

A HORIZONTAL-STRIP LLT POLYNOMIAL IS DETERMINED BY ITS WEIGHTED GRAPH

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Abstract. We prove that two horizontal-strip LLT polynomials are equal if the associated weighted graphs defined by the author in a previous paper are isomorphic. This provides a sufficient condition for equality of horizontal-strip LLT polynomials and yields a well-defined LLT polynomial indexed by a weighted graph. We use this to prove some new relations between LLT polynomials and we explore a connection with extended chromatic symmetric functions.

Keywords. Chromatic symmetric function, LLT polynomial, Hall–Littlewood polynomial, interval graph, Schur function, weighted graph

Mathematics Subject Classifications. 05E05, 05E10, 05C15

1. Introduction

LLT polynomials have been studied extensively in algebraic combinatorics and representation theory. Horizontal-strip LLT polynomials generalize the Hall–Littlewood polynomials, which are the Frobenius series of cohomology rings of certain subsets of the flag variety [Hai02]. The Shuffle Theorem [CM18] of Carlsson and Mellit describes an LLT expansion of the Frobenius series of the space of diagonal harmonics. The Extended Delta Theorem of Blasiak, Haiman, Morse, Pun, and Seelinger [BHM⁺23] generalizes this to an LLT expansion of an infinite series of GL_m characters using an action of the Schiffmann algebra on the space of symmetric functions. LLT polynomials appear positively in an expansion of Macdonald polynomials [HHL05], which implies that Macdonald polynomials are Schur-positive. LLT polynomials also arise in the representation theory of the quantum affine algebra [LLT97] and of regular semisimple Hessenberg varieties via their connection to chromatic quasisymmetric functions [GP16, SW12].

If λ is a sequence of single cells, then the unicellular LLT polynomial $G_\lambda(\mathbf{x}; q)$ can be expressed as a sum over arbitrary colourings of a unit interval graph $\Gamma(\lambda)$ associated to λ . Huh,

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Nam, and Yoo [HNY20] proved a combinatorial Schur expansion of $G_\lambda(\mathbf{x}; q)$ whenever $\Gamma(\lambda)$ is a “melting lollipop” and Alexandersson conjectured [Ale21] and then proved with Sulzgruber [AS22] a combinatorial elementary symmetric function expansion of $G_\lambda(\mathbf{x}; q+1)$ in terms of acyclic orientations of $\Gamma(\lambda)$. An equality of unicellular LLT polynomials $G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q)$ is equivalent [CM18] to an equality of the corresponding chromatic quasisymmetric functions $X_{\Gamma(\lambda)}(\mathbf{x}; q) = X_{\Gamma(\mu)}(\mathbf{x}; q)$ introduced by Shareshian and Wachs [SW16]. Therefore, LLT polynomials are intimately connected to longstanding conjectures about equalities of chromatic symmetric functions, which is an area of active research [APZ14, APCSZ21, Cre22].

If λ is a sequence of rows, then the horizontal-strip LLT polynomial $G_\lambda(\mathbf{x}; q)$ can be expressed as a sum over certain colourings of a unit interval graph $\tilde{\Gamma}(\lambda)$ with some decorated edges and Alexandersson and Sulzgruber’s result [AS22] generalizes to this setting. In [Tom21] the author defined an alternative generalization of $\Gamma(\lambda)$ to a weighted interval graph $\Pi(\lambda)$. That paper gives a combinatorial Schur expansion of $G_\lambda(\mathbf{x}; q)$ whenever $\Pi(\lambda)$ is triangle-free and shows that the largest power of q in $G_\lambda(\mathbf{x}; q)$ is the total edge weight of $\Pi(\lambda)$.

The main result of this paper, Theorem 2.7, states that the horizontal-strip LLT polynomial $G_\lambda(\mathbf{x}; q)$ is determined by the weighted graph $\Pi(\lambda)$. In other words, if $\Pi(\lambda) \cong \Pi(\mu)$, then $G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q)$. In particular, this implies that if Π is a weighted graph arising from this construction, then there corresponds a well-defined LLT polynomial $G_\Pi(\mathbf{x}; q)$. In Section 3, we prove Theorem 2.7 modulo a technical result, Lemma 3.23, whose proof we postpone to Section 5. In Section 4, we explore a connection between $G_\Pi(\mathbf{x}; q)$ and the extended chromatic symmetric functions $X_\Pi(\mathbf{x})$ associated to weighted graphs that were defined by Crew and Spirkl [CS20] and whose relations were considered in [AWvW21].

2. Background

A *composition* α is a finite sequence of positive integers $\alpha = \alpha_1 \cdots \alpha_\ell$. We denote by $\ell(\alpha)$ the *length* of α and by convention, we set $\alpha_i = 0$ if $i > \ell$. A *partition* σ is a composition that is weakly decreasing, that is $\sigma_1 \geq \cdots \geq \sigma_\ell$. We also define the integer $n(\sigma) = \sum_i (i-1)\sigma_i$. If σ and τ are partitions with $\sigma_i \geq \tau_i$ for every i , then the corresponding *skew diagram* is

$$\lambda = \sigma/\tau = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i \geq 1, \tau_i + 1 \leq j \leq \sigma_i\}. \quad (2.1)$$

If τ is the empty partition, we write σ instead of σ/\emptyset . The elements of λ are called *cells* and the *content* of a cell $u = (i, j) \in \lambda$ is the integer $c(u) = j - i$. We primarily work with *rows* and we assume the contents are nonnegative, so that they are skew diagrams of the form

$$R = a/b = \{(1, j) \in \mathbb{N} \times \mathbb{N} : b + 1 \leq j \leq a\} \quad (2.2)$$

for some $a \geq b \geq 0$. We denote by $l(R) = b$ and $r(R) = a - 1$ the smallest and largest contents of cells of R . Note that $l(R)$ is the content of the leftmost cell of R , not the length of R , which is $|R| = r(R) - l(R) + 1$. We also denote by $R^+ = (a+1)/(b+1)$ and $R^- = (a-1)/(b-1)$ the rows obtained by shifting R right or left respectively by one cell. A *semistandard Young tableau* (SSYT) of shape λ is a function $T : \lambda \rightarrow \{1, 2, 3, \dots\}$ that satisfies $T_{i,j} \leq T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$, where we write $T_{i,j}$ instead of $T((i, j))$. A *multiskew partition* is a sequence

of skew diagrams $\lambda = (\lambda^{(1)}, \dots, \lambda^{(n)})$. We say that λ is *unicellular* if each $\lambda^{(i)}$ is a single cell and in keeping with the terminology of Alexandersson and Sulzgruber [AS22], we say that λ is a *horizontal-strip* if each $\lambda^{(i)}$ is a row. If λ is a horizontal-strip, we denote by λ the partition determined by the row lengths of λ and we define $n(\lambda) = n(\lambda)$. We denote by

$$\text{SSYT}_\lambda = \{T = (T^{(1)}, \dots, T^{(n)}) : T^{(i)} \in \text{SSYT}_{\lambda^{(i)}}\} \tag{2.3}$$

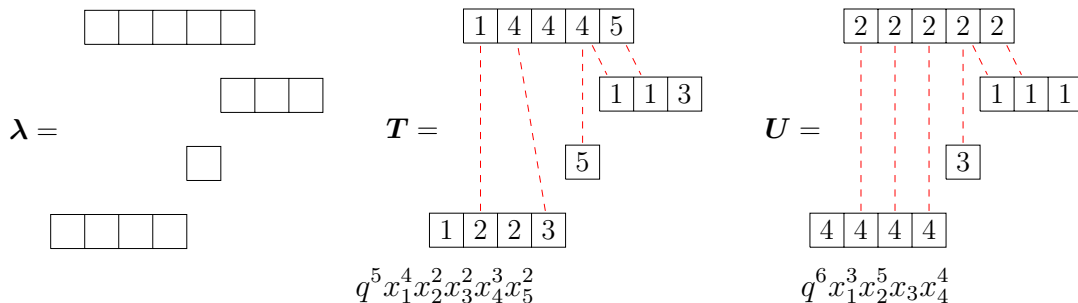
the set of sequences of SSYTs of shape λ . Two entries $T^{(i)}(u)$ and $T^{(j)}(v)$ of T with $i < j$ form an *inversion* if either

- $c(u) = c(v)$ and $T^{(i)}(u) > T^{(j)}(v)$, or
- $c(u) = c(v) + 1$ and $T^{(i)}(u) < T^{(j)}(v)$.

We denote by $\text{inv}(T)$ the number of inversions of T and we define $\mathbf{x}^T = x_1^{\#\text{ of }1\text{'s}} x_2^{\#\text{ of }2\text{'s}} \dots$. Now we define the *LLT polynomial*

$$G_\lambda(\mathbf{x}; q) = \sum_{T \in \text{SSYT}_\lambda} q^{\text{inv}(T)} \mathbf{x}^T. \tag{2.4}$$

Example 2.1. The multiskew partition $\lambda = (4/0, 5/4, 8/5, 6/1)$, two SSYTs T and U of shape λ with their inversions marked by dashed lines, and the corresponding monomials of the LLT polynomial $G_\lambda(\mathbf{x}; q)$ are given below. Because λ is a horizontal-strip, we draw it so that cells of the same content are aligned vertically.



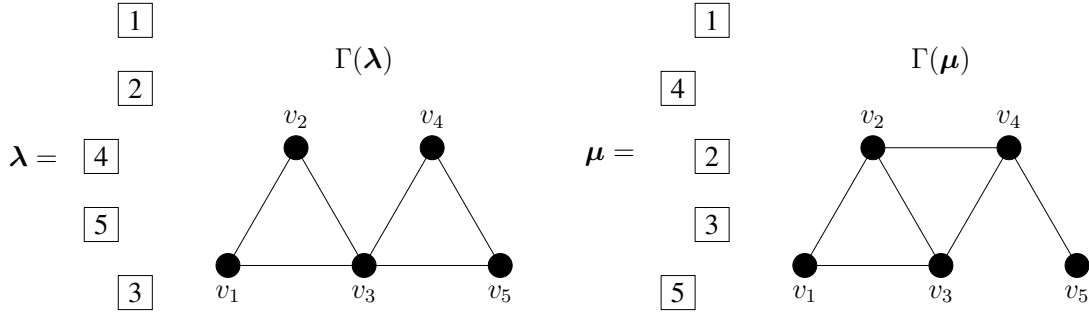
The LLT polynomial $G_{(4/0,5/4,8/5,6/1)}(\mathbf{x}; q)$ can be expanded in the Schur function basis as

$$\begin{aligned} G_\lambda(\mathbf{x}; q) = & q^6 s_{5431} + q^6 s_{544} + q^6 s_{5521} + 2q^6 s_{553} + q^6 s_{6331} + 2q^6 s_{6421} + (3q^6 + q^5) s_{643} \\ & + 2q^6 s_{6511} + (4q^6 + q^5) s_{652} + (2q^6 + q^5) s_{661} + (q^6 + q^5) s_{7321} + (q^6 + 2q^5) s_{733} \\ & + (q^6 + 2q^5) s_{7411} + (2q^6 + 5q^5) s_{742} + (2q^6 + 6q^5) s_{751} + 4q^5 s_{76} + q^5 s_{8221} \\ & + (2q^5 + q^4) s_{8311} + (4q^5 + 2q^4) s_{832} + (5q^5 + 5q^4) s_{841} + (3q^5 + 4q^4) s_{85} + 2q^4 s_{9211} \\ & + 3q^4 s_{922} + (7q^4 + 2q^3) s_{931} + (5q^4 + 3q^3) s_{94} + q^3 s_{(10)111} + 6q^3 s_{(10)21} \\ & + (6q^3 + q^2) s_{(10)3} + 3q^2 s_{(11)11} + 5q^2 s_{(11)2} + 3q s_{(12)1} + s_{(13)}. \end{aligned}$$

LLT polynomials are symmetric functions [LLT97, Theorem 6.1] and moreover are *Schur-positive*, [HHLU05, Theorem 3.1.3] [LT00] [GH07, Corollary 6.9], meaning that they can be expressed as a linear combination of Schur functions where the coefficients are polynomials in q with positive coefficients. Finding a combinatorial formula for this Schur expansion is a major open problem of active research [Ale21, AP18, AS22, AU20, HNY20, Tom21].

When $\lambda = (\lambda^{(1)}, \dots, \lambda^{(n)})$ is unicellular, we can associate to λ a labelled graph $\Gamma(\lambda)$ with vertices v_1, \dots, v_n as follows. We label the cells of λ as $1, 2, 3, \dots$ in *reverse content reading order*, meaning in order of decreasing content and from top to bottom along constant content lines. Then vertices v_i and v_j are joined by an edge if it is possible for entries in cells i and j to form an inversion. This approach was effective in finding combinatorial formulas for unicellular LLT polynomials [Ale21, AP18, HNY20]. Graphs arising from this construction are called *unit interval graphs* and they have several equivalent characterizations [Gar07].

Example 2.2. The unicellular multiskew partitions $\lambda = (2/1, 1/0, 1/0, 2/1, 2/1)$ and $\mu = (1/0, 2/1, 2/1, 1/0, 2/1)$ are given below with their cells labelled in reverse content reading order, along with the associated labelled graphs $\Gamma(\lambda)$ and $\Gamma(\mu)$.



When λ is unicellular, a tableau $T \in \text{SSYT}_\lambda$ is precisely a (possibly not proper) colouring $\kappa : \Gamma(\lambda) \rightarrow \mathbb{N}$ and an inversion of T is an *ascent* of κ , namely an edge $(v_i, v_j) \in E(\Gamma(\lambda))$ with $i < j$ and $\kappa(v_i) < \kappa(v_j)$. Therefore, we can express $G_\lambda(\mathbf{x}; q)$ in terms of $\Gamma(\lambda)$ as

$$G_\lambda(\mathbf{x}; q) = \sum_{\substack{\kappa: \Gamma(\lambda) \rightarrow \mathbb{N} \\ \kappa \text{ arbitrary}}} q^{\text{asc}(\kappa)} \mathbf{x}^\kappa. \quad (2.5)$$

If we restrict the sum (2.5) to *proper colourings*, meaning that $\kappa(v_i) \neq \kappa(v_j)$ whenever $(v_i, v_j) \in E(\Gamma(\lambda))$, then we have the *chromatic quasisymmetric function* $X_{\Gamma(\lambda)}(\mathbf{x}; q)$. Unicellular LLT polynomials and chromatic quasisymmetric functions are related by a change of variables, namely the plethystic relationship [CM18, Proposition 3.4]

$$X_{\Gamma(\lambda)}(\mathbf{x}; q) = (q-1)^{-|\Gamma(\lambda)|} G_\lambda[\mathbf{x}(q-1); q], \quad (2.6)$$

which in particular implies that if λ and μ are unicellular, then

$$X_{\Gamma(\lambda)}(\mathbf{x}; q) = X_{\Gamma(\mu)}(\mathbf{x}; q) \text{ if and only if } G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q). \quad (2.7)$$

For example, the unicellular multiskew partitions λ and μ from Example 2.2 have the same LLT polynomials and their labelled unit interval graphs have the same chromatic quasisymmetric

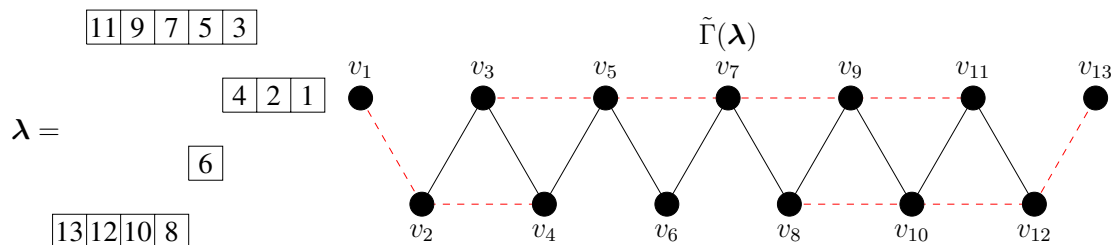
functions. Because of this relationship, unicellular LLT polynomials may be a fruitful approach to the Stanley–Stembridge conjecture that chromatic symmetric functions of unit interval graphs are e -positive, or its refinement, the Shareshian–Wachs conjecture, which asks for a combinatorial elementary function expansion of the chromatic quasisymmetric function.

