

SUB-FIBONACCI BEHAVIOR IN NUMERICAL SEMIGROUP ENUMERATION

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Abstract. In 2013, Zhai proved that most numerical semigroups of a given genus have depth at most 3 and that the number n_g of numerical semigroups of a genus g is asymptotic to $S\varphi^g$, where S is some positive constant and $\varphi \approx 1.61803$ is the golden ratio. In this paper, we prove exponential upper and lower bounds on the factors that cause n_g to deviate from a perfect exponential, including the number of semigroups with depth at least 4. Among other applications, these results imply the sharpest known asymptotic bounds on n_g and shed light on a conjecture by Bras-Amorós (2008) that $n_g \geq n_{g-1} + n_{g-2}$. Our main tools are the use of Kunz coordinates, introduced by Kunz (1987), and a result by Zhao (2011) bounding weighted graph homomorphisms.

Keywords. Numerical semigroup, genus, Kunz coordinate, graph homomorphism

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1. Introduction

A *numerical semigroup* Λ is an additive submonoid of the nonnegative integers \mathbb{N}_0 with finite complement $\mathbb{N}_0 \setminus \Lambda$. The *genus* $g(\Lambda)$ of a numerical semigroup is defined to be the size of $\mathbb{N}_0 \setminus \Lambda$.

Over the past 15 years, significant progress has been made in understanding the number n_g of numerical semigroups with genus g . Perhaps most significantly, in 2013 Zhai proved the following theorem, resolving two conjectures of Bras-Amorós [BA08]:

Theorem 1.1 (Zhai [Zha13]). *We have $n_g \sim S\varphi^g$, where S is some positive constant and $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803$ is the golden ratio.*

However, the following conjecture remains unresolved:

Conjecture 1.2 (Bras-Amorós [BA08]). *For $g \geq 2$, $n_g \geq n_{g-1} + n_{g-2}$.*

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In fact, it is not even known whether $n_g \geq n_{g-1}$ holds for all $g \geq 1$; while Theorem 1.1 implies that $n_g \geq n_{g-1}$ for sufficiently large g , the bounds involved do not make it feasible to manually verify $n_g \geq n_{g-1}$ for small g .

Key properties of numerical semigroups also include the *multiplicity* $m(\Lambda) = \min(\Lambda \setminus \{0\})$, *conductor* $c(\Lambda) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteq \Lambda\}$, and *Frobenius number* $f(\Lambda) = c(\Lambda) - 1$. Recently, Eliahou and Fromentin [EF20] also introduced the *depth* $q(\Lambda) = \lceil c(\Lambda)/m(\Lambda) \rceil$. (From now on, we will omit the argument Λ if it is clear from context.) For example, the numerical semigroup $\mathbb{N}_0 \setminus \{1, 2, 3, 4, 5, 7, 9, 10, 13\}$ has genus 9, multiplicity 6, conductor 14, Frobenius number 13, and depth 3.

Let t_g be the number of numerical semigroups of genus g satisfying $q \leq 3$ and let $\hat{n}_g = n_g - t_g$ be the number of numerical semigroups of genus g with $q > 3$. This split is useful for enumerative purposes since “most” numerical semigroups satisfy $q \leq 3$, as proven by Zhai:

Theorem 1.3 (Zhai [Zha13]). *We have $\lim_{g \rightarrow \infty} \frac{t_g}{n_g} = 1$. In particular, $t_g \sim S\varphi^g$.*

Recently, Eliahou and Fromentin proved the following bounds:

Theorem 1.4 (Eliahou–Fromentin [EF20]). *For $g \geq 3$, $t_{g-1} + t_{g-2} \leq t_g \leq t_{g-1} + t_{g-2} + t_{g-3}$.*

In this paper, we study the asymptotics of the quantities \hat{n}_g and $s_g = t_g - t_{g-1} - t_{g-2}$. Our main result is the following bound:

Theorem 1.5. *Let $r_1 = \limsup_{g \rightarrow \infty} s_g^{1/g}$ and $r_2 = \limsup_{g \rightarrow \infty} \hat{n}_g^{1/g}$. Then $r_{1.51} \leq r_1 \leq r_2 \leq r_{1.54}$, where $r_{1.51} \approx 1.51519$ and $r_{1.54} \approx 1.54930$ are algebraic numbers defined in Section 2.*

The two most important tools in the proof of Theorem 1.5 are the use of Kunz coordinates and a result by Zhao giving bounds on graph homomorphisms.

One notable aspect of our proof is that it is independent of Theorems 1.1 and 1.3. In fact, it is easy to see that Theorem 1.3 is a consequence of Theorem 1.5. Moreover, it is possible to use Theorem 1.5 to prove the following bound on n_g :

Theorem 1.6. *Let $S = \frac{\varphi}{\sqrt{5}}(1 + \sum_{g \geq 3} s_g \varphi^{-g})$. Then $\limsup_{g \rightarrow \infty} |n_g - S\varphi^g|^{1/g} \leq r_{1.54}$.*

In particular, Theorem 1.6 implies that $|n_g - S\varphi^g| = o(\varphi^g)$, so it is a strengthening of Theorem 1.1 and provides an alternative expression for the constant S . (For a comparison of this expression with others, see Remark 7.1.) Therefore, an additional contribution of this paper is to give alternative proofs of Theorems 1.1 and 1.3.

We also have the following result, which may be seen as another generalization of Theorem 1.3:

Theorem 1.7. *Fix some $0 < \varepsilon < 1$. Then, if $n_{g,\varepsilon}$ is the number of numerical semigroups of genus g with $f > (2 + \varepsilon)m$, we have $\limsup_{g \rightarrow \infty} n_{g,\varepsilon}^{1/g} < \varphi$.*

More generally, our approach also provides an important framework for answering statistical questions about a “typical” numerical semigroup of a given genus. For example, we show in Theorem 6.2 that, for most numerical semigroups Λ of a given genus, the quantity $f(\Lambda) - 2m(\Lambda)$

is close to zero, and we determine its limiting distribution, strengthening results of Kaplan and Ye [KY13] and Singhal [Sin22]. It is likely possible to extend these techniques to other statistics concerning numerical semigroups.

Finally, we develop and implement algorithms that evaluate s_g , and thus t_g , for all $g \leq 95$, a large improvement upon [EF20], which only computes t_g for $g \leq 65$. Combining these results with known values for n_g for $g \leq 72$ [BAFG20, OEI23, A007323], we are able to make conjectures for the asymptotics of \hat{n}_g and s_g that open up an intriguing avenue for future study. Future work in this area could lead to a proof or disproof of Conjecture 1.2 for sufficiently large g .

Comparison with the work of Bacher

A recent preprint of Bacher [Bac21], written independently of this work, reproves Theorems 1.1 and 1.3 in a manner similar to what we do here. Like this paper, Bacher focuses on Kunz coordinates and defines stressed words (though he calls them “NSG-compositions with maximum 3 ending with a part of maximal size”). Moreover, Bacher uses the same framework of bounding growth rates by looking at the convergence of generating functions, and proves both the lower bound $r_1 \geq r_{1.51}$, in the language of Theorem 1.5, and the result that $S = \frac{\varphi}{\sqrt{5}}(1 + \sum_{g \geq 3} s_g \varphi^{-g})$. Moreover, he proposes the same algorithm for computing s_g as we do in Section 7, although he only computes s_g up to $g = 50$.

Our work improves upon that of Bacher in two main ways. For one, Bacher appears to be solely interested in proving that r_1 and r_2 , in the language of Theorem 1.5, are strictly less than φ , and thus makes several “inefficient” arguments that simplify the analysis but mask the true growth rate. The second difference is that without Zhao’s lemma bounding graph homomorphisms, Bacher bounds the relevant objects “manually”, which becomes rather complicated even when the underlying graph structure is relatively simple; loosely speaking, in the language of Section 5, Bacher goes as far as the case $k = 3$, which already involves rather involved computations concerning crystallographic reflection groups and transfer matrices. This is enough to prove $r_1, r_2 < \varphi$, but it is unclear if this technique can be pushed further. Our work, in contrast, uses the language of graph homomorphisms to sidestep most of these complexities.

Outline

Section 2 introduces various definitions and lemmas used in the rest of the paper, while Section 3 develops the theory of Kunz coordinates and objects we call stressed words. The proof of Theorem 1.5 is spread across Section 4 and Section 5. Section 6 discusses various applications of our results to the statistics of numerical semigroups, including the proofs of Theorems 1.6 and 1.7. Finally, Section 7 discusses the computation of s_g , numerical results, and subsequent conjectures.

2. Preliminaries

2.1. Notation and conventions

The set of nonnegative integers is denoted \mathbb{N}_0 . For a positive integer a , let $[a] = \{1, 2, \dots, a\}$. For real numbers $a < b$, it will also be useful to define the set $(a, b] = \{n \in \mathbb{Z} \mid n \geq 1, a < n \leq b\}$. Observe that all elements of $(a, b]$ are positive integers.

The Fibonacci numbers are denoted F_n , with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. There is also an explicit formula of $F_n = \frac{1}{\sqrt{5}}(\varphi^n - (-\varphi)^{-n})$.

We define the constants $r_{1.51} \approx 1.51519$ and $r_{1.54} \approx 1.54930$ so that $1/r_{1.51}$ and $1/r_{1.54}$ are the unique positive zeros of the polynomials

$$x^3(x+1)(x^2+x+1) - 1 \quad \text{and} \quad x^4(x+1)^2(x^4+2x^3+x^2+1) - 1,$$

respectively. Alternatively, $r_{1.51}$ and $r_{1.54}$ can be described as the unique positive zeros of the polynomials

$$x^6 - x^3 - 2x^2 - 2x - 1 \quad \text{and} \quad x^{10} - x^6 - 2x^5 - 2x^4 - 4x^3 - 6x^2 - 4x - 1.$$

2.2. Bounding weighted graph homomorphisms

In this section we state Zhao's graph homomorphism lemma and reframe it in a form which will be useful in Section 5.

All mentions of graphs in this paper are confined to this section. Here, all graphs are assumed to be finite, undirected, and with no multiple edges. However, they may contain loops unless otherwise stated. For any graph G , we let $V(G)$ denote the vertices of G and $E(G)$ denote the edges of G . Let $K_{a,b}$ be the complete bipartite graph with parts of size a and b .

Following [Zha11], we define a graph H , possibly with loops, to be a *threshold graph* if there is some function $g: V(H) \rightarrow \mathbb{R}$ and real number t such that for (possibly equal) $u, v \in V(H)$, we have $uv \in E(H)$ if and only if $g(u) + g(v) \leq t$.

For graphs G and H , let $\text{Hom}(G, H)$ be the set of graph homomorphisms¹ from G to H . Given an assignment of a nonnegative weight $z(v)$ to each vertex v of H and a homomorphism $f \in \text{Hom}(G, H)$, let $z(f) = \prod_{v \in V(G)} z(f(v))$. Finally, let $\text{hom}_z(G, H) = \sum_{f \in \text{Hom}(G, H)} z(f)$. We can now state the following result:

Theorem 2.1 (Zhao [Zha11, Cor. 7.6]). *If G is a loop-free, k -regular graph with n vertices and H is a threshold graph, then $\text{hom}_z(G, H) \leq \text{hom}_z(K_{k,k}, H)^{n/(2k)}$ for all nonnegative weight assignments z .*

Of use to us will be the following corollary:

¹Recall that a graph homomorphism between two graphs G and H is a function $f: V(G) \rightarrow V(H)$ such that for every $uv \in E(G)$, we have $f(u)f(v) \in E(H)$.

