e., the 1-skeleton of the secondary polytope of an order polytope, does not appear to have been studied in detail and motivates our work.

Our contributions in this paper add to the literature on order polytopes and further the study of the general space of regular triangulations of order polytopes. Specifically, we investigate circuits, flips, and regular triangulations of order polytopes arising from a certain class of posets, called generalized snake posets. These posets are constructed recursively by adding a square face at the bottom and gluing it to an edge of the previous square. First, we prove results regarding the volumes of their corresponding order polytopes. In particular, for generalized snake posets of the same rank, Theorem 3.11 characterizes those with minimal and maximal normalized volumes.

Next, we turn our attention to the poset Q , whose lattice of upper order ideals is a generalized snake poset P , and study the combinatorial properties of the corresponding order polytope $\mathcal{O}(Q)$. Thus, the vertices of $\mathcal{O}(Q)$ are given by the elements of P . In Theorem 4.5, we prove that there exists a bijection between the set of nonempty connected induced subgraphs associated to the faces of P and the set of circuits of $\mathcal{O}(Q)$. As a consequence, we obtain Theorem 4.12, which states that every vertex of the secondary polytope of $\mathcal{O}(Q)$ corresponds to a unimodular triangulation. The combinatorial characterization of the circuits also implies that the canonical triangulation of $\mathcal{O}(Q)$ admits the same number of flips as there are faces in P , see Theorem 5.1. Then, in Theorem 5.2 we determine that the flip graph of the order polytope $\mathcal{O}(Q)$, when P is a ladder, is in fact the Cayley graph of a symmetric group.

Finally, we introduce an action on the vertices of P given by the so-called twists. It extends to an action on regular triangulations, and in Theorem 5.8 we prove that twists of a canonical triangulation of $\mathcal{O}(Q)$ are again regular triangulations. Moreover, twists preserve circuits of $\mathcal{O}(Q)$, and hence they commute with applying flips, see Theorem 5.6. In particular, this implies that twists give an action on the component of the flip graph of $\mathcal{O}(Q)$ containing all regular triangulations.

Because our posets are strongly planar, by work of Mészáros, Morales, and Striker [MMS19], their order polytopes are unimodularly equivalent to flow polytopes for directed acyclic graphs associated to the posets. While there are many triangulations known for flow polytopes [DKK12], the structure of secondary polytopes for flow polytopes have not been widely studied. Thus, the results above can also be interpreted in the context of circuits and triangulations for certain flow polytopes. In this work, our focus is on using the structure of order polytopes and having results about flow polytopes as implied corollaries, with the exception of the proof of Theorem 5.8, where we use known results regarding regular triangulations of flow polytopes to make an observation regarding order polytopes.

The article is organized as follows. In Section 2, we review some background and establish notation for triangulations and order polytopes. In Section 3, we introduce the family of generalized snake posets P and study volumes of their corresponding order polytopes. The characterization of circuits of the order polytope $\mathcal{O}(Q)$ of the poset of upper order ideals of P is given in Section 4. Section 5 is devoted to introducing twists, and then proving four theorems regarding twists, flips, and triangulations of $\mathcal{O}(Q)$. Lastly, in Section 6 we conclude the paper with conjectures for future work.

2. Background and Notation

2.1. Triangulations

Our primary focus in this paper is the study of triangulations for a particular family of order polytopes. We begin by providing the necessary background for triangulations following the presentation in De Loera, Rambau, and Santos [DLRS10, Section 2.4].

Definition 2.1. Given a point configuration $\mathbf{A} \subseteq \mathbb{R}^d$, let $\text{conv}(\mathbf{A})$ denote the convex hull of \mathbf{A} . A *triangulation* of \mathbf{A} is a collection \mathcal{T} of d -simplices all of whose vertices are points in \mathbf{A} that satisfies the following two properties:

1. The union of all of these simplices equals $\text{conv}(\mathbf{A})$. (*Union Property*)
2. Any pair of these simplices intersects in a (possibly empty) common face. (*Intersection Property*)

A triangulation is *unimodular* if every simplex has normalized volume one. A triangulation of a point configuration $\mathbf{A} \subseteq \mathbb{R}^d$ is *regular* if it can be obtained by projecting the lower envelope of a lifting of \mathbf{A} from \mathbb{R}^{d+1} .

Example 2.2. Consider the polytope generated by the convex hull of the points $(0, 0, 0, 0)$, $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 1, 1, 0)$, and $(1, 1, 1, 1)$. A triangulation of this point configuration consists of the simplices:

$$\begin{aligned}\sigma_{4231} &= \text{conv}\{(0, 0, 0, 0), (1, 0, 0, 0), (1, 0, 1, 0), (1, 1, 1, 0), (1, 1, 1, 1)\} \\ \sigma_{4321} &= \text{conv}\{(0, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)\}.\end{aligned}$$

Definition 2.3. A point configuration \mathbf{A} with index set J has *corank one* if and only if it has an affine dependence relation $\sum_{j \in J} \lambda_j \mathbf{v}_j = 0$ with $\sum_{j \in J} \lambda_j = 0$ that is unique up to multiplication by a constant. This affine dependence partitions J into three subsets:

$$J_+ := \{j \in J : \lambda_j > 0\}, J_0 := \{j \in J : \lambda_j = 0\}, \text{ and } J_- := \{j \in J : \lambda_j < 0\}.$$

In the case when \mathbf{A} has corank one, J_+ and J_- are the only disjoint subsets of J with the property that their relative interiors intersect at the point

$$\sum_{j \in J_+} \lambda_j \mathbf{v}_j = \sum_{j \in J_-} |\lambda_j| \mathbf{v}_j,$$

where the λ_j are assumed to be normalized so that $\sum_{j \in J_+} \lambda_j = \sum_{j \in J_-} |\lambda_j| = 1$. The set $J_+ \cup J_-$ is called a *circuit* in J and the pair (J_+, J_-) is called the *oriented circuit*, or *Radon partition*, of \mathbf{A} .

Definition 2.4. Let \mathbf{A} be a point configuration with index set J . In general, a subset Z of J is a *circuit* if it is a minimal dependent set (that is, it is dependent but every proper subset is independent). Let (Z_+, Z_-) be a partition of Z , such that $\text{conv}(Z_+) \cap \text{conv}(Z_-)$ is nonempty. The partition (Z_+, Z_-) is called an *oriented circuit*. We say the circuit is of *type* $(|Z_+|, |Z_-|)$.

From the circuits we can generate triangulations by using flips to locally transform one triangulation into another.

Lemma 2.5. [DLRS10, Lemma 2.4.2] *Let \mathbf{A} be a point configuration of corank one and $J = J_+ \cup J_0 \cup J_-$ be its label set, partitioned by the unique oriented circuit of \mathbf{A} . Then the following are the only two triangulations of \mathbf{A} :*

$$\mathcal{T}_+ = \{J \setminus \{j\} : j \in J_+\}, \text{ and } \mathcal{T}_- = \{J \setminus \{j\} : j \in J_-\}.$$

Example 2.6. We return to Example 2.2, where we considered the convex hull of the corank one point configuration

$$\begin{aligned} \mathbf{v}_0 &= (0, 0, 0, 0), & \mathbf{v}_1 &= (1, 0, 0, 0), & \mathbf{v}_2 &= (1, 1, 0, 0), \\ \mathbf{v}_3 &= (1, 0, 1, 0), & \mathbf{v}_4 &= (1, 1, 1, 0), & \mathbf{v}_5 &= (1, 1, 1, 1). \end{aligned}$$

Since $\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$, then an oriented circuit is $J_+ = \{1, 4\}$ and $J_- = \{2, 3\}$. The circuit type is $(2, 2)$ and the two triangulations are:

$$\begin{aligned} \mathcal{T}_+ &= \{J \setminus \{j\} : j \in J_+\} = \{\{0, 2, 3, 4, 5\}, \{0, 1, 2, 3, 5\}\} \\ \mathcal{T}_- &= \{J \setminus \{j\} : j \in J_-\} = \{\{0, 1, 3, 4, 5\}, \{0, 1, 2, 4, 5\}\}. \end{aligned}$$

A triangulation of \mathbf{A} is a simplicial complex on \mathbf{A} . Recall that an (*abstract*) *simplicial complex* Δ on a set X is a collection of subsets of X such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of a simplicial complex are called *faces* and a subcomplex Δ' of Δ is a subcollection of Δ which is also a simplicial complex. The *link* of a face $\sigma \in \Delta$ is the simplicial complex

$$\text{lk}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta \text{ and } \sigma \cap \tau = \emptyset\}.$$

If Δ and Δ' are simplicial complexes, then their *join* is $\Delta * \Delta' = \{\sigma \cup \sigma' : \sigma \in \Delta \text{ and } \sigma' \in \Delta'\}$.

Theorem 2.7. [DLRS10, Theorem 4.4.1] Let \mathcal{T}_1 and \mathcal{T}_2 be two triangulations of a point configuration \mathbf{A} . Then \mathcal{T}_1 and \mathcal{T}_2 differ by a flip if and only if there is a circuit Z of \mathbf{A} such that

- (i) They contain, respectively, the two triangulations \mathcal{T}_Z^+ and \mathcal{T}_Z^- of Z .
- (ii) All the maximal simplices of \mathcal{T}_Z^+ and \mathcal{T}_Z^- have the same link L in \mathcal{T}_1 .
- (iii) Removing the subcomplex $\mathcal{T}_Z^+ * L$ from \mathcal{T}_1 and replacing it by $\mathcal{T}_Z^- * L$ gives \mathcal{T}_2 .

Two triangulations of \mathbf{A} are *adjacent* if they differ by a flip. The set of all triangulations of \mathbf{A} , under adjacency by flips, forms the *graph of triangulations*, or *flip graph*, of \mathbf{A} .

Example 2.8. Continuing from Example 2.6 we demonstrate a flip from \mathcal{T}_+ to \mathcal{T}_- . We have

$$\mathcal{T}_+ = \{\{0, 2, 3, 4, 5\}, \{0, 1, 2, 3, 5\}\} \in \mathcal{T}.$$

The circuit $Z = \{1, 2, 3, 4\}$ has triangulations $\mathcal{T}_Z^+ = \{\{2, 3, 4\}, \{1, 2, 3\}\}$ and $\mathcal{T}_Z^- = \{\{1, 3, 4\}, \{1, 2, 4\}\}$. The link of the simplices is $L = \{\{\mathbf{v}_0, \mathbf{v}_5\}\}$, so flipping at the circuit supported at Z gives the triangulation

$$\mathcal{T}_Z^- * L = \{\{0, 1, 3, 4, 5\}, \{0, 1, 2, 4, 5\}\} = \mathcal{T}_-.$$

In Sections 4 and 5, we will take a look at the *secondary polytope* whose vertices are in bijection with regular triangulations of a point configuration. Recall that we can define for each triangulation of a point configuration \mathbf{A} a GKZ-vector. As stated in the following definition, the convex hull of the GKZ-vectors for \mathbf{A} is the secondary polytope. See De Loera, Rambau, and Santos [DLRS10, Section 5.1] for a further discussion of secondary polytopes and GKZ-vectors.

Definition 2.9 (Secondary Polytope). For a point configuration \mathbf{A} the secondary polytope of \mathbf{A} is $\text{conv}\{\varphi_{\mathbf{A}}(\mathcal{T}) \mid \mathcal{T} \text{ triangulation of } \mathbf{A}\}$, where $\varphi_{\mathbf{A}}(\mathcal{T})$ represents the GKZ-vector of \mathcal{T} in \mathbf{A} .

The flip graph, which is the graph of all triangulations connected by flips, is in general not connected, but the flip graph of regular triangulations is connected and contains the 1-skeleton of the secondary polytope as a spanning subgraph [DLRS10, Theorem 5.3.1].

2.2. Order polytopes

Let P be a partially ordered set on the set of elements $[d] := \{1, \dots, d\}$. We abuse notation and write P to denote the elements of P . The *order polytope* of P , introduced by Stanley [Sta86], is defined as

$$\mathcal{O}(P) = \{\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d : x_i \leq x_j \text{ for } i <_P j\}.$$

See Example 2.10. An *upper order ideal* of P , also called a *filter*, is a set $A \subseteq P$ such that if $i \in A$ and $i <_P j$, then $j \in A$. Let $J(P)$ denote the poset of upper order ideals of P , ordered by reverse inclusion. We use $\langle p_1, \dots, p_k \rangle$ to denote the upper order ideal generated by elements $p_1, \dots, p_k \in P$. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ denote the standard basis vectors of \mathbb{R}^d . For an upper

order ideal $A \in J(P)$, define the *characteristic vector* $\mathbf{v}_A := \sum_{i \in A} \mathbf{e}_i$. The vertices of $\mathcal{O}(P)$ are given by

$$V(\mathcal{O}(P)) = \{\mathbf{v}_A : A \in J(P)\}.$$

Define a hyperplane $\mathcal{H}_{i,j} = \{\mathbf{x} \in \mathbb{R}^d : x_i = x_j\}$ for $1 \leq i < j \leq d$. The set of all such hyperplanes, called the d -dimensional braid arrangement of type A, induces a triangulation \mathcal{T} of $\mathcal{O}(P)$ known as the *canonical triangulation*, which has the following three fundamental properties:

1. \mathcal{T} is unimodular,
2. the maximal simplices are in bijection with the linear extensions of P , so the normalized volume of the order polytope is

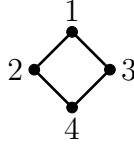
$$\text{vol}(\mathcal{O}(P)) = \# \text{ of linear extensions of } P, \text{ and}$$

3. the simplex corresponding to a linear extension (a_1, \dots, a_d) of P is

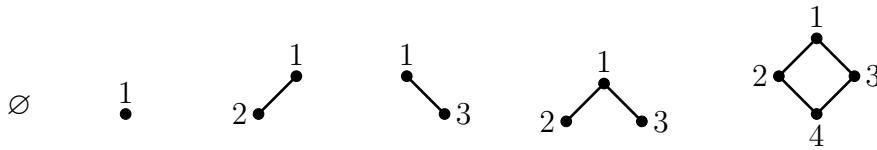
$$\sigma_{a_1, \dots, a_d} = \{\mathbf{x} \in [0, 1]^d : x_{a_1} \leq x_{a_2} \leq \dots \leq x_{a_d}\},$$

with vertex set $\{\mathbf{0}, \mathbf{e}_{a_d}, \mathbf{e}_{a_{d-1}} + \mathbf{e}_{a_d}, \dots, \mathbf{e}_{a_1} + \dots + \mathbf{e}_{a_d} = \mathbf{1}\}$.