When $\lambda = (R_1, \dots, R_n)$ is a horizontal-strip, one generalization of the above construction is to include a set S of special edges that record when cells of λ are in the same row. Then we can express the LLT polynomial in terms of this decorated graph $\tilde{\Gamma}(\lambda)$ as

$$G_\lambda(\mathbf{x}; q) = \sum_{\substack{\kappa: \tilde{\Gamma}(\lambda) \rightarrow \mathbb{N} \\ \kappa(v_i) \leq \kappa(v_j) \text{ if } i < j, (v_i, v_j) \in S}} q^{\text{asc}(\kappa)} \mathbf{x}^\kappa. \tag{2.8}$$

This approach was successfully employed by Alexandersson and Sulzgruber [AS22].

Example 2.3. The horizontal-strip $\lambda = (4/0, 5/4, 8/5, 6/1)$ and its associated decorated labelled graph $\tilde{\Gamma}(\lambda)$ are given below.



We will be interested in the following *weighted graph* $\Pi(\lambda)$ that the author defined as an alternative generalization of $\Gamma(\lambda)$ to the horizontal-strip case.

Definition 2.4. [Tom21, Definition 3.1, Definition 3.2] Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip. For $1 \leq i < j \leq n$ we define the integers

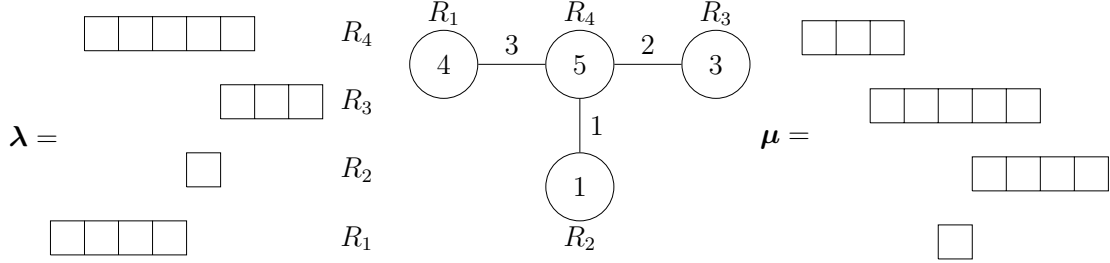
$$M(R_i, R_j) = \begin{cases} |R_i \cap R_j| & \text{if } l(R_i) \leq l(R_j), \\ |R_i \cap R_j^+| & \text{if } l(R_i) > l(R_j). \end{cases} \tag{2.9}$$

We then define a weighted graph $\Pi(\lambda)$ whose vertices are the rows of λ . The weight of a row R_i is the number of cells $|R_i|$ and rows R_i and R_j with $i < j$ are joined by an edge of weight $M(R_i, R_j)$. By convention, we omit edges of weight zero.

Remark 2.5. In [Tom21, Definition 3.2] the author also defined an ordering of the vertices of $\Pi(\lambda)$ based on the content reading order of the rightmost cells in the rows of λ . However, because our main Theorem states that the corresponding LLT polynomial does not depend on this labelling, we will not be interested in it. Also note that if λ is unicellular, then the graph $\Pi(\lambda)$ is simply $\Gamma(\lambda)$ with all vertex and edge weights one.

Example 2.6. The horizontal-strip $\lambda = (4/0, 5/4, 8/5, 6/1)$ and the weighted graph $\Pi(\lambda)$ are given below. We have $M(R_1, R_4) = 3$, $M(R_2, R_4) = 1$, and $M(R_3, R_4) = 2$. We have also

drawn the horizontal-strip $\mu = (5/4, 9/5, 7/2, 3/0)$, whose weighted graph $\Pi(\mu)$ is isomorphic to $\Pi(\lambda)$.



We can now state our main Theorem. We will state the key Lemma 3.23 in Section 3 and we will show how it implies Theorem 2.7. We will then prove Lemma 3.23 in Section 5.

Theorem 2.7. *Let λ and μ be horizontal-strips. If the weighted graphs $\Pi(\lambda)$ and $\Pi(\mu)$ are isomorphic, then the LLT polynomials $G_\lambda(\mathbf{x}; q)$ and $G_\mu(\mathbf{x}; q)$ are equal. In other words,*

$$\text{if } \Pi(\lambda) \cong \Pi(\mu), \text{ then } G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q). \quad (2.10)$$

Example 2.8. The horizontal-strips $\lambda = (4/0, 5/4, 8/5, 6/1)$ and $\mu = (5/4, 9/5, 7/2, 3/0)$ in Example 2.6 have isomorphic weighted graphs, so it follows from Theorem 2.7 that $G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q)$.

We define a weighted graph Π to be *admissible* if it arises from our construction, in other words if $\Pi \cong \Pi(\lambda)$ for some horizontal-strip λ . Then an alternative restatement of Theorem 2.7 is that if Π is admissible, then there is a well-defined LLT polynomial $G_\Pi(\mathbf{x}; q)$ given by $G_\Pi(\mathbf{x}; q) = G_\lambda(\mathbf{x}; q)$, where λ is any horizontal-strip with $\Pi \cong \Pi(\lambda)$.

Theorem 2.7 justifies our choice to associate to λ this mysterious weighted graph $\Pi(\lambda)$. We now mention some advantages of this definition.

1. [Tom21, Theorem 4.6] If $\Pi(\lambda)$ is triangle-free, then there is a combinatorial Schur expansion of $G_\lambda(\mathbf{x}; q)$ defined purely in terms of $\Pi(\lambda)$, but not λ itself.
2. [Tom21, Example 4.10] As an example of (1), if we let $P_{ab}(M)$ denote a graph consisting of two vertices of weights a and b , joined by an edge of weight M , where we may assume that $a \geq b \geq M$ without loss of generality, then the Schur function expansion of the LLT polynomial $G_{P_{ab}(M)}(\mathbf{x}; q)$ can be expressed in terms of a , b , and M as

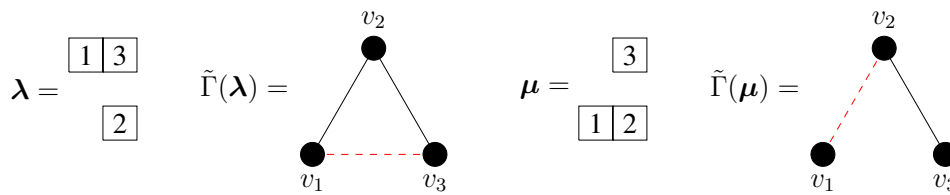
$$G_{P_{ab}(M)} = \sum_{k=0}^b q^{\min\{M,k\}} s_{(a+b-k)k} = s_{(a+b)} + qs_{(a+b-1)1} + \cdots + q^M s_{(a+b-M)M} + \cdots + q^M s_{ab}. \quad (2.11)$$

3. [Tom21, Lemma 3.15] If λ and μ are horizontal-strips that differ by switching a pair of adjacent rows, then we have

$$G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q) \text{ if and only if } \Pi(\lambda) \cong \Pi(\mu). \tag{2.12}$$

In other words, equalities of LLT polynomials in this case are characterized by the associated weighted graphs.

4. The graph $\tilde{\Gamma}(\lambda)$ contains essentially the same information as the horizontal-strip λ , while the coarser invariant $\Pi(\lambda)$ is more useful for detecting equalities of LLT polynomials. For example, consider $\lambda = (2/1, 2/0)$ and $\mu = (2/0, 2/1)$ below.



Note that $\tilde{\Gamma}(\lambda)$ has two normal edges and one special edge, while $\tilde{\Gamma}(\mu)$ has one normal edge and one special edge. Meanwhile, we have $\Pi(\lambda) \cong \Pi(\mu) \cong P_{21}(1)$, so by (2.11) we have $G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q) = s_3 + qs_{21}$.

We saw in Example 2.2 that even in the unicellular case, the converse to Theorem 2.7 does not hold because there are horizontal-strips λ and μ with $G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q)$ but $\Pi(\lambda) \not\cong \Pi(\mu)$. However, we hope that such cases underlie a deep connection to equalities of chromatic symmetric functions, which are an area of active research [AWvW21, APZ14, APCSZ21, Cre22]. We explore this connection in Theorem 4.12 and Theorem 4.18 in Section 4.

3. Proof of Theorem 2.7

In this section we prove Theorem 2.7, modulo a technical result, Lemma 3.23, whose proof we postpone to Section 5. The general idea is to rewrite λ and μ in a more convenient form while preserving their weighted graphs and LLT polynomials. The following definition will be convenient to describe the relationship between rows within a fixed horizontal-strip.

Definition 3.1. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip and $1 \leq i, j \leq n$ with $i \neq j$. We define $M_{i,j}(\lambda)$ to be the number of edges in $\Pi(\lambda)$ joining R_i and R_j , that is

$$M_{i,j}(\lambda) = \begin{cases} M(R_i, R_j) & \text{if } i < j, \\ M(R_j, R_i) & \text{if } i > j. \end{cases} \tag{3.1}$$

We abbreviate $M_{i,j}(\lambda)$ as $M_{i,j}$ if the context is clear. Note that

$$0 \leq M_{i,j} \leq \min\{|R_i|, |R_j|\}. \tag{3.2}$$

We also take this opportunity to formally define an isomorphism of weighted graphs.

Definition 3.2. Let $\lambda = (R_1, \dots, R_n)$ and $\mu = (S_1, \dots, S_n)$ be horizontal-strips. An *isomorphism of weighted graphs*, denoted $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$, is a permutation

$$\varphi : \{1, \dots, n\} \rightarrow \{1, \dots, n\} \quad (3.3)$$

that satisfies $|R_i| = |S_{\varphi_i}|$ and $M_{i,j}(\lambda) = M_{\varphi_i, \varphi_j}(\mu)$ for all i and j . The weighted graphs $\Pi(\lambda)$ and $\Pi(\mu)$ are *isomorphic*, denoted $\Pi(\lambda) \cong \Pi(\mu)$, if there exists such an isomorphism. We say that λ and μ are *similar* if $\Pi(\lambda) \cong \Pi(\mu)$ and $G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q)$ and we denote by $\mathcal{S}(\lambda)$ the set of horizontal-strips that are similar to λ .

Remark 3.3. Theorem 2.7 states that $\Pi(\lambda) \cong \Pi(\mu)$ implies that $G_\lambda(\mathbf{x}; q) = G_\mu(\mathbf{x}; q)$, but until we have proven this it will be convenient to temporarily define this concept of similarity.

Example 3.4. In Example 2.6, we have that $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$, where φ is the permutation given by $\varphi_1 = 2$, $\varphi_2 = 1$, $\varphi_3 = 4$, and $\varphi_4 = 3$.

Note that similarity is an equivalence relation. We now describe some operations we can perform on a horizontal-strip while preserving similarity.

Proposition 3.5. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip.

1. Let $\lambda^+ = (R_1^+, \dots, R_n^+)$ and $\lambda^- = (R_1^-, \dots, R_n^-)$ be the horizontal-strips obtained by translating all rows right by one cell or left by one cell respectively. Then we have $\lambda^+, \lambda^- \in \mathcal{S}(\lambda)$.
2. Define the cycle of λ to be $\kappa(\lambda) = (R_2, \dots, R_n, R_1^-)$. Then we have $\kappa(\lambda) \in \mathcal{S}(\lambda)$.
3. For a sufficiently large integer N , define the rotation of λ to be $N-\lambda = (N-R_n, \dots, N-R_1)$, where $N-R = \{(1, N-j) : j \in R\}$. Then we have $N-\lambda \in \mathcal{S}(\lambda)$.

Remark 3.6. In (3), we take N to be sufficiently large simply because we assume that our cells have nonnegative content. Because of (1), the precise value of N will not matter to us.

Proof of Proposition 3.5.

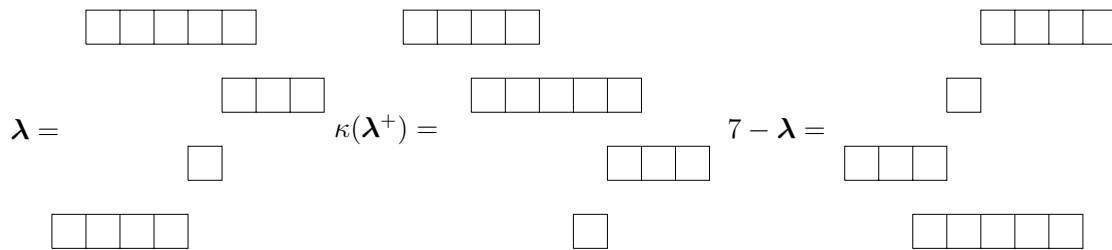
1. This follows directly from the definition because $\Pi(\lambda)$ and $G_\lambda(\mathbf{x}; q)$ only depend on the relative positions of the rows of λ .
2. This follows directly from the definition because $M(R_i, R_j) = M(R_j, R_i^-)$ and because the condition for a cell $u \in R_i$ for $i \geq 2$ to make an inversion with a cell $v = (1, j') \in R_1$ in λ is exactly the condition for u to make an inversion with the cell $v^- = (1, j'-1) \in R_1^-$ in $\kappa(\lambda)$.

3. It follows directly from the definition that $M(N - R_i, N - R_j) = M(R_j, R_i)$ and therefore $\Pi(N - \lambda) \cong \Pi(\lambda)$. If we restrict to a finite set of variables (x_1, \dots, x_k) , then by associating a tableau $\mathbf{T} = (T^{(1)}, \dots, T^{(n)}) \in \text{SSYT}_\lambda$ to a tableau $-\mathbf{T} = (-T^{(1)}, \dots, -T^{(n)}) \in \text{SSYT}_{N-\lambda}$ by setting $-T_{1,j}^{(i)} = k + 1 - T_{1,N-j}^{(n+1-i)}$, we have

$$G_{N-\lambda}(x_1, \dots, x_k; q) = G_\lambda(x_k, \dots, x_1; q) \tag{3.4}$$

and because LLT polynomials are symmetric, it follows that $G_{N-\lambda}(\mathbf{x}; q) = G_\lambda(\mathbf{x}; q)$. \square

Example 3.7. Let $\lambda = (4/0, 5/4, 8/5, 6/1)$ as in Example 2.6. The cycle $\kappa(\lambda)$ has a negative content, so for convenience we will translate right. Then the cycle $\kappa(\lambda^+) = (6/5, 9/6, 7/2, 4/0)$ and the rotation $7 - \lambda = (7/2, 3/0, 4/3, 7/4)$ are drawn below.



Definition 3.8. We say that rows R and R' commute, denoted $R \leftrightarrow R'$, if we have $M(R, R') = M(R', R)$, and otherwise we write $R \not\leftrightarrow R'$.

Lemma 3.9. [Tom21, Lemma 3.15] Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip. If $R_i \leftrightarrow R_{i+1}$, then $(R_1, \dots, R_{i+1}, R_i, \dots, R_n) \in \mathcal{S}(\lambda)$.

We now examine the integers $M_{i,j}(\lambda)$ in more detail, particularly their relationship to commutation. The following Proposition is a straightforward calculation.

Proposition 3.10. [Tom21, Proposition 3.8] Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip and let $1 \leq i, j \leq n$. Without loss of generality, assume that $l(R_i) \leq l(R_j)$.