Corollary 2.2. Fix

- a finite set V of real numbers;
- a nonnegative weight $z(v)$ for every $v \in V$;
- a real number t ;
- positive integers d and k ;
- a subset $A \subseteq \mathbb{Z}/d\mathbb{Z}$ of size k , where $\mathbb{Z}/d\mathbb{Z}$ is the abelian group of integers modulo d .

Let \mathcal{F} be the set of functions $f: \mathbb{Z}/d\mathbb{Z} \rightarrow V$ such that $f(a) + f(b) \geq t$ for all a, b with $a + b \in A$. Furthermore, let \mathcal{X} be the set of tuples $(a_1, \dots, a_k, b_1, \dots, b_k) \in V^{2k}$ with $\min(a_1, \dots, a_k) + \min(b_1, \dots, b_k) \geq t$. Then

$$\sum_{f \in \mathcal{F}} \prod_{a \in \mathbb{Z}/d\mathbb{Z}} z(f(a)) \leq \left(\sum_{(a_1, \dots, a_k, b_1, \dots, b_k) \in \mathcal{X}} \prod_{i=1}^k z(a_i)z(b_i) \right)^{d/(2k)}.$$

Proof. Let H be the graph with vertex set V and an edge from v to v' if $v + v' \geq t$; this is clearly a threshold graph (set $g(v) = -v$). Now, construct a graph G with vertex set $\mathbb{Z}/d\mathbb{Z} \times \{1, 2\}$ by connecting an edge between (a, i) and (b, j) , for $a, b \in \mathbb{Z}/d\mathbb{Z}$ and $i, j \in \{1, 2\}$, if and only if both of the following conditions are satisfied:

1. $a + b \in A$;
2. $a \neq b$ and $i = j$, or $a = b$ and $i \neq j$.

It is not hard to show that G is k -regular, and moreover the second condition implies that G is loop-free.

Applying Theorem 2.1, we now find that $\text{hom}_z(G, H) \leq \text{hom}_z(K_{k,k}, H)^{d/k}$. Graph homomorphisms from $K_{k,k}$ to H are given exactly by the elements of \mathcal{X} , and in fact

$$\text{hom}_z(K_{k,k}, H) = \sum_{(a_1, \dots, a_k, b_1, \dots, b_k) \in \mathcal{X}} \prod_{i=1}^k z(a_i)z(b_i).$$

Therefore we only need to show that

$$\left(\sum_{f \in \mathcal{F}} \prod_{a \in \mathbb{Z}/d\mathbb{Z}} z(f(a)) \right)^2 \leq \text{hom}_z(G, H).$$

To do this, take arbitrary $f_1, f_2 \in \mathcal{F}$ and define the function $h: V(G) \rightarrow V$ by setting $h(a, i) = f_i(a)$. We claim that $h \in \text{Hom}(G, H)$. To show this, note that if $a + b \in A$, $a \neq b$, and $i = j$, we have $h(a, i) + h(b, j) = f_i(a) + f_i(b) \geq t$. On the other hand, if $a + b \in A$, $a = b$, and $i \neq j$, then we have $h(a, i) + h(b, j) = \frac{1}{2}(f_i(a) + f_i(b) + f_j(a) + f_j(b)) \geq t$.

It is evident that the map $(f_1, f_2) \mapsto h$ is injective. Therefore

$$\sum_{f_1, f_2 \in \mathcal{F}} \prod_{a \in \mathbb{Z}/d\mathbb{Z}} z(f_1(a))z(f_2(a)) \leq \sum_{h \in \text{Hom}(G, H)} \prod_{a \in \mathbb{Z}/d\mathbb{Z}} z(h(a, 1))z(h(a, 2)) = \text{hom}_z(G, H),$$

which concludes the proof. \square

Remark 2.3. If we define G' as the graph with vertex set $\mathbb{Z}/d\mathbb{Z}$ and vertices a and b connected if and only if $a + b \in A$, then Corollary 2.2 can be rephrased as $\text{hom}_z(G', H) \leq \text{hom}_z(K_{k,k}, H)^{d/(2k)}$, though this does not immediately follow from Theorem 2.1 since G' can have loops. The trick used above to sidestep this issue can be generalized to prove Theorem 2.1 even when G can have loops. (Here, a loop contributes 1 to the degree of a vertex.)

3. Kunz and Stressed Words

3.1. Kunz words

Consider a numerical semigroup Λ with multiplicity m . Since $m \in \Lambda$, by additive closure we have $\Lambda + m \subseteq \Lambda$. In particular, for each $0 \leq i < m$, there is some nonnegative integer k_i such that

$$\{n \in \Lambda \mid n \equiv i \pmod{m}\} = \{k_i m + i, (k_i + 1)m + i, \dots\}.$$

Since $0 \in \Lambda$, we always have $k_0 = 0$. However, by the minimality of m , we have $k_i > 0$ for all other i . To capture this data, define the *Kunz coordinate vector* $\mathcal{K}(\Lambda) = (k_1, k_2, \dots, k_{m-1})$. These indices provide an important representation of Λ :

Theorem 3.1 (Kunz [Kun87], Rosales et al. [RGSGB02]). *The map \mathcal{K} is a bijection between numerical semigroups of multiplicity m and $(m-1)$ -tuples of positive integers $(k_1, k_2, \dots, k_{m-1})$ such that $k_i + k_j \geq k_{i+j}$ and $k_i + k_j + 1 \geq k_{i+j-m}$ for all indices i, j for which these inequalities are defined.*

A common method by which to interpret this result is to imagine these tuples as the integral points inside a convex polyhedron in $(m-1)$ -dimensional space. However, in this paper we will make something of a psychological shift to instead consider Kunz coordinate vectors as words drawn from an alphabet of positive integers.

Definition 3.2. A (possibly empty) word $w = w_1 w_2 \cdots w_\ell$ of positive integers is *Kunz* if $w_i + w_j \geq w_{i+j}$ and $w_i + w_j + 1 \geq w_{i+j-\ell-1}$ for all i, j for which these inequalities are defined. If we additionally have $w_i \leq q$ for all i , the word w is said to be *q-Kunz*.

Thus, \mathcal{K} is a bijection between numerical semigroups and Kunz words.

Example 3.3. The Kunz word of \mathbb{N}_0 is the empty word. If $\Lambda = \mathbb{N}_0 \setminus \{1, 2, 3, 4, 5, 7, 9, 10, 13\}$, then $\mathcal{K}(\Lambda) = 31221$.

Given this bijection, one can define the multiplicity, genus, depth, conductor, and Frobenius number of a Kunz word as that of its corresponding numerical semigroup. These properties can also be described more naturally in terms of the word itself:

Proposition 3.4. *Let Λ be a numerical semigroup and let $w = w_1w_2 \cdots w_\ell = \mathcal{K}(\Lambda)$ be its Kunz word. Then*

- $m(w) = \ell + 1$;
- $g(w) = \sum_{i \in [\ell]} w_i$;
- $q(w) = \max_{i \in [\ell]} w_i$, or 0 if w is empty.
- $f(w) = (q - 1)m + j$, where j is maximal such that $w_j = q$, or $j = 0$ if w is empty.

Proof. If $\Lambda = \mathbb{N}_0$, which corresponds to w being empty, we have $m = 1, g = 0, q = 0$ and $f = -1$, which is consistent with the expressions for m, g, q, f . Henceforth assume $\Lambda \neq \mathbb{N}_0$.

The fact that $m = \ell + 1$ follows from the definition of \mathcal{K} . Also, from the definition of \mathcal{K} we find

$$\mathbb{N}_0 \setminus \Lambda = \bigcup_{i \in [\ell]} \{i, m + i, \dots, (w_i - 1)m + i\}.$$

Thus, $g = |\mathbb{N}_0 \setminus \Lambda| = \sum_{i \in [\ell]} w_i$ and $f = \max(\mathbb{N}_0 \setminus \Lambda) = (\max_{i \in [\ell]} w_i - 1)m + j$. Finally, $q = \lceil c/m \rceil = \max_{i \in [\ell]} w_i - 1 + \lceil \frac{j+1}{m} \rceil = \max_{i \in [\ell]} w_i$. □

In particular, q -Kunz words of genus g correspond to semigroups of genus g and depth at most q .

3.2. 2-Kunz words

As a warmup, we will first demonstrate how Kunz words can enumerate semigroups of depth at most 2.

Consider a word $w = w_1w_2 \cdots w_\ell$ on the alphabet $\{1, 2\}$. For all i, j , we find $w_i + w_j \geq 2 \geq w_{i+j}$ and $w_i + w_j + 1 \geq 3 \geq w_{i+j-\ell-1}$, so w is automatically 2-Kunz.

Proposition 3.5 ([Zha10]). *The number of numerical semigroups of genus g and depth at most 2 is F_{g+1} .*

Proof. Let the number of 2-Kunz words of genus g be a_g . We can calculate $a_0 = 1 = F_1$ and $a_1 = 1 = F_2$. Moreover, every nonempty 2-Kunz word can be generated from a shorter 2-Kunz word by appending a 1 or a 2, so for $g > 1$ we find $a_g = a_{g-1} + a_{g-2}$. By induction, $a_g = F_{g+1}$. □

3.3. 3-Kunz words

Consider a word $w = w_1w_2 \cdots w_\ell$ on the alphabet $\{1, 2, 3\}$. For all i, j , we find $w_i + w_j + 1 \geq 3 \geq w_{i+j-\ell-1}$, so w is 3-Kunz if and only if there do not exist indices i and j with $w_i = w_j = 1$ and $w_{i+j} = 3$.

Proposition 3.6. (a) *If w is nonempty and 3-Kunz, then the word w' formed by deleting the last character in w is also 3-Kunz.*

(b) *If w is 3-Kunz, so are the words $w1$ and $w2$. (Here $w1$ refers to the word created by appending a 1 to the end of w , and similarly for $w2$.)*

Proof. If w' is not 3-Kunz, then there are i, j with $w'_i = w'_j = 1$ and $w'_{i+j} = 3$. But then $w_i = w_j = 1$ and $w_{i+j} = 3$, contradicting the fact that w is 3-Kunz.

Now suppose $v = w1$ is not 3-Kunz; call its length ℓ . If $v_i = v_j = 1$ and $v_{i+j} = 3$, then since $v_\ell = 1$ we find $i, j, i + j < \ell$. Thus $w_i = w_j = 1$ and $w_{i+j} = 3$, again contradicting the fact that w is 3-Kunz. Similar reasoning proves that $w2$ is 3-Kunz. \square

At this point, we will now give a short proof of Theorem 1.4.