Example 2.10 (Order polytope and triangulations). Let P be the diamond poset



Then $\mathcal{O}(P) = \{(x_1, x_2, x_3, x_4) \in [0, 1]^4 : x_4 \leq x_2 \leq x_1 \text{ and } x_4 \leq x_3 \leq x_1\}$. The six upper order ideals of P are



so $\mathcal{O}(P)$ is the convex hull of the points $(0, 0, 0, 0)$, $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 1, 1, 0)$, and $(1, 1, 1, 1)$. The poset P has two linear extensions, namely $4, 2, 3, 1$, and $4, 3, 2, 1$. The canonical triangulation of $\mathcal{O}(P)$ then consists of the following simplices:

$$\sigma_{4231} = \text{conv}\{(0, 0, 0, 0), (1, 0, 0, 0), (1, 0, 1, 0), (1, 1, 1, 0), (1, 1, 1, 1)\}$$

$$\sigma_{4321} = \text{conv}\{(0, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)\}.$$

3. Generalized snake posets

We introduce the family of generalized snake posets $P(\mathbf{w})$, which are distributive lattices with width two, and give a recursive formula for the normalized volume of the order polytope of $P(\mathbf{w})$. For generalized snake posets of the same rank, we characterize those with minimal and maximal normalized volumes.

Definition 3.1. For $n \in \mathbb{Z}_{\geq 0}$, a *generalized snake word* is a word of the form $\mathbf{w} = w_0 w_1 \cdots w_n$ where $w_0 = \varepsilon$ is the empty letter and w_i is in the alphabet $\{L, R\}$ for $i = 1, \dots, n$. The *length* of the word is n , which is the number of letters in $\{L, R\}$.

Definition 3.2. Given a generalized snake word $\mathbf{w} = w_0 w_1 \cdots w_n$, we define the *generalized snake poset* $P(\mathbf{w})$ recursively in the following way:

- $P(w_0) = P(\varepsilon)$ is the poset on elements $\{0, 1, 2, 3\}$ with cover relations $1 \prec 0, 2 \prec 0, 3 \prec 1$ and $3 \prec 2$.
- $P(w_0 w_1 \cdots w_n)$ is the poset $P(w_0 w_1 \cdots w_{n-1}) \cup \{2n + 2, 2n + 3\}$ with the added cover relations $2n + 3 \prec 2n + 1, 2n + 3 \prec 2n + 2$, and

$$\begin{cases} 2n + 2 \prec 2n - 1, & \text{if } n = 1 \text{ and } w_n = L, \text{ or } n \geq 2 \text{ and } w_{n-1} w_n \in \{RL, LR\}, \\ 2n + 2 \prec 2n, & \text{if } n = 1 \text{ and } w_n = R, \text{ or } n \geq 2 \text{ and } w_{n-1} w_n \in \{LL, RR\}. \end{cases}$$

In this definition, the minimal element of the poset $P(\mathbf{w})$ is $\hat{0} = 2n + 3$, and the maximal element of the poset is $\hat{1} = 0$.

If $\mathbf{w} = w_0 w_1 \cdots w_n$ is a generalized snake word of length n , then $P(\mathbf{w})$ is a distributive lattice of width two and rank $n + 2$. We point out two special cases of generalized snake posets. For the length n word $\varepsilon LRLR \cdots$, $S_n := P(\varepsilon LRLR \cdots)$ is the *snake poset*, and for the length n word $\varepsilon LLLL \cdots$, $\mathcal{L}_n := P(\varepsilon LLLL \cdots)$ is the *ladder poset*. For an example, refer to Figure 3.1.

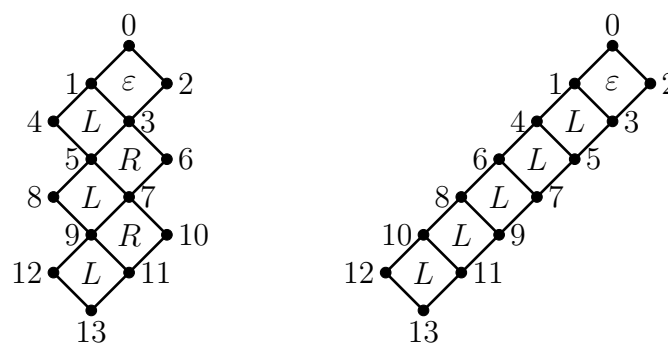


Figure 3.1: The snake poset $S_5 = P(\varepsilon LRLRL)$ and the ladder poset $\mathcal{L}_5 = P(\varepsilon LLLLL)$.

In this article, we consider the generalized snake posets in two contexts. In the next subsection, we consider the order polytope of the generalized snake posets, $\mathcal{O}(P(\mathbf{w}))$. More precisely,

we give a recursive formula for the volume and obtain tight lower and upper bounds for the volumes of $\mathcal{O}(P(\mathbf{w}))$ when \mathbf{w} is of fixed length. In the remaining sections, we study the order polytope of a related poset $Q_{\mathbf{w}}$, which is the poset of join-irreducibles of the generalized snake poset.

3.1. Volume of the order polytope of generalized snake posets

Recall that the volume of an order polytope $\mathcal{O}(P)$ is determined by the number of linear extensions of the poset P . Thus, to study the volume of $\mathcal{O}(P(\mathbf{w}))$ we consider the recursive structure of the poset of upper order ideals of $P(\mathbf{w})$. Because of the definition of the generalized snake poset $P(\mathbf{w})$, the minimal element of $J(P(\mathbf{w}))$ is $\widehat{0} = \langle 2n+3 \rangle = P(\mathbf{w})$ and the maximal element is $\widehat{1} = \emptyset$.

Lemma 3.3. *Let $\mathbf{w} = w_0 w_1 \cdots w_n$ be a generalized snake word. If $k \geq 0$ is the largest index such that $w_k \neq w_n$, then $J(P(\mathbf{w}))$ is equal to*

$$J(P(w_0 w_1 \cdots w_{n-1})) \cup \{\langle 2n+3 \rangle, \langle 2n+2 \rangle, \langle 2n+2, 2k+2 \rangle\} \cup \{\langle 2n+2, 2k+2i+1 \rangle\}_{i=1}^{n-k}.$$

Proof. First note that $P(\mathbf{w}) = P(w_0 w_1 \cdots w_{n-1}) \cup \{2n+2, 2n+3\}$, where $2n+3 \prec 2n+1$, $2n+3 \prec 2n+2$, and $2n+2 \prec 2n$ or $2n+2 \prec 2n-1$. One can see that $J(P(w_0 w_1 \cdots w_{n-1}))$ is contained in $J(P(\mathbf{w}))$. The added elements $2n+3$ and $2n+2$ generate the upper order ideals $\langle 2n+3 \rangle$ and $\langle 2n+2 \rangle$, respectively. Since $2n+3$ is comparable with every other element of $P(\mathbf{w})$, it is not in the minimal generating set of any other upper order ideal. The only elements of $P(w_0 w_1 \cdots w_{n-1})$ which are not comparable with $2n+2$ are $2k+2$ and $\{2k+2i+1\}_{i=1}^{n-k}$. Hence, each pair $\{2n+2, 2k+2\}$ and $\{2n+2, 2k+2i+1\}_{i=1}^{n-k}$ generates an upper order ideal of $P(\mathbf{w})$. Since $2n+1 \prec \cdots \prec 2k+5 \prec 2k+3 \prec 2k+2$, no additional minimal generating sets of upper order ideals are possible. \square

Remark 3.4. Thus, we see that $J(P(\mathbf{w}))$ can be constructed by adding a chain of $n-k+3$ elements to the bottom of $J(P(w_0 w_1 \cdots w_{n-1}))$. In the Hasse diagram for $J(P(\mathbf{w}))$, this corresponds to drawing a strip of $n-k+1$ squares. See Figure 3.2 for an illustration.

Notice that in the strip of the $n-k+1$ newly added squares in $J(P(\mathbf{w}))$, the lowest square (consisting of the four elements $\langle 2n+2, 2n+1 \rangle, \langle 2n+1 \rangle, \langle 2n+2, 2n-1 \rangle, \langle 2n, 2n-1 \rangle$) lies directly below the topmost square of $J(P(\mathbf{w}))$ (consisting of the four elements $\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \langle 1, 2 \rangle$). Hence from Lemma 3.3, we see that the Hasse diagram of $J(P(\mathbf{w}))$ contains exactly n squares which are lined up directly below the topmost square of $J(P(\mathbf{w}))$. We will refer to these squares as the *central squares* of $J(P(\mathbf{w}))$. Swapping every letter from R to L and vice versa in \mathbf{w} corresponds to reflecting $J(P(\mathbf{w}))$ about this central line of squares.

The normalized volume of the order polytope $\mathcal{O}(P(\mathbf{w}))$ can be computed by a recursive formula involving Catalan numbers.

Definition 3.5. For $m \geq 0$, the m -th Catalan number is $\text{Cat}(m) = \frac{1}{m+1} \binom{2m}{m}$.

The Catalan number $\text{Cat}(m)$ enumerates *Dyck paths*, which are lattice paths from $(0, 0)$ to (m, m) that do not fall below the line $y = x$.

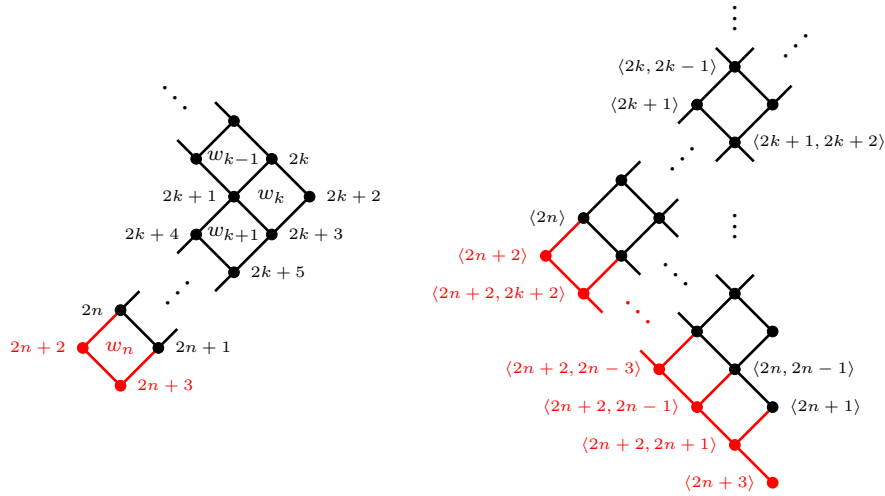


Figure 3.2: An illustration of Lemma 3.3. On the left is a portion of a generalized snake poset $P(\mathbf{w})$ and on the right is the corresponding poset of upper order ideals $J(P(\mathbf{w}))$. To construct $J(P(\mathbf{w}))$ from $J(P(w_0 \cdots w_{n-1}))$ is to add $n - k + 3$ elements, with cover relations shown in red in the Hasse diagram on the right.

Theorem 3.6. For $n \geq 0$, let $\mathbf{w} = w_0 w_1 \cdots w_n$ be a generalized snake word. If $k \geq 0$ is the largest index such that $w_k \neq w_n$, then the normalized volume v_n of $\mathcal{O}(P(\mathbf{w}))$ is given recursively by

$$v_n = \text{Cat}(n - k + 1)v_k + (\text{Cat}(n - k + 2) - 2 \cdot \text{Cat}(n - k + 1))v_{k-1}$$

with $v_{-1} = 1$ and $v_0 = 2$.

Proof. The normalized volume of $\mathcal{O}(P(\mathbf{w}))$ is the number of linear extensions of $P(\mathbf{w})$, and the set of linear extensions of $P(\mathbf{w})$ is in bijection with the set of maximal chains in $J(P(\mathbf{w}))$, so we enumerate the latter.

Let $c(p_1, \dots, p_j)$ denote the number of maximal chains in $J(P(\mathbf{w}))$ which contain the elements p_1, \dots, p_j . Each maximal chain in $J(P(\mathbf{w}))$ contains at least one of $\langle 2k, 2k - 1 \rangle$ or $\langle 2k + 1, 2k + 2 \rangle$, as can be seen in Figure 3.2, so the total number of maximal chains in $J(P(\mathbf{w}))$ is then $c(\emptyset)$ and is given by

$$c(\emptyset) = c(\langle 2k + 1, 2k + 2 \rangle) + c(\langle 2k, 2k - 1 \rangle) - c(\langle 2k + 1, 2k + 2 \rangle, \langle 2k, 2k - 1 \rangle).$$

Note that $c(\langle 2k + 1, 2k + 2 \rangle)$ is the product of the number of maximal chains in the interval $[\langle 2n + 3 \rangle, \langle 2k + 1, 2k + 2 \rangle]$ and the number of maximal chains in the interval $[\langle 2k + 1, 2k + 2 \rangle, \emptyset]$. There are v_k many maximal chains in $[\langle 2k + 1, 2k + 2 \rangle, \emptyset]$, and the maximal chains in $[\langle 2n + 3 \rangle, \langle 2k + 1, 2k + 2 \rangle]$ are counted by $\text{Cat}(n - k + 1)$, as they can be viewed as Dyck paths. Therefore, $c(\langle 2k + 1, 2k + 2 \rangle) = \text{Cat}(n - k + 1)v_k$. Similarly, one sees that $c(\langle 2k, 2k - 1 \rangle) = \text{Cat}(n - k + 2)v_{k-1}$.

Finally, $c(\langle 2k + 1, 2k + 2 \rangle, \langle 2k, 2k - 1 \rangle)$ is given by $2\text{Cat}(n - k + 1)v_{k-1}$, as there are two ways to form a maximal chain in $J(P(\mathbf{w}))$ from a maximal chain in $[\langle 2k, 2k - 1 \rangle, \emptyset]$ and a maximal chain in $[\langle 2n + 3 \rangle, \langle 2k + 1, 2k + 2 \rangle]$. Therefore,

$$c(\emptyset) = \text{Cat}(n - k + 1)v_k + \text{Cat}(n - k + 2)v_{k-1} - 2\text{Cat}(n - k + 1)v_{k-1}. \quad \square$$

Focusing our attention on the snake poset $S_n = P(\varepsilon LRLR \dots)$, the letters alternate so we have $n - k = 1$ at every step, which leads to the following corollary.

Corollary 3.7. *The normalized volume of $\mathcal{O}(S_n)$ with $n \geq 0$ is given recursively by*

$$v_n = 2v_{n-1} + v_{n-2},$$

with $v_{-1} = 1$ and $v_0 = 2$. These are the Pell numbers. \square

In the case of the ladder poset $\mathcal{L}_n = P(\varepsilon LLLL \dots)$, we have $k = 0$ at every step, and hence we have the following well-known result as a corollary.

Corollary 3.8. *The normalized volume of $\mathcal{O}(\mathcal{L}_n)$ with $n \geq 0$ is given by*

$$v_n = \text{Cat}(n + 2).$$

Remark 3.9. The order polytope $\mathcal{O}(P(\mathbf{w}))$ is integrally equivalent to a flow polytope by the work of Mészáros, Morales, and Striker [MMS19]. In particular, Corollary 3.8 appears in the context of flow polytopes as the volume of the “caracol flow polytope” [BGDH⁺19, MM19]. Explicit flow polytope analogues of Theorem 3.6 and Corollary 3.7 do not seem to appear in the literature.

We end this section by showing that the normalized volume of an order polytope $\mathcal{O}(P(\mathbf{w}))$ of a generalized snake poset is bounded above and below by the volume of the order polytope of the ladder poset and the snake poset, respectively.