1. If $r(R_i) < l(R_j) - 1$, then $M(R_i, R_j) = M(R_j, R_i) = 0$, so $R_i \leftrightarrow R_j$.
2. If $l(R_i) = l(R_j)$ or $r(R_j) \leq r(R_i)$, then $M(R_i, R_j) = M(R_j, R_i) = \min\{|R_i|, |R_j|\}$, so $R_i \leftrightarrow R_j$.
3. Otherwise, we have $l(R_i) < l(R_j) \leq r(R_i) + 1 \leq r(R_j)$, and

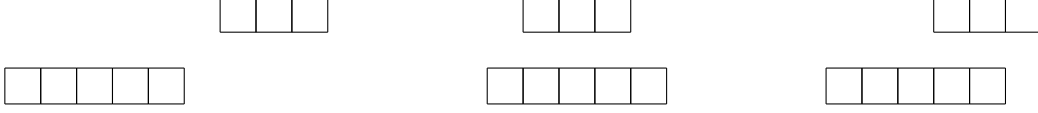
$$M(R_i, R_j) = r(R_i) - l(R_j) + 1 \text{ and } M(R_j, R_i) = r(R_i) - l(R_j) + 2, \text{ so } R_i \not\leftrightarrow R_j. \tag{3.5}$$

In particular, we have

$$M_{i,j} = r(R_i) - l(R_j) + 1 + \chi(i > j), \tag{3.6}$$

where $\chi(i > j) = 1$ if $i > j$ and 0 otherwise.

Example 3.11. The three cases of Proposition 3.10 are illustrated below. The pairs on the left and the middle commute and the pair on the right does not. As a visual description, we have that two rows commute if and only if they are either disjoint and separated by at least one cell, or if one is contained in the other.



Remark 3.12. Informally, once we fix the edge weight $M_{i,j}$ between two rows R_i and R_j , if $R_i \leftrightarrow R_j$, then there is some flexibility in their positions because they need only be disjoint (and separated by at least one cell) or one is contained in the other, while if $R_i \leftrightarrow R_j$, then by (3.6) their relative positions are specifically constrained.

Corollary 3.13. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip.

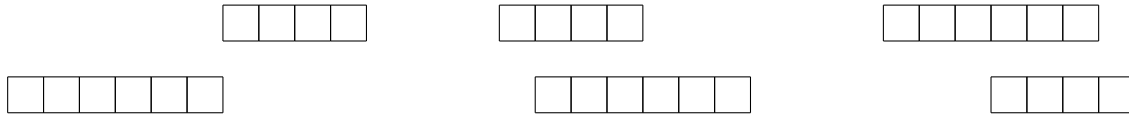
1. If $R_i \leftrightarrow R_j$, then $M_{i,j}$ is either 0 or $\min\{|R_i|, |R_j|\}$. Equivalently, if $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$, then $R_i \leftrightarrow R_j$.
2. If $R_i \leftrightarrow R_j$, then we either have $l(R_i) < l(R_j)$ and $r(R_i) < r(R_j)$, or we have $l(R_i) > l(R_j)$ and $r(R_i) > r(R_j)$.
3. Suppose that $i < j$ and $R_i \leftrightarrow R_j$. If $M_{i,j} = 0$, then $l(R_j) = r(R_i) + 1$, so in particular $l(R_i) < l(R_j)$. If $M_{i,j} = |R_i|$, then $r(R_j) = r(R_i) - 1$, while if $M_{i,j} = |R_j|$, then $l(R_j) = l(R_i) - 1$, so in particular we have $l(R_j) < l(R_i)$ in both cases.

Proof.

1. Assuming without loss of generality that $l(R_i) \leq l(R_j)$, then all of the possibilities are enumerated in Proposition 3.10 and we have $R_i \leftrightarrow R_j$ only when $M_{i,j} = 0$ or $\min\{|R_i|, |R_j|\}$.
2. Assuming without loss of generality that $l(R_i) \leq l(R_j)$, then all of the possibilities are enumerated in Proposition 3.10 and we have $R_i \leftrightarrow R_j$ only when $l(R_i) < l(R_j)$ and $l(R_j) \leq r(R_i) + 1 \leq r(R_j)$, so in particular $r(R_i) < r(R_j)$.
3. By (3.5), if $M_{i,j} = M(R_i, R_j) = 0$, then if $l(R_j) < l(R_i)$ we would have $M(R_j, R_i) = M(R_i, R_j) - 1 = -1$, contradicting (3.2), so we must have $l(R_i) < l(R_j)$ and $M_{i,j} = 0 = r(R_i) - l(R_j) + 1$. Similarly, by (3.5), if $M_{i,j} = M(R_i, R_j) = \min\{|R_i|, |R_j|\}$, then if $l(R_i) < l(R_j)$ we would have $M(R_j, R_i) = \min\{|R_i|, |R_j|\} + 1$, contradicting (3.2), so we must have $l(R_j) < l(R_i)$ and $M_{i,j} = r(R_j) - l(R_i) + 2$. The result now follows by noting that $|R_i| = r(R_i) - l(R_i) + 1$ and $|R_j| = r(R_j) - l(R_j) + 1$. \square

Remark 3.14. Note that the converse to (1) does not hold. It is possible to have $M(R_i, R_j) = 0$ and $R_i \leftrightarrow R_j$ as below left. However, (3) tells us that if this occurs, the higher row must be to the right. Similarly, it is possible to have $M(R_i, R_j) = \min\{|R_i|, |R_j|\}$ and $R_i \leftrightarrow R_j$ as below

middle or below right. However, (3) tells us that if this occurs, the higher row must be to the left.



We now show how we can use cycling and commuting to prove Theorem 2.7 in a special case. Recall that if $\lambda = (R_1, \dots, R_n)$, then we have

$$M(\lambda) = \sum_{1 \leq i < j \leq n} M(R_i, R_j) \leq \sum_{1 \leq i < j \leq n} \min\{|R_i|, |R_j|\} = n(\lambda). \tag{3.7}$$

Lemma 3.15. *Theorem 2.7 holds if $M(\lambda) = n(\lambda)$.*

Proof. Let $\lambda = (R_1, \dots, R_n)$. By (3.7), we have that if $M(\lambda) = n(\lambda)$, then $M(R_i, R_j) = \min\{|R_i|, |R_j|\}$ for all $1 \leq i < j \leq n$. Recall that we denote by λ the partition determined by the row lengths of λ . We now show that the horizontal-strip $H(\lambda) = (\lambda_1/0, \dots, \lambda_n/0)$ is similar to λ , meaning that the LLT polynomial $G_\lambda(\mathbf{x}; q)$ only depends on λ and therefore only on the weighted graph $\Pi(\lambda)$. By translating, we may assume without loss of generality that $\min\{l(R_i) : 1 \leq i \leq n\} = 0$, and suppose that $l(R_a) = 0$. Because $M_{i,j} = \min\{|R_i|, |R_j|\}$ for every i, j , we have by Proposition 3.10, Part 2 an upper bound $l(R_j) \leq r(R_a) + 1 = |R_a|$, so let us further assume that λ has $\sum_{i=1}^n l(R_i)$ minimal among all horizontal-strips similar to λ .

We now claim that $l(R_i) = 0$ for every $1 \leq i \leq n$. If not, let j be such that $l(R_j) \geq 1$ is maximal. By Corollary 3.13, Part 3, if $i < j$ and $R_i \leftrightarrow R_j$, then because $M_{i,j} = \min\{|R_i|, |R_j|\}$, we must have $l(R_j) < l(R_i)$, contradicting maximality of $l(R_j)$, so we must have $R_i \leftrightarrow R_j$ for every $i < j$. By Proposition 3.5, we can now commute and cycle to find that $(R_1, \dots, R_n, R_j^-) \in \mathcal{S}(\lambda)$, contradicting minimality of $\sum_{i=1}^n l(R_i)$. Therefore, we indeed have $l(R_i) = 0$ for every $1 \leq i \leq n$, so by Proposition 3.10, Part 2, we have $R_i \leftrightarrow R_j$ for every $1 \leq i, j \leq n$ and by commuting once again we have $H(\lambda) \in \mathcal{S}(\lambda)$. This completes the proof. \square

Example 3.16. Figure 3.1 illustrates the idea of the proof of Lemma 3.15. The row R_3 , which has $l(R_3)$ maximal, commutes with all rows below, so by commuting and cycling, we can move it to the left. Continuing in this way, the horizontal-strip λ is shown to be similar to $H(4432)$ on the right.

Remark 3.17. In this case where $M(\lambda) = n(\lambda)$, the LLT polynomial $G_\lambda(\mathbf{x}; q)$ is the transformed modified Hall–Littlewood polynomial $\tilde{H}_\lambda(\mathbf{x}; q)$, which has many connections in algebraic combinatorics [Hag07, Hai02].

There is another useful linear relationship between LLT polynomials, which we can think of as a deletion-contraction relation. It will be convenient for us to rearrange this relation as follows, so that we add an edge and contract, rather than delete an edge and contract.

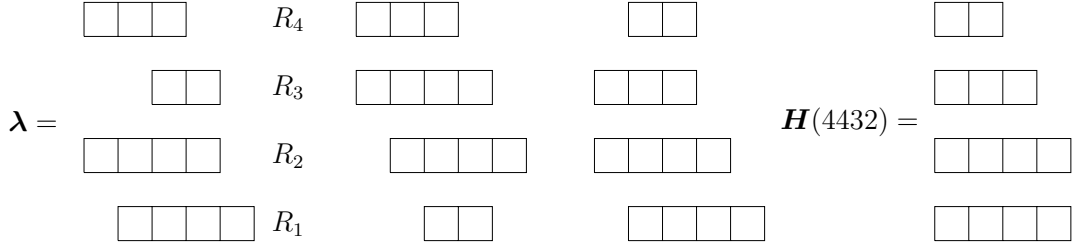


Figure 3.1: An example of using commuting and cycling to show that $\mathbf{H}(\lambda) \in \mathcal{S}(\lambda)$.

Lemma 3.18. [Tom21, Lemma 3.17] Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip with $l(R_{i+1}) > l(R_i)$ and $R_i \leftrightarrow R_{i+1}$. Define the horizontal-strips

$$\begin{aligned} \lambda' &= (R_1, \dots, R_{i+1}, R_i, \dots, R_n) \text{ and} \\ \lambda'' &= (R_1, \dots, R_i \cup R_{i+1}, R_i \cap R_{i+1}, \dots, R_n). \end{aligned} \quad (3.8)$$

Then we have

$$G_\lambda(\mathbf{x}; q) = \frac{1}{q} G_{\lambda'}(\mathbf{x}; q) + \frac{q-1}{q} G_{\lambda''}(\mathbf{x}; q). \quad (3.9)$$

Note that by Proposition 3.10, Part 1, the condition $R_i \leftrightarrow R_{i+1}$ implies that $r(R_i) \geq \ell(R_{i+1}) - 1$, so the row $R_i \cup R_{i+1}$ makes sense.

We now describe the weighted graphs associated to these horizontal-strips λ' and λ'' .

Lemma 3.19. [Tom21, Proposition 3.18] Let R_1, R_2 , and R be rows with $R_1 \leftrightarrow R_2$ and $l(R_2) > l(R_1)$ and let $M = M(R_1, R_2)$, $M_1 = M(R_1, R)$, and $M_2 = M(R_2, R)$. Then

$$M(R_2, R_1) = M + 1, \quad (3.10)$$

$$M(R_1 \cap R_2, R) = \min\{M, M_1, M_2\}, \text{ and} \quad (3.11)$$

$$M(R_1 \cup R_2, R) = \min\{|R|, \max\{M_1, M_2, M_1 + M_2 - M\}\}. \quad (3.12)$$

Remark 3.20. Lemma 3.19 describes exactly how the weighted graphs of the horizontal-strips in (3.8) can be obtained from $\Pi = \Pi(\lambda)$. The weighted graph Π' is obtained from Π by increasing the weight of the edge (R_i, R_{i+1}) by one. The weighted graph Π'' is obtained from Π by replacing R_i and R_{i+1} by new vertices $R_i \cap R_{i+1}$ and $R_i \cup R_{i+1}$ of weights M and $|R_i| + |R_{i+1}| - M$ respectively, joined by an edge of weight M , and joining them to each other vertex R by edges of weights given in (3.11) and (3.12). In particular, we have

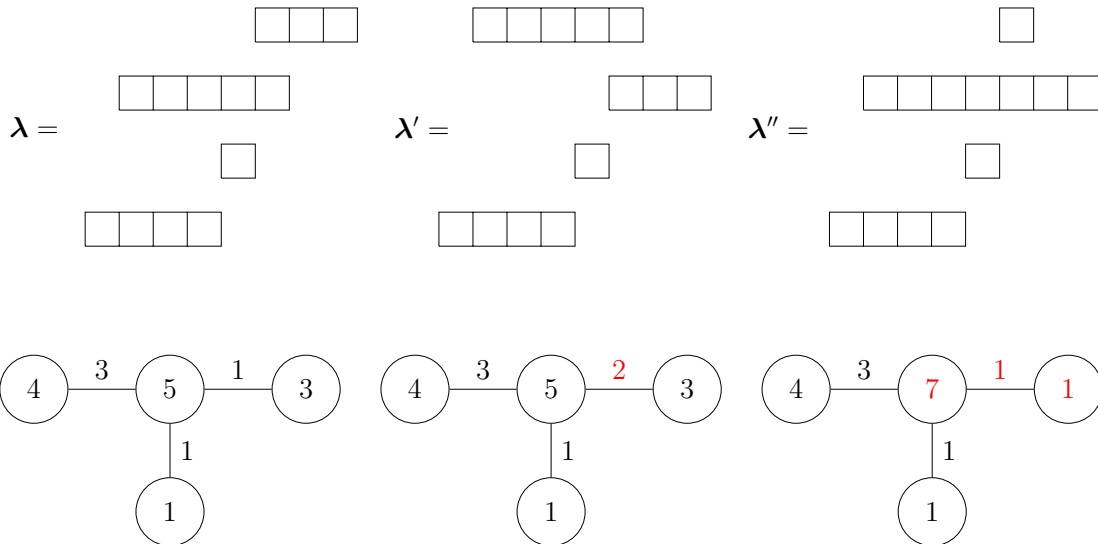
$$n(\lambda') = n(\lambda), \quad M(\lambda') = M(\lambda) + 1, \text{ and } n(\lambda'') < n(\lambda), \quad (3.13)$$

so the relation (3.9) will allow us to perform induction on $n(\lambda)$ and on $n(\lambda) - M(\lambda)$.

Example 3.21. Let $\lambda = (4/0, 5/4, 6/1, 8/5)$ and note that $l(R_4) > l(R_3)$ and $R_3 \leftrightarrow R_4$. Therefore, letting $\lambda' = (4/0, 5/4, 8/5, 6/1)$ and $\lambda'' = (4/0, 5/4, 8/1, 6/5)$, we have that

$$G_\lambda(\mathbf{x}; q) = \frac{1}{q} G_{\lambda'}(\mathbf{x}; q) + \frac{q-1}{q} G_{\lambda''}(\mathbf{x}; q). \quad (3.14)$$

The horizontal-strips λ , λ' , and λ'' , and their weighted graphs $\Pi(\lambda)$, $\Pi(\lambda')$, and $\Pi(\lambda'')$ are given below. We can think of $\Pi(\lambda)$ and $\Pi(\lambda'')$ as a deletion and contraction of $\Pi(\lambda')$.



Our strategy is to replace λ and μ by similar horizontal-strips to which we can apply (3.9). We make the following definition.