Proof of Theorem 1.4. The number t_g counts the number of 3-Kunz words of genus g . By Proposition 3.6, those that end with 1 are precisely the words $w1$ where w is any 3-Kunz word of genus $g - 1$, of which there are t_{g-1} . Similarly, there are exactly t_{g-2} such words that end in 2. The words that end in 3 must all be of the form $w3$ for some 3-Kunz word w of genus $g - 3$, but not all words of this form may necessarily be 3-Kunz. Hence, $t_{g-1} + t_{g-2} \leq t_g \leq t_{g-1} + t_{g-2} + t_{g-3}$, as desired. \square

This proof motivates the definition of a stressed word.

Definition 3.7. A word of positive integers is *stressed* if it is 3-Kunz and ends in 3.

It is apparent from the above proof that the number of stressed words of genus $g \geq 3$ is $t_g - t_{g-1} - t_{g-2}$, a quantity we will define to be s_g . To be consistent with the idea that s_g counts stressed words of a given genus, we also define $s_0 = s_1 = s_2 = 0$.

Example 3.8. The stressed words of length at most 3 are 3, 23, 33, 123, 213, 313, 223, 233, 323, and 333.

Proposition 3.9. *Kunz words of depth 3 are in bijection with pairs of stressed words and 2-Kunz words, with the bijection given by concatenation.*

Proof. Suppose $w = w_1 w_2 \cdots w_\ell$ is Kunz of depth 3. If i is maximal such that $w_i = 3$, then Proposition 3.6 implies $w_1 w_2 \cdots w_i$ is stressed and $w_{i+1} w_{i+2} \cdots w_\ell$ is 2-Kunz. Moreover, if w is stressed and v is 2-Kunz, then Proposition 3.6 yields that the concatenation of w and v is Kunz of depth 3. These two operations are clearly inverse to each other. \square

Corollary 3.10. *For all $g \geq 0$, $t_g = F_{g+1} + \sum_{i=1}^g s_i F_{g+1-i}$.*

Definition 3.11. Let w be a 3-Kunz word. If $q(w) = 3$, Proposition 3.9 associates a stressed word with w , which we call the *prefix* of w . If $q(w) \leq 2$, we define the prefix of w to be the empty word.

4. First Bounds on Growth Rates

In this section, we prove the first two bounds of Theorem 1.5, namely that $r_1 \geq r_{1.51}$ and $r_1 \leq r_2$, deferring the proof that $r_2 \leq r_{1.54}$, by far the most involved part, to Section 5.

4.1. A lower bound on r_1

Since w is 3-Kunz if and only if there are no i, j with $w_i = w_j = 1$ and $w_{i+j} = 3$, a sufficient condition for a word of length ℓ to be 3-Kunz is that $w_i > 1$ for all $i \leq \ell/2$. In particular, if the length is $2k + 1$ for $k \geq 0$, a word is stressed if the first k characters are in the set $\{2, 3\}$, the next k are in the set $\{1, 2, 3\}$, and the last is 3. Therefore, if we define the generating function

$$f(x) = \sum_{k \geq 0} x^3(x^2 + x^3)^k(x + x^2 + x^3)^k = \frac{x^3}{1 - x^3(x + 1)(x^2 + x + 1)} = \sum_{g \geq 0} s'_g x^g,$$

we find that $s'_g \leq s_g$.

We now claim that $\lim_{g \rightarrow \infty} (s'_g)^{1/g} = r_{1.51}$. This is a standard result in the theory of generating functions; a more detailed exposition can be found in many standard texts, e.g. [Wil94, Ch. 5]. In our case, observe that $1 - x^3(x + 1)(x^2 + x + 1)$ has a unique zero of smallest magnitude, namely $1/r_{1.51}$, which happens to be simple. Therefore, by the theory of partial fractions, $f(x)$ can be written as $\tilde{f}(x) + \frac{C}{x - (1/r_{1.51})}$, where $\tilde{f}(x)$ has a radius of convergence strictly greater than $1/r_{1.51}$ and C is some nonzero real constant. The result follows immediately.

4.2. Proof that $r_1 \leq r_2$

We first prove a useful lemma.

Lemma 4.1. *If $g \geq 7$, then $s_g \leq 2\hat{n}_{g-2} + 3\hat{n}_{g-3} + 2\hat{n}_{g-4} + \hat{n}_{g-5}$.*

Proof. If $w = w_1w_2 \cdots w_{\ell-1}3$ is stressed of genus g and has length $\ell \geq 3$, let $\Phi(w) = 4w_2w_3 \cdots w_{\ell-2}$. We claim that $\Phi(w)$ is Kunz. Indeed, for $i, j \in [\ell - 2]$ with $i + j \leq \ell - 2$, we have $\Phi(w)_i + \Phi(w)_j \geq w_i + w_j \geq w_{i+j} = \Phi(w)_{i+j}$. Also, if $i + j \geq \ell + 1$, then $\Phi(w)_i + \Phi(w)_j + 1 \geq 3 \geq \Phi(w)_{i+j-(\ell-1)}$. Lastly, if $i + j = \ell$, then $\Phi(w)_i + \Phi(w)_j \geq w_i + w_j \geq w_{i+j} = 3 = \Phi(w)_1 - 1$.

For g a positive integer and $a, b \in [3]$, let $\mathcal{S}(g, a, b)$ be the set of stressed words of genus g and of the form $w = aw_2 \cdots w_{\ell-2}b3$, where $\ell \geq 3$ is arbitrary. Since every stressed word of genus $g \geq 7$ has length at least 3, and $\mathcal{S}(g, 1, 1)$ is empty, we know that for $g \geq 7$ we have

$$s_g = \sum_{\substack{a, b \in [3] \\ (a, b) \neq (1, 1)}} |\mathcal{S}(g, a, b)|.$$

On the other hand, Φ , when restricted to each of the $\mathcal{S}(g, a, b)$, is injective, and produces a Kunz word of genus $g + 1 - a - b$ and depth 4. Therefore $|\mathcal{S}(g, a, b)| \leq \hat{n}_{g+1-a-b}$. The result follows. □

To finish, we note that by the definition of r_2 , for all $\varepsilon > 0$ there is some g_0 such that $\hat{n}_g < (r_2 + \varepsilon)^g$ for all $g > g_0$. (If r_2 were infinite the statement that $r_1 \leq r_2$ would be tautological.) Therefore, for every $\varepsilon > 0$ we find that for sufficiently large g ,

$$s_g < 2(r_2 + \varepsilon)^{g-2} + 3(r_2 + \varepsilon)^{g-3} + 2(r_2 + \varepsilon)^{g-4} + (r_2 + \varepsilon)^{g-5} < 8(r_2 + \varepsilon)^g.$$

Thus $r_1 \leq r_2 + \varepsilon$ for all ε , implying that $r_1 \leq r_2$.

5. The Main Upper Bound

5.1. Introduction and outline of proof

In this section we will prove that $r_2 \leq r_{1.54}$, finishing the proof of Theorem 1.5. Our general technique will be to use the fact that $1/r_2$ is the radius of convergence of the generating function $\sum_{g \geq 1} \hat{n}_g x^g$. In particular, we will be done if we can show that if $|x| < r_{1.54}^{-1}$, then the sum $\sum_{g \geq 1} \hat{n}_g x^g$ converges. Since $\hat{n}_g \geq 0$ for all g , we may additionally assume that x is a positive real number. It will be further convenient to assume that $x > 5/8$.

At this point, we let k be a positive integer that we will determine at a later time. It will be true that as x approaches $r_{1.54}^{-1}$, the value of k needed for the proof to work will become arbitrarily large.

Define the k -dense depth q' of a Kunz word to be the largest positive integer such that there are at least k^2 occurrences of q' in the word, or 0 if such a q' does not exist. Note that q' is at most the depth q . For positive integers q, q', ℓ, p with $q' \leq q$ and $p \leq \ell$, define the following sets:

$$\begin{aligned} \mathcal{S}_q(\ell) &= \{\text{Kunz words of depth } q \text{ and length } \ell\}, \\ \mathcal{S}_{q,q'}(\ell) &= \{\text{Kunz words of depth } q, k\text{-dense depth } q', \text{ and length } \ell\}, \\ \mathcal{S}_{q,q'}(\ell, p) &= \{\text{words } w \in \mathcal{S}_{q,q'}(\ell) \text{ such that } w_p = q'\}. \end{aligned}$$

Moreover, given a word $w = w_1 w_2 \cdots w_\ell$ consisting of positive integers, we define its weight $z(w) = x^{\sum_{i \in [\ell]} w_i}$. In this section, we will work with several (finite) sets of such words with names containing the symbol \mathcal{S} , like those defined above. As a notational shorthand, we denote the sum of the weights of the elements of such sets by switching the \mathcal{S} into a \mathcal{W} . For example,

$$\mathcal{W}_q(\ell) = \sum_{w \in \mathcal{S}_q(\ell)} z(w).$$

The main structure of the proof is to break up the sum $\sum_{g \geq 1} \hat{n}_g x^g = \sum_{\ell \geq 1} \sum_{q \geq 4} \mathcal{W}_q(\ell)$ into five cases, as follows:

- I. All words with $q \geq 9$.
- II. Words with $q' \leq 1$ and $4 \leq q \leq 8$.
- III. Words with $q' = 2$ and $4 \leq q \leq 8$.
- IV. Words with $q' = 3$ and $4 \leq q \leq 8$.
- V. Words with $4 \leq q' \leq q \leq 8$.

Specifically, Case I establishes that $\sum_{\ell \geq 1} \sum_{q \geq 9} \mathcal{W}_q(\ell)$ converges, after which it will suffice to show that for the finitely many remaining pairs (q, q') , the sum $\sum_{\ell \geq 1} \mathcal{W}_{q,q'}(\ell)$ converges, which is the content of the remaining cases. For each (q, q') , we will show that the sum converges for sufficiently large k , meaning that by making k large enough, all of these sums will converge, which will show the desired result.

For Cases III, IV, and V, the general strategy will be to use the pigeonhole principle to find occurrences of q' that are close together in the word, and then to apply a “standardization map” to transform the word into objects that can either be handled directly, as in Case III, or bounded using Corollary 2.2, as in Cases IV and V. These techniques will be developed in two interlude subsections.

The only part of the proof that is critically dependent on the condition $x < r_{1.54}^{-1}$ is Case IV. The remainder of the proof can be easily modified to work with any $x < r_{1.51}^{-1}$.

5.2. Case I: $q \geq 9$

The objective of this section is to find a q_0 such that we can prove that $\sum_{\ell=1}^{\infty} \sum_{q=q_0}^{\infty} \mathcal{W}_q(\ell)$ converges. To see this, pick some $p \in [\ell]$ and consider all Kunz words $w = w_1 w_2 \cdots w_\ell$ such that $w_p = q \geq q_0$. Since w is Kunz, it can be easily shown that $w_i + w_j \geq w_p - 1 \geq q_0 - 1$ whenever $i + j \equiv p \pmod{\ell + 1}$. The map $i \mapsto p - i \pmod{\ell + 1}$ is an involution on $[\ell] \setminus \{p\}$, so we conclude that the sum of the weights of all Kunz words of this form is at most $x^q \alpha^a \beta^b$ where a and b are the number of cycles of length 1 and 2 created by this involution, respectively, and

$$\alpha = \sum_{2i \geq q_0 - 1} x^i \text{ and } \beta = \sum_{i+j \geq q_0 - 1} x^{i+j}.$$

Since $\alpha^2 \leq \beta$, this is bounded above by $x^q \beta^{\frac{a}{2} + b} = x^q \beta^{\frac{\ell - 1}{2}}$. Therefore, summing over all q and p ,

$$\sum_{q=q_0}^{\infty} \mathcal{W}_q(\ell) \leq \ell \left(\sum_{q \geq q_0} x^q \right) \beta^{\frac{\ell - 1}{2}}.$$

This expression converges when summed over all ℓ if and only if $\beta < 1$. Computations with software show that $q_0 = 9$ satisfies this condition.