Let W_n denote the set of generalized snake words of length n . For $i = 1, \dots, n$, define a *swap operation* $f_i : W_n \rightarrow W_n$ by letting $f_i(\mathbf{w})$ be the word obtained from \mathbf{w} by swapping all letters with indices greater than or equal to i to the opposite letter.

Lemma 3.10. *Let \mathbf{w} be a generalized snake word. Then*

$$\text{vol}(\mathcal{O}(P(f_i(\mathbf{w})))) \leq \text{vol}(\mathcal{O}(P(\mathbf{w})))$$

whenever $w_{i-1} = w_i$ or $i = 1$. Furthermore, equality occurs only when $i = 1$.

Proof. Consider the maximal chains in $J(P(\mathbf{w}))$ and for the moment assume $i \neq 1$. Without loss of generality assume that $w_i = w_{i-1} = L$ (the case $w_i = w_{i-1} = R$ is symmetric).

Let $A = \langle 2i, 2i - 1 \rangle$ and $B = \langle 2i - 2, 2i - 3 \rangle$. In light of Remark 3.4 we see that $J(P(f_i(\mathbf{w})))$ consists of a union of the upper order ideal generated by A in $J(P(\mathbf{w}))$ and a reflected lower order ideal generated by B . Figure 3.4 provides an illustration.

In $J(P(f_i(\mathbf{w})))$, all maximal chains contain at least one of A or B . Maximal chains containing A in $J(P(\mathbf{w}))$ corresponds bijectively to chains containing A in $J(P(f_i(\mathbf{w})))$ by reflecting

the portion of the chain after A . Similarly, the chains containing B are in bijection in both posets. Since $J(P(\mathbf{w}))$ contains maximal chains that pass through neither A nor B , there are strictly more maximal chains in $J(P(\mathbf{w}))$. Hence, $\text{vol}(\mathcal{O}(P(f_i(\mathbf{w})))) \leq \text{vol}(\mathcal{O}(P(\mathbf{w})))$.

Finally, in the case $i = 1$, $P(\mathbf{w})$ and $P(f_1(\mathbf{w}))$ are isomorphic via a reflection, and so their posets of upper order ideals are isomorphic. \square

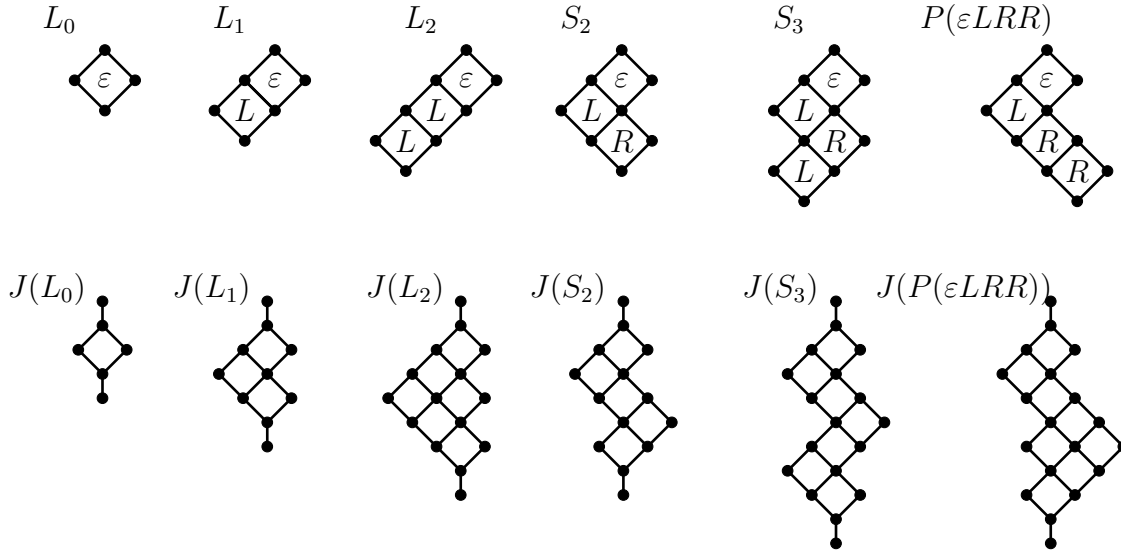


Figure 3.3: Some generalized snake posets (top row) with their corresponding posets of upper order ideals (bottom row). From left to right, the number of linear extensions of these generalized snake posets (or maximal chains in the posets of upper order ideals) are 2, 5, 14, 12, 29, and 33.

Theorem 3.11. For any generalized snake word $\mathbf{w} = w_0w_1 \cdots w_n$ of length n ,

$$\text{vol } \mathcal{O}(S_n) \leq \text{vol } \mathcal{O}(P(\mathbf{w})) \leq \text{vol } \mathcal{O}(\mathcal{L}_n).$$

Proof. First, we show that $\text{vol } \mathcal{O}(S_n) \leq \text{vol } \mathcal{O}(P(\mathbf{w}))$. Let $2 \leq i_1 < i_2 < \cdots < i_k \leq n$ be the set of indices such that $w_{i_j} = w_{i_{j-1}}$. Applying a swap operation at any index strictly smaller than i_j yields a word whose letters indexed by i_j and $i_j - 1$ are still the same. Then for any $j \in [k]$, the letters indexed by i_j and $i_j - 1$ are also the same in the word $f_{i_{j-1}}f_{i_{j-2}} \cdots f_{i_1}(\mathbf{w})$. By Lemma 3.10, we can conclude that

$$\text{vol } \mathcal{O}(P(f_{i_k}f_{i_{k-1}} \cdots f_{i_1}(\mathbf{w}))) \leq \text{vol } \mathcal{O}(P(\mathbf{w})).$$

Moreover, by the construction of i_j 's and the definition of the swap operation, no two adjacent letters with indices up to i_j are the same in $f_{i_j}f_{i_{j-1}} \cdots f_{i_1}(\mathbf{w})$. This shows that $P(f_{i_k}f_{i_{k-1}} \cdots f_{i_1}(\mathbf{w}))$ equals $P(\epsilon LRLR \dots) = S_n$ or $P(\epsilon RLRL \dots) = f_1(S_n)$. By Lemma 3.10, applying f_1 does not change the volume of the order polytope, so we conclude that $\text{vol } \mathcal{O}(S_n) \leq \text{vol } \mathcal{O}(P(\mathbf{w}))$.

Now, we show the second part of the inequality that $\text{vol } \mathcal{O}(P(\mathbf{w})) \leq \text{vol } \mathcal{O}(\mathcal{L}_n)$. Let $2 \leq i_1 < i_2 < \dots < i_k \leq n$ be the set of indices such that $w_{i_j} \neq w_{i_j-1}$. Then for any $j \in [k]$ the letters in $f_{i_{j-1}} \dots f_{i_1}(\mathbf{w})$ with indices strictly smaller than i_j are the same, but the letters with indices i_j and $i_j - 1$ are different. Then the letters in $f_{i_j} f_{i_{j-1}} \dots f_{i_1}(\mathbf{w})$ with indices i_j and $i_j - 1$ are the same and Lemma 3.10 implies that

$$\text{vol } \mathcal{O}(P(\mathbf{w})) \leq \text{vol } \mathcal{O}(P(f_{i_k} f_{i_{k-1}} \dots f_{i_1}(\mathbf{w}))).$$

Furthermore, by construction all letters in $f_{i_k} f_{i_{k-1}} \dots f_{i_1}(\mathbf{w})$ are the same so its generalized snake poset equals $P(\varepsilon LLL \dots) = \mathcal{L}_n$ or $P(\varepsilon RRR \dots) = f_1(\mathcal{L}_n)$. By the same reasoning as above we conclude that $\text{vol } \mathcal{O}(P(\mathbf{w})) \leq \text{vol } \mathcal{O}(\mathcal{L}_n)$. \square

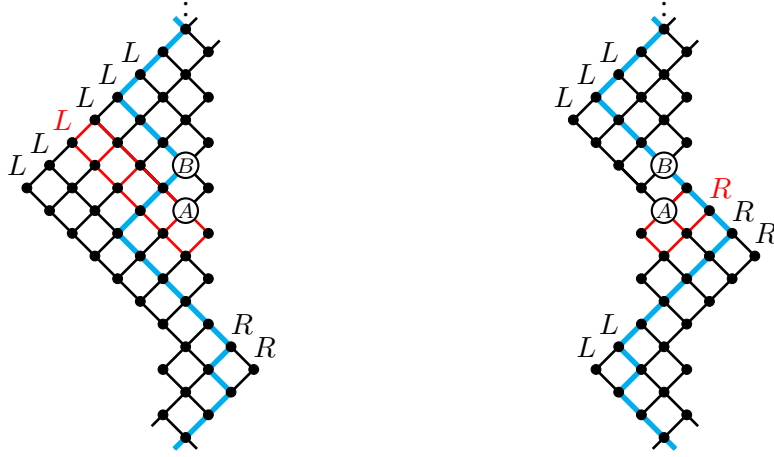


Figure 3.4: On the left is a snippet of a poset $J(P(\mathbf{w}))$ where \mathbf{w} contains the sequence $\dots RLLL LLLRRL \dots$. On the right is the corresponding snippet of $J(P(f_i(\mathbf{w})))$, where i is the index of the red L in \mathbf{w} . The corresponding portion in $f_i(\mathbf{w})$ is $\dots RLLL RRRLLR \dots$. The blue paths demonstrate the bijective correspondence between maximal chains in $J(P(\mathbf{w}))$ through A or B with maximal chains in $J(P(f_i(\mathbf{w})))$.

4. A combinatorial interpretation of circuits

In the remainder of this article, we study the properties of the order polytope of a poset $Q_{\mathbf{w}}$ whose lattice of upper order ideals is $\widehat{P}(\mathbf{w})$, which is defined to be the generalized snake poset $P(\mathbf{w})$ with $\widehat{0}$ and $\widehat{1}$ adjoined. When \mathbf{w} is clear from context we write \widehat{P} .

Given $\mathbf{w} = w_0 w_1 \dots w_n$, $\widehat{P} = \widehat{P}(\mathbf{w})$ is a distributive lattice with order $2n+6$ because \widehat{P} does not contain a copy of the smallest non-modular lattice with five elements and does not contain a sublattice isomorphic to a three-element antichain with a $\widehat{0}$ and $\widehat{1}$ added. Let $Q_{\mathbf{w}} = \text{Irr}_{\wedge}(\widehat{P})$ denote the poset of meet-irreducibles of \widehat{P} . For reference on meet-irreducibles, see [Sta97, Chapter 3]. Heuristically, $\text{Irr}_{\wedge}(\widehat{P})$ is obtained from \widehat{P} by removing $\widehat{1}$, and every vertex which

is at the bottom of a bounded face in the Hasse diagram. See Figure 4.1. By the fundamental theorem of finite distributive lattices, $\widehat{P} \cong J(Q_{\mathbf{w}})$, where $J(Q_{\mathbf{w}})$ is the lattice of upper order ideals of $Q_{\mathbf{w}}$, ordered by reverse inclusion.

We construct a graph $G = G(\mathbf{w})$ associated to $\widehat{P} = \widehat{P}(\mathbf{w})$ as follows. If $\mathbf{w} = w_0w_1 \cdots w_n$, the vertex set of G is $V(G) = \{w_0, w_1, \dots, w_n\}$. The edge set of G is given by

$$E(G) = \{(w_i, w_{i+1}) \mid i = 0, \dots, n - 1\} \cup \{(w_i, w_{i+2}) \mid w_iw_{i+1}w_{i+2} \text{ is given by } xLR \text{ or } xRL \text{ where } x \in \{\varepsilon, L, R\}\}.$$

In other words, G consists of the path of length n on the vertices w_0, \dots, w_n , with a 3-cycle for each turn LLR or RRL in \mathbf{w} . See Figure 4.1. We denote the set of nonempty connected induced subgraphs of $G(\mathbf{w})$ by $\mathcal{G}(\mathbf{w})$.

The Hasse diagram of $\widehat{P}(\mathbf{w})$ can be embedded on the plane so that its edges are non-crossing where each bounded face of the embedded Hasse diagram has degree 4 given by the length of the cycle bounding the face. We call these bounded faces the *squares* of $\widehat{P}(\mathbf{w})$.

There is a one-to-one correspondence between the squares of $\widehat{P}(\mathbf{w})$ and the letters of \mathbf{w} by realizing $G = G(\mathbf{w})$ as follows. Consider each square in the Hasse diagram $\text{Hasse}(\widehat{P})$ as a vertex, then form an edge between squares when they intersect in the plane, as shown in Figure 4.1. To each vertex w_i of G , we denote by $\text{Sq}(w_i)$ the four elements of \widehat{P} contained in the 4-cycle which bounds the face of $\text{Hasse}(\widehat{P})$ corresponding to w_i .

Remark 4.1. The volume of $\mathcal{O}(Q_{\mathbf{w}})$ equals the number of maximal chains in $\widehat{P}(\mathbf{w})$ or, equivalently, in $P(\mathbf{w})$. By [Pro20, Section 4], maximal chains in $P(\mathbf{w})$ are in bijection with perfect matchings of the Hasse diagram of $P(\mathbf{w}^*)$, where \mathbf{w}^* denotes the dual of \mathbf{w} . Informally, $P(\mathbf{w}^*)$ is obtained from $P(\mathbf{w})$ by replacing three consecutive squares that form a ladder by three squares that form a bend and vice versa. Perfect matchings of $P(\mathbf{w}^*)$ have been extensively studied because they play an important role in the theory of cluster algebras and their total number can be computed via explicit formulas involving continued fractions [cS18, Theorem 3.4] or certain admissible sequences [BFG⁺18, Theorem 4.6].

Next, we study the circuits of the vertices of the order polytope $\mathcal{O}(Q_{\mathbf{w}})$. Understanding this for arbitrary words \mathbf{w} is a challenge, therefore we instead restrict our attention in this section to the following set of words.

Definition 4.2. Let \mathcal{V} denote the subset of words which do not contain the substring LRL or RRL .

Theorem 4.5 shows that for $\mathbf{w} \in \mathcal{V}$, circuits in the vertices of $\mathcal{O}(Q_{\mathbf{w}})$ have a combinatorial interpretation as the nonempty connected induced subgraphs of the graph $G(\mathbf{w})$.

Lemma 4.3. Let $\mathbf{w} \in \mathcal{V}$ be a generalized snake word of length n . The poset $Q_{\mathbf{w}}$ has order $n + 4$.

Proof. A non- $\widehat{1}$ element of \widehat{P} is meet-irreducible if and only if it is not the minimum element in a square of $\widehat{P}(\mathbf{w})$. There are $n + 1$ squares, thus $|\text{Irr}_{\wedge}(\widehat{P})| = 2n + 6 - 1 - (n + 1) = n + 4$. \square

Lemma 4.4. Let $\mathbf{w} \in \mathcal{V}$ be a generalized snake word of length n . A circuit of the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$ cannot contain the zero vector \mathbf{v}_{\emptyset} or $\mathbf{v}_{Q_{\mathbf{w}}} = (1, 1, \dots, 1)$.