Definition 3.22. Let λ and μ be horizontal-strips with $\Pi(\lambda) \cong \Pi(\mu)$. A *good substitute* for (λ, μ) is a pair of horizontal-strips (λ', μ') , where $\lambda' = (R_1, \dots, R_n) \in \mathcal{S}(\lambda)$ and $\mu' = (S_1, \dots, S_n) \in \mathcal{S}(\mu)$ satisfy

$$l(R_1) < l(R_2), R_1 \leftrightarrow R_2, l(S_1) < l(S_2), S_1 \leftrightarrow S_2, \tag{3.15}$$

and $\varphi_1 = 1$ and $\varphi_2 = 2$, where $\varphi : \Pi(\lambda') \xrightarrow{\sim} \Pi(\mu')$. A single horizontal-strip λ is *good* if for any horizontal-strip μ such that $\Pi(\lambda) \cong \Pi(\mu)$, there is a good substitute for (λ, μ) .

We now state a key Lemma. The proof is quite technical so we postpone it to Section 5.

Lemma 3.23. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip with $n(\lambda) - M(\lambda) \geq 1$. Suppose that λ satisfies the condition that

$$\begin{aligned} & \text{Theorem 2.7 holds for horizontal-strips } \lambda' \text{ and } \mu' \text{ with either} \\ & n(\lambda') < n(\lambda), \text{ or with } n(\lambda') = n(\lambda) \text{ and } M(\lambda') > M(\lambda). \end{aligned} \tag{3.16}$$

Then λ is good.

We are now ready to prove that our key Lemma implies our main Theorem.

Proof of Theorem 2.7 assuming Lemma 3.23. We use induction on $n(\lambda)$. If $n(\lambda) = 0$, then λ has only one row and the result follows from Proposition 3.5, Part 1, so assume that $n(\lambda) \geq 1$ and that Theorem 2.7 holds for horizontal-strips λ' and μ' with $n(\lambda') < n(\lambda)$. We also use

induction on $n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda})$. If $n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda}) = 0$, then the result follows from Lemma 3.15, so assume that $n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda}) \geq 1$ and that Theorem 2.7 holds for horizontal-strips $\boldsymbol{\lambda}'$ and $\boldsymbol{\mu}'$ with $n(\boldsymbol{\lambda}') = n(\boldsymbol{\lambda})$ and $n(\boldsymbol{\lambda}') - M(\boldsymbol{\lambda}') < n(\boldsymbol{\lambda}) - M(\boldsymbol{\lambda})$. This is exactly the condition (3.16), so assuming Lemma 3.23, we have that $\boldsymbol{\lambda}$ is good.

Now by replacing $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ and $\boldsymbol{\mu} = (S_1, \dots, S_n)$ by a good substitute as necessary, we may assume that $l(R_1) < l(R_2)$, $R_1 \leftrightarrow R_2$, $l(S_1) < l(S_2)$, $S_1 \leftrightarrow S_2$, and $\varphi_1 = 1$ and $\varphi_2 = 2$, where $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$. Consider the horizontal-strips

$$\boldsymbol{\lambda}' = (R_2, R_1, R_3, \dots, R_n) \text{ and } \boldsymbol{\lambda}'' = (R_1 \cup R_2, R_1 \cap R_2, R_3, \dots, R_n) \quad (3.17)$$

and similarly define $\boldsymbol{\mu}'$ and $\boldsymbol{\mu}''$. Lemma 3.19 describes exactly how to construct $\Pi(\boldsymbol{\lambda}')$ and $\Pi(\boldsymbol{\lambda}'')$ from $\Pi(\boldsymbol{\lambda})$, and therefore we have $\Pi(\boldsymbol{\lambda}') \cong \Pi(\boldsymbol{\mu}')$ and $\Pi(\boldsymbol{\lambda}'') \cong \Pi(\boldsymbol{\mu}'')$. By (3.13), our induction hypothesis implies that $G_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}'}(\boldsymbol{x}; q)$ and $G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}''}(\boldsymbol{x}; q)$. Therefore, by (3.9), we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = \frac{1}{q}G_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) + \frac{q-1}{q}G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q) = \frac{1}{q}G_{\boldsymbol{\mu}'}(\boldsymbol{x}; q) + \frac{q-1}{q}G_{\boldsymbol{\mu}''}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q). \quad (3.18)$$

This completes the proof. \square

4. A connection to extended chromatic symmetric functions

Theorem 2.7 tells us that if Π is an admissible weighted graph, then there is a well-defined horizontal-strip LLT polynomial $G_{\Pi}(\boldsymbol{x}; q)$ given by setting $G_{\Pi}(\boldsymbol{x}; q) = G_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}; q)$, where $\boldsymbol{\lambda}$ is any horizontal-strip with $\Pi \cong \Pi(\boldsymbol{\lambda})$. We now apply Theorem 2.7 by exploring a connection between $G_{\Pi}(\boldsymbol{x}; q)$ and the extended chromatic symmetric function defined by Crew and Spirkl [CS20].

Definition 4.1. [CS20, Equation 1] Let G be a vertex-weighted graph. The *extended chromatic symmetric function* of G is

$$X_G(\boldsymbol{x}) = \sum_{\substack{\kappa: G \rightarrow \mathbb{N} \\ \kappa \text{ proper}}} \prod_{v \in G} x_{\kappa(v)}^{w(v)}, \quad (4.1)$$

where $w(v)$ denotes the weight of the vertex v .

Their motivation was to establish a deletion-contraction relation [CS20, Lemma 2], which exists for the chromatic polynomial of a graph but not for the chromatic symmetric function. Aliniaiefard, Wang, and van Willigenburg used this deletion-contraction relation to prove some equalities of extended chromatic symmetric functions [AWvW21, Theorem 4.12, Theorem 7.3]. In this section, we will use our deletion-contraction relation in Lemma 3.18 to extend these results to equalities of horizontal-strip LLT polynomials.

It will be convenient to first define the following operations and relations on compositions.

Definition 4.2. Let $\alpha = \alpha_1 \cdots \alpha_n$ and $\beta = \beta_1 \cdots \beta_m$ be compositions. The *reverse* of α is $\alpha^{\text{rev}} = \alpha_n \cdots \alpha_1$. The *concatenation* and *near-concatenation* of α and β are

$$\alpha \cdot \beta = \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m \text{ and } \alpha \odot \beta = \alpha_1 \cdots \alpha_{n-1} (\alpha_n + \beta_1) \beta_2 \cdots \beta_m. \quad (4.2)$$

The *composition* of α and β , defined by Billera, Thomas, and van Willigenburg [BTvW06, Section 3] is

$$\alpha \circ \beta = \beta^{\circ\alpha_1} \dots \beta^{\circ\alpha_n}, \tag{4.3}$$

where $\beta^{\circ k}$ denotes the k -fold near-concatenation of β . We say that β is a *coarsening* of α or alternatively that α is a *refinement* of β , denoted $\alpha \prec \beta$, if β can be obtained from α by summing adjacent parts. We also define the multiset $\mathcal{M}(\alpha) = \{\lambda(\beta) : \alpha \prec \beta\}$, where $\lambda(\beta)$ denotes the partition determined by rearranging the parts of β in weakly decreasing order.

There is a bijection between the set C^N of compositions α with sum N and subsets of $\{1, \dots, N-1\}$ given by taking the partial sums of α , other than N itself. Under this bijection, a coarsening corresponds to a subset so the partially ordered set (C^N, \prec) is (anti-)isomorphic to the boolean lattice B_{N-1} . In particular, by Möbius inversion, if A is an abelian group and $f, g : C^N \rightarrow A$, then

$$f(\alpha) = \sum_{\alpha \prec \beta} g(\beta) \text{ if and only if } g(\alpha) = \sum_{\alpha \prec \beta} (-1)^{\ell(\alpha) - \ell(\beta)} f(\beta). \tag{4.4}$$

Example 4.3. Consider the compositions $\alpha = 21231$, $\beta = 23121$, $\delta = 12$, and $\gamma = 21$. We have that $\delta^{\text{rev}} = \gamma$,

$$\delta \circ \gamma = \gamma^{\circ 1} \cdot \gamma^{\circ 2} = \gamma \cdot (\gamma \odot \gamma) = 21 \cdot (21 \odot 21) = 21 \cdot 231 = 21231 = \alpha, \text{ and} \tag{4.5}$$

$$\gamma \circ \gamma = \gamma^{\circ 2} \cdot \gamma^{\circ 1} = (\gamma \odot \gamma) \cdot \gamma = (21 \odot 21) \cdot 21 = 231 \cdot 21 = 23121 = \beta. \tag{4.6}$$

Some coarsenings of α include 3231, obtained by summing the first two parts, and 54, obtained by summing the first three parts and the last two parts. We have that

$$\mathcal{M}(\alpha) = \left\{ \begin{array}{l} 32211, 5211, 4221, 3321, 3321, 62, 621, \\ 531, 531, 432, 432, 81, 72, 63, 54, 9 \end{array} \right\} = \mathcal{M}(\beta). \tag{4.7}$$

Billera, Thomas, and van Willigenburg found the following characterization of when we have $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$ in terms of their operation \circ .

Theorem 4.4. [BTvW06, Theorem 4.1] *Let α and β be compositions. We have $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$ if and only if there are factorizations*

$$\alpha = \delta^{(1)} \circ \dots \circ \delta^{(k)} \text{ and } \beta = \gamma^{(1)} \circ \dots \circ \gamma^{(k)} \tag{4.8}$$

so that every $\gamma^{(i)}$ is either $\delta^{(i)}$ or $\delta^{(i)\text{rev}}$.

Example 4.5. We saw that the compositions α and β from Example 4.3 can be factorized as $\alpha = \delta \circ \gamma$ and $\beta = \delta^{\text{rev}} \circ \gamma$, so it follows from Theorem 4.4 that $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

Their motivation was to classify equalities of ribbon Schur functions.

Definition 4.6. Let α be a composition. The *ribbon Schur function* $r_\alpha(\mathbf{x})$ is the skew Schur function indexed by the skew diagram whose i -th row has α_i cells and where adjacent rows overlap in exactly one column. When α has a single part k , then $r_\alpha(\mathbf{x})$ is the *complete homogeneous symmetric function* $h_k(\mathbf{x})$. If $\lambda = \lambda_1 \dots \lambda_\ell$ is a partition, we also set $h_\lambda(\mathbf{x}) = h_{\lambda_1}(\mathbf{x}) \dots h_{\lambda_\ell}(\mathbf{x})$.

Theorem 4.7. [BTvW06, Equation 2.2, Proposition 2.1, Theorem 2.6] *The ribbon Schur functions satisfy the relation*

$$r_\alpha(\mathbf{x})r_\beta(\mathbf{x}) = r_{\alpha\cdot\beta}(\mathbf{x}) + r_{\alpha\circ\beta}(\mathbf{x}). \tag{4.9}$$

By iterating (4.9), we have

$$h_{\lambda(\alpha)}(\mathbf{x}) = r_{\alpha_1}(\mathbf{x}) \cdots r_{\alpha_\ell}(\mathbf{x}) = \sum_{\alpha \prec \beta} r_\beta(\mathbf{x}) \tag{4.10}$$

and therefore, by (4.4), we have

$$r_\alpha(\mathbf{x}) = \sum_{\alpha \prec \beta} (-1)^{\ell(\alpha) - \ell(\beta)} h_{\lambda(\beta)}(\mathbf{x}) = \sum_{\lambda \in \mathcal{M}(\alpha)} (-1)^{\ell(\alpha) - \ell(\lambda)} h_\lambda(\mathbf{x}). \tag{4.11}$$

In particular, because the $h_\lambda(\mathbf{x})$ are linearly independent, we have $r_\alpha(\mathbf{x}) = r_\beta(\mathbf{x})$ if and only if $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

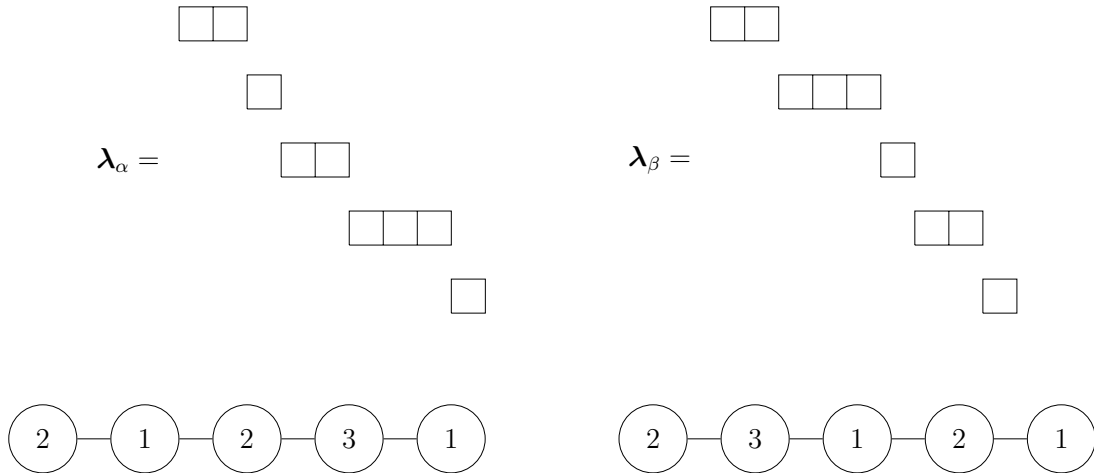
Aliniaiefard, Wang, and van Willigenburg found a similar result for the extended chromatic symmetric functions of vertex-weighted paths. These graphs can be indexed by compositions.

Definition 4.8. Let $\alpha = \alpha_1 \cdots \alpha_n$ be a composition. Define P_α to be the weighted graph with vertices $\{v_1, \dots, v_n\}$ where v_i has weight α_i , and with edges (v_i, v_{i+1}) of weight one for $1 \leq i \leq n - 1$. When α has a single part k , then the extended chromatic symmetric function $X_{P_k}(\mathbf{x})$ is the power sum symmetric function $p_k(\mathbf{x})$. If $\lambda = \lambda_1 \cdots \lambda_\ell$ is a partition, we also set $p_\lambda(\mathbf{x}) = p_{\lambda_1}(\mathbf{x}) \cdots p_{\lambda_\ell}(\mathbf{x})$.

Proposition 4.9. Let $\alpha = \alpha_1 \cdots \alpha_n$ be a composition of N and consider the horizontal-strip $\lambda_\alpha = (R_1, \dots, R_n)$, where $R_i = (\sum_{j=1}^{n-i+1} \alpha_j) / (\sum_{j=1}^{n-i} \alpha_j)$. Then $\Pi(\lambda_\alpha) \cong P_\alpha$.

Proof. We have that $|R_i| = \alpha_{n-i+1}$ is the weight of v_{n-i+1} , $M(R_i, R_{i+1}) = r(R_{i+1}) - l(R_i) + 2 = 1$, and $M(R_i, R_j) = 0$ for $j \geq i + 2$. □

Example 4.10. Let $\alpha = 21231$ and $\beta = 23121$ as in Example 4.3. The horizontal-strips $\lambda_\alpha = (9/8, 8/5, 5/3, 3/2, 2/0)$ and $\lambda_\beta = (9/8, 8/6, 6/5, 5/2, 2/0)$ and the paths P_α and P_β are given below. Because the (nonzero) edge weights are all one, they are not written.