5.3. Case II: $q' \leq 1$

Fix some $q' \leq 1$ and $4 \leq q \leq 8$ and consider some $w \in \mathcal{S}_{q,q'}(\ell, p)$.

Again, it is easy to see that $w_i + w_j \geq w_p - 1 = q - 1$ for all $i + j \equiv p \pmod{\ell + 1}$, so by summing over all $i \neq p$ we find that $2 \sum_{i \neq p} w_i \geq (q - 1)(\ell - 1)$, meaning that the average of all the w_i is greater than $\frac{q-1}{2} \geq \frac{3}{2} > q'$.

On the other hand, there are at most $(q - q')k^2$ integers larger than q' in any word in $\mathcal{S}_{q,q'}$, so the average of all the w_i is bounded above by

$$\frac{(q - q')k^2 \cdot q + (\ell - (q - q')k^2) \cdot q'}{\ell} = q' + \frac{(q - q')^2 k^2}{\ell}.$$

Therefore, $\mathcal{S}_{q,q'}(\ell)$ is empty for sufficiently large ℓ . Therefore $\sum_{\ell=1}^{\infty} \mathcal{W}_{q,q'}(\ell)$ converges².

²This same argument works whenever $q > 2q' + 1$, but using this fact for $q' \geq 2$ is unnecessary for our purposes as we can get the high- q cases “for free” in our later arguments.

5.4. Interlude I: Pigeonhole trick

For $d, p \in [\ell]$, let

$$\begin{aligned} \mathcal{S}_{q,q'}(\ell, d, p) &= \{\text{words } w \in \mathcal{S}_{q,q'}(\ell, p) \text{ with at least } k \text{ values of } i \in (d - \ell/k, d] \text{ with } w_i = q'\}, \\ \mathcal{S}'_{q,q'}(\ell, d, p) &= \{\text{words } w \in \mathcal{S}_{q,q'}(\ell, d, p) \text{ such that } s \notin \mathcal{S}_{q,q'}(\ell, d', p) \text{ for all } d < d' < \ell\}. \end{aligned}$$

Lemma 5.1. *If $q' > 0$, then*

$$\mathcal{S}_{q,q'}(\ell) = \bigcup_{d=1}^{\ell} \bigcup_{p=1}^{\ell} \mathcal{S}'_{q,q'}(\ell, d, p).$$

In particular,

$$\mathcal{W}_{q,q'}(\ell) \leq \sum_{d=1}^{\ell} \sum_{p=1}^{\ell} \mathcal{W}'_{q,q'}(\ell, d, p).$$

Proof. Obviously, $\mathcal{S}_{q,q'}(\ell) = \bigcup_{p=1}^{\ell} \mathcal{S}_{q,q'}(\ell, p)$. For $w \in \mathcal{S}_{q,q'}(\ell, p)$, by the definition of q' there are at least k^2 values of i with $w_i = q'$. The k intervals $(0, \ell/k], (\ell/k, 2\ell/k], \dots, ((k-1)\ell/k, \ell]$ cover $[\ell]$, so one of them must have at least k such values of i . Therefore, there is some d such that $w \in \mathcal{S}_{q,q'}(\ell, d, p)$. By taking d to be maximal, we find that $w \in \mathcal{S}'_{q,q'}(\ell, d, p)$. \square

Proposition 5.2. *If $w \in \mathcal{S}'_{q,q'}(\ell, d, p)$, there are at most k^2 values of $i > d$ with $w_i = q'$.*

Proof. If not, then ignore all the $i \leq d$ with $w_i = q'$ and apply the same argument in Lemma 5.1 to find some d' such that there are at least k values of $i \in (d' - \ell/k, d']$ and $i > d$ with $w_i = q'$. Such a d' must be greater than d , and we also have $w \in \mathcal{S}_{q,q'}(\ell, d', p)$, which is a contradiction. \square

5.5. Interlude II: Standardization

Now, we aim to convert the set $\mathcal{S}'_{q,q'}(\ell, d, p)$ into a form that is easier to work with by jettisoning most of the “extraneous material”, like all the occurrences of numbers greater than q' . To do this, we first define the following operation:

Definition 5.3. Given $w \in \mathcal{S}'_{q,q'}(\ell, d, p)$ let $I \subseteq [\ell]$ be the set of indices i such that at least one of the following holds:

- $i \in (d - \ell/k, d]$,
- $i \in [d]$ and $w_i > q'$, or
- $i > d$ and $w_i \geq q'$.

Then, define $w^\circ = w_1^\circ w_2^\circ \cdots w_\ell^\circ$ so that

- if $i \in I$, then $w_i^\circ = q'$ if $i \leq d$ and $w_i^\circ = \max(q' - 1, 2)$ if $i > d$;
- otherwise, $w_i^\circ = w_i$.

Definition 5.4. If A is a subset of $(d - \ell/k, d]$ of size k , let $\mathcal{S}_{q,q'}^\circ(\ell, d, p, A)$ be the image of this map $w \mapsto w^\circ$ over all $w \in \mathcal{S}'_{q,q'}(\ell, d, p)$ such that $w_i = q'$ for all $i \in A$.

Proposition 5.5. A word $w \in \mathcal{S}_{q,q'}^\circ(\ell, d, p, A)$ satisfies the following conditions:

- $w_i \in [q']$ for all $i \in [\ell]$ and $w_i \leq \max(q' - 1, 2)$ for all $i > d$,
- $w_i + w_j \geq q'$ for all $i, j \in [d]$ with $i + j \in A \pmod{d}$ (where we treat $[d]$ as $\mathbb{Z}/d\mathbb{Z}$),
- $w_i + w_j \geq 3$ for all $i, j \in [\ell]$ such that $i + j \equiv p \pmod{\ell + 1}$.

Proof. Let $w \in \mathcal{S}'_{q,q'}(\ell, d, p)$ be such that $w_i = q'$ for all $i \in A$. It suffices to check the conditions for w° .

The first condition is easy. For the second, note that if $i \in I$ or $j \in I$ then the condition is true. Otherwise, we know that $i, j \leq d - \ell/k$, so $i + j \leq 2d - \ell/k$, implying that $i + j \in A$. Therefore $w_i^\circ + w_j^\circ = w_i + w_j \geq w_{i+j} = q'$.

To check the third condition, again note that if $i \in I$ or $j \in I$ then the condition is true. Otherwise, if $i + j = p$, then $w_i^\circ + w_j^\circ = w_i + w_j \geq w_{i+j} = q \geq 4$. If $i + j = p + \ell + 1$, then $w_i^\circ + w_j^\circ = w_i + w_j \geq w_{i+j-\ell-1} - 1 = q - 1 \geq 3$. \square

The map $w \mapsto w^\circ$ does not lose too much information:

Lemma 5.6. If q, q', ℓ, d, p, A are as defined above,

$$\mathcal{W}'_{q,q'}(\ell, d, p) \leq \ell^{10k^2} 1000^{10k^2 + \lceil \ell/k \rceil} \sum_A \mathcal{W}_{q,q'}^\circ(\ell, d, p, A).$$

Proof. Every word $w \in \mathcal{S}'_{q,q'}(\ell, d, p)$ has some subset $A \subseteq (d - \ell/k, d]$ of size k such that $w_i = q'$ for all $i \in A$. (In fact there may be multiple such A .) Thus, $w^\circ \in \mathcal{S}_{q,q'}^\circ(\ell, d, p, A)$ for some A . We now claim that for any word w' in the range of $w \mapsto w^\circ$,

$$\sum_{\substack{w \in \mathcal{S}'_{q,q'}(\ell, d, p) \\ w^\circ = w'}} z(w) \leq \ell^{10k^2} 1000^{10k^2 + \lceil \ell/k \rceil} z(w').$$

Then, by summing over all w' , it will follow that

$$\begin{aligned} \mathcal{W}'_{q,q'}(\ell, d, p) &\leq \ell^{10k^2} 1000^{10k^2 + \lceil \ell/k \rceil} \sum_{w' \in \bigcup_A \mathcal{S}_{q,q'}^\circ(\ell, d, p, A)} z(w') \\ &\leq \ell^{10k^2} 1000^{10k^2 + \lceil \ell/k \rceil} \sum_A \mathcal{W}_{q,q'}^\circ(\ell, d, p, A). \end{aligned}$$

Given some $w' = w^\circ$, the word w can be reconstructed using only the information of I and a choice of $w_i \in [q]$ for all $i \in I$. The set I can be decomposed as $(d - \ell/k, d] \cup I'$, where I' is the set of indices i such that $i \leq d - \ell/k$ and $w_i > q'$, or $i > d$ and $w_i \geq q'$. If $q' < a \leq q$, then by the definition of q' there exist at most k^2 values of i with $w_i = a$. Moreover, by Proposition 5.2 there are at most k^2 values of $i \in I'$ with $w_i = q'$. Therefore $|I'| \leq (q - q' + 1)k^2 \leq 10k^2$.

Since $d \notin I'$, we conclude that the number of choices for I , which is entirely determined by I' , is at most³

$$\binom{\ell-1}{0} + \binom{\ell-1}{1} + \cdots + \binom{\ell-1}{10k^2} \leq \ell^{10k^2}.$$

Moreover, this bound on $|I'|$ implies that $|I| \leq 10k^2 + \lceil \ell/k \rceil$. Now, we fix I , the sum of the weights of the words produced by, for every $i \in I$, replacing w'_i with an arbitrary element of $[q]$ is

$$z(w') \prod_{i \in I} \frac{x + x^2 + \cdots + x^q}{x^{w'_i}} \leq z(w') (qx^{1-q'})^{|I|} \leq z(w') 1000^{10k^2 + \lceil \ell/k \rceil},$$

where we have used the fact that $qx^{1-q'} \leq 8 \cdot (8/5)^7 \leq 1000$. Summing over all I proves the bound. \square

We make one last definition in this section:

Definition 5.7. For $i, j \in [\ell]$, we say $i \sim j$ if and only if $i \neq j$ and $i + j \equiv p \pmod{\ell + 1}$.

The relation \sim breaks $[\ell]$ into some number of pairs, with the only elements not in a pair being p or any element i with $2i \equiv p \pmod{\ell + 1}$, of which there are at most 3 in total.