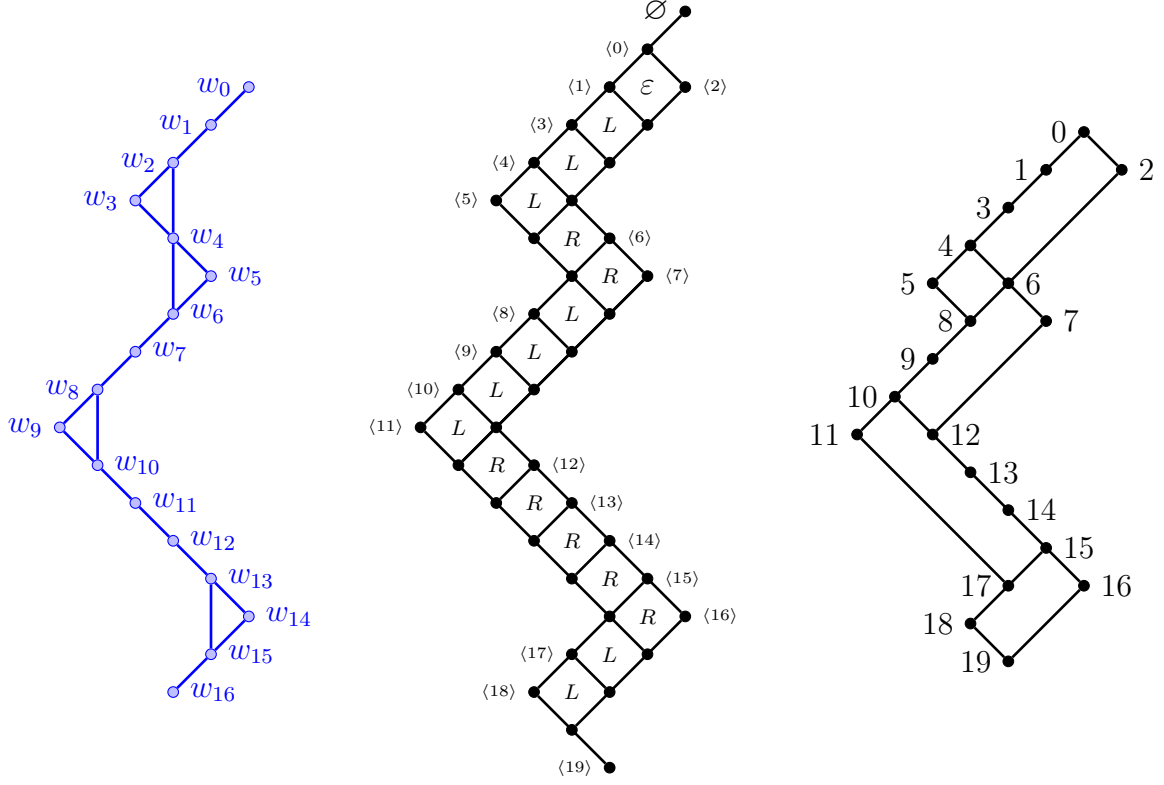


Figure 4.1: In the center is the lattice $\widehat{P}(\mathbf{w})$ for $\mathbf{w} = \varepsilon L^3 R^2 L^4 R^5 L^2$. Its poset of meet-irreducibles $Q_{\mathbf{w}} = \text{Irr}_{\wedge}(\widehat{P})$ is shown to the right, and the associated graph $G(\mathbf{w})$ is shown to the left.

Proof. Observe that \mathbf{v}_{\emptyset} is the zero vector in $\mathbb{R}^{|Q_{\mathbf{w}}|}$, so it cannot be in a minimal dependent set. Also, $\mathbf{v}_{Q_{\mathbf{w}}}$ is the vector of all ones in $\mathbb{R}^{|Q_{\mathbf{w}}|}$, and in particular is the only vertex whose $|Q_{\mathbf{w}}|$ -th coordinate is nonzero, so it also cannot be in a minimal dependent set. \square

Let $\mathbf{w} = w_0 w_1 \dots w_n$ and let $H \in \mathcal{G}(\mathbf{w})$. For an upper order ideal A of $Q_{\mathbf{w}}$, we consider A to be a point in $J(Q_{\mathbf{w}}) \cong \widehat{P}$. We say A is *compatible* with H if A is an element of an odd number of squares in $\{\text{Sq}(w_i) : w_i \in H\}$. Consider the map $\Gamma : \mathcal{G}(\mathbf{w}) \rightarrow \mathcal{C}(Q_{\mathbf{w}})$ from the set $\mathcal{G}(\mathbf{w})$ of nonempty connected induced subgraphs of $G(\mathbf{w})$ to the set $\mathcal{C}(Q_{\mathbf{w}})$ of circuits of the vertex set of the order polytope $\mathcal{O}(Q_{\mathbf{w}})$ where $\Gamma(H)$ is defined to be the set of all \mathbf{v}_A such that A is compatible with H . See Figure 4.2 for an illustration.

Theorem 4.5. *Let $\mathbf{w} \in \mathcal{V}$ be a generalized snake word of length n . The map $\Gamma : \mathcal{G}(\mathbf{w}) \rightarrow \mathcal{C}(Q_{\mathbf{w}})$ is a bijection.*

Proof. We split the proof into two parts.

Part I: $\Gamma(H)$ is a circuit. We shall show by induction on $|H|$ that $\Gamma(H)$ is a minimal dependent set of vertices of $\mathcal{O}(Q_{\mathbf{w}})$. First, suppose $H = \{w_k\}$ is a single vertex of G for some $k = 0, \dots, n$.

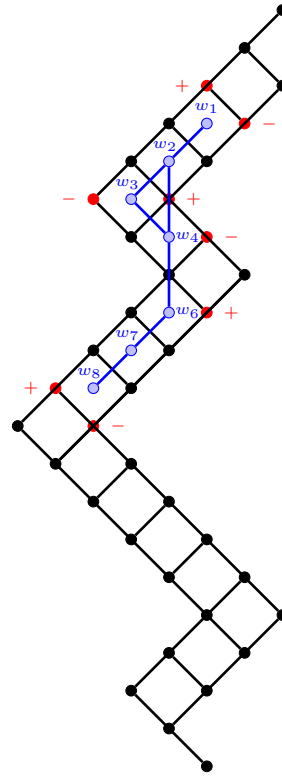


Figure 4.2: The connected induced subgraph H of $G(\mathbf{w})$, depicted in blue, is embedded in $\text{Hasse}(\widehat{P})$. The elements in the circuit $\Gamma(H)$ are depicted in red.

Then for some upper order ideal A of $Q_{\mathbf{w}}$ and distinct incomparable elements $x, y \in Q_{\mathbf{w}}$, we have

$$\Gamma(H) = \{\mathbf{v}_B : B \in \text{Sq}(w_k)\} = \{\mathbf{v}_A, \mathbf{v}_{A \cup \{x\}}, \mathbf{v}_{A \cup \{y\}}, \mathbf{v}_{A \cup \{x, y\}}\}.$$

As

$$\begin{aligned} \mathbf{v}_{A \cup \{x\}} &= \mathbf{v}_A + \mathbf{e}_x, \\ \mathbf{v}_{A \cup \{y\}} &= \mathbf{v}_A + \mathbf{e}_y, \\ \mathbf{v}_{A \cup \{x, y\}} &= \mathbf{v}_A + \mathbf{e}_x + \mathbf{e}_y, \end{aligned}$$

then $\mathbf{v}_A - \mathbf{v}_{A \cup \{x\}} - \mathbf{v}_{A \cup \{y\}} + \mathbf{v}_{A \cup \{x, y\}} = \mathbf{0}$, and $\Gamma(H)$ is a circuit.

Next, suppose $H = \{w_{i_1}, \dots, w_{i_k}\}$ is a connected induced subgraph of G , with the assumption that $i_1 < \dots < i_k$ and $k \geq 2$. Observe that H contains the path w_{i_1}, \dots, w_{i_k} . We will inductively assign signs to the squares of \widehat{P} that contain vertices of H . Start by defining $\text{sgn}(\text{Sq}(w_{i_1})) = 1$. For $j = 2, \dots, k$,

$$\text{sgn}(\text{Sq}(w_{i_j})) = \begin{cases} \text{sgn}(\text{Sq}(w_{i_{j-1}})), & \text{if } i_j - i_{j-1} = 1, \\ -\text{sgn}(\text{Sq}(w_{i_{j-1}})), & \text{if } i_j - i_{j-1} = 2. \end{cases}$$

We note that since H is a connected subgraph, then $\text{Sq}(w_{i_j}) \cap \text{Sq}(w_{i_{j+1}}) \neq \emptyset$. If

$$\text{Sq}(w_{i_j}) = \{A_{i_j}, A_{i_j} \cup \{x_{i_j}\}, A_{i_j} \cup \{y_{i_j}\}, A_{i_j} \cup \{x_{i_j}, y_{i_j}\}\},$$

then

$$\sigma_{i_j} := \mathbf{v}_{A_{i_j}} - \mathbf{v}_{A_{i_j} \cup \{x_{i_j}\}} - \mathbf{v}_{A_{i_j} \cup \{y_{i_j}\}} + \mathbf{v}_{A_{i_j} \cup \{x_{i_j}, y_{i_j}\}} = \mathbf{0}$$

is an affine dependence relation on the vertices of $\text{Sq}(w_{i_j})$. Thus,

$$\sum_{j=1}^k \text{sgn}(\text{Sq}(w_{i_j})) \cdot \sigma_{i_j} = \mathbf{0}. \quad (4.1)$$

Because of the definition of the sgn function, the terms which occur in the left hand side of this expression with nonzero coefficient are indexed precisely by the elements of \widehat{P} which are compatible with H , i.e., $\Gamma(H)$. Thus, $\Gamma(H)$ is a dependent set with an affine dependence relation (4.1).

Having shown $\Gamma(H)$ is dependent, it remains to show that $\Gamma(H)$ is also minimal, i.e., a circuit. We will use induction on $|H| = k$. If $k = 1$, then it is straightforward to verify that $\text{Sq}(w_{i_1})$ is a circuit. Assume that $k > 1$. We consider three cases. For the first case, suppose that $i_{k-1} + 1 = i_k$ and $(w_{i_{k-2}}, w_{i_k})$ is not an edge in H . Thus, it follows that for some upper order ideal A and elements a, b, c , we have

$$\text{Sq}(w_{i_{k-1}}) = \{A, A \cup \{b\}, A \cup \{a\}, A \cup \{a, b\}\}$$

and

$$\text{Sq}(w_{i_k}) = \{A \cup \{b\}, A \cup \{a, b\}, A \cup \{b, c\}, A \cup \{a, b, c\}\}.$$

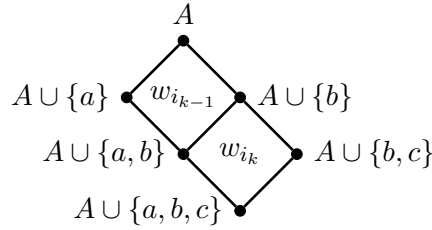


Figure 4.3: The first case, with $i_{k-1} + 1 = i_k$ and $(w_{i_{k-2}}, w_{i_k})$ is not an edge in H .

In $\Gamma(H)$, the only vectors supported on the c -coordinate are $\mathbf{v}_{A \cup \{b, c\}}$ and $\mathbf{v}_{A \cup \{a, b, c\}}$. If we restrict the vectors in $\{\mathbf{v}_C : C \in \Gamma(H)\}$ to the coordinates in $A \cup \{a, b\}$, then we obtain the vectors $\{\mathbf{v}_C : C \in \Gamma(H \setminus w_{i_k})\}$. By induction, these vectors form a circuit with a unique minimal dependence where the coefficients of $\mathbf{v}_{A \cup \{b\}}$ and $\mathbf{v}_{A \cup \{a, b\}}$ are equal and opposite in sign. Thus, this is the only potential dependence (up to scaling) for $\{\mathbf{v}_C : C \in \Gamma(H)\}$, where the coefficients of $\mathbf{v}_{A \cup \{b\}}$ and $\mathbf{v}_{A \cup \{a, b\}}$ become the coefficients of $\mathbf{v}_{A \cup \{b, c\}}$ and $\mathbf{v}_{A \cup \{a, b, c\}}$ respectively. It is immediate that this choice of coefficients is a dependence with all non-zero coefficients, and thus $\Gamma(H)$ is a circuit.

For the second case, suppose that $i_{k-1} + 2 = i_k$, i.e., the last edge in H is the vertical edge of a triangle in G . Thus, it follows that for some upper order ideal A and elements a, b, c, d , we have

$$\text{Sq}(w_{i_{k-1}}) = \{A, A \cup \{b\}, A \cup \{a\}, A \cup \{a, b\}\}$$

and

$$\text{Sq}(w_{i_k}) = \{A \cup \{a, b\}, A \cup \{a, b, c\}, A \cup \{a, b, d\}, A \cup \{a, b, c, d\}\}.$$

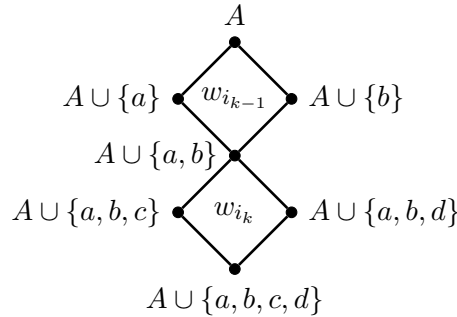


Figure 4.4: Case two, with $i_{k-1} + 2 = i_k$.

In $\Gamma(H)$, the only vectors supported on the d -coordinate are $\mathbf{v}_{A \cup \{a, b, c, d\}}$ and $\mathbf{v}_{A \cup \{a, b, d\}}$. Thus, in any dependence for $\Gamma(H)$, these two vectors have coefficients that are equal in magnitude and opposite in sign. The only vectors supported on the c -coordinate are $\mathbf{v}_{A \cup \{a, b, c\}}$ and $\mathbf{v}_{A \cup \{a, b, c, d\}}$. Thus, these vectors have coefficients that are equal in magnitude and opposite in sign in any dependence for $\Gamma(H)$. Summing these three vectors with these equal and opposite coefficients yields the vector $\mathbf{v}_{A \cup \{a, b\}}$ (scaled by the same coefficient). Thus, any dependence on $\Gamma(H)$ arises from a dependence on $\Gamma(H \setminus w_{i_k})$. By induction, this set is a circuit with a unique minimal dependence. This unique minimal dependence induces a unique minimal dependence on $\Gamma(H)$, for which every coefficient is non-zero.