Theorem 4.11. [AWvW21, Equation 2, Equation 3, Theorem 4.12] *The extended chromatic symmetric functions satisfy the relation*

$$X_{P_\alpha}(\mathbf{x})X_{P_\beta}(\mathbf{x}) = X_{P_{\alpha\beta}}(\mathbf{x}) + X_{P_{\alpha\circ\beta}}(\mathbf{x}). \tag{4.12}$$

By iterating (4.12), we have

$$p_{\lambda(\alpha)}(\mathbf{x}) = X_{P_{\alpha_1}}(\mathbf{x}) \cdots X_{P_{\alpha_\ell}}(\mathbf{x}) = \sum_{\alpha \prec \beta} X_{P_\beta}(\mathbf{x}) \tag{4.13}$$

and therefore, by (4.4), we have

$$X_{P_\alpha}(\mathbf{x}) = \sum_{\alpha \prec \beta} (-1)^{\ell(\alpha)-\ell(\beta)} p_{\lambda(\beta)}(\mathbf{x}) = \sum_{\lambda \in \mathcal{M}(\alpha)} (-1)^{\ell(\alpha)-\ell(\lambda)} p_\lambda(\mathbf{x}). \tag{4.14}$$

In particular, because the $p_\lambda(\mathbf{x})$ are linearly independent, we have $X_{P_\alpha}(\mathbf{x}) = X_{P_\beta}(\mathbf{x})$ if and only if $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

We now show the analogous result for horizontal-strip LLT polynomials of vertex-weighted paths.

Theorem 4.12. *The horizontal-strip LLT polynomials satisfy the relation*

$$G_{P_\alpha}(\mathbf{x}; q)G_{P_\beta}(\mathbf{x}; q) = \frac{1}{q}G_{P_{\alpha\beta}}(\mathbf{x}; q) + \frac{q-1}{q}G_{P_{\alpha\circ\beta}}(\mathbf{x}; q). \tag{4.15}$$

By iterating (4.15), we have

$$h_{\lambda(\alpha)}(\mathbf{x}) = G_{P_{\alpha_1}}(\mathbf{x}; q) \cdots G_{P_{\alpha_\ell}}(\mathbf{x}; q) = \sum_{\alpha \prec \beta} \frac{(q-1)^{\ell(\alpha)-\ell(\beta)}}{q^{\ell(\alpha)-1}} G_{P_\beta}(\mathbf{x}; q) \tag{4.16}$$

and rearranging, we have

$$q^{\ell(\alpha)-1}(q-1)^{-\ell(\alpha)} h_{\lambda(\alpha)}(\mathbf{x}) = \sum_{\alpha \prec \beta} (q-1)^{-\ell(\beta)} G_{P_\beta}(\mathbf{x}; q). \tag{4.17}$$

Therefore, by (4.4), we have

$$(q-1)^{-\ell(\alpha)} G_{P_\alpha}(\mathbf{x}; q) = \sum_{\alpha \prec \beta} q^{\ell(\beta)-1}(q-1)^{-\ell(\beta)} h_{\lambda(\beta)}(\mathbf{x}) \tag{4.18}$$

and rearranging again, we have

$$G_{P_\alpha}(\mathbf{x}; q) = \sum_{\alpha \prec \beta} q^{\ell(\beta)-1}(q-1)^{\ell(\alpha)-\ell(\beta)} h_{\lambda(\beta)}(\mathbf{x}) = \sum_{\lambda \in \mathcal{M}(\alpha)} q^{\ell(\lambda)-1}(q-1)^{\ell(\alpha)-\ell(\lambda)} h_\lambda(\mathbf{x}). \tag{4.19}$$

In particular, because the $h_\lambda(\mathbf{x})$ are linearly independent, we have $G_{P_\alpha}(\mathbf{x}; q) = G_{P_\beta}(\mathbf{x}; q)$ if and only if $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

Proof. We need only prove the first statement (4.15). Let $\lambda_\alpha = (R_1, \dots, R_n)$ and $\lambda_\beta = (S_1, \dots, S_m)$ be the horizontal-strips defined in Proposition 4.9. Suppose that α has sum N and consider the horizontal-strips

$$\begin{aligned} \lambda &= (N + S_1, \dots, N + S_{m-1}, R_1, N + S_m, R_2, \dots, R_n), \\ \lambda' &= (N + S_1, \dots, N + S_{m-1}, N + S_m, R_1, R_2, \dots, R_n), \text{ and} \\ \lambda'' &= (N + S_1, \dots, N + S_{m-1}, (N + S_{m-1}) \cup R_1, (N + S_{m-1}) \cap R_1, R_2, \dots, R_n). \end{aligned} \quad (4.20)$$

Note that λ was constructed by translating λ_β , concatenating, and switching rows $(R_t, N + S_{t'})$ with $(t, t') \neq (1, m)$, which commute by Proposition 3.10, Part 1 because $\ell(N + S_{t'}) > r(R_t) + 1$. Therefore, we have that $\Pi(\lambda)$ is the disjoint union of P_α and P_β and so $G_\lambda(\mathbf{x}; q) = G_{P_\alpha}(\mathbf{x}; q)G_{P_\beta}(\mathbf{x}; q)$. Lemma 3.19 describes exactly how the weighted graphs $\Pi(\lambda')$ and $\Pi(\lambda'')$ are constructed and we see that $\Pi(\lambda') \cong P_{\alpha \cdot \beta}$ and $\Pi(\lambda'') \cong P_{\alpha \odot \beta}$. Because $\ell(N + S_m) = r(R_1) + 1 > l(R_1)$, we have $R_1 \leftrightarrow (N + S_m)$ by Proposition 3.10, Part 3 and now (4.15) follows from Lemma 3.18. \square

We take a moment to summarize these results.

Corollary 4.13. *Let α and β be compositions. The following are equivalent.*

1. $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$
2. $\alpha = \delta^{(1)} \circ \dots \circ \delta^{(k)}$ and $\beta = \gamma^{(1)} \circ \dots \circ \gamma^{(k)}$, with every $\gamma^{(i)} = \delta^{(i)}$ or $\delta^{(i) \text{ rev}}$
3. $r_\alpha(\mathbf{x}) = r_\beta(\mathbf{x})$
4. $X_{P_\alpha}(\mathbf{x}) = X_{P_\beta}(\mathbf{x})$
5. $G_{P_\alpha}(\mathbf{x}; q) = G_{P_\beta}(\mathbf{x}; q)$

Proof. We have (1) \iff (2) by Theorem 4.4, (1) \iff (3) by Theorem 4.7, (1) \iff (4) by Theorem 4.11, and (1) \iff (5) by Theorem 4.12. \square

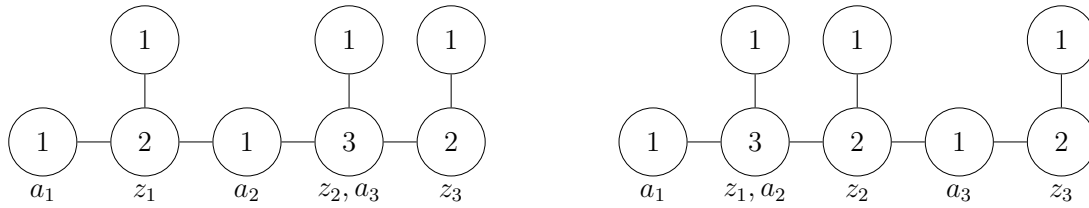
Aliniaefard, Wang, and van Willigenburg generalize Theorem 4.11 to the following construction.

Definition 4.14. [AWvW21, Definition 7.1] Let G and H be vertex-weighted graphs with distinguished (not necessarily distinct) vertices $a_G, z_G \in G$ and $a_H, z_H \in H$. The *concatenation* of G and H , denoted $G \cdot H$, is the disjoint union of G and H with an edge joining z_G and a_H . The *near-concatenation* of G and H , denoted $G \odot H$, is the graph $G \cdot H$ with the edge (z_G, a_H) contracted. If $\alpha = \alpha_1 \cdots \alpha_n$ is a composition, the *composition* of α and G is the graph

$$\alpha \circ G = G^{\odot \alpha_1} \cdots G^{\odot \alpha_n}, \quad (4.21)$$

where $G^{\odot k}$ denotes the k -fold near-concatenation of G .

Example 4.15. For any composition α , we have $\alpha \circ P_1 \cong P_\alpha$. For $G = P_{121}$ with $a = v_1$ and $z = v_2$, the graphs $12 \circ G$ and $21 \circ G$ are given below.



Note that it follows from the definition that

$$(\alpha \cdot \beta) \circ G = (\alpha \circ G) \cdot (\beta \circ G) \text{ and } (\alpha \odot \beta) \circ G = (\alpha \circ G) \odot (\beta \circ G). \tag{4.22}$$

Theorem 4.16. [AWvW21, Theorem 7.3] Let G be a vertex-weighted graph with distinguished vertices a and z . If $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$, then $X_{P_{\alpha \circ G}}(\mathbf{x}) = X_{P_{\beta \circ G}}(\mathbf{x})$. Moreover, if G is simple and connected, then the converse holds.

We prove a similar result for horizontal-strip LLT polynomials, but we require a condition on the distinguished vertices to ensure that the resulting weighted graphs be admissible. Recall that *content reading order* is the total ordering on cells by increasing content and from bottom to top along constant content lines.

Definition 4.17. Let $\Pi = \Pi(\lambda)$ be an admissible weighted graph. Let a and z be the vertices corresponding to the rows containing the first and last cells of λ in content reading order. Then we define concatenation, near-concatenation, and composition as in Definition 4.14 with respect to these distinguished vertices a and z .

Theorem 4.18. Let $\Pi = \Pi(\lambda)$ be an admissible weighted graph. For any composition α , the weighted graph $\alpha \circ \Pi$ is admissible, and if $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$, then $G_{\alpha \circ \Pi}(\mathbf{x}; q) = G_{\beta \circ \Pi}(\mathbf{x}; q)$.

Proof. Let $\Pi_1 = \Pi(\lambda)$ and $\Pi_2 = \Pi(\mu)$ be admissible weighted graphs. We first show that $\Pi_1 \cdot \Pi_2$ and $\Pi_1 \odot \Pi_2$ are admissible, which proves the first statement. By cycling and translating, we may assume without loss of generality that $\lambda = (R_1, \dots, R_n)$ has a unique cell $u \in R_1$ of maximal content $N - 1$ and $\mu = (S_1, \dots, S_m)$ has a unique cell $v \in S_m$ of minimal content 0. Then by translating μ and concatenating, we can consider the horizontal-strip $(R_1, \dots, R_n, N + S_1, \dots, N + S_m)$, whose weighted graph is the disjoint union $\Pi_1 \sqcup \Pi_2$. Consider the horizontal-strips

$$\begin{aligned} \lambda &= (N + S_1, \dots, N + S_{m-1}, R_1, N + S_m, R_2, \dots, R_n), \\ \lambda' &= (N + S_1, \dots, N + S_{m-1}, N + S_m, R_1, R_2, \dots, R_n), \text{ and} \\ \lambda'' &= (N + S_1, \dots, N + S_{m-1}, (N + S_{m-1}) \cup R_1, (N + S_{m-1}) \cap R_1, R_2, \dots, R_n). \end{aligned} \tag{4.23}$$

Note that λ was constructed by switching rows $(R_t, N + S_{t'})$ with $(t, t') \neq (1, m)$, which commute by Proposition 3.10, Part 1 because $\ell(N + S_{t'}) > r(R_t) + 1$. Therefore, we have that $\Pi(\lambda) \cong \Pi_1 \sqcup \Pi_2$ and so $G_\lambda(\mathbf{x}; q) = G_{\Pi_1}(\mathbf{x}; q)G_{\Pi_2}(\mathbf{x}; q)$. Lemma 3.19 describes exactly how the weighted graphs $\Pi(\lambda')$ and $\Pi(\lambda'')$ are constructed and we see that $\Pi(\lambda') \cong \Pi_1 \cdot \Pi_2$

and $\Pi(\lambda'') \cong \Pi_1 \odot \Pi_2$, so indeed these weighted graphs are admissible. Because $\ell(N + S_m) = r(R_1) + 1 > l(R_1)$, we have $R_1 \leftrightarrow (N + S_m)$ by Proposition 3.10, Part 3 and now by Lemma 3.18 we have

$$G_{\Pi_1}(\mathbf{x}; q)G_{\Pi_2}(\mathbf{x}; q) = \frac{1}{q}G_{\Pi_1 \cdot \Pi_2}(\mathbf{x}; q) + \frac{q-1}{q}G_{\Pi_1 \odot \Pi_2}(\mathbf{x}; q). \quad (4.24)$$

Therefore, by (4.22), for compositions α and β , the horizontal-strip LLT polynomials satisfy the relation

$$G_{\alpha \circ \Pi}(\mathbf{x}; q)G_{\beta \circ \Pi}(\mathbf{x}; q) = \frac{1}{q}G_{(\alpha \cdot \beta) \circ \Pi}(\mathbf{x}; q) + \frac{q-1}{q}G_{(\alpha \odot \beta) \circ \Pi}(\mathbf{x}; q). \quad (4.25)$$

By iterating (4.25), we have

$$G_{\Pi \circ \alpha_1}(\mathbf{x}; q) \cdots G_{\Pi \circ \alpha_\ell}(\mathbf{x}; q) = \sum_{\alpha \prec \beta} \frac{(q-1)^{\ell(\alpha) - \ell(\beta)}}{q^{\ell(\alpha) - 1}} G_{\beta \circ \Pi}(\mathbf{x}; q) \quad (4.26)$$

and rearranging, we have

$$q^{\ell(\alpha) - 1} (q-1)^{-\ell(\alpha)} \prod_{i=1}^{\ell} G_{\Pi \circ \alpha_i}(\mathbf{x}; q) = \sum_{\alpha \prec \beta} (q-1)^{-\ell(\beta)} G_{\beta \circ \Pi}(\mathbf{x}; q). \quad (4.27)$$

Therefore, by (4.4), we have

$$(q-1)^{-\ell(\alpha)} G_{\alpha \circ \Pi}(\mathbf{x}; q) = \sum_{\alpha \prec \beta} q^{\ell(\beta) - 1} (q-1)^{-\ell(\beta)} \prod_{i=1}^{\ell} G_{\Pi \circ \beta_i}(\mathbf{x}; q) \quad (4.28)$$

and rearranging again, we have

$$\begin{aligned} G_{\alpha \circ \Pi}(\mathbf{x}; q) &= \sum_{\alpha \prec \beta} q^{\ell(\beta) - 1} (q-1)^{\ell(\alpha) - \ell(\beta)} \prod_{i=1}^{\ell} G_{\Pi \circ \beta_i}(\mathbf{x}; q) \\ &= \sum_{\lambda \in \mathcal{M}(\alpha)} q^{\ell(\lambda) - 1} (q-1)^{\ell(\alpha) - \ell(\lambda)} \prod_{i=1}^{\ell} G_{\Pi \circ \lambda_i}(\mathbf{x}; q). \end{aligned} \quad (4.29)$$

In particular, if $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$, then $G_{\alpha \circ \Pi}(\mathbf{x}; q) = G_{\beta \circ \Pi}(\mathbf{x}; q)$. \square

5. Proof of Lemma 3.23

In this section we prove Lemma 3.23, which completes the proof of Theorem 2.7. The general idea will be to use rotating, cycling, and commuting to move rows into the desired position. We will be able to do this unless there is a *noncommuting path*, which is a concept we will introduce in Definition 5.4. In Lemma 5.15 and Lemma 5.17, we will describe the structure of a minimal noncommuting path very precisely, which will allow us to rule out several cases. We begin with the following definition, which expresses a relationship between rows that will be convenient to consider.

Definition 5.1. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip. We write $R_i \prec R_j$ if $M_{i,j} = |R_i|$ and $R_i \not\prec R_j$ otherwise. We also write $R_i \succsim R_j$ to mean that $R_i \prec R_j$ and $R_j \not\prec R_i$.

Proposition 5.2. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip.

1. Suppose that $i < j$. Then we have $R_i \prec R_j$ if and only if either $R_i \subseteq R_j$ or $R_i \subseteq R_j^+$.
2. Suppose that $i > j$. Then we have $R_i \prec R_j$ if and only if either $R_i \subseteq R_j$ or $R_i \subseteq R_j^-$.
3. We have $R_i \prec R_j$ and $R_i \leftrightarrow R_j$ if and only if $R_i \subseteq R_j$.