5.6. Case III: $q' = 2$

Let S be the set of indices i such that $i \sim j$ for some j . For all $w \in \mathcal{S}_{q,2}^\circ(\ell, d, p, A)$, we have that $w_i \leq 2$ for all i and $w_i + w_j \geq 3$ for all i, j with $i \sim j$. Therefore

$$\mathcal{W}_{q,2}^\circ(\ell, d, p, A) = (2x^3 + x^4)^{|S|/2} (x + x^2)^{\ell - |S|}.$$

Since $\ell - |S| \leq 3$, and $\sqrt{2r_{1.54}^{-3} + r_{1.54}^{-4}} \leq 0.9$, we find that

$$\mathcal{W}_{q,2}^\circ(\ell, d, p, A) \leq 0.9^{\ell-3} 2^3 \leq 100 \cdot 0.9^\ell.$$

Therefore, after applying Lemma 5.6 and summing over at most ℓ^k possible values for A , ℓ possible values for d , and ℓ possible values for p , we get

$$\mathcal{W}_{q,2}(\ell) \leq 100 \cdot \ell^{10k^2+k+2} 1000^{10k^2 + \lceil \ell/k \rceil} 0.9^\ell.$$

Summing over all ℓ , we find that this converges if and only if $1000^{1/k} \cdot 0.9 < 1$, which is true for sufficiently large k .

5.7. Case IV: $q' = 3$

Under the relation \sim , we may decompose the set $[\ell]$ into five sets:

- S_1 , the set of elements $i \leq d$ not paired with an element greater than d ,
- S_2 , the set of elements $i \leq d$ paired with an element greater than d ,

³We are using the fact that for positive integers a and b , $\binom{a-1}{0} + \cdots + \binom{a-1}{b} \leq 1 + (a-1) + \cdots + (a-1)^b \leq a^b$.

- S_3 , the set of elements $i > d$ paired with an element in $[d]$,
- S_4 , the set of elements $i > d$ paired with an element greater than d ,
- S_5 , the set of elements $i > d$ not paired with any element.

Now, every element $w \in \mathcal{S}_{q,3}^\circ(\ell, d, p, A)$ can be generated via the following process:

- Choose $w_1, w_2, \dots, w_d \in [3]$ so that $w_i + w_j \geq 3$ for all i, j with $i + j \in A \pmod{d}$.
- For all $i \in S_2$ and $j \in S_3$ with $i \sim j$, choose $w_j \in [2]$ so that $w_i + w_j \geq 3$.
- For all $i, j \in S_4$ with $i \sim j$, choose $w_i, w_j \in [2]$ so that $w_i + w_j \geq 3$.
- For all $i \in S_5$, choose $w_i \in [3]$ arbitrarily.

Now, if we let

$$u(a) = \sum_{\substack{b \in [2] \\ a+b \geq 3}} x^{a+b} \quad \text{and} \quad u'(a) = x^a,$$

we find that the weighted sum of all the words produced this way is

$$\left(\sum_{w_1, w_2, \dots, w_d} \prod_{i \in S_1} u'(w_i) \prod_{i \in S_2} u(w_i) \right) (2x^3 + x^4)^{|S_4|/2} (x + x^2 + x^3)^{|S_5|},$$

where the sum is over all $w_1, w_2, \dots, w_d \in [3]$ that satisfy $w_i + w_j \geq 3$ for all i, j with $i + j \in A \pmod{d}$. As a result, since $(2x^3 + x^4)^{1/2} \leq 0.9$ and $(x + x^2 + x^3)^{|S_5|} \leq 1.5^{|S_5|} \leq 10$, we conclude that

$$\mathcal{W}_{q,3}^\circ(\ell, d, p, A) \leq \left(\sum_{w_1, w_2, \dots, w_d} \prod_{i \in [d]} u''(w_i) \right) \cdot 10 \cdot 0.9^{|S_4|},$$

where we let $u''(a) = \max(u(a), u'(a))$. The first term may be bounded using Corollary 2.2⁴, which yields that it is at most

$$\left(\sum_{(a_1, \dots, a_k, b_1, \dots, b_k) \in \mathcal{X}} \prod_{i=1}^k u''(a_i) u''(b_i) \right)^{d/(2k)},$$

where \mathcal{X} is the set of tuples $(a_1, \dots, a_k, b_1, \dots, b_k) \in [3]^{2k}$ that satisfy $\min(a_1, \dots, a_k) + \min(b_1, \dots, b_k) \geq 3$. We may decompose $\mathcal{X} = (\{2, 3\}^k \times [3]^k) \cup ([3]^k \times \{2, 3\}^k)$, so

$$\sum_{(a_1, \dots, a_k, b_1, \dots, b_k) \in \mathcal{X}} \prod_{i=1}^k u''(a_i) u''(b_i) \leq 2(u''(1) + u''(2) + u''(3))^k (u''(2) + u''(3))^k.$$

⁴We apply Corollary 2.2 with the parameters $V = [3]$, $z(v) = u''(v)$, and $t = 3$. The parameters d, k , and A are the same.

Since $x > 5/8$, we have $x + x^2 > 1$, from which we find $u''(1) = \max(x^3, x) = x$, $u''(2) = \max(x^2(x + x^2), x) = x^3(x + 1)$, and $u''(3) = \max(x^3(x + x^2), x^3) = x^4(x + 1)$. So

$$(u''(1) + u''(2) + u''(3))(u''(2) + u''(3)) = x^4(x + 1)^2(x^4 + 2x^3 + x^2 + 1).$$

Call this quantity y ; by the definition of $r_{1.54}$, we know that since $x < r_{1.54}^{-1}$, we have $y < 1$. Also, it will be useful to note that since $x > 5/8$, we have $y > 0.9^4$.

Putting everything together, we conclude that

$$\mathcal{W}_{q,3}^\circ(\ell, d, p, A) \leq 2^{d/(2k)} y^{d/2} \cdot 10 \cdot 0.9^{|S_4|} \leq 2^{\ell/(2k)} y^{d/2+|S_4|/4} \cdot 10.$$

We know that

$$\ell - 3 \leq |S_1| + |S_2| + |S_3| + |S_4| \leq 2d + |S_4|,$$

so in fact

$$\mathcal{W}_{q,3}^\circ(\ell, d, p, A) \leq 10 \cdot 2^{\ell/(2k)} y^{(\ell-3)/4}.$$

After applying Lemma 5.6 and summing over d, p, A , we conclude that

$$\mathcal{W}_{q,3}(\ell) \leq 10 \cdot \ell^{10k^2+k+2} 1000^{10k^2+\lceil \ell/k \rceil} \cdot 2^{\ell/(2k)} y^{(\ell-3)/4}.$$

Summing over all ℓ , this converges if $1000^{1/k} \cdot 2^{1/(2k)} \cdot y^{1/4} < 1$. Since $y < 1$, this is true for sufficiently large k .

5.8. Case V: $4 \leq q' \leq 8$

Definition 5.8. If ℓ and q are positive integers and A is a subset of $[\ell]$ of size k , let $\mathcal{S}_q^\circ(\ell, A)$ consist of words $w_1 w_2 \cdots w_\ell$ such that $w_i \in [q]$ for all i , and $w_i + w_j \geq q$ whenever $i + j \in A \pmod{\ell}$.

Lemma 5.9. If $q' \geq 4$,

$$\mathcal{W}_{q,q'}^\circ(\ell, d, p, A) \leq x^{-(q'-1)\lceil \ell/k \rceil} \mathcal{W}_{q'}^\circ(d, A) \mathcal{W}_{q'-1}^\circ(\ell + \lceil \ell/k \rceil - d, \ell + 1 - A).$$

Proof. For $w \in \mathcal{S}_{q,q'}^\circ(\ell, d, p, A)$, let $\Phi_1(w) = w_1 w_2 \cdots w_d$ and $\Phi_2(w) = w_\ell w_{\ell-1} \cdots w_{d+1} (q' - 1)^{\lceil \ell/k \rceil}$, where $(q' - 1)^a$ denotes $q' - 1$ repeated a times. We claim that $\Phi_1(w) \in \mathcal{S}_{q'}^\circ(d, A)$ and $\Phi_2(w) \in \mathcal{S}_{q'-1}^\circ(\ell + \lceil \ell/k \rceil - d, \ell + 1 - A)$.

The fact that $\Phi(w) \in \mathcal{S}_{q'}^\circ(d, A)$ follows immediately from Proposition 5.5. For the second claim, we first note that Proposition 5.5 implies that $\Phi_2(w)_i \leq q' - 1$ for all i . To show the second condition, let $d' = d - \lceil \ell/k \rceil$, suppose $w = v^\circ$, and take $i, j \in [\ell - d']$ so that $i + j \in \ell + 1 - A \pmod{\ell - d'}$. We wish to show that $\Phi_2(w)_i + \Phi_2(w)_j \geq q' - 1$. If $i > \ell - d$, then $\Phi_2(w)_i = q' - 1$, so the condition is automatically satisfied. We are similarly done if $j > \ell - d$, so assume $i, j \in [\ell - d]$, so that $\Phi_2(w)_i = w_{\ell+1-i}$ and $\Phi_2(w)_j = w_{\ell+1-j}$. If $w_{\ell+1-i} \neq v_{\ell+1-i}$, then $w_{\ell+1-i} = q' - 1$, so the condition is true in this case. Similarly, we are done if $w_{\ell+1-j} \neq v_{\ell+1-j}$. Now, since $\max A \leq d$, we have $\min(\ell + 1 - A) \geq \ell + 1 - d$. Since $(\ell + 1 - d) + (\ell - d') > 2(\ell - d) \geq i + j$, it must be true that $i + j \in \ell + 1 - A$. Therefore $w_{\ell+1-i} + w_{\ell+1-j} = v_{\ell+1-i} + v_{\ell+1-j} \geq v_{\ell+1-i-j} - 1 = q' - 1$, as desired.

Therefore, the map $\Phi = (\Phi_1, \Phi_2)$ is an injection

$$\mathcal{S}_{q,q'}^\circ(\ell, d, p, A) \rightarrow \mathcal{S}_{q'}^\circ(d, A) \times \mathcal{S}_{q'-1}^\circ(\ell + \lceil \ell/k \rceil - d, \ell + 1 - A).$$

Also, we have that $z(\Phi_1(w))z(\Phi_2(w)) = x^{(q'-1)\lceil \ell/k \rceil} z(w)$, so the lemma follows. \square

Lemma 5.10. For a given $3 \leq q \leq 8$ there is a constant k_0 independent of ℓ and A such that

$$\mathcal{W}_q^\circ(\ell, A) \leq 0.99^\ell$$

whenever $k > k_0$.