For the third case, suppose that $i_{k-1} + 1 = i_k$ and $(w_{i_{k-2}}, w_{i_k})$ is an edge in H , i.e., that $w_{i_{k-2}}, w_{i_{k-1}}$, and w_{i_k} form a triangle in H . Thus, it follows that for some upper order ideal A and elements a, b, c, d , we have

$$\text{Sq}(w_{i_{k-2}}) = \{A, A \cup \{b\}, A \cup \{a\}, A \cup \{a, b\}\}$$

and

$$\text{Sq}(w_{i_{k-1}}) = \{A \cup \{b\}, A \cup \{a, b\}, A \cup \{b, c\}, A \cup \{a, b, c\}\}$$

and

$$\text{Sq}(w_{i_k}) = \{A \cup \{a, b\}, A \cup \{a, b, c\}, A \cup \{a, b, d\}, A \cup \{a, b, c, d\}\}.$$

In $\Gamma(H)$, the only vectors supported on the d -coordinate are $\mathbf{v}_{A \cup \{a, b, c, d\}}$ and $\mathbf{v}_{A \cup \{a, b, d\}}$. Thus, in any dependence for $\Gamma(H)$, these two vectors have coefficients that are equal in magnitude and opposite in sign. Because $A \cup \{a, b, c\}$ is not compatible with H , the only non-zero vectors supported on the c -coordinate are $\mathbf{v}_{A \cup \{b, c\}}$ and $\mathbf{v}_{A \cup \{a, b, c, d\}}$. Thus, these vectors have coefficients that are equal in magnitude and opposite in sign in any dependence for $\Gamma(H)$. Summing these three

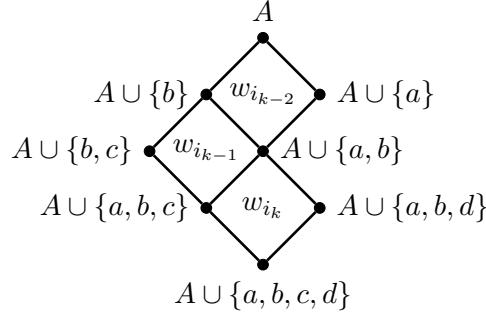


Figure 4.5: Case three, with $i_{k-1} + 1 = i_k$ and $(w_{i_{k-2}}, w_{i_k})$ is an edge in H .

vectors with these equal and opposite coefficients yields the vector $\mathbf{v}_{A \cup \{b\}}$ (scaled by the same coefficient). Thus, any dependence on $\Gamma(H)$ arises from a dependence on $\Gamma(H \setminus \{w_{i_{k-1}}, w_{i_k}\})$. By induction, this set is a circuit with a unique minimal dependence. This unique minimal dependence induces a unique minimal dependence on $\Gamma(H)$, for which every coefficient is non-zero.

Part II: Γ is bijective. Having established that Γ is well-defined, we next show that Γ is injective. Suppose $\Gamma(H) = \Gamma(K)$ but $H \neq K$. Since H and K are induced subgraphs, then this means $V(H) \neq V(K)$. Without loss of generality, suppose $w_m \in H$ but $w_m \notin K$. Since H and K are each connected, then w_m must occur either at one of the ends of the main path $(w_{i_1}, \dots, w_{i_k})$ of H , or is a corner of a triangle in H , where w_i is a corner if w_{i-1} and $w_{i+1} \in H$. In either case, this implies $\Gamma(H) \setminus \Gamma(K) \neq \emptyset$, a contradiction.

To see that Γ surjective, we induct on the length of \mathbf{w} . If $\mathbf{w} = w_0 w_1 \cdots w_n$, we define $\ell(\mathbf{w}) = n$. Consider $\ell(\mathbf{w}) = 0$ so that $\mathbf{w} = \varepsilon$. See Figure 4.6.



Figure 4.6: The base case with $\mathbf{w} = \varepsilon$.

The only circuit arises from $\text{Sq}(\varepsilon)$:

$$\mathbf{v}_{\langle 0 \rangle} - \mathbf{v}_{\langle 1 \rangle} - \mathbf{v}_{\langle 2 \rangle} + \mathbf{v}_{\langle 1,2 \rangle} = (1, 0, 0, 0) - (1, 1, 0, 0) - (1, 0, 1, 0) + (1, 1, 1, 0) = (0, 0, 0, 0).$$

Suppose the map $\Gamma : \mathcal{G}(\mathbf{w}) \rightarrow \mathcal{C}(Q_{\mathbf{w}})$ is surjective for all \mathbf{w} where $\ell(\mathbf{w}) \leq n - 1$.

Now, let $\mathbf{u} = w_0 \cdots w_{n-1} \in \mathcal{V}$ and $\mathbf{w} = \mathbf{u} w_n \in \mathcal{V}$. Suppose $\text{Sq}(w_n) = \{A, B, C, D\}$ so that $A \prec B$, $A \prec C$, $B \prec D$, $C \prec D$, and C and B are incomparable. Without loss of generality, there are two cases to consider; $w_{n-1} = w_n = R$, or $w_{n-1} = L$ and $w_n = R$. These cases are shown in Figure 4.7.

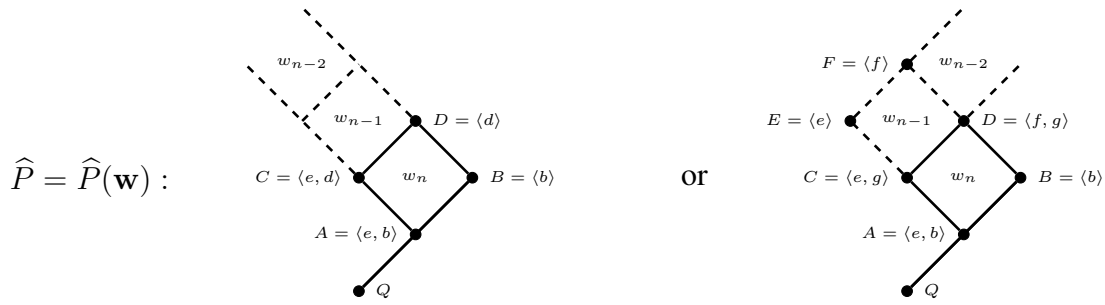


Figure 4.7: The two cases in the induction step.

In both cases, A and C are not meet-irreducible, while B is meet-irreducible. In the case $w_n = w_{n-1}$, D is meet-irreducible, so we have $B = \langle b \rangle$, $D = \langle d \rangle$, $A = \langle e, b \rangle$ and $C = \langle e, d \rangle$ for some $b, d, e \in Q$. In the case $w_n \neq w_{n-1}$, D is not meet-irreducible. If $\text{Sq}(w_{n-1}) = \{C, D, E, F\}$ with $E \prec F$, then we have $B = \langle b \rangle$, $E = \langle e \rangle$, $F = \langle f \rangle$, $A = \langle e, b \rangle$, $C = \langle e, d \rangle$ and $D = \langle f, d \rangle$ for some $b, d, e, f \in Q$. See the picture on the left in Figure 4.7.

Let $T = \text{Irr}_\wedge(\widehat{P}(\mathbf{u}))$. If $\gamma \in \mathcal{C}(Q_{\mathbf{w}})$ is a circuit that does not contain \mathbf{v}_A or \mathbf{v}_B , then γ is a circuit of the vertex set of $\mathcal{O}(T)$. By the induction hypothesis, there exists a connected induced subgraph $H \in \mathcal{G}(\mathbf{u}) \subset \mathcal{G}(\mathbf{w})$ such that $\Gamma(H) = \gamma$.

We shall show that for any circuit γ that contains \mathbf{v}_A or \mathbf{v}_B , there exists a connected induced subgraph $H \subseteq G$ such that $\Gamma(H) = \gamma$. First, observe that \mathbf{v}_A or \mathbf{v}_B are the only vertices in $\mathcal{O}(Q_{\mathbf{w}})$ whose b -th coordinate is nonzero, aside from $\mathbf{v}_{Q_{\mathbf{w}}}$ (which by Lemma 4.4 we know cannot be contained in any circuit). Thus, if one of these vectors is in γ , then they must both be in γ . This also implies that any dependency relation involving \mathbf{v}_A and \mathbf{v}_B must be of the form $\alpha(\mathbf{v}_A - \mathbf{v}_B) + R = \mathbf{0}$ for some nonzero $\alpha \in \mathbb{R}$ and where R is a linear combination of the elements of $\gamma \setminus \{\mathbf{v}_A, \mathbf{v}_B\}$. Second, since $\mathbf{v}_A \in \gamma$ and $A = \langle e, b \rangle$, then γ must contain a vertex of the form $\mathbf{v}_{\langle e \rangle}$ or $\mathbf{v}_{\langle e, x \rangle}$ for some $x \in Q_{\mathbf{w}}$.

Consider the case $w_n = w_{n-1} = R$, so that $C = \langle e, d \rangle$ and $D = \langle d \rangle$. If $\mathbf{v}_C \in \gamma$, then $\mathbf{v}_D \in \gamma$ as well, since these are the only vertices in $\mathcal{O}(T)$ whose d -th coordinate is nonzero. Then by minimality γ must be the circuit $\{A, B, C, D\}$ as

$$\mathbf{v}_A - \mathbf{v}_B - \mathbf{v}_C + \mathbf{v}_D = \mathbf{0}.$$

In this case $H = \{w_n\}$ and $\Gamma(H) = \gamma$.

Otherwise, γ does not contain \mathbf{v}_C nor \mathbf{v}_D , so suppose γ gives rise to a minimal dependence relation of the form

$$\alpha \mathbf{v}_A - \alpha \mathbf{v}_B + R = \mathbf{0}$$

for some nonzero $\alpha \in \mathbb{R}$, and R is a nonzero linear combination of vertices of $\mathcal{O}(T)$ that does not involve \mathbf{v}_A or \mathbf{v}_B . As $\mathbf{v}_A - \mathbf{v}_B = \mathbf{v}_C - \mathbf{v}_D$, then

$$\alpha \mathbf{v}_C - \alpha \mathbf{v}_D + R = \mathbf{0}$$

is another dependence relation which is minimal, because the initial dependence relation was minimal. This new dependence relation consists of a set of vertices $\gamma' \subseteq \mathcal{O}(T)$, so by the induc-

tion hypothesis, there is a connected induced subgraph $H' \in \mathcal{G}(\mathbf{u})$ such that $\Gamma(H') = \gamma'$. From this, it follows that the connected induced subgraph $H = H' \cup \{w_n\} \in \mathcal{G}(\mathbf{w})$ satisfies $\Gamma(H) = \gamma$.

Next, we consider the case $w_{n-1} = L$ and $w_n = R$. We have $C = \langle e, g \rangle$, $D = \langle f, g \rangle$, $E = \langle e \rangle$ and $F = \langle f \rangle$. See the picture on the right in Figure 4.7. We consider four cases; in each we will find an H with $\Gamma(H) = \gamma$, concluding the proof.

- (i) Case $\mathbf{v}_C, \mathbf{v}_D \in \gamma$: This implies γ contains $\{A, B, C, D\}$, which is a circuit. Thus, it must be that $\gamma = \{A, B, C, D\}$ and hence $\Gamma(\{w_n\}) = \gamma$.
- (ii) Case $\mathbf{v}_C \in \gamma, \mathbf{v}_D \notin \gamma$: Since $\mathbf{v}_A = \mathbf{v}_F + \mathbf{e}_g + \mathbf{e}_e + \mathbf{e}_b$, $\mathbf{v}_B = \mathbf{v}_F + \mathbf{e}_g + \mathbf{e}_b$, and $\mathbf{v}_C = \mathbf{v}_F + \mathbf{e}_g + \mathbf{e}_e$, then γ must contain \mathbf{v}_F since \mathbf{v}_F is the only other vertex in $\mathcal{O}(T)$ whose f -th entry is nonzero. In this case, γ gives rise to a minimal dependence relation of the form

$$\alpha(\mathbf{v}_A - \mathbf{v}_B - \mathbf{v}_C + \mathbf{v}_F) + R = \mathbf{0}$$

for some nonzero $\alpha \in \mathbb{R}$, and R is a nonzero linear combination of vertices of $\mathcal{O}(T)$ that does not involve $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C, \mathbf{v}_D, \mathbf{v}_E$, or \mathbf{v}_F . As $\mathbf{v}_A - \mathbf{v}_B - \mathbf{v}_C = -\mathbf{v}_D$, then

$$\alpha(\mathbf{v}_F - \mathbf{v}_D) + R = \mathbf{0}$$

is another minimal dependence relation consisting of vertices $\gamma' \subseteq \mathcal{O}(T)$ so there exists $H' \in \mathcal{G}(\mathbf{u})$ such that $\Gamma(H') = \gamma'$, and it follows that $\Gamma(H' \cup \{w_n\}) = \gamma$.

- (iii) Case $\mathbf{v}_C \notin \gamma, \mathbf{v}_F \in \gamma$: By a similar analysis as above, it must be that γ is the circuit $\{A, B, E, F\}$, since

$$\mathbf{v}_A - \mathbf{v}_B - \mathbf{v}_E + \mathbf{v}_F = \mathbf{0},$$

and H is the subgraph induced on $\{w_{n-1}, w_n\}$, where $\Gamma(H) = \gamma$.

- (iv) Case $\mathbf{v}_C, \mathbf{v}_F \notin \gamma$: Since $\mathbf{v}_C \notin \gamma$, we must have $\mathbf{v}_E \in \gamma$ as $\mathbf{v}_A = \mathbf{v}_B + \mathbf{e}_e$ and \mathbf{v}_E is the only other vertex in $\mathcal{O}(T)$ whose e -th coordinate is nonzero. Since $A, B, E < F$, then the f -th entry of each of $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_E$ are nonzero. This implies that \mathbf{v}_D is in γ since the only other vertex in $\mathcal{O}(T)$ whose f -th entry is nonzero is \mathbf{v}_F .

Thus, γ gives rise to a minimal dependence relation of the form

$$\alpha(\mathbf{v}_A - \mathbf{v}_B + \mathbf{v}_D - \mathbf{v}_E) + R = \mathbf{0}$$

for some nonzero $\alpha \in \mathbb{R}$, and R is a nonzero linear combination of vertices of $\mathcal{O}(T)$ that does not involve $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C, \mathbf{v}_D, \mathbf{v}_E$, or \mathbf{v}_F . As $\mathbf{v}_A - \mathbf{v}_B + \mathbf{v}_D = \mathbf{v}_C$, then

$$\alpha(\mathbf{v}_C - \mathbf{v}_E) + R = \mathbf{0}$$

is another minimal dependence relation consisting of vertices $\gamma' \subseteq \mathcal{O}(T)$, so there exists $H' \in \mathcal{G}(\mathbf{u})$ such that $\Gamma(H') = \gamma'$, and $\Gamma(H' \cup \{w_n\}) = \gamma$. \square

Remark 4.6. Theorem 4.5 does not hold for a generalized snake word \mathbf{w} outside of \mathcal{V} . Computational evidence suggests that the size of $\mathcal{G}(\mathbf{w})$ is an upper bound for the number of circuits of $\mathcal{O}(Q_{\mathbf{w}})$.

Next, we obtain a number of corollaries about the structure of the circuits in the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$.

Corollary 4.7. *Let $\mathbf{w} \in \mathcal{V}$. A circuit Z with partition (Z_+, Z_-) in the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$ has an affine dependence relation of the form*

$$\sum_{j \in Z_+} \mathbf{v}_j = \sum_{j \in Z_-} \mathbf{v}_j.$$

In particular, $|Z_-| = |Z_+|$.