Proof.

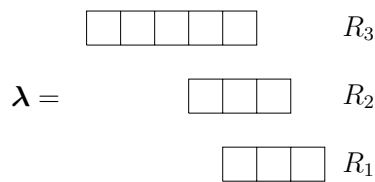
1. If $R_i \prec R_j$, then by definition we have

$$M_{i,j} = |R_i| = \begin{cases} |R_i \cap R_j| & \text{if } l(R_i) \leq l(R_j), \\ |R_i \cap R_j^+| & \text{if } l(R_i) > l(R_j), \end{cases} \tag{5.1}$$

and therefore we must have $R_i \subseteq R_j$ or $R_i \subseteq R_j^+$. Conversely, if $R_i \subseteq R_j^+$, then $l(R_i) > l(R_j)$ and $M_{i,j} = |R_i|$ by Proposition 3.10, Part 2, while if $R_i \not\subseteq R_j^+$ and $R_i \subseteq R_j$, then $l(R_i) = l(R_j)$ and again $M_{i,j} = |R_i|$ by Proposition 3.10, Part 2.

2. This follows from the previous part by considering a rotation of λ .
3. If $R_i \subseteq R_j$, then $R_i \prec R_j$ by the previous parts and we have $l(R_j) \leq l(R_i) \leq r(R_i) \leq r(R_j)$, so $R_i \leftrightarrow R_j$ by Proposition 3.10, Part 2. Conversely, if $i < j$, $R_i \prec R_j$, and $R_i \not\subseteq R_j$, then by Part 1 we have $R_i \subseteq R_j^+$, so $l(R_j) < l(R_i) \leq r(R_j) + 1 \leq r(R_i)$ and $R_i \leftrightarrow R_j$ by Proposition 3.10, Part 3. The case where $i > j$ follows by rotating. \square

Example 5.3. Let $\lambda = (R_1, R_2, R_3) = (7/4, 6/3, 5/0)$ as in the example below. We have $M(R_1, R_2) = 3 = |R_1| = |R_2|$, so $R_1 \prec R_2$ and $R_2 \prec R_1$. We have $M(R_1, R_3) = 2 < |R_1|$, so $R_1 \not\prec R_3$, and we have $M(R_2, R_3) = 3 = |R_2| < |R_3|$, so $R_2 \succsim R_3$.



Informally, we can think of the relation $R_i \prec R_j$ as being very similar to the relation $R_i \subseteq R_j$, except that we may need to shift a row by one cell. Because of this possible shift, the relation \prec is not transitive. In the above example, we have $R_1 \prec R_2$ and $R_2 \prec R_3$, but $R_1 \not\prec R_3$.

The following concept will be important to define the potential obstruction to commuting.

Definition 5.4. A sequence of $n \geq 3$ rows (R_1, \dots, R_n) is a *noncommuting path* from R_1 to R_n if $R_i \leftrightarrow R_{i+1}$ for every $1 \leq i \leq n - 1$. A noncommuting path is *minimal* if there is no subsequence of rows $(R_1 = R_{i_1}, R_{i_2}, \dots, R_{i_k} = R_n)$ with $i_1 < i_2 < \dots < i_k$ and $3 \leq k < n$ that forms a noncommuting path from R_1 to R_n .

In particular, if (R_1, \dots, R_n) is a minimal noncommuting path, then we have $R_i \leftrightarrow R_j$ for every i, j with $1 < |j - i| < n - 1$. Because we require a noncommuting path to have length at least 3, we cannot conclude that $R_1 \leftrightarrow R_n$ in a minimal noncommuting path.

Our next Lemma shows that, given two rows R_i and R_j of λ with $i < j$, either there is a minimal noncommuting path in λ from R_i to R_j , or we may assume that $j = i + 1$.

Lemma 5.5. *Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip and let $1 \leq i < j \leq n$. Then one of the following holds.*

1. *There is a minimal noncommuting path $(R_i = R_{i_1}, \dots, R_{i_k} = R_j)$ in λ .*
2. *There is a horizontal-strip $\mu = (S_1, \dots, S_n) \in \mathcal{S}(\lambda)$ and an isomorphism of weighted graphs $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$ such that $l(S_{\varphi_i}) = l(R_i)$, $l(S_{\varphi_j}) = l(R_j)$, and $\varphi_j = \varphi_i + 1$.*

Proof. We use induction on $j - i$. If $j - i = 1$, then the second possibility holds by simply taking $\lambda = \mu$, so assume that $j - i \geq 2$. If $R_j \leftrightarrow R_{j-1}$, then by commuting we have $(R_1, \dots, R_j, R_{j-1}, \dots, R_n) \in \mathcal{S}(\lambda)$ and we are done by our induction hypothesis on $j - i$, so we may assume that $R_j \not\leftrightarrow R_{j-1}$. Similarly, if $R_{j-1} \leftrightarrow R_t$ for every $i \leq t \leq j - 2$, then by commuting we would have $(R_1, \dots, R_{j-1}, R_i, \dots, R_j, \dots, R_n) \in \mathcal{S}(\lambda)$ and we are again done by induction. So we may assume that $R_{j-1} \not\leftrightarrow R_t$ for some $i \leq t \leq j - 2$, and continuing in this way there must be a noncommuting path in λ from R_i to R_j . Finally, if this noncommuting path is not minimal, then it contains a minimal one. \square

Examples of minimal noncommuting paths are given in Figure 5.1. Informally, our next goal is to show that all minimal noncommuting paths look like these examples. Specifically, our goal is to prove Lemma 5.15, which describes what a minimal noncommuting path may look like, and Lemma 5.17, which describes the extent to which our weighted graph determines the structure of a minimal noncommuting path. This will allow us to prove Corollary 5.19, which proves Lemma 3.23 in many cases. We will first prove some elementary Propositions.

Proposition 5.6. *Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip.*

1. *Suppose that $R_i \leftrightarrow R_j$, $R_j \leftrightarrow R_k$, $R_i \leftrightarrow R_k$, and that the integers $l(R_j) - l(R_i)$ and $l(R_k) - l(R_j)$ have the same sign. Then $M_{i,k} = 0$.*
2. *Suppose that $R_i \leftrightarrow R_j$, $R_j \leftrightarrow R_k$, $R_i \leftrightarrow R_k$, and that the integers $l(R_j) - l(R_i)$ and $l(R_k) - l(R_j)$ have opposite signs. Then $R_i \prec R_k$ or $R_k \prec R_i$.*
3. *Suppose that $R_i \leftrightarrow R_j$, $R_i \leftrightarrow R_k$, and $R_j \leftrightarrow R_k$. Then $R_j \prec R_i$ if and only if $R_k \prec R_i$.*

Note that in (1) and (2), because $R_i \leftrightarrow R_j$ and $R_j \leftrightarrow R_k$, the integers $l(R_j) - l(R_i)$ and $l(R_k) - l(R_j)$ are nonzero by Proposition 3.10, Part 2.

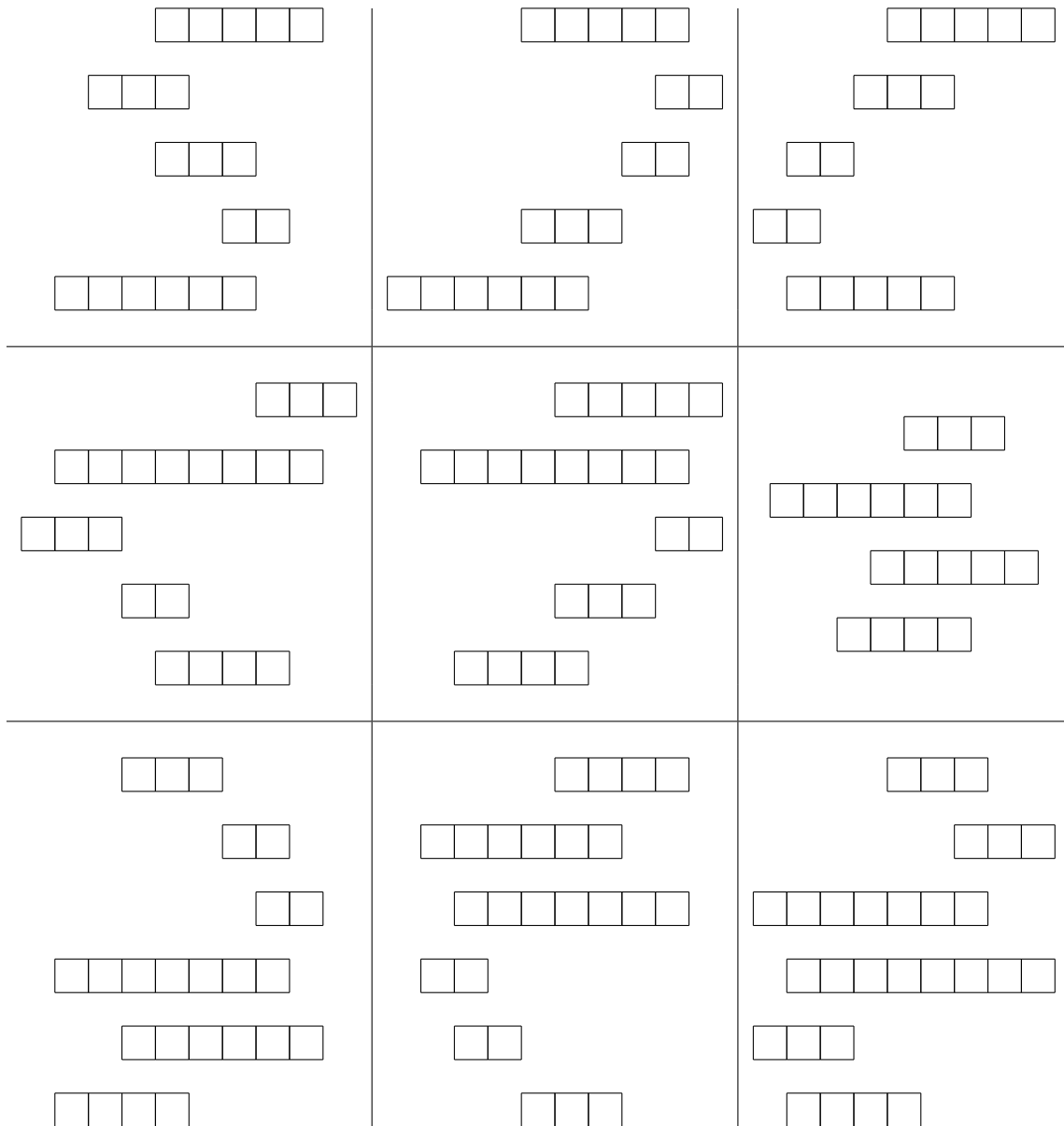


Figure 5.1: Some examples of minimal noncommuting paths.

Proof of Proposition 5.6.

1. Without loss of generality, we may assume that $l(R_i) < l(R_j) < l(R_k)$, and then because $R_i \leftrightarrow R_j$ and $R_j \leftrightarrow R_k$, by Corollary 3.13, Part 2, we must have $r(R_i) < r(R_j) < r(R_k)$ as well. But now we cannot have $R_i \subseteq R_k$ or $R_k \subseteq R_i$, so by Proposition 5.2, Part 3, we cannot have $R_i \prec R_k$ or $R_k \prec R_i$. Therefore, because $R_i \leftrightarrow R_k$, by Corollary 3.13, Part 1, we must have $M_{i,k} = 0$.
2. By rotating, we may assume without loss of generality that $l(R_i) \leq l(R_j) - 1$ and $l(R_k) \leq l(R_j) - 1$. Because $R_i \leftrightarrow R_j$ and $R_k \leftrightarrow R_j$, by Proposition 3.10, Part 1, we have $r(R_i) \geq l(R_j) - 1$ and $r(R_k) \geq l(R_j) - 1$. But now $M_{i,k} > 0$ and because $R_i \leftrightarrow R_k$, we have by Corollary 3.13, Part 1, that $R_i \prec R_k$ or $R_k \prec R_i$.
3. By symmetry, it suffices to prove that $R_j \prec R_i$ implies $R_k \prec R_i$. Suppose that $R_j \prec R_i$, and then because $R_i \leftrightarrow R_j$, by Proposition 5.2, Part 1, we have $R_j \subseteq R_i$, that is $l(R_i) \leq l(R_j) \leq r(R_j) \leq r(R_i)$. Suppose that $l(R_k) < l(R_i)$. By Proposition 3.10 Parts 1 and 2, if $r(R_k) < l(R_i) - 1 \leq l(R_j) - 1$, then $R_j \leftrightarrow R_k$, if $l(R_i) - 1 \leq r(R_k) \leq r(R_i) - 1$, then $R_i \leftrightarrow R_k$, and if $r(R_k) \geq r(R_i) \geq r(R_j)$, then again $R_j \leftrightarrow R_k$, a contradiction in all cases, so we must have $l(R_k) \geq l(R_i)$. Similarly, by rotating, we must have $r(R_k) \leq r(R_i)$, so we have $R_k \subseteq R_i$ and $R_k \prec R_i$. \square

Proposition 5.7. *Let $\lambda = (R_1, \dots, R_n)$ be a minimal noncommuting path.*

1. *If $R_i \prec R_j$ and $R_j \prec R_i$ for some $j \geq i + 2$, then we must have $n = 3$ or $n = 4$ and $j = i + 2$. In particular, if $n \geq 5$, then $R_i \prec R_j$ implies that in fact $R_i \prec_{\neq} R_j$.*
2. *If $R_j \prec R_i$ for some $j \geq i + 2$, then $R_k \prec R_i$ for every $k \geq i + 2$, with the possible exception of $k = n$ if $i = 1$. Similarly, if $R_j \prec R_i$ for some $j \leq i - 2$, then $R_k \prec R_i$ for every $k \leq i - 2$, with the possible exception of $k = 1$ if $i = n$.*
3. *Suppose that $R_j \prec_{\neq} R_i$ for some $i \geq j + 2$, that i is minimal with these two properties, and that $i \neq n$ if $j = 1$. Then either $l(R_{i-1}) > \dots > l(R_j)$ and $l(R_i) < l(R_{i-1})$, or $l(R_{i-1}) < \dots < l(R_j)$ and $l(R_i) > l(R_{i-1})$.*

Remark 5.8. Figure 5.8 shows that if $n \leq 4$, then it is possible to have a minimal noncommuting path with $R_1 \prec R_3$ and $R_3 \prec R_1$. If $n \geq 5$, then this will not happen because there will be some $R_t \leftrightarrow R_{t'}$ with $t' \geq t + 2$ and $(t, t') \neq (1, n)$, contradicting minimality. Informally, (2) states that if we are contained in R_i , then we must remain stuck in R_i and (3) states that if R_j is contained in some minimal R_i , then we must move in the same direction until R_i .

Proof of Proposition 5.7.

1. Because $j \geq i + 2$, we have $R_i \leftrightarrow R_j$ by minimality and therefore $R_i = R_j$ by Proposition 5.2. Now $R_{i+1} \leftrightarrow R_j$, so by minimality of the noncommuting path we must have $j = i + 2$. Similarly, if $i \geq 2$, then $R_{i-1} \leftrightarrow R_j$, so we must then have $i = 2$ and $j = n = 4$; if $j \leq n - 1$, then $R_i \leftrightarrow R_{j+1}$, so we must have $i = 1$ and $j = 3 = n - 1$ so again $n = 4$; and otherwise we have $i = 1$ and $j = 3 = n$.