Proof. An application of Corollary 2.2⁵ tells us that

$$\mathcal{W}_q^\circ(\ell, A) \leq \left(\sum_{(a_1, \dots, a_k, b_1, \dots, b_k) \in \mathcal{X}} x^{\sum_{i=1}^k a_i + b_i} \right)^{\ell/(2k)},$$

where \mathcal{X} is the set of tuples $(a_1, \dots, a_k, b_1, \dots, b_k) \in [q]$ with the condition $\min(a_1, \dots, a_k) + \min(b_1, \dots, b_k) \geq q$. Each element of \mathcal{X} must have an integer $1 \leq a \leq q - 1$ such that $a_i \geq a$ and $b_i \geq q - a$ for all i , so

$$\sum_{(a_1, \dots, a_k, b_1, \dots, b_k) \in \mathcal{X}} x^{\sum_{i=1}^k a_i + b_i} \leq \sum_{a=1}^{q-1} \left(\sum_{b=a}^q x^b \right)^k \left(\sum_{b=q-a}^q x^b \right)^k.$$

In particular, if we let

$$h_q(x) = \max_{a \in [q-1]} \left(\sum_{b=a}^q x^b \right) \left(\sum_{b=q-a}^q x^b \right),$$

we have

$$\sum_{a=1}^{q-1} \left(\sum_{b=a}^q x^b \right)^k \left(\sum_{b=q-a}^q x^b \right)^k \leq (q-1)h_q(x)^k.$$

As a result, we find that

$$\mathcal{W}_q^\circ(\ell, A) \leq ((q-1)h_q(x)^k)^{\ell/(2k)} = ((q-1)^{1/(2k)}h_q(x)^{1/2})^\ell.$$

To show that this is less than 0.99^ℓ for large k , we only need to show that $h_q(x) < 0.99^2$. Since $h_q(x)$ is increasing in x , this may be shown by verifying $h_q(r_{1.54}^{-1}) < 0.99^2$ for all $3 \leq q \leq 8$. \square

To finish, we note that by combining Lemma 5.9 and Lemma 5.10 we find that for sufficiently large k ,

$$\mathcal{W}_{q,q'}^\circ(\ell, d, p, A) \leq x^{-(q'-1)\lceil \ell/k \rceil} 0.99^{\ell + \lceil \ell/k \rceil} \leq 100^{\lceil \ell/k \rceil} 0.99^\ell,$$

so, applying Lemma 5.6 and summing over d, p, A , we get

$$\mathcal{W}_{q,q'}(\ell) \leq \ell^{10k^2+k+2} 1000^{10k^2+\lceil \ell/k \rceil} 100^{\lceil \ell/k \rceil} 0.99^\ell.$$

The sum over all ℓ converges as long as $100000^{1/k} \cdot 0.99 < 1$, which is true for large enough k .

⁵We set $V = [q]$, $z(v) = x^v$, $t = q$, and $d = \ell$. The parameters k and A are the same.

6. Statistics of Numerical Semigroups

In this section we will apply our results to questions regarding counts of numerical semigroups.

6.1. Proof of Theorem 1.6

Since $F_n = \frac{1}{\sqrt{5}}(\varphi^n - (-\varphi)^{-n})$, by applying Corollary 3.10 we have

$$\begin{aligned} n_g - S\varphi^g &= \hat{n}_g + F_{g+1} + \sum_{i=1}^g s_i F_{g+1-i} - \frac{\varphi^{g+1}}{\sqrt{5}} - \sum_{i=1}^{\infty} s_i \frac{\varphi^{g+1-i}}{\sqrt{5}} \\ &= \hat{n}_g - \frac{(-\varphi)^{-(g+1)}}{\sqrt{5}} - \sum_{i=1}^g s_i \frac{(-\varphi)^{-(g+1-i)}}{\sqrt{5}} - \sum_{i=g+1}^{\infty} s_i \frac{\varphi^{g+1-i}}{\sqrt{5}}. \end{aligned}$$

It suffices to prove that for every $\varepsilon > 0$, the absolute value of each of these terms is bounded above by $(r_{1.54} + \varepsilon)^g$ for large g .

For the first term, this follows immediately from Theorem 1.5. The second term decreases to 0 as g grows, so it is certainly bounded by $(r_{1.54} + \varepsilon)^g$. For the third, note that

$$\left| \sum_{i=1}^g s_i \frac{(-\varphi)^{-(g+1-i)}}{\sqrt{5}} \right| \leq \sum_{i=1}^g s_i.$$

By Theorem 1.5, we may assume $s_i < (r_{1.54} + \varepsilon/2)^i$ for large i , so there is some constant C such that

$$\sum_{i=1}^g s_i < C + \sum_{i=1}^g (r_{1.54} + \varepsilon/2)^i < C + \frac{(r_{1.54} + \varepsilon/2)^{g+1}}{r_{1.54} + \varepsilon/2 - 1},$$

which is indeed less than $(r_{1.54} + \varepsilon)^g$ for large g . For the fourth, note that if g is sufficiently large,

$$\sum_{i=g+1}^{\infty} s_i \frac{\varphi^{g+1-i}}{\sqrt{5}} < \sum_{i=g+1}^{\infty} (r_{1.54} + \varepsilon/2)^i \varphi^{g+1-i} = (r_{1.54} + \varepsilon/2)^{g+1} \sum_{j=0}^{\infty} \left(\frac{r_{1.54} + \varepsilon/2}{\varphi} \right)^j.$$

As long as $r_{1.54} + \varepsilon/2 < \varphi$, this last sum converges to a constant independent of g , so this last term is also bounded by $(r_{1.54} + \varepsilon)^g$. This concludes the proof.

6.2. Proof of Theorem 1.7

Proposition 6.1. *Suppose w is a 3-Kunz word such that $f(w) > (2 + \varepsilon)m(w)$. Then the genus of the prefix (in the sense of Definition 3.11) of w is at least $\frac{\varepsilon}{3}g(w)$.*

Proof. Let v be the prefix of w ; call its length ℓ . Since $f(w) > 2m(w)$, we must have $q(w) = 3$, so by Proposition 3.4, $f(w) = 2m(w) + \ell$. Thus, we find that $g(v) > \ell > \varepsilon m(w) > \frac{\varepsilon}{3}g(w)$. \square

By Proposition 6.1 and mimicking the proof of Corollary 3.10 we find that

$$n_{g,\varepsilon} \leq \hat{n}_g + \sum_{i=\lceil \frac{\varepsilon}{3}g \rceil}^g s_i F_{g+1-i}.$$

It can be easily proved by induction that $F_k < \varphi^k$ for all $k \geq 0$. Moreover, by Theorem 1.5 we know that for sufficiently large g , we have $\hat{n}_g, s_g < 1.55^g$. Therefore, for sufficiently large g ,

$$n_{g,\varepsilon} < 1.55^g + \sum_{i=\lceil \frac{\varepsilon}{3}g \rceil}^g \varphi^{g+1} \left(\frac{1.55}{\varphi}\right)^i < 1.55^g + \varphi^{g+1} \frac{(1.55/\varphi)^{\frac{\varepsilon}{3}g}}{1 - 1.55/\varphi}.$$

Therefore

$$\limsup_{g \rightarrow \infty} n_{g,\varepsilon}^{1/g} \leq \varphi \left(\frac{1.55}{\varphi}\right)^{\varepsilon/3}.$$

6.3. The statistic $f - 2m$

The strategy of decomposing a numerical semigroup into a stressed word and 2-Kunz word, as in Proposition 3.9, is also useful in answering various questions regarding the properties of a typical numerical semigroup. For example, we have the following result:

Theorem 6.2. *Let \mathcal{S}_ℓ be the set of stressed words of length ℓ and let Λ_g be a random numerical semigroup of genus g (under the uniform distribution). Then, for any integer k ,*

$$\lim_{g \rightarrow \infty} \mathbb{P}[f(\Lambda_g) - 2m(\Lambda_g) = k] = \begin{cases} \frac{1}{\sqrt{5} \cdot S} \varphi^k & k < 0, \\ 0 & k = 0, \\ \frac{1}{\sqrt{5} \cdot S} \sum_{w \in \mathcal{S}_k} \varphi^{1-g(w)} & k > 0. \end{cases}$$

Since one can show that

$$\sum_{k < 0} \frac{1}{\sqrt{5} \cdot S} \varphi^k + \sum_{k > 0} \frac{1}{\sqrt{5} \cdot S} \sum_{w \in \mathcal{S}_k} \varphi^{1-g(w)} = 1,$$

this result implies that the random variable $f(\Lambda_g) - 2m(\Lambda_g)$ has a limiting distribution that can be computed in terms of stressed words. This is a refinement of the following result by Singhal:

Theorem 6.3 (Singhal [Sin22, Thm. 8]). *For all $\varepsilon > 0$, there is a positive integer N such that $\lim_{g \rightarrow \infty} \mathbb{P}[|f(\Lambda_g) - 2m(\Lambda_g)| > N] < \varepsilon$.*

Singhal’s result was itself a strengthening of the following result of Kaplan and Ye:

Theorem 6.4 (Kaplan and Ye [KY13, Thm. 4]). *For all $\varepsilon > 0$,*

$$\lim_{g \rightarrow \infty} \mathbb{P}[2 - \varepsilon < f(\Lambda_g)/m(\Lambda_g) < 2 + \varepsilon] = 1.$$

Proof of Theorem 6.2. For each integer k and $g > 0$ let $n_{g,k}$ be the number of numerical semigroups of genus g satisfying $f - 2m = k$. Since for every numerical semigroup Λ we have $m(\Lambda) \in \Lambda$ and $f(\Lambda) \notin \Lambda$, we find $n_{g,0} = 0$. Thus the theorem is true for $k = 0$. Recall the result in Proposition 3.4 that $f = (q - 1)m + j$, where j is maximal such that $w_j = q$. Thus, we want to count semigroups such that $j = k + (3 - q)m$.

Suppose $k < 0$. Then any semigroup with $f - 2m = k$ satisfies $q < 3$. If $q \leq 1$, there is at most one semigroup of genus g . If $q = 2$, then we have that $j = m + k$, so we need to count the number of words $w_1 w_2 \cdots w_{m-1}$ such that $w_i \in [2]$, $w_{m+k} = 2$, and $w_i = 1$ for all $i > m + k$. Removing the last $-k$ entries, which are completely determined, we wish to count 2-Kunz words of genus $g - k - 1$, of which there are F_{g+k} if $g \geq -k$, by Proposition 3.5. Therefore, for large g , we have $F_{g+k} \leq n_{g,k} \leq F_{g+k} + 1$, implying that

$$\lim_{g \rightarrow \infty} \frac{n_{g,k}}{n_g} = \lim_{g \rightarrow \infty} \frac{\frac{1}{\sqrt{5}} \varphi^{g+k}}{S \varphi^g} = \frac{\varphi^k}{\sqrt{5} \cdot S}.$$

Now suppose $k > 0$. Since $j = k + (3 - q)m$ and $j < m$ we must have $q \geq 3$. For a semigroup of depth 3, the quantity j is simply the length of the prefix of the associated Kunz word (in the sense of Definition 3.11). Thus, in light of Proposition 3.9, the number of semigroups of depth 3 with $f - 2m = k$ is, for sufficiently large g , given by $\sum_{w \in \mathcal{S}_k} F_{g+1-g(w)}$. This implies that for large g ,

$$\sum_{w \in \mathcal{S}_k} F_{g+1-g(w)} \leq n_{g,k} \leq \hat{n}_g + \sum_{w \in \mathcal{S}_k} F_{g+1-g(w)}.$$

Therefore,

$$\lim_{g \rightarrow \infty} \frac{n_{g,k}}{n_g} = \lim_{g \rightarrow \infty} \frac{\frac{1}{\sqrt{5}} \sum_{w \in \mathcal{S}_k} \varphi^{g+1-g(w)}}{S \varphi^g} = \frac{\sum_{w \in \mathcal{S}_k} \varphi^{1-g(w)}}{\sqrt{5} \cdot S}.$$

This concludes the proof. \square

This technique can likely be extended to answer related statistical questions.