Proof. By proof of Theorem 4.5, a circuit $Z = \Gamma(H)$ for some nonempty connected induced subgraph H of $G(\mathbf{w})$. Moreover, an affine dependence relation for Z is given in (4.1), where by construction every vertex of Z appears with coefficient ± 1 . This shows that there is an affine dependence relation for Z as in the statement of the corollary. Furthermore, we conclude that $|Z_-| = |Z_+|$, because the dependence is affine. \square

Corollary 4.8. *Let $H = \{w_{i_1}, \dots, w_{i_k}\}$ be a connected induced subgraph of G induced by the subword $w_{i_1} \cdots w_{i_k}$ of $\mathbf{w} = w_0 \cdots w_n \in \mathcal{V}$ such that $i_1 < \cdots < i_k$. Suppose $H' = H \cup \{w_{i_j}\}$ is a connected induced subgraph of G such that $i_k < i_j$. Then*

- (a) *If $w_{i_j} = w_{i_k}$, then $|\Gamma(H')| = |\Gamma(H)|$.*
- (b) *If $w_{i_j} \neq w_{i_k}$, then $|\Gamma(H')| = |\Gamma(H)| + 2$.*

In the case where $H = \{\varepsilon\}$, $|\Gamma(H')| = |\Gamma(H)|$. Thus, the smallest circuits in the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$ have four vertices. The largest circuits have $4 + 2t$ vertices where t is the number of turns (an occurrence of LLR or RRL) in \mathbf{w} .

Using the bijection of Theorem 4.5, we can recursively compute the number of circuits in the vertex set of $\mathcal{O}(Q_{\mathbf{w}})$.

Corollary 4.9. *Let $\mathbf{u} = w_0 \cdots w_{n-1} \in \mathcal{V}$ and $\mathbf{w} = \mathbf{u}w_n \in \mathcal{V}$. Let N_k be the number of connected induced subgraphs of $G(\mathbf{u})$ that contain w_k but not w_{k+1} . Then $|\mathcal{G}(\varepsilon)| = 1$, $|\mathcal{G}(\varepsilon w_1)| = 3$, and*

- (a) *If $w_n = w_{n-1}$, then $|\mathcal{G}(\mathbf{w})| = |\mathcal{G}(\mathbf{u})| + N_{n-1} + 1$.*
- (b) *If $w_n \neq w_{n-1}$, then $|\mathcal{G}(\mathbf{w})| = |\mathcal{G}(\mathbf{u})| + N_{n-1} + N_{n-2} + 1$.*

Proof.

- (a) If $w_n = w_{n-1}$, then $\deg_{G(\mathbf{w})} w_n = 1$. Thus, for any connected induced subgraph H of $G(\mathbf{w})$ that contains w_n , the connected induced subgraph $H' = H \setminus \{w_n\}$ contains w_{n-1} . From the proof of Theorem 4.5, every circuit supported on the squares corresponding to H has a corresponding circuit supported on the squares corresponding to H' . The claim follows as N_{n-1} counts the subgraphs of $G(\mathbf{u})$ which contain w_{n-1} and 1 counts the subgraph $\{w_n\}$.

- (b) If $w_n = w_{n-1}$, then $\deg_{G(\mathbf{w})} w_n = 2$. Thus, for any connected induced subgraph H of $G(\mathbf{w})$ that contains w_n , the connected induced subgraph $H' = H \setminus \{w_n\}$ contains at least one of w_{n-2} and w_{n-1} . From the proof of Theorem 4.5, every circuit supported on the squares corresponding to H has a corresponding circuit supported on the squares corresponding to H' . The claim follows as N_{n-2} is the number of connected induced subgraphs of $G(\mathbf{u})$ which contain w_{n-2} but not w_{n-1} , N_{n-1} is the number of connected induced subgraphs of $G(\mathbf{u})$ which contain w_{n-1} , and 1 counts the subgraph $\{w_n\}$.

□

Remark 4.10. When $\mathbf{w} = \varepsilon R R L L R R L L \dots$, the poset $Q_{\mathbf{w}} = P(\varepsilon R L R L R L \dots) = S_k$ is the snake poset. The number of circuits of the order polytope of the snake poset is equal to the number of nonempty connected induced subgraphs of the graph TS_{2k+1} , defined as follows. For odd n , let TS_n denote the graph on n vertices formed by taking the the path graph on n vertices P_n and adding edges $(2i-1, 2i+1)$ for $i = 1, \dots, (n-1)/2$. The graph TS_n is called a *triangular snake graph*; see [Gal98] and the references therein for additional information about triangular snakes.

The properties of circuits imply the following results regarding triangulations of $\mathcal{O}(Q_{\mathbf{w}})$.

Lemma 4.11. *Let $\mathbf{w} \in \mathcal{V}$. If two triangulations of the polytope $\mathcal{O}(Q_{\mathbf{w}})$ are connected by a flip, then they have the same number of simplices.*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2$ be a pair of triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ that differ by a flip at circuit Z . Then $\mathcal{T}_1, \mathcal{T}_2$ are related as in Theorem 2.7. By Corollary 4.7, we have $|Z_-| = |Z_+|$, which implies that the two triangulations $\mathcal{T}_Z^-, \mathcal{T}_Z^+$ of Z have the same number of simplices. Therefore, because by definition the link L is disjoint from both $\mathcal{T}_Z^-, \mathcal{T}_Z^+$, we see that $\mathcal{T}_1 = \mathcal{T}_Z^+ * L$ and $\mathcal{T}_2 = \mathcal{T}_Z^- * L$ also have the same number of simplices. This shows that the number of simplices in a triangulation does not change when performing a flip. □

Theorem 4.12. *For $\mathbf{w} \in \mathcal{V}$, every vertex of the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$ is a unimodular triangulation. Thus, every triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ is unimodular.*

Proof. The vertices of a secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$ correspond to regular triangulations of $\mathcal{O}(Q_{\mathbf{w}})$, see [DLRS10, Theorem 5.1.9]. Moreover, all regular triangulations are connected by flips [DLRS10, Theorem 5.3.1]. In particular, because the canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ is regular and unimodular, then Lemma 4.11 implies that all regular triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ are also unimodular. Finally, note that for a lattice polytope P , the following conditions are equivalent: all full-dimensional simplices formed from the lattice points in P are unimodular; all triangulations of P are unimodular; all regular triangulations of P are unimodular; all placing triangulations of P are unimodular. It is straightforward that each condition in this list implies the next. To show that all placing triangulations being unimodular implies that all full-dimensional simplices are unimodular, note that any full-dimensional simplex can be used as the initial simplex in a placing triangulation, and thus we get a circle of equivalences. Hence, all triangulations are unimodular. □

5. Flips and a twist action on triangulations

In this section we will take a deeper look at the 1-skeleton of the secondary polytope of $\mathcal{O}(Q_w)$. Starting from the canonical triangulation, we will see that for a length k word there are exactly $k + 1$ flips, where a single flip corresponds to a local move along an edge in the flip graph. As a consequence, we fully determine the flip graph of regular triangulations in the special case of the ladder. We will also introduce the notion of twists which act globally by inducing automorphisms on the flip graph.

5.1. Theorems regarding twists, flips, and triangulations

Using the notation from Section 4, let w be a generalized snake word in \mathcal{V} and consider the associated poset Q_w . In this section, our goal is to prove four theorems about flips of regular triangulations for $\mathcal{O}(Q_w)$. We state the four theorems below; all undefined terms and proofs will be given in later subsections. First, we classify the flips that can be made from the canonical triangulation of $\mathcal{O}(Q_w)$.

Theorem 5.1. *Let $w \in \mathcal{V}$ have length k . The canonical triangulation of $\mathcal{O}(Q_w)$ admits exactly $k + 1$ flips.*

As an application, we determine the flip graph of regular triangulations for the special case of a ladder. When $w = \varepsilon L^{n-1}$, $\widehat{P} \setminus \{\widehat{0}, \widehat{1}\}$ is the product of a $(n + 1)$ -chain and a 2-chain. Thus the next result is a rephrasing of the well-known result that the secondary polytope of the Cartesian product of an n -simplex and 1-simplex is an n -dimensional permutahedron [GOT18, Section 16.7.1].

Theorem 5.2. *Let $w = \varepsilon L^{n-1}$, and $Q_w = \text{Irr}_\wedge(\widehat{P}(w))$. The flip graph of triangulations of $\mathcal{O}(Q_w)$ is the Cayley graph of the symmetric group \mathfrak{S}_{n+1} with the simple transpositions as the generating set.*

Third, we introduce the following group. Let $\widehat{P} = \widehat{P}(w)$ be defined as in the previous section. We can then think of \widehat{P} as being made up of $\widehat{0}, \widehat{1}$, and ladders $\mathcal{L}^1, \dots, \mathcal{L}^t$ for $t \geq 1$ defined as follows. Given the vertices w_0, \dots, w_k of $G(w)$, let w_{i_1} be the first index such that there is an edge from w_{i_1} to w_{i_1+2} . Then \mathcal{L}^1 is the ladder in \widehat{P} induced by the elements of $\cup_{j=0}^{i_1+1} \text{Sq}(w_j)$. Let w_{i_2} be the next vertex where there is an edge from w_{i_2} to w_{i_2+2} . Then \mathcal{L}^2 is the ladder in \widehat{P} induced by the elements of $\cup_{j=i_1+1}^{i_2+1} \text{Sq}(w_j)$. Inductively define \mathcal{L}^i in a similar fashion. Note that by definition these ladders are disjoint except that $\mathcal{L}^i \cap \mathcal{L}^{i+1}$ is a single square corresponding to a corner box in \widehat{P} . That is, $\mathcal{L}^i \cap \mathcal{L}^{i+1}$ comes from the underlined letter $\dots R\underline{R}L \dots$ or $\dots L\underline{L}R \dots$ in the expression for w . Moreover, we index the ladders so that y , the top element of \mathcal{L}^1 , is covered by $\widehat{1}$ in \widehat{P} . That is, $y \prec \widehat{1}$. Since w avoids subwords LRL and RLR , each \mathcal{L}^i , for $1 < i < t$, consists of at least three squares and $\mathcal{L}^1, \mathcal{L}^t$ consist of at least two squares, except for the case where $w = \varepsilon$, in which case we have one square and one ladder. For example, in Figure 4.2 the poset \widehat{P} consists of five ladders $\mathcal{L}^1, \dots, \mathcal{L}^5$ made up of 4, 3, 5, 6, and 3 boxes respectively.

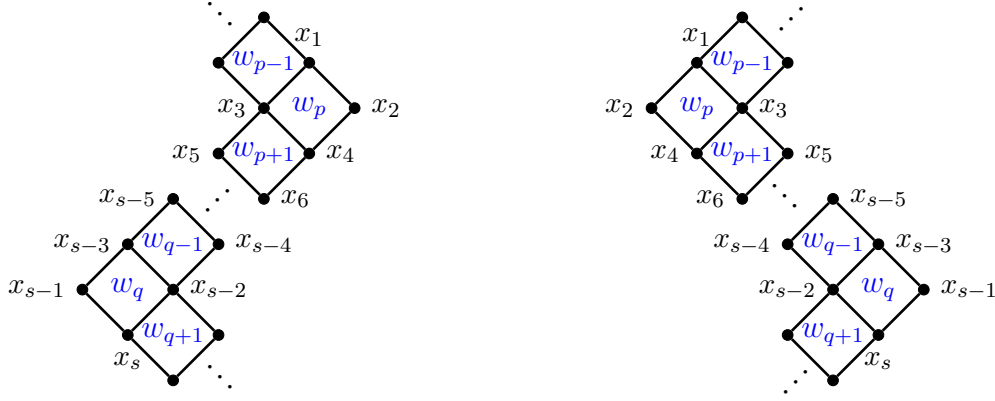


Figure 5.1: Ladder \mathcal{L}^i in \widehat{P} containing boxes with labels w_p, \dots, w_q , where $w_p < w_{p+1} < \dots < w_q$. The left (right) represents the case where $w_q = L$ ($w_q = R$).

Let V_0 denote the set of vertices of \widehat{P} . Next, we define a collection of certain permutations on elements of V_0 . Consider the ladder \mathcal{L}^i for $i \in [t]$ in the poset \widehat{P} . Then \mathcal{L}^i has the following structure up to a reflection of \widehat{P} in a vertical axis. Label the vertices of \mathcal{L}^i as x_1, \dots, x_s for some even integer s as in Figure 5.1. In the case where $\mathbf{w} = \varepsilon$, we resolve the ambiguity of the labeling by choosing the convention that the left and right elements in the antichain of the square have labels x_2 and x_3 respectively.

Definition 5.3. Given a ladder \mathcal{L}^i , define $\tau_i \in \mathfrak{S}_{|V_0|}$ to be the permutation of V_0 such that for $v \in V_0$,

$$\tau_i(v) = \begin{cases} x_{j-1}, & \text{if } v = x_j \text{ and } j \in [s] \text{ is even,} \\ x_{j+1}, & \text{if } v = x_j \text{ and } j \in [s] \text{ is odd,} \\ v, & \text{otherwise.} \end{cases}$$

Hence, τ_i acts on V_0 by reflecting the vertices of \mathcal{L}^i across a diagonal and fixing the remaining vertices. The next lemma says that the set of τ_i for $i \in [t]$ generate a commutative subgroup of $\mathfrak{S}_{|V_0|}$.

Lemma 5.4. For all $\tau_i, \tau_r \in \mathfrak{S}_{|V_0|}$, the following properties hold.

(a) $\tau_i^2 = 1$

(b) $\tau_i \tau_r = \tau_r \tau_i$

Proof. Part (a) follows directly from the definition of τ_i . Part (b) is straightforward when $|r - i| \geq 2$ because the ladders $\mathcal{L}^r, \mathcal{L}^i$ have no vertices in common. The case $r = i$ follows from part (a), so it suffices to consider the case $r = i + 1$. Moreover, it is enough to check the action of the τ 's on the vertices of the square in $\mathcal{L}^i \cap \mathcal{L}^{i+1}$. Label the vertices of this square as x_a, x_b, x_c, x_d and consider the computations $\tau_i \tau_{i+1}$ and $\tau_{i+1} \tau_i$, as shown in Figure 5.2. This shows that $\tau_i \tau_{i+1} = \tau_{i+1} \tau_i$ as desired. \square

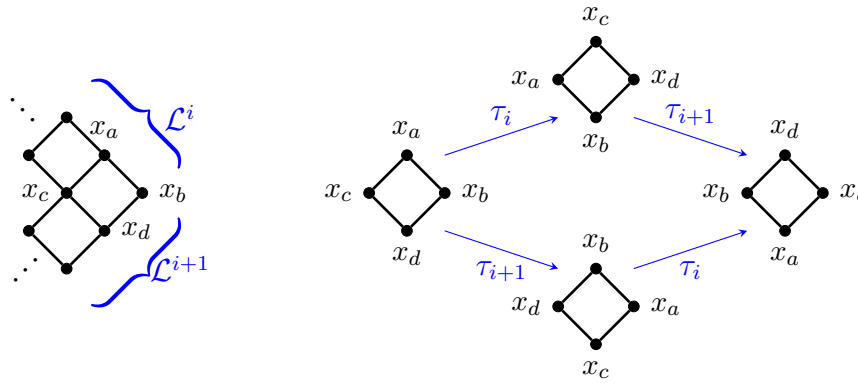


Figure 5.2: The action of τ_i and τ_{i+1} commute.