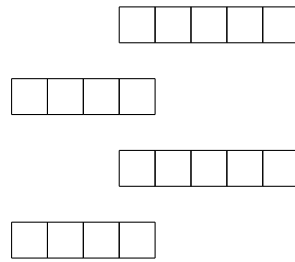


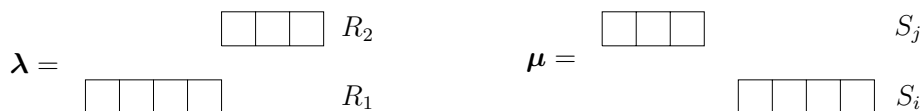
Figure 5.2: A minimal noncommuting path with $R_1 < R_3$ and $R_3 < R_1$.

2. If $i + 2 \leq k \leq n - 1$, $R_k < R_i$, and $k + 1 \neq n$ in the case of $i = 1$, then because $R_i \leftrightarrow R_k$, $R_i \leftrightarrow R_{k+1}$, and $R_k \leftrightarrow R_{k+1}$, by Proposition 5.6, Part 3, we have $R_k < R_i$ if and only if $R_{k+1} < R_i$, so the first statement follows by induction on k and the second statement follows by rotating.
3. Recall that because $R_j < R_i$ and $R_j \leftrightarrow R_i$, by Proposition 5.2 we must have $l(R_i) \leq l(R_j)$ and $r(R_i) \geq r(R_j)$. Suppose that either $l(R_{j+1}) > l(R_j)$ and $l(R_{t+1}) < l(R_t)$ for some minimal $j + 1 \leq t \leq i - 2$, or $l(R_{j+1}) < l(R_j)$ and $l(R_{t+1}) > l(R_t)$ for some minimal $j + 1 \leq t \leq i - 2$, so in particular, $n \geq 5$. Then by Proposition 5.6, Part 2, we have either $R_{t-1} < R_{t+1}$ or $R_{t+1} < R_{t-1}$. If $R_{t-1} < R_{t+1}$, then by the previous two parts we have $R_j \not\prec R_{t+1}$, contradicting minimality of i . If $R_{t+1} < R_{t-1}$, then by the previous two parts we have $R_i \not\prec R_{t-1}$, but now by Proposition 5.2, Part 3, we have $R_j \subsetneq R_i \subsetneq R_{t-1}$, so $R_j \leftrightarrow R_{t-1}$, $t - 1 \geq j + 2$, and $R_j \not\prec R_{t-1}$, again contradicting minimality of i . Therefore, we must have either $l(R_{i-1}) > \dots > l(R_j)$ and $l(R_i) \leq l(R_j) < l(R_{i-1})$, or $l(R_{i-1}) < \dots < l(R_j)$ and $r(R_i) \geq r(R_j) > r(R_{i-1})$, so $l(R_i) > l(R_{i-1})$ by Corollary 3.13, Part 2. \square

Proposition 5.9. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip with $R_1 \leftrightarrow R_2$ and let $\mu = (S_1, \dots, S_n)$ be a horizontal-strip with $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$. Let $i = \varphi_1$ and $j = \varphi_2$. If $l(R_2) > l(R_1)$, also assume that $l(S_j) > l(S_i)$. Then

$$l(S_j) - l(S_i) \geq l(R_2) - l(R_1) \text{ with equality only if } i < j. \tag{5.2}$$

Remark 5.10. Informally, Proposition 5.9 states that the leftmost possible position of S_j , given the weighted graph, occurs when S_j is above S_i and $S_i \leftrightarrow S_j$. The following example demonstrates the necessity of the hypothesis that if $l(R_2) > l(R_1)$, then $l(S_j) > l(S_i)$.



Proof of Proposition 5.9. We calculate directly, using (3.6) and noting that $|R| = r(R) - l(R) + 1$. If $l(R_2) < l(R_1)$, then the statement holds unless $l(S_j) < l(S_i)$, and because $R_1 \leftrightarrow R_2$, we have by Corollary 3.13, Part 3, that $M_{1,2}(\lambda) = M_{i,j}(\mu) > 0$. By Proposition 3.10, Parts 2 and 3, we

have $M_{1,2}(\boldsymbol{\lambda}) = r(R_2) - l(R_1) + 2$ and we either have $M_{i,j}(\boldsymbol{\mu}) = r(S_j) - l(S_i) + 1 + \chi(i < j)$, or $M_{i,j}(\boldsymbol{\mu}) = \min\{|S_i|, |S_j|\} \leq r(S_j) - l(S_i) + 1 + \chi(i < j)$. Therefore we have

$$\begin{aligned} l(S_j) - l(S_i) &= r(S_j) - |S_j| + 1 - l(S_i) \geq M_{i,j}(\boldsymbol{\mu}) - |S_j| - \chi(i < j) \\ &\geq M_{1,2}(\boldsymbol{\lambda}) - |R_2| - 1 = r(R_2) - |R_2| + 1 - l(R_1) = l(R_2) - l(R_1), \end{aligned} \quad (5.3)$$

with equality only if $i < j$.

Similarly, if $l(R_2) > l(R_1)$, then by hypothesis we have $l(S_j) > l(S_i)$, and because $R_1 \leftrightarrow R_2$, we have by Corollary 3.13, Part 3, that $M_{i,j}(\boldsymbol{\mu}) < \min\{|S_i|, |S_j|\}$. By Proposition 3.10, Parts 1 and 3, we have $M_{1,2}(\boldsymbol{\lambda}) = r(R_1) - l(R_2) + 1$ and we either have

$$M_{i,j}(\boldsymbol{\mu}) = r(S_i) - l(S_j) + 1 + \chi(i < j) \quad \text{or} \quad M_{i,j}(\boldsymbol{\mu}) = 0 \geq r(S_i) - l(S_j) + 1 + \chi(i < j).$$

Therefore we have

$$\begin{aligned} l(S_j) - l(S_i) &= l(S_j) - r(S_i) - 1 + |S_i| \geq |S_i| - M_{i,j}(\boldsymbol{\mu}) + \chi(i > j) \\ &\geq |R_1| - M_{1,2}(\boldsymbol{\lambda}) = l(R_2) - r(R_1) - 1 + |R_1| = l(R_2) - l(R_1), \end{aligned} \quad (5.4)$$

with equality only if $i < j$. □

Proposition 5.11. *Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a horizontal-strip with $l(R_1) < l(R_3)$ and $R_1 \leftrightarrow R_3$. Let $\boldsymbol{\mu} = (S_1, \dots, S_n)$ be a horizontal-strip and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ such that $\varphi_1 < \varphi_3$, $l(S_{\varphi_1}) < l(S_{\varphi_3})$, and $S_{\varphi_1} \leftrightarrow S_{\varphi_3}$. Then if $l(R_2) > l(R_1)$ and $R_1 \leftrightarrow R_2$, then $l(S_{\varphi_2}) > l(S_{\varphi_1})$. Similarly, if $l(R_3) > l(R_2)$ and $R_2 \leftrightarrow R_3$, then $l(S_{\varphi_3}) > l(S_{\varphi_2})$.*

Remark 5.12. Informally, Proposition 5.11 describes the extent to which rows R_1 and R_3 with $l(R_1) < l(R_3)$ and $R_1 \leftrightarrow R_3$, given the weighted graph data, determine the relative horizontal position of another row R_2 .

Proof of Proposition 5.11. We first suppose for a contradiction that $l(R_2) > l(R_1)$ and $R_1 \leftrightarrow R_2$, but $l(S_{\varphi_2}) \leq l(S_{\varphi_1})$. Because $l(R_2) > l(R_1)$ and $R_1 \leftrightarrow R_2$, we have by Corollary 3.13, Part 2, that $r(R_2) > r(R_1)$ and by Corollary 3.13, Part 3, that $M_{1,2}(\boldsymbol{\lambda}) < \min\{|R_1|, |R_2|\}$. Therefore, we must have $M_{\varphi_1, \varphi_2}(\boldsymbol{\mu}) < \min\{|S_{\varphi_1}|, |S_{\varphi_2}|\}$ and by Proposition 3.10, Part 2, we have that $l(S_{\varphi_2}) < l(S_{\varphi_1}) < l(S_{\varphi_3})$ and $r(S_{\varphi_2}) < r(S_{\varphi_1}) < r(S_{\varphi_3})$. In particular, we have $M_{\varphi_2, \varphi_3}(\boldsymbol{\mu}) < \min\{|S_{\varphi_2}|, |S_{\varphi_3}|\}$, so $M_{2,3}(\boldsymbol{\lambda}) < \min\{|R_2|, |R_3|\}$ and $l(R_2) \neq l(R_3)$.

Now if $l(R_2) < l(R_3)$, then by (3.5) we have

$$M_{2,3}(\boldsymbol{\lambda}) = r(R_2) - l(R_3) + 1 > r(R_1) - l(R_3) + 1 = M_{1,3}(\boldsymbol{\lambda}), \quad (5.5)$$

but if $r(S_{\varphi_2}) < l(S_{\varphi_3}) - 1$, then $M_{\varphi_2, \varphi_3}(\boldsymbol{\mu}) = 0 \leq M_{\varphi_1, \varphi_3}(\boldsymbol{\mu})$, and if $r(S_{\varphi_2}) \geq l(S_{\varphi_3}) - 1$, then

$$M_{\varphi_2, \varphi_3}(\boldsymbol{\mu}) = r(S_{\varphi_2}) - l(S_{\varphi_3}) + 1 + \chi(\varphi_2 > \varphi_3) \leq r(S_{\varphi_1}) - l(S_{\varphi_3}) + 1 = M_{\varphi_1, \varphi_3}(\boldsymbol{\mu}), \quad (5.6)$$

a contradiction in either case. Similarly, if $l(R_2) > l(R_3)$, then by (3.5) we have

$$M_{2,3}(\boldsymbol{\lambda}) = r(R_3) - l(R_2) + 2 > r(R_1) - l(R_2) + 1 = M_{1,2}(\boldsymbol{\lambda}), \quad (5.7)$$

but if $r(S_{\varphi_2}) < l(S_{\varphi_3}) - 1$, then $M_{\varphi_2, \varphi_3}(\boldsymbol{\mu}) = 0 \leq M_{\varphi_1, \varphi_2}(\boldsymbol{\mu})$, and if $r(S_{\varphi_2}) \geq l(S_{\varphi_3}) - 1$, then

$$M_{\varphi_2, \varphi_3}(\boldsymbol{\mu}) = r(S_{\varphi_2}) - l(S_{\varphi_3}) + 1 + \chi(\varphi_2 > \varphi_3) \leq r(S_{\varphi_2}) - l(S_{\varphi_3}) + 1 = M_{\varphi_1, \varphi_2}(\boldsymbol{\mu}), \quad (5.8)$$

a contradiction in either case. This proves the first claim and the second follows by rotating. □

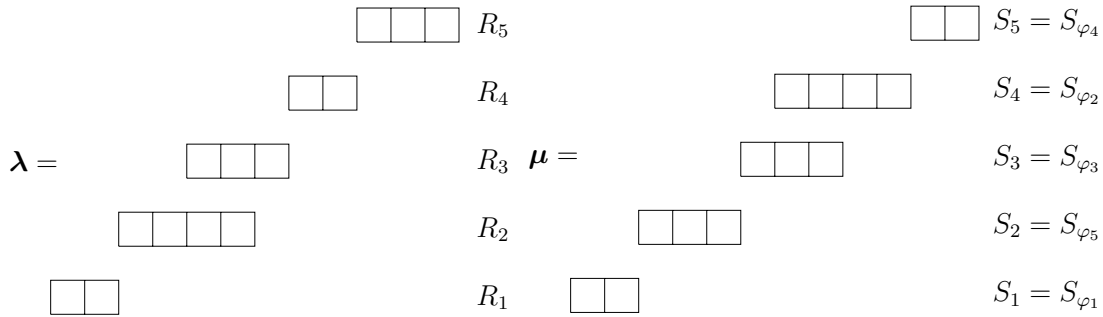


Figure 5.3: A minimal noncommuting path $\lambda = (R_1, \dots, R_5)$ with $l(R_5) > \dots > l(R_1)$ and a horizontal-strip $\mu = (S_1, \dots, S_5)$ with $\Pi(\lambda) \cong \Pi(\mu)$.

Proposition 5.13. Let $\lambda = (R_1, \dots, R_n)$ be a minimal noncommuting path with $l(R_n) > \dots > l(R_1)$. Let $\mu = (S_1, \dots, S_n)$ be a horizontal-strip, $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$, and let i be such that $l(S_i)$ is minimal and j be such that $l(S_j)$ is maximal. Then

$$r(S_j) - l(S_i) \geq r(R_n) - l(R_1) \text{ with equality only if } i = 1 \text{ and } j = n. \tag{5.9}$$

Example 5.14. Proposition 5.13 describes a situation like the one in Figure 5.3. Informally, when the rows of λ move to the right, we could have $M_{t,t+1}(\lambda) = 0$ and the corresponding rows could be permuted in μ . This is a crucial example to keep in mind. We may not have $l(S_{\varphi_{t+1}}) > l(S_{\varphi_t})$ for every t , but we can deduce about the position of the rightmost row S_j .

Proof of Proposition 5.13. Note that we cannot directly apply Proposition 5.9 because we do not know that $l(S_{\varphi_{t+1}}) > l(S_{\varphi_t})$ for every t . Instead, we will reorder the rows of μ and compute directly. By Corollary 3.13, Part 3, we have $M_{t,t+1}(\lambda) < \min\{|R_t|, |R_{t+1}|\}$, so $M_{\varphi_t, \varphi_{t+1}}(\mu) < \min\{|S_{\varphi_t}|, |S_{\varphi_{t+1}}|\}$ and in particular the $l(S_{\varphi_t})$ are distinct. Let σ be the permutation that sorts the rows of μ so that

$$l(S_i) = l(S_{\sigma_1}) < l(S_{\sigma_2}) < \dots < l(S_{\sigma_{n-1}}) < l(S_{\sigma_n}) = l(S_j). \tag{5.10}$$

By Proposition 5.6, Part 1, we have $M_{t,t'}(\lambda) = 0$ if $|t' - t| \geq 2$. Now by (3.6), we have

$$\begin{aligned} r(S_j) - l(S_i) &= l(S_{\sigma_n}) - l(S_{\sigma_1}) + |S_{\sigma_n}| - 1 & (5.11) \\ &= l(S_{\sigma_{n-1}}) - l(S_{\sigma_1}) + |S_{\sigma_n}| + |S_{\sigma_{n-1}}| - M_{\sigma_{n-1}, \sigma_n}(\mu) - 1 + \chi(\sigma_{n-1} > \sigma_n) \\ &\dots = \sum_{t=1}^n |S_{\sigma_t}| - M(\mu) - 1 + \sum_{t=1}^{n-1} \chi(\sigma_t > \sigma_{t+1}) \\ &\geq \sum_{t=1}^n |R_t| - M(\lambda) - 1 \\ &\dots = l(R_{n-1}) - l(R_1) + |R_n| + |R_{n-1}| - M_{n-1, n}(\lambda) - 1 \\ &= r(R_n) - l(R_1) + |R_n| - 1 = r(R_n) - l(R_1), \end{aligned}$$

with equality only if $i = \sigma_1 < \cdots < \sigma_n = j$, so $i = 1$ and $j = n$. \square

We now describe the structure of a minimal noncommuting path (R_1, \dots, R_n) where we have $l(R_1) < l(R_n)$ and $R_1 \leftrightarrow R_n$. Informally, such a minimal noncommuting path must look loosely like one of the examples in Figure 5.1.