7. Computational Results, Conjectures, and Further Directions

7.1. Enumerating stressed words

A stressed word of length ℓ can be considered as a partition of the set $[\ell - 1]$ into three sets A, B, C representing the locations of ones, twos, and threes, such that $\ell \notin A + A$ and C is disjoint from $A + A$. Such a word would have genus $3 + |A| + 2|B| + 3|C| = 2\ell + 1 - |A| + |C|$. In particular, for a given ℓ and A with $\ell \notin A + A$, the number of stressed words of genus $2\ell + 1 - |A| + k$ is $\binom{|\ell - 1| \setminus (A \cup (A + A))}{k}$.

For a given ℓ , it is relatively straightforward to iterate through the $3^{\lceil \ell/2 \rceil - 1}$ possibilities for A by choosing at most one element from each of the sets $\{1, \ell - 1\}, \{2, \ell - 2\}, \dots, \{\lceil \ell/2 \rceil - 1, \ell + 1 - \lceil \ell/2 \rceil\}$. By recording the number of possibilities for A with a given value of $|A|$ and $|\ell - 1| \setminus (A \cup (A + A))$, the stressed words of a given length can be enumerated by

ℓ	Sum	ℓ	Sum	ℓ	Sum	ℓ	Sum	ℓ	Sum	ℓ	Sum
1	0.236068	11	2.067270	21	3.288559	31	3.903892	41	4.167027	51	4.269349
2	0.381966	12	2.182360	22	3.355010	32	3.933559	42	4.178757	52	4.273803
3	0.618034	13	2.368955	23	3.454253	33	3.976294	43	4.196208	53	4.280242
4	0.763932	14	2.478676	24	3.509096	34	4.001513	44	4.206072	54	4.283812
5	1.005025	15	2.640734	25	3.593944	35	4.038160	45	4.220026	55	4.289037
6	1.145898	16	2.737483	26	3.643206	36	4.058413	46	4.228224	56	4.291988
7	1.380047	17	2.886712	27	3.713999	37	4.089100	47	4.240024		
8	1.517814	18	2.970980	28	3.755414	38	4.106684	48	4.246500		
9	1.731797	19	3.102391	29	3.817255	39	4.131525	49	4.256156		
10	1.862340	20	3.177471	30	3.851635	40	4.145897	50	4.261649		

Table 7.1: Partial sums of $\sum_{w \text{ stressed}} \varphi^{-g(w)}$ considering all stressed words of length at most ℓ . This table is largely equivalent to Table 2 in [Zha10], by Remark 7.1. All results with $\ell > 46$ are new.

genus. Moreover, if A is built recursively, the set $S = [\ell - 1] \setminus (A \cup (A + A))$ can be memoized, so that if an element is added to A , the set S can be updated in $O(\ell)$ time.

Using this technique, we wrote a computer program (available at <https://github.com/zhdag/stressed>) to enumerate all stressed words of length at most 56 or genus at most 95. While the latter computation required the additional consideration of the range $57 \leq \ell \leq 62$, the program only considered possible A that were sufficiently large to create words of genus at most 95, drastically reducing the search space. Another minor optimization was the observation that for the case when $k = 0$, the number $|\ell - 1 \setminus (A \cup (A + A))|$ did not need to be computed; in particular, this idea led to the $\ell = 62$ case being dropped entirely. In total, these computations required approximately 6.5 hours of single-core processor time on the author’s laptop.

Remark 7.1. The observations in this section allow our relation $S = \frac{\varphi}{\sqrt{5}}(1 + \sum_{g \geq 3} s_g \varphi^{-g})$ to be related to several expressions for S found elsewhere in the literature. If we fix ℓ and A , the sum of $\varphi^{-g(w)}$ over all stressed words w of length ℓ and ones in locations given by A is

$$\begin{aligned} \varphi^{-(2\ell+1-|A|)}(1 + \varphi^{-1})^{|\ell-1 \setminus (A \cup (A + A))|} &= \varphi^{-2\ell-1+|A|+|\ell-1 \setminus (A \cup (A + A))|} \\ &= \varphi^{-\ell-2+|A|-|\ell \cap (A \cup (A + A))|}. \end{aligned}$$

Summing over all A and ℓ , we obtain an expression for S that matches that in [Zha10, Thm. 3.11]. Another expression for S is given by O’Dorney [O’D13, Lems. 10, 11]⁶, who shows that $S = \frac{\varphi}{\sqrt{5}}(1 + \sum_{k \geq 1} a_k \varphi^{-(k+2)})$, where a_k is the number of strongly descended semigroups of genus $2k + 1$ and efficacy $k + 1$. This too can be related to the above two sums, and a brief sketch of the correspondence is as follows: associate a pair (ℓ, A) with a Kunz word $w_1 w_2 \cdots w_{\ell+1-|A|+2|\ell \cap (A \cup (A + A))|}$, where $w_i = 2$ for $i \in [\ell] \setminus A$ and $w_i = 1$ for all other i .

7.2. Results

Our first result is a new lower bound of the quantity S . Table 7.1 shows the partial sums of the sum $\sum_{w \text{ stressed}} \varphi^{-g(w)}$, considering all stressed words of length at most ℓ . If we use the $\ell = 56$

⁶Lemma 11 in [O’D13] has a minor error and should instead state $C = S\varphi^2$.

g	s_g	t_g	\hat{n}_g	g	s_g	t_g	\hat{n}_g
0	0	1	0	48	398937594	35227607540	3032888834
1	0	1	0	49	620308837	57443335681	4756701071
2	0	2	0	50	964299016	93635242237	7455057891
3	1	4	0	51	1498722966	152577300884	11675899900
4	0	6	1	52	2328886172	248541429293	18273725810
5	1	11	1	53	3618215600	404736945777	28580512964
6	3	20	3	54	5619924806	658898299876	44671692245
7	2	33	6	55	8725957048	1072361202701	69779534158
8	4	57	10	56	13542732051	1744802234628	108935597479
9	9	99	19	57	21008277551	2838171714880	169969266940
10	12	168	36	58	32573278946	4615547228454	265059561556
11	20	287	56	59	50480678072	7504199621406	413144466289
12	32	487	105	60	78197851828	12197944701688	643658549663
13	50	824	177	61	121086932116	19823231255210	1002326746843
14	84	1395	298	62	187445618110	32208621575008	1560141961678
15	132	2351	506	63	290118087627	52321970917845	2427273997885
16	208	3954	852	64	448979969989	84979572462842	3774618610486
17	331	6636	1409	65	694763898132	137996307278819	5867177646731
18	526	11116	2351	66	1074945125010	224050824866671	9115752259043
19	841	18593	3871	67	1662803652299	363709935797789	14156971708484
20	1333	31042	6354	68	2571392093291	590332152757751	21977155500049
21	2145	51780	10414	69	3975074031374	958017162586914	34103955827937
22	3401	86223	17023	70	6142744228748	1554492059573413	52902754596745
23	5314	143317	27646	71	9489097657132	2521998319817459	82034862865123
24	8396	237936	44892	72	14653792397521	4091144171788393	127165544752421
25	13279	394532	72692	73	22623678356496	6635766169962348	—
26	20952	653420	117412	74	34921396940988	10761831738691729	—
27	33029	1080981	189286	75	53896060190628	17451493968844705	—
28	51927	1786328	304702	76	83170721269779	28296496428806213	—
29	81527	2948836	489003	77	128330590081463	45876320987732381	—
30	128102	4863266	783507	78	197980950188515	74370798366727109	—
31	201700	8013802	1252986	79	305373013675616	120552492368135106	—
32	317461	13194529	2000541	80	470903017814500	195394193752676715	—
33	498911	21707242	3188964	81	725955935033818	316672642055845639	—
34	782868	35684639	5076448	82	1118822820960919	513185658629483273	—
35	1226255	58618136	8069065	83	1723818676661774	831582119361990686	—
36	1919070	96221845	12810655	84	2655337007585162	1347423114999059121	—
37	3000905	157840886	20317403	85	4089504559715309	2183094738920765116	—
38	4687213	258749944	32189863	86	6297510121544894	3536815364041369131	—
39	7315975	423906805	50944640	87	9696835494675713	5729606938456809960	—
40	11419861	694076610	80537674	88	14929924846668923	9281352227344848014	—
41	17833383	1135816798	127176042	89	22984927653735759	15033944093455393733	—
42	27857264	1857750672	200605850	90	35381018243394036	24350677339043635783	—
43	43511423	3037078893	316112953	91	54452928150272573	39439074360649302089	—
44	67908811	4962738376	497663200	92	83788019525979642	63873539719218917514	—
45	105857661	8105674930	782811886	93	128897984242944822	103441512064111164425	—
46	164837336	13233250642	1230383006	94	19825140524002501	167513303188854084440	—
47	256493732	21595419304	1932426198	95	304861061425644202	271259676314390893067	—

Table 7.2: Known values for s_g , t_g , and \hat{n}_g . All results with $g > 65$ are new.

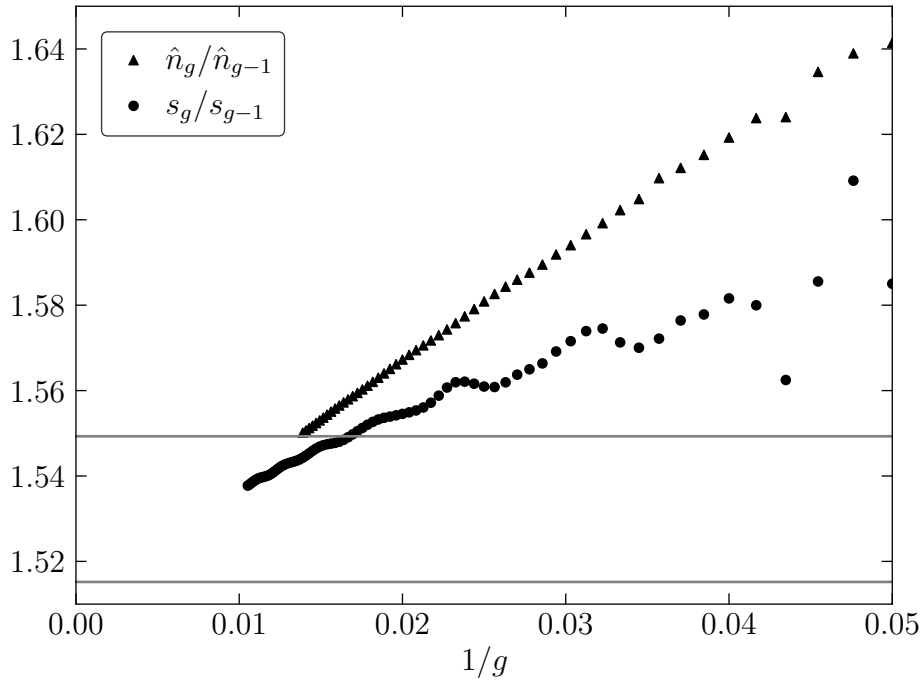


Figure 7.1: A plot of the ratios \hat{n}_g/\hat{n}_{g-1} and s_g/s_{g-1} against $1/g$. The horizontal lines are located at $r_{1.51}$ and $r_{1.54}$.

partial sum as a lower bound, we obtain the following bound:

Proposition 7.2. $S > 3.8293$.