Definition 5.5. Let $\mathfrak{T}(\mathbf{w})$ denote the subgroup of $\mathfrak{S}_{|V_0|}$ generated by the set of the τ_i 's. We call $\mathfrak{T}(\mathbf{w})$ the twist group of $\widehat{P}(\mathbf{w})$. Elements of $\mathfrak{T}(\mathbf{w})$ are called *twists* and the elements τ_i are called *elementary twists*.

Note that by Lemma 5.4, $\mathfrak{T}(\mathbf{w}) = \langle \tau_i \mid i \in [t] \rangle$ is isomorphic to \mathbb{Z}_2^t . The fact that elementary twists commute will be an important factor in several proofs in this section. As the next theorem demonstrates, the twist group acts on the component of the flip graph of triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ containing the canonical triangulation, and flips are preserved by twists. A priori, a simplex σ in the triangulation T after twisting becomes a collection of vertices $\tau(\sigma)$ of $\mathcal{O}(Q_{\mathbf{w}})$ that may or may not also form a simplex. Hence, a twist $\tau(T)$ of a triangulation T is a collection of subsets of vertices obtained by applying the twist τ to every simplex in T , so $\tau(T)$ is not necessarily a triangulation. However, in the case when twisting results in a triangulation, the following theorem says that twists and flips behave well with each other.

Recall that if Z is a circuit in $\mathcal{O}(Q_{\mathbf{w}})$ and T is a triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ that admits a flip using Z , then $T = T_Z^+$ and T_Z^- are the triangulations related by flips at Z .

Theorem 5.6. Let $\mathbf{w} \in \mathcal{V}$, $Q_{\mathbf{w}} = \text{Irr}_{\wedge}(\widehat{P}(\mathbf{w}))$, and let T and $\tau(T)$ be two triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ where τ is a twist. If $T = T_Z^+$ can be flipped at circuit Z and $\tau(T_Z^+) = \tau(T_Z^+)_{\tau(Z)}^+$, then $\tau(T_Z^+)_{\tau(Z)}^- = \tau(T_Z^-)$. In other words, the following diagram commutes.

$$\begin{array}{ccc}
 T_Z^+ & \xrightarrow{\text{flip in } Z} & T_Z^- \\
 \text{twist} \downarrow & & \downarrow \text{twist} \\
 \tau(T_Z^+) = \tau(T_Z^+)_{\tau(Z)}^+ & \xrightarrow{\text{flip in } \tau(Z)} & \tau(T_Z^+)_{\tau(Z)}^- = \tau(T_Z^-)
 \end{array}$$

Corollary 5.7. Let $\mathbf{w} \in \mathcal{V}$, $Q_{\mathbf{w}} = \text{Irr}_{\wedge}(\widehat{P}(\mathbf{w}))$, and let T and $\tau(T)$ be two triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ where τ is a twist. Then T and $\tau(T)$ admit the same number of flips.

Proof. As will be seen later, a twist τ is an involution on $J(Q_{\mathbf{w}})$ that yields an involution on circuits of $\mathcal{O}(Q_{\mathbf{w}})$. Thus, we can apply τ to $\tau(T)$ and recover T . This shows that there is a bijective correspondence between flips from T and flips from $\tau(T)$. \square

Lastly, we show that twists of the canonical triangulation are also regular triangulations.

Theorem 5.8. *Let $w \in \mathcal{V}$ and $Q_w = \text{Irr}_\wedge(\widehat{P}(w))$. The canonical triangulation \mathcal{T}_w of $\mathcal{O}(Q_w)$ is a regular triangulation, and for any twist τ , $\tau(\mathcal{T}_w)$ is also a regular triangulation.*

For each of these theorems, we have dedicated one subsection that follows to their proof.

5.2. Proof of Theorem 5.1

For the canonical triangulation of $\mathcal{O}(Q_w)$, simplices correspond to maximal chains in $J(Q_w)$. In order for a circuit Z to be supported on a flip in the canonical triangulation, then either \mathcal{T}_+ or \mathcal{T}_- must be a subcomplex of the canonical triangulation. Consider the circuits corresponding to a single square $\text{Sq}(w_i) = \{A, A \cup \{a\}, A \cup \{b\}, A \cup \{a, b\}\}$ in $J(Q_w)$. If we assign the left $A \cup \{a\}$ and right $A \cup \{b\}$ vertices of the square to be have negative signs in the circuit, and we assign positive signs to the upper vertex A and the lower vertex $A \cup \{a, b\}$, then the canonical triangulation contains \mathcal{T}_- , where \mathcal{T}_- consists of the two triangles each formed by the positive vertices and one of the negative vertices in $\text{Sq}(w_i)$. Further, for each of these triangles, the link in the canonical triangulation is the set of all chains contained in $\{x \in J(Q_w) \mid x < A \cup \{a, b\}\} \cup \{x \in J(Q_w) \mid x > A\}$, and thus \mathcal{T}_- can be flipped at $\text{Sq}(w_i)$. However, \mathcal{T}_+ is not contained in the canonical triangulation, because there is not a maximal chain in $J(Q_w)$ that runs through the two negative vertices, which form an antichain in $J(Q_w)$. Thus, for each of the $k + 1$ squares in $J(Q_w)$, we can flip the corresponding circuit.

If we have a circuit in $J(Q_w)$ that does not come from a square, then we consider two cases. If the circuit is of size four, then the corresponding connected induced subgraph has vertices $w_{i_1}, w_{i_1+1}, \dots, w_{i_1+r}$. In this case, because there are elements of $J(Q_w)$ strictly between the elements of the circuit and they lie in different ranks, the links are different for the two faces in \mathcal{T}_- . For example, suppose that $\text{Sq}(w_{i_1})$ consists of the upper order ideals $A, A \cup \{a\}, A \cup \{c\}$, and $A \cup \{a, c\}$. Suppose also that $\text{Sq}(w_{i_1+r})$ consists of the upper order ideals $B, B \cup \{b\}, B \cup \{c\}$, and $B \cup \{b, c\}$. See Figure 5.3 for an example. Then the upper order ideals $A, A \cup \{c\}, B \cup \{b, c\}$ form a triangle in \mathcal{T}_- , as does $A, B \cup \{b\}, B \cup \{b, c\}$. Note that $B \cup \{c\}$ is in the link of the first triangle but not the second. A similar argument holds in general, and thus this circuit does not support a flip.

For a circuit of size greater than four, the word defining the corresponding induced subgraph either contains a subword of the form $w_{i_j}w_{i_j+2}$ or contains a turn, i.e., contains one of RRL, LLR, RLL , or LRR . Consider the first case, where a subword of the form $w_{i_j}w_{i_j+2}$ is present. See Figure 4.2 for an example with w_4w_6 . In this case, it is straightforward to verify that there exist at least two incomparable pairs in the circuit, where the elements of each pair share the same sign in the circuit and each pair has a different sign. Thus, both \mathcal{T}_+ and \mathcal{T}_- have simplices containing antichains, and hence the canonical triangulation does not contain any triangulations arising from circuits of this type.

Next consider the case where the induced subgraph contains a turn, i.e., contains one of RRL, LLR, RLL , or LRR . Again see Figure 4.2 for an example with $w_2w_3w_4$. In this case, the existence of a turn implies that there are two antichains $\{A, B\}$ and $\{B, C\}$ where each upper order ideal has the same sign in the circuit. Again, this implies that both \mathcal{T}_+ and \mathcal{T}_- have simplices

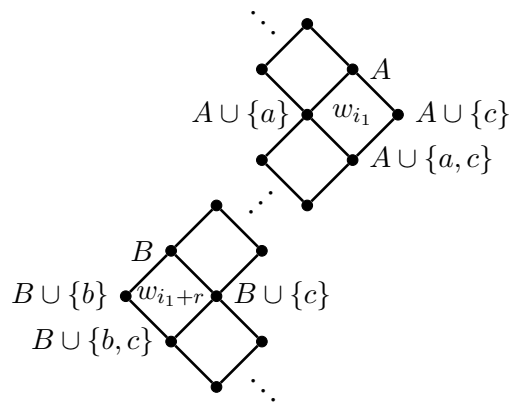


Figure 5.3: A circuit failing the link condition.

containing antichains, and hence the canonical triangulation does not contain any triangulations arising from circuits of this type.

Thus, the canonical triangulation admits only flips in circuits formed by the squares, and each of the resulting triangulations is distinct, from which the result follows.

5.3. Proof of Theorem 5.2

Let $w = \varepsilon L^{n-1}$. Then $\widehat{P}(w) \setminus \{\hat{0}, \hat{1}\}$ is the ladder with n squares, and the graph $G(w)$ associated to $\widehat{P}(w)$ is the path graph with n vertices. By Theorem 4.5, the circuits of the vertex set of the order polytope $\mathcal{O}(Q_w)$ are in bijection with the nonempty connected induced subgraphs of $G(w)$, so in this context, every circuit has exactly 4 elements, corresponding to some nonempty connected subgraph of the path graph with n vertices. Since the circuits never contain $\hat{0}$ or $\hat{1}$, then without loss of generality we only need to concern ourselves with the remaining $2(n+1)$ vertices of $\widehat{P}(w) \setminus \{\hat{0}, \hat{1}\}$, which we label as $[n+1] \cup [(n+1)']$. See Figure 5.4.

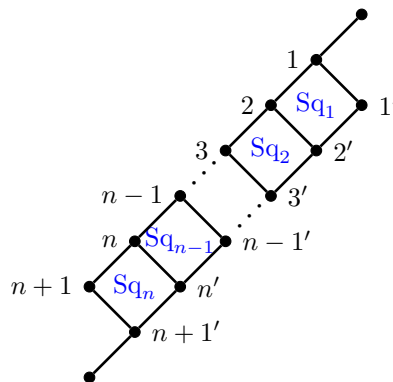


Figure 5.4: The n -ladder $J(Q_w) = \widehat{P}(w)$ of Theorem 5.2.

There are $\binom{n+1}{2}$ circuits; explicitly, the circuits are $Z_{i,j} = \{i, i', j, j'\}$ for $1 \leq i < j \leq n+1$. In particular, it follows from Theorem 5.1 that the circuits which support a flip in the canonical triangulation $\mathcal{T}_{\mathbf{w}}$ are $Z_{i,i+1}$ for $i = 1, \dots, n$.

Next, we define maps on the labels of $\widehat{P}(\mathbf{w}) \setminus \{\hat{0}, \hat{1}\}$. Given $1 \leq i \leq n$, if the labels on the four vertices of the i -th square Sq_i are a, a', b, b' , then π_i swaps a with b , and a' with b' . In other words, π_i permutes places, not values. See Figure 5.5. Compare this to the twist maps which permute values, not places. It is clear from this definition that π_i acts as a simple transposition on the labels of $\widehat{P}(\mathbf{w}) \setminus \{\hat{0}, \hat{1}\}$ so that $\pi_i^2 = 1$, and moreover, π_1, \dots, π_n generate the symmetric group \mathfrak{S}_{n+1} .

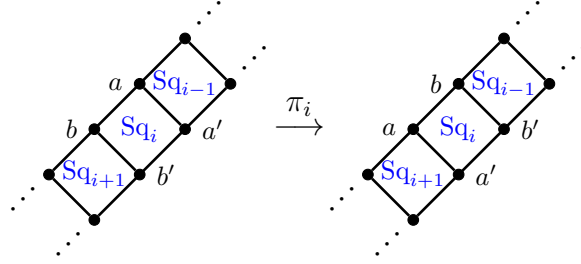


Figure 5.5: The action of π_i on the labels of $\widehat{P}(\varepsilon L^{n-1})$.

Lemma 5.9. *Let $\mathcal{T}_{\mathbf{w}}$ denote the canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$. Let \mathcal{U}_i denote the triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ that differs from $\mathcal{T}_{\mathbf{w}}$ by the flip supported at the circuit $Z_{i,i+1}$ for $1 \leq i < n$. Then the simplices of \mathcal{U}_i are the maximal chains of the poset $\pi_i \cdot \widehat{P}(\mathbf{w})$. Thus, \mathcal{U}_i is a canonical triangulation of an order polytope, and hence is regular.*

Proof. Let

$$Z_{i,i+1} = ((Z_{i,i+1})_+, (Z_{i,i+1})_-) = (\{i, (i+1)'\}, \{i+1, i'\})$$

be the oriented circuit so that the canonical triangulation $\mathcal{T}_{\mathbf{w}}$ contains $\mathcal{T}_{\mathcal{Z}}^-$ but not $\mathcal{T}_{\mathcal{Z}}^+$. The $n+1$ maximal chains in $\widehat{P}(\mathbf{w})$ are of the form

$$C_i : 1, 2, \dots, i, i', \dots, n', (n+1)'$$

for $i = 1, \dots, n+1$. To make a flip from $\mathcal{T}_{\mathbf{w}}$ to \mathcal{U}_i , each maximal chain in $\widehat{P}(\mathbf{w})$ containing $\{i, i+1, (i+1)'\}$ has that triplet replaced by $\{i, i+1, i'\}$, and every maximal chain in $\widehat{P}(\mathbf{w})$ containing $\{i, i', (i+1)'\}$ has that triplet replaced by $\{i+1, i', (i+1)'\}$. So the flip affects only the two chains C_i and C_{i+1} , where effectively, $(i+1)'$ is replaced by i' , and i is replaced by i' . Thus the simplices of \mathcal{U}_i are precisely the maximal chains in the poset $\pi_i \cdot \widehat{P}(\mathbf{w})$. \square

We will denote the triangulation \mathcal{U}_i by $\pi_i \mathcal{T}_{\mathbf{w}}$.

Lemma 5.10. *Let \mathcal{T} be a regular triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ whose simplices are maximal chains in a labeled n -ladder poset $\widehat{P}_{\mathcal{T}}$. There are exactly n regular triangulations that differ from \mathcal{T} by a circuit flip, and each of these triangulations have simplices which are the maximal chains in $\pi_i \widehat{P}_{\mathcal{T}}$ for $i = 1, \dots, n$.*

Proof. Proceed by induction on the number of flips away from the canonical triangulation \mathcal{T}_w . The base case follows from Lemma 5.9. Suppose $\mathcal{T} = \pi_{i_\ell} \cdots \pi_{i_1} \mathcal{T}_c$ for some (reduced) sequence of transpositions $\pi_{i_\ell}, \dots, \pi_{i_1} \in \mathfrak{S}_{n+1}$. Let $\pi = \pi_{i_\ell} \cdots \pi_{i_1}$. The circuits of \mathcal{T} are then

$$Z_{\pi^{-1}(j), \pi^{-1}(k)} = \{\pi^{-1}(j), \pi^{-1}(j'), \pi^{-1}(k), \pi^{-1}(k')\}$$

for $1 \leq j < k \leq n + 1$. In particular, for $k = 1, \dots, n$, the four elements of the circuit $Z_{\pi^{-1}(k), \pi^{-1}(k+1)}$ are labels on the square $\text{Sq}_{\pi^{-1}(k)}$ of the poset $\widehat{P}_{\mathcal{T}}$, and these are the only circuits which support a flip as they are the only ones whose links are the same for the two faces in \mathcal{T}_Z^- . So a flip in this circuit yields a triangulation $\pi\mathcal{T}$ whose simplices are maximal chains in the poset $\pi \cdot \widehat{P}_{\mathcal{T}}$. \square

Since the transpositions π_1, \dots, π_n generate \mathfrak{S}_{n+1} , the proof of Theorem 5.2 now follows.