A crude extrapolation suggests that the true value of S lies between 3.85 and 3.86.

We will note, due to Remark 7.1, that the contents of Table 7.1 are largely equivalent to Table 2 in [Zha10], which computes this sum up to $\ell = 46$, concluding that $S > 3.78$. As such, our computations yield 10 additional terms.

A larger improvement on existing knowledge is obtained by the calculation of s_g for all $g \leq 95$ (see Table 7.2). Using the fact that $s_g = t_g - t_{g-1} - t_{g-2}$, we are then able to deduce the value of t_g for all $g \leq 95$. To our best knowledge, the highest g for which t_g had been previously calculated is $g = 65$ [EF20], so this computation adds 30 previously unknown terms to the sequence.

Combining these results for t_g with known values of n_g for $g \leq 72$ [BAFG20, OEI23, A007323] yields values for \hat{n}_g for $g \leq 72$. While not attempted in this paper, it may be possible to construct algorithms for evaluating \hat{n}_g that are significantly faster than algorithms for computing n_g , considering that $\lim_{g \rightarrow \infty} \hat{n}_g/n_g = 0$. If this is possible, then new values for \hat{n}_g may be combined with values of t_g computed in this paper to yield further terms of the sequence n_g .

7.3. Conjectural asymptotics

Theorem 1.5 bounds the growth rates r_1 and r_2 within an interval with endpoints $r_{1.51}$ and $r_{1.54}$, but does not determine them exactly. To analyze the numerical evidence, in Figure 7.1 we plot

the ratios \hat{n}_g/\hat{n}_{g-1} and s_g/s_{g-1} against $1/g$. We use $1/g$ instead of the more traditional g in order to better visualize the process $g \rightarrow \infty$. An extrapolation of the data suggests not only that $r_1 = r_2 = r_{1.51}$, but that $\lim_{g \rightarrow \infty} \hat{n}_g/\hat{n}_{g-1} = \lim_{g \rightarrow \infty} s_g/s_{g-1} = r_{1.51}$. As mentioned in Section 5, all that would be needed to show that $r_1 = r_2 = r_{1.51}$ would be an improvement of the arguments in Section 5.7, whereas proving $\lim_{g \rightarrow \infty} \hat{n}_g/\hat{n}_{g-1} = \lim_{g \rightarrow \infty} s_g/s_{g-1} = r_{1.51}$ would require additional techniques beyond analyzing the convergence of generating functions.

On the other hand, it appears unlikely that s_g and \hat{n}_g are asymptotic to exponential functions, since the ratios \hat{n}_g/\hat{n}_{g-1} and s_g/s_{g-1} appear to have a nonzero slope in terms of $1/g$; in other words, there appear to be constants α and α' (possibly equal) such that $\hat{n}_g/\hat{n}_{g-1} = r_{1.51}(1 + \alpha/g + o(1/g))$ and $s_g/s_{g-1} = r_{1.51}(1 + \alpha'/g + o(1/g))$. This suggests that $\hat{n}_g/r_{1.51}^g$ and $s_g/r_{1.51}^g$ grow polynomially in g with exponents α and α' . In summary, we may condense our above discussion into the following three successively stronger conjectures, where each implies the previous:

Conjecture 7.3. (a) $\lim_{g \rightarrow \infty} \hat{n}_g^{1/g} = \lim_{g \rightarrow \infty} s_g^{1/g} = r_{1.51}$.

(b) $\lim_{g \rightarrow \infty} \hat{n}_g/\hat{n}_{g-1} = \lim_{g \rightarrow \infty} s_g/s_{g-1} = r_{1.51}$.

(c) There exist constants α and α' (possibly equal) such that $\hat{n}_g = g^{\alpha+o(1)}r_{1.51}^g$ and $s_g = g^{\alpha'+o(1)}r_{1.51}^g$.

Computational evidence suggests that if α and α' exist, then they are both between 1.6 and 1.7.

7.4. Remarks on Conjecture 1.2

We conclude this paper by discussing Conjecture 1.2 in light of our results, which may be written as the statement that $s_g + \hat{n}_g - \hat{n}_{g-1} - \hat{n}_{g-2} \geq 0$. While our results bound $|s_g + \hat{n}_g - \hat{n}_{g-1} - \hat{n}_{g-2}| < (r_{1.54} + o(1))^g$, they do not determine its sign. Nevertheless, this bound indicates that the validity of the conjecture is in a sense determined by phenomena that lie “underneath” the prevailing Fibonacci behavior of n_g .

Our work with s_g and \hat{n}_g also hints at a possible method for proving or disproving Conjecture 1.2. The first step lies in proving part (b) of Conjecture 7.3, which will imply that $\hat{n}_{g-1} + \hat{n}_{g-2} - \hat{n}_g \sim (r_{1.51}^{-1} + r_{1.51}^{-2} - 1)\hat{n}_g$. Assuming this result, the truth of Conjecture 1.2 will then be highly dependent on the ratio \hat{n}_g/s_g , which seems to approach some real constant A as g goes to infinity (see Figure 7.2). In particular, assuming that A exists, we then find that

$$s_g + \hat{n}_g - \hat{n}_{g-1} - \hat{n}_{g-2} = (A^{-1} - (r_{1.51}^{-1} + r_{1.51}^{-2} - 1) + o(1))\hat{n}_g.$$

Thus, if $A > (r_{1.51}^{-1} + r_{1.51}^{-2} - 1)^{-1} \approx 10.465$, Conjecture 1.2 is false for sufficiently large g , and if $A < (r_{1.51}^{-1} + r_{1.51}^{-2} - 1)^{-1}$, then Conjecture 1.2 is true for sufficiently large g . The most intriguing possibility would be if we had $A = (r_{1.51}^{-1} + r_{1.51}^{-2} - 1)^{-1}$, which would give no information as to whether Conjecture 1.2 is true. If this were the case, it would strongly suggest a deep connection between the numbers \hat{n}_g and $\hat{n}_{g-1} + \hat{n}_{g-2}$ in spite of the fact that the growth rate of \hat{n}_g is less than φ . Such a connection could take the form of a novel operation on numerical semigroups.

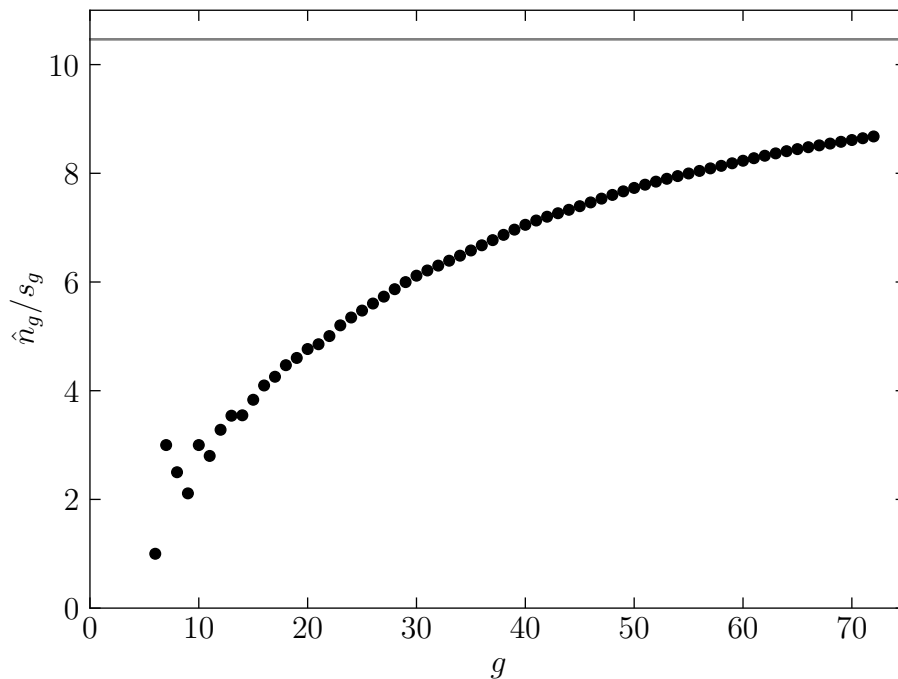


Figure 7.2: A plot of \hat{n}_g/s_g . The horizontal line is located at $(r_{1.51}^{-1} + r_{1.51}^{-2} - 1)^{-1} \approx 10.465$.

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References

- [BA08] Maria Bras-Amorós. Fibonacci-like behavior of the number of numerical semi-groups of a given genus. *Semigroup Forum*, 76(2):379–384, 2008. doi:10.1007/s00233-007-9014-8.
- [Bac21] Roland Bacher. Generic numerical semigroups. 2021. arXiv:2105.04200.
- [BAFG20] Maria Bras-Amorós and Julio Fernández-González. The right-generators descendant of a numerical semigroup. *Math. Comp.*, 89(324):2017–2030, 2020. doi:10.1090/mcom/3502.
- [EF20] Shalom Eliahou and Jean Fromentin. Gapsets and numerical semigroups. *J. Combin. Theory Ser. A*, 169, 2020. doi:10.1016/j.jcta.2019.105129.

- [Kun87] Ernst Kunz. *Über die Klassifikation numerischer Halbgruppen*, volume 11 of *Regensburger Mathematische Schriften*. Universität Regensburg, Fachbereich Mathematik, Regensburg, 1987.
- [KY13] Nathan Kaplan and Lynnelle Ye. The proportion of Weierstrass semigroups. *J. Algebra*, 373:377–391, 2013. doi:10.1016/j.jalgebra.2012.09.041.
- [O’D13] Evan O’Dorney. Degree asymptotics of the numerical semigroup tree. *Semigroup Forum*, 87(3):601–616, 2013. doi:10.1007/s00233-013-9486-7.
- [OEI23] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2023. URL: <http://oeis.org>.
- [RGSGB02] J. C. Rosales, P. A. García-Sánchez, J. I. García-García, and M. B. Branco. Systems of inequalities and numerical semigroups. *J. Lond. Math. Soc. (2)*, 65(3):611–623, 2002. doi:10.1112/S0024610701003052.
- [Sin22] Deepesh Singhal. Distribution of genus among numerical semigroups with fixed Frobenius number. *Semigroup Forum*, 104(3):704–723, 2022. doi:10.1007/s00233-022-10282-6.
- [Wil94] Herbert S. Wilf. *generatingfunctionology*. Academic Press, Inc., Boston, MA, 2nd edition, 1994.
- [Zha10] Yufei Zhao. Constructing numerical semigroups of a given genus. *Semigroup Forum*, 80(2):242–254, 2010. doi:10.1007/s00233-009-9190-9.
- [Zha11] Yufei Zhao. The bipartite swapping trick on graph homomorphisms. *SIAM J. Discrete Math.*, 25(2):660–680, 2011. doi:10.1137/100800415.
- [Zha13] Alex Zhai. Fibonacci-like growth of numerical semigroups of a given genus. *Semigroup Forum*, 86(3):634–662, 2013. doi:10.1007/s00233-012-9456-5.