5.4. Proof of Theorem 5.6

We use the notation from Definition 5.5. We can naturally extend the action of $\mathfrak{T}(w)$ on V_0 to the action of $\mathfrak{T}(w)$ on subsets of V_0 . The following lemma states that the twist group also acts on the circuits of the vertices of $\mathcal{O}(Q_w)$.

Lemma 5.11. *Let $w \in \mathcal{V}$, $Z = (Z_+, Z_-)$ be a circuit on V_0 , and $\tau \in \mathfrak{T}(w)$. Then $\tau(Z) := (\tau(Z_+), \tau(Z_-))$ is also a circuit on V_0 .*

Proof. Because $\mathfrak{T}(w)$ is abelian, it suffices to show that $\tau_k(Z)$ is a circuit for every k . We use the labeling of the squares and vertices of \mathcal{L}^k given in Figure 5.1. Also, by Theorem 4.5 we have $Z = \Gamma(H_{w'})$, where $H_{w'}$ is an induced connected subgraph of $G(w)$ corresponding to a subword w' of w .

Observe that if Z does not contain any vertex of \mathcal{L}^k then $\tau_k(Z) = Z$ and the lemma holds. If every vertex of Z is also a vertex of \mathcal{L}^k , then Z consists of four vertices and we have $Z = (\{x_i, x_j\}, \{x_{i+1}, x_{j-1}\})$ up to interchanging Z_+, Z_- for some i odd and j even. Then,

$$\tau_k(Z) = (\{x_{i+1}, x_{j-1}\}, \{x_i, x_j\}) = (Z_-, Z_+)$$

which is the same circuit as Z . It remains to consider the case where Z contains vertices both in \mathcal{L}^k and outside of \mathcal{L}^k . In this case, it must be that $Z = \Gamma(H_{w'})$ where w' contains one or both of w_{p-1} and w_{q+1} .

If w' contains w_{p-1} and no letter in \mathcal{L}^k , then τ_k applied to Z replaces x_1, x_3 with x_2, x_4 respectively. Hence, $\tau_k(Z) = \Gamma(H_{w'w_p})$ is again a circuit. Similar computation holds if w' contains w_{q-1} and no letter in \mathcal{L}^k . Since $\tau_k^2 = 1$ by Lemma 5.4(a), this also resolves the case when w' ends in $w_{p-1}w_p$ or starts with w_qw_{q+1} .

The cases when w' contains w_{p-1}, w_{p+1} but not w_{q+1} or w' contains w_{q-1}, w_{q+1} but not w_{p-1} follow similarly to the case when w' contains both w_{p-1} and w_{q+1} ; therefore, we only provide a detailed proof for the latter case. Suppose that w' contains both w_{p-1} and w_{q+1} . Then w' contains w_{p+1}, \dots, w_{q-1} and it may or may not contain each of w_p, w_q . We treat the situation

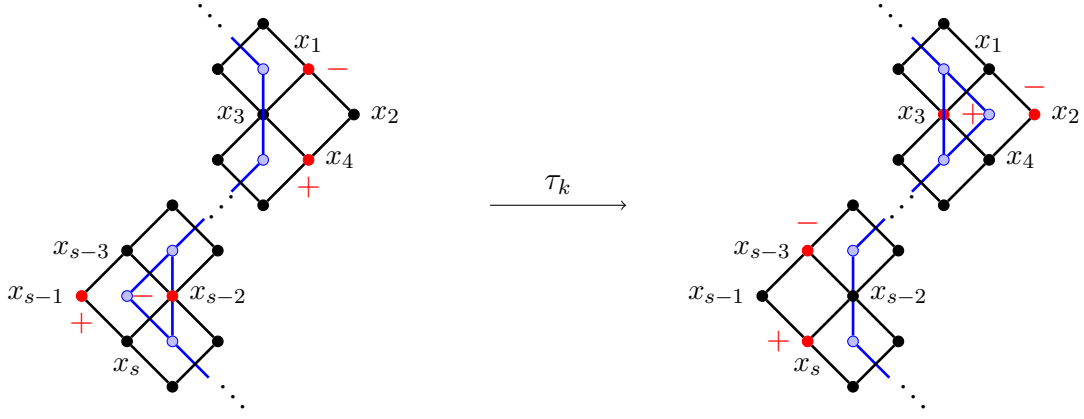


Figure 5.6: Two cases for proof of Lemma 5.11.

when \mathbf{w}' contains w_q but not w_p , and the other possibilities follow similarly. Up to interchanging Z_+ and Z_- , we have the following situation where $x_1, x_{s-2} \in Z_-$ and $x_{s-1}, x_4 \in Z_+$ and no other vertex in \mathcal{L}^k appears in the circuit Z . This is depicted in the left-hand side of Figure 5.6.

In this case,

$$\tau_k(Z_-) = (Z_- \setminus \{x_1, x_{s-2}\}) \cup \{x_2, x_{s-3}\}$$

and

$$\tau_k(Z_+) = (Z_+ \setminus \{x_4, x_{s+1}\}) \cup \{x_3, x_s\}.$$

In particular, $\tau_k(Z) = \Gamma(H_{\mathbf{w}''})$ where \mathbf{w}'' is obtained from \mathbf{w}' by adding w_p and removing w_q . Therefore, $\tau_k(Z)$ is a circuit. The remaining cases are proved in a similar fashion. This completes the proof of the lemma. \square

Proof of Theorem 5.6. Let $\mathcal{T} = \mathcal{T}_Z^+$ and \mathcal{T}_Z^- be triangulations related by a flip at the circuit Z . Let $\mathcal{T}|_Z$ denote the restriction of \mathcal{T} to the circuit Z . Since we can flip \mathcal{T} at Z , the links of simplices of $\mathcal{T}|_Z$ in \mathcal{T} match. By Lemma 5.11, the twist $\tau(Z)$ is also a circuit, so the links of $\tau(\mathcal{T})|_{\tau(Z)}$ in $\tau(\mathcal{T})$ are obtained from the links of $\mathcal{T}|_Z$ in \mathcal{T} by applying the permutation τ . Thus, they also match. Therefore, we obtain two triangulations related by a flip through $\tau(Z)$, denoted $\tau(\mathcal{T}) = \tau(\mathcal{T})_{\tau(Z)}^+$ and $\tau(\mathcal{T})_{\tau(Z)}^-$. It suffices to show that $\tau(\mathcal{T})_{\tau(Z)}^-$ and $\tau(\mathcal{T}_Z^-)$ are the same as sets.

Every full-dimensional simplex σ in $\mathcal{T} = \mathcal{T}_Z^+$ that does not contain a simplex supported on Z remains a simplex in \mathcal{T}_Z^- . Hence $\tau(\mathcal{T}_Z^-)$ contains $\tau(\sigma)$ as a subset. Also, $\tau(\sigma)$ is a simplex of $\tau(\mathcal{T}) = \tau(\mathcal{T})_{\tau(Z)}^+$ that does not a simplex supported on $\tau(Z)$, so it remains a simplex in $\tau(\mathcal{T})_{\tau(Z)}^-$ after the flip. Every full-dimensional simplex σ in \mathcal{T} that contains a simplex supported on Z becomes $(\sigma \setminus \{\sigma_+\}) \cup \{\sigma_-\}$ after the flip, for some appropriate pair of subsets σ_+, σ_- obtained from Z by removing a single element in Z_+, Z_- respectively. Similarly, $\tau(\sigma)$ contains a simplex supported on $\tau(Z)$, and after the flip supported at the circuit $\tau(Z)$ it becomes $(\tau(\sigma) \setminus \{\tau(\sigma_+)\}) \cup \{\tau(\sigma_-\})$ which equals $\tau((\sigma \setminus \{\sigma_+\}) \cup \{\sigma_-\})$. \square

5.5. Proof of Theorem 5.8

We thank an anonymous referee for highlighting the fact that the Hasse diagrams of the posets Q_w are strongly planar, and thus the work of Mészáros, Morales and Striker [MMS19] shows that the order polytope $\mathcal{O}(Q_w)$ is integrally equivalent to a flow polytope. An earlier version of this article contained a much longer direct proof of the regularity results, but a more elegant proof is possible through the connection with flow polytopes and the work of Danilov, Karzanov and Koshevoy [DKK12], which we now explain. We follow the definitions and notations as set forth in [DKK12] and [MMS19].

Given a directed acyclic graph G , a *framing* of G is a collection of linear orders on the incoming and outgoing edge sets at every non-source and non-sink vertex of G . For any directed acyclic graph with a fixed framing, Danilov, Karzanov and Koshevoy constructed a triangulation of the flow polytope \mathcal{F}_G , and showed that the triangulation is regular and unimodular [DKK12, Theorem 2].

A poset Q is *strongly planar* if, after adjoining $\hat{0}$ and $\hat{1}$, the Hasse diagram of $Q \cup \{\hat{0}, \hat{1}\}$ has a planar embedding such that its y -coordinates respect the order of the poset. Let G_Q denote the *truncated dual graph* [MMS19, Section 3.3] arising from the Hasse diagram of $Q \cup \{\hat{0}, \hat{1}\}$. Then Theorem 3.14 of Mészáros, Morales and Striker states that the order polytope $\mathcal{O}(Q)$ is integrally equivalent to the flow polytope \mathcal{F}_{G_Q} .

Since Q_w is strongly planar, then by Theorem 1.3 of Mészáros, Morales and Striker, the canonical triangulation \mathcal{T}_w of $\mathcal{O}(Q_w)$ maps to the Danilov–Karzanov–Koshevoy triangulation of the flow polytope $\mathcal{F}_{G_{Q_w}}$ with the planar framing [MMS19, Section 6.2] of the graph G_{Q_w} under an integral equivalence map. Since the planar-framed triangulation of $\mathcal{F}_{G_{Q_w}}$ is regular, then so is the canonical triangulation of $\mathcal{O}(Q_w)$.

The twist group $\mathfrak{T}(w) = \langle \tau_i \mid i \in [t] \rangle$ acts on the canonical triangulation \mathcal{T}_w of $\mathcal{O}(Q_w)$, so it is natural to ask how $\mathfrak{T}(w)$ acts on the planar-framed triangulation of $\mathcal{F}_{G_{Q_w}}$. If $\hat{P} = \hat{P}(w)$ has t ladders, then the truncated dual G_{Q_w} has t vertices (one for each ladder of \hat{P}) together with an additional source vertex a and a sink vertex b . In particular, for $i \in [t]$, if the i -th ladder of \hat{P} is a ladder of the form L^m (respectively R^m), then there are exactly two edges (a, i) (respectively two edges (i, b)) in G_{Q_w} .

In the flow polytope setting, the twist τ_i transposes the framing on the pair of edges in G_{Q_w} of the form (a, i) or (i, b) , and this pair of edges is determined by the two vertices at the top of the i -th ladder of \hat{P} . Hence, for any twist τ , $\tau(\mathcal{T}_w)$ corresponds to a framed triangulation of $\mathcal{F}_{G_{Q_w}}$. By Theorem 2 of Danilov, Karzanov and Koshevoy, we can conclude that the triangulation $\tau(\mathcal{T}_w)$ is regular.

Corollary 5.12. *For $w \in \mathcal{V}$, the component of the flip graph of $\mathcal{O}(Q_w)$ containing all regular triangulations admits a \mathbb{Z}_2^t action given by twists.*

Proof. By Theorem 5.8, a twist of a canonical triangulation is a regular triangulation of $\mathcal{O}(Q_w)$, so both triangulations lie in the same connected component of the flip graph. Any two triangulations in this component are connected by a sequence of flips, hence Theorem 5.6 allows us to extend the action of twists on all triangulations in this component. Moreover, this action respects

the edge structure of the flip graph. This implies that this component admits a \mathbb{Z}_2^t action given by twists as claimed. \square

6. Future Directions

We conclude with several conjectures. Throughout this article, we studied generalized snake posets $\widehat{P}(\mathbf{w})$, with our main goal being to study the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$.

In Theorem 5.2, when $\mathbf{w} = \varepsilon L^{n-1}$ and $\widehat{P}(\mathbf{w})$ is the n -ladder, we saw that the 1-skeleton of the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$ is the Cayley graph of \mathfrak{S}_{n+1} , which is an n -regular graph. Furthermore, Corollary 5.7 showed that each triangulation which is generated by applying twists to the canonical triangulation admits the same number of flips as the canonical triangulation. In light of these results, along with a limited number of other examples, we conjecture that the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$ is simple, and the degree of each vertex is equal to the dimension of the secondary polytope.

Conjecture 6.1. For $\mathbf{w} \in \mathcal{V}$, the flip graph of regular triangulations for $\mathcal{O}(Q_{\mathbf{w}})$ is k -regular, where k is the dimension of the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$.

In the case when $\widehat{P}(\mathbf{w})$ is the n -ladder and its secondary polytope is a permutohedron, Lemma 5.10 implies that the dual graph of every triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ is the same as the dual graph of the canonical triangulation. When $\widehat{P}(\mathbf{w})$ contains a turn, our computations support the following conjecture.

Conjecture 6.2. If $J(Q_{\mathbf{w}}) = \widehat{P}(\mathbf{w})$ contains a turn, then $\mathcal{O}(Q_{\mathbf{w}})$ has a regular triangulation whose dual graph is not isomorphic to the dual graph of the canonical triangulation.

Naturally, the next case to study in-depth is when $\widehat{P}(\mathbf{w})$ is nearly a ladder. We have verified the following conjecture for $n = 3, 4, 5, 6, 7$.

Conjecture 6.3. If $J(Q_{\mathbf{w}}) = \widehat{P}(\mathbf{w})$, where $\mathbf{w} = \varepsilon LR^{n-2}$ for $n \geq 3$, then the number of triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ whose dual graph is isomorphic to the dual graph of the canonical triangulation is $4n(n-2)!$.

We know that, by Theorem 4.12, all triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ are unimodular. Moreover, all of our computations support the following conjecture.

Conjecture 6.4. If $\mathbf{w} \in \mathcal{V}$, all triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ are regular.

When $\mathbf{w} \in \mathcal{V}$, a twist of a canonical triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ again yields a regular triangulation, by Theorem 5.8. Therefore, if the above conjecture holds, we obtain an action of the twist group on the set of all (regular) triangulations. Hence, the number of triangulations would be divisible by the order of the twist group. In the special case when $Q_{\mathbf{w}} = S_n$ the twist group has order 2^{n+1} . We make the following conjecture about the precise number of regular triangulations of $\mathcal{O}(S_n)$ where there appears to be a relationship between regular triangulations of $\mathcal{O}(S_n)$ and odd Catalan numbers. We have verified this conjecture for $n = 1, 2, 3$.

Conjecture 6.5. The number of regular triangulations of $\mathcal{O}(S_n)$ is $2^{n+1} \cdot \text{Cat}(2n+1)$.

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