Lemma 5.15. *Let $\lambda = (R_1, \dots, R_n)$ be a minimal noncommuting path with $n \geq 4$, $l(R_1) < l(R_n)$, and $R_1 \leftrightarrow R_n$.*

1. *Suppose that there is no $i \geq 3$ for which $R_1 \not\approx R_i$ and that there is no $j \leq n - 2$ for which $R_n \not\approx R_j$. Then one of the following holds.*

- *We have $l(R_2) > l(R_1)$, $l(R_{n-1}) < \cdots < l(R_2)$, and $l(R_n) > l(R_{n-1})$.*
- *We have $l(R_{n-1}) > \cdots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq n - 2$, and $l(R_n) < l(R_{n-1})$.*
- *We have $l(R_2) < l(R_1)$, $l(R_n) > \cdots > l(R_2)$, and $R_t \prec R_1$ for $3 \leq t \leq n - 1$.*

2. *Now suppose that $R_1 \not\approx R_i$ for some minimal $i \geq 3$, and that in fact $i = n - 1$. Then $l(R_n) > l(R_{n-1})$ and one of the following holds.*

- *We have $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$.*
- *We have $l(R_{i-1}) > \cdots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq i - 1$, and $l(R_i) < l(R_{i-1})$.*
- *We have $n = 4$, $l(R_2) > l(R_1)$, $l(R_3) < l(R_2)$, and $R_4 \not\approx R_2$.*

3. *Now suppose that $R_1 \not\approx R_i$ for some minimal $i \geq 3$, and that $i \leq n - 2$. Then $R_n \not\approx R_i$, so $R_n \not\approx R_j$ for some maximal $i \leq j \leq n - 2$. Additionally, one of the following holds.*

- *We have $l(R_{i-1}) < \cdots < l(R_1)$ and $l(R_i) > l(R_{i-1})$.*
- *We have $i = j = 3$, $l(R_2) > l(R_1)$, $R_n \not\approx R_2$, and $l(R_3) < l(R_2)$.*

Similarly, one of the following holds.

- *We have $l(R_n) < \cdots < l(R_{j+1})$ and $l(R_{j+1}) > l(R_j)$.*
- *We have $i = j = n - 2$, $l(R_n) > l(R_{n-1})$, $R_1 \not\approx R_{n-1}$, and $l(R_{n-1}) < l(R_{n-2})$.*

Finally, we must have either $j = i$, or $j = i + 1$ and $l(R_j) < l(R_i)$.

Remark 5.16. If the hypothesis of (1) does not hold, then $R_1 \not\approx R_i$ for some $i \geq 3$ or $R_n \not\approx R_j$ for some $j \leq n - 2$. By rotating, we may assume that $R_1 \not\approx R_i$ for some $i \geq 3$, and by Corollary 3.13, Part 3, we cannot have $R_1 \prec R_n$, so $i \leq n - 1$. Therefore (2) and (3) cover all of the cases we will need.

Proof of Lemma 5.15. Note that because $n \geq 4$, we must have $R_2 \leftrightarrow R_n$ by minimality.

Suppose that there is no $i \geq 3$ for which $R_1 \not\sim R_i$ and that there is no $j \leq n - 2$ for which $R_n \not\sim R_j$. Additionally, suppose that $l(R_2) > l(R_1)$. Because $l(R_1) < l(R_n)$, by Proposition 5.6, Part 2, we must have $R_2 \prec R_n$ or $R_n \prec R_2$, so by our hypothesis we must have $R_2 \prec R_n$ and by Proposition 5.7, Part 2, that $R_t \prec R_n$ for $2 \leq t \leq n - 2$. Note that if $R_2 \prec R_k$ for any $4 \leq k \leq n - 1$, then we would have $R_1 \prec R_k$ by Proposition 5.7, Part 2, and because $n \geq 5$, $R_1 \not\sim R_k$ by Proposition 5.7 Part 1, contradicting our hypothesis. Therefore, by Proposition 5.7, Part 3, we have either $l(R_{n-1}) < \dots < l(R_2)$ and $l(R_n) > l(R_{n-1})$ and the first possibility holds, or we have $l(R_{n-1}) > \dots > l(R_2) > l(R_1)$ and $l(R_n) < l(R_{n-1})$ and the second possibility holds. Now suppose that $l(R_2) < l(R_1)$. If $l(R_n) > l(R_{n-1})$, then by rotating we can reduce to the case where $l(R_2) > l(R_1)$ and it follows that the third possibility holds, so it remains to consider the case where $l(R_n) < l(R_{n-1})$. Because $l(R_{n-1}) > l(R_n) > r(R_2) \geq l(R_2)$, we have $R_2 \leftrightarrow R_{n-1}$ and $n \geq 5$, so Proposition 5.7, Part 1 applies. Also, we must have $l(R_{t+1}) > l(R_t)$ for some minimal $2 \leq t \leq n - 2$. By Proposition 5.6, Part 2, we must have $R_{t-1} \prec R_{t+1}$ or $R_{t+1} \prec R_{t-1}$. However, by Proposition 5.7, Part 2, if $R_{t-1} \prec R_{t+1}$, then $R_1 \not\sim R_{t+1}$, and if $R_{t+1} \prec R_{t-1}$, then $R_n \not\sim R_{t-1}$, contradicting our hypothesis in both cases.

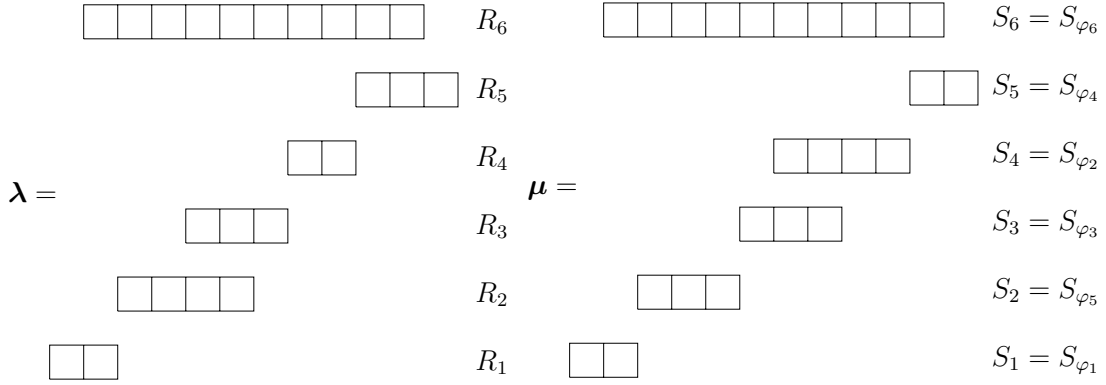
Now suppose that $R_1 \not\sim R_i$ for some minimal $i \geq 3$, and that in fact $i = n - 1$. Then $l(R_{n-1}) \leq l(R_1) < l(R_n)$ by Proposition 5.2. By Proposition 5.7, Part 2, we must have either $l(R_{i-1}) < \dots < l(R_1)$ and $l(R_i) > l(R_{i-1})$ and the first possibility holds, or we have $l(R_{i-1}) > \dots > l(R_1) \geq l(R_i)$ and $l(R_i) < l(R_{i-1})$. Then because $l(R_n) > l(R_1)$, by Proposition 5.6, Part 2, we must have either $R_2 \prec R_n$, in which case we have $R_t \prec R_n$ for $2 \leq t \leq n - 2 = i - 1$ by Proposition 5.7, Part 2, and the second possibility holds, or we have $R_n \not\sim R_2$. In this case, if $n \geq 5$ then $R_i \prec R_2$ and $R_2 \prec R_i$ by Proposition 5.7, Part 2, but this is impossible by Proposition 5.7, Part 1, so we must have $n = 4$ and the third possibility holds.

Now suppose that $R_1 \not\sim R_i$ for some minimal $i \geq 3$, and that $i \leq n - 2$, so in particular, $n \geq 5$ and Proposition 5.7, Part 1 applies. Then because $R_1 \leftrightarrow R_n$, by Proposition 5.6, Part 3, we have $R_n \not\sim R_i$, so $R_n \not\sim R_j$ for some maximal $i \leq j \leq n - 2$. By Proposition 5.7, Part 3 we must have either $l(R_{i-1}) < \dots < l(R_1)$ and $l(R_i) > l(R_{i-1})$, in which case the first possibility holds, or we have $l(R_{i-1}) > \dots > l(R_1)$ and $l(R_i) < l(R_{i-1})$. But in this case, because $l(R_n) > l(R_1)$, by Proposition 5.6, Part 2, we have $R_2 \prec R_n$ or $R_n \prec R_2$. If $R_2 \prec R_n$, then by Proposition 5.7, Part 2, we would have $R_j \prec R_n$, which is impossible by definition of j , so we must have $R_n \not\sim R_2$. If $j \geq 4$, then we would have $R_j \prec R_2$ by Proposition 5.7, Part 2, but because $R_n \prec R_j$ and $R_1 \leftrightarrow R_n$, by Proposition 5.6, Part 3, we have $R_1 \prec R_j$ and $R_2 \prec R_j$ by Proposition 5.7, Part 2 again, which is impossible by Proposition 5.7, Part 1. Therefore, we must have $i = j = 3$ and the second possibility holds. This proves the first claim and the second follows by rotating. Finally, if $j \geq i + 2$, then by Proposition 5.7, Part 2, we would have $R_i \prec R_j$ and $R_j \prec R_i$, which is impossible by Proposition 5.7, Part 1, so we must have either $j = i$ or $j = i + 1$. If $j = i + 1$, then by Proposition 5.7, Part 2, we have $R_{i-1} \prec R_j$ and now $l(R_j) \leq l(R_{i-1}) < l(R_i)$ by Proposition 5.2. This completes the proof. \square

We now describe the image of a minimal noncommuting path under an isomorphism of weighted graphs.

Lemma 5.17. *Let $\lambda = (R_1, \dots, R_n)$ be a minimal noncommuting path with $l(R_1) < l(R_n)$ and $R_1 \leftrightarrow R_n$. Let $\mu = (S_1, \dots, S_n)$ and $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$ satisfy $\varphi_1 = 1$, $l(S_1) < l(S_{\varphi_n})$, and $S_1 \leftrightarrow S_{\varphi_n}$. Then $\varphi_n = n$.*

Example 5.18. Informally, Lemma 5.17 describes a situation like the one below. The conditions $l(R_1) < l(R_6)$, $R_1 \leftrightarrow R_6$, $l(S_1) < l(S_{\varphi_6})$, and $S_1 \leftrightarrow S_{\varphi_6}$ fix the horizontal positions of R_1 , R_6 , S_1 , and S_{φ_6} . Then if we have a minimal noncommuting path in λ from R_1 to R_6 , the row S_{φ_6} must be above the other rows in μ . Note that the intermediate rows can be permuted as in this example.



Proof of Lemma 5.17. By translating all rows, we may assume without loss of generality that $l(R_1) = l(S_1)$, and then by (3.6) we have $l(R_n) = l(S_{\varphi_n})$ as well. The idea is to repeatedly apply Proposition 5.9 and Proposition 5.13 to write the inequality $l(S_{\varphi_n}) \geq l(R_n)$, for which equality holds only if $\varphi_n = n$. Because equality does indeed hold, we will conclude that $\varphi_n = n$.

Case 0: We have $n = 3$. By Proposition 5.11, if $l(R_2) > l(R_1)$, then $l(S_{\varphi_2}) > l(S_1)$, so by Proposition 5.9 we have $l(S_{\varphi_2}) \geq l(R_2)$. By Proposition 5.11 again, if $l(R_3) > l(R_2)$, then $l(S_{\varphi_3}) > l(S_{\varphi_2})$, so by Proposition 5.9 we have $l(S_{\varphi_3}) \geq l(R_3)$ with equality only if $\varphi_3 > \varphi_2$. Because equality does indeed hold, we must have $\varphi_3 > \varphi_2$ and therefore $\varphi_3 = 3$.

We may now assume that $n \geq 4$, so it remains to consider the several possibilities outlined in Lemma 5.15. Cases 1a and 1b illustrate the main ideas.

Case 1: There is no $i \geq 3$ for which $R_1 \prec R_i$ and there is no $j \leq n - 2$ for which $R_n \prec R_j$.

Case 1a: We have $l(R_2) > l(R_1)$, $l(R_{n-1}) < \dots < l(R_2)$, and $l(R_n) > l(R_{n-1})$. By Proposition 5.11, we have $l(S_{\varphi_2}) > l(S_1)$, so by Proposition 5.9 we have $l(S_{\varphi_2}) \geq l(R_2)$. By Proposition 5.9 again, we have $l(S_{\varphi_{n-1}}) \geq l(R_{n-1})$ with equality only if $\varphi_{n-1} > \dots > \varphi_2$. By Proposition 5.11, we have $l(S_{\varphi_n}) > l(S_{\varphi_{n-1}})$, so by Proposition 5.9 we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if we also have $\varphi_n > \varphi_{n-1}$. Because equality does indeed hold, we must have $\varphi_n > \varphi_{n-1} > \dots > \varphi_2$, so $\varphi_n = n$.

Case 1b: We have $l(R_{n-1}) > \dots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq n-2$, and $l(R_n) < l(R_{n-1})$. As in Case 1a, by Proposition 5.11, we have $l(S_{\varphi_2}) > l(S_1)$, so by Proposition 5.9 we have $l(S_{\varphi_2}) \geq l(R_2)$. However, we must now be careful because we need not have $l(S_{\varphi_{n-1}}) > \dots > l(S_{\varphi_2})$, so we take a slightly different approach. Let $k \in \{\varphi_t : 2 \leq t \leq n-1\}$ be such that $l(S_k)$ is maximal. By Proposition 5.13, we have $r(S_k) \geq r(R_{n-1})$ with equality only if $k = n-1$. If $R_{n-1} \not\prec R_n$, then we must have $k = \varphi_{n-1}$ because $S_{\varphi_t} \prec S_{\varphi_n}$ for every $2 \leq t \leq n-1$ except for S_k . Now by Proposition 5.9, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_{n-1} = k = n-1$. Because equality does indeed hold, we must have $\varphi_n = n$. On the other hand, if $R_{n-1} \prec R_n$, then $S_k \prec S_{\varphi_n}$ and by Corollary 3.13, Part 3, we have $r(S_{\varphi_n}) \geq r(S_k) - 1 \geq r(R_{n-1}) - 1 = r(R_n)$ with equality only if $\varphi_n > k = n-1$. Because equality does indeed hold, we must have $\varphi_n = n$.

Case 1c: We have $l(R_2) < l(R_1)$, $l(R_n) > \dots > l(R_2)$, and $R_t \prec R_1$ for $3 \leq t \leq n-1$. By rotating, the conclusion follows from Case 1b.

We now assume that $R_1 \not\prec R_i$ for some minimal $i \geq 3$ or $R_n \not\prec R_j$ for some maximal $j \leq n-2$. By rotating, we may assume that $R_1 \not\prec R_i$ for some minimal $i \geq 3$. Note that by Corollary 3.13, Part 3, we cannot have $R_1 \prec R_n$, so we have $i \leq n-1$. It remains to consider the cases where $i = n-1$ and where $i \leq n-2$.

Case 2: We have $R_1 \not\prec R_i$ for some minimal $i \geq 3$, and in fact $i = n-1$.

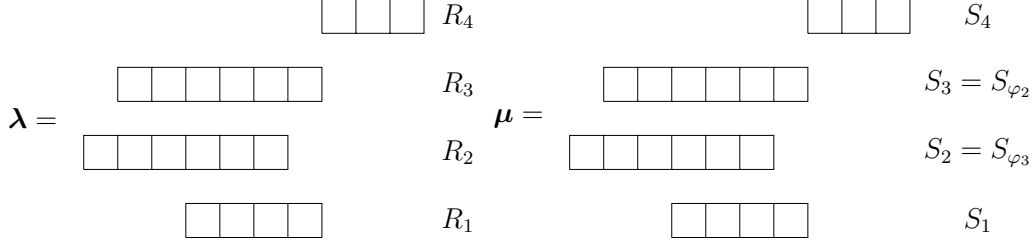
Case 2a: We have $l(R_{i-1}) < \dots < l(R_1)$ and $l(R_i) > l(R_{i-1})$. By Proposition 5.9, we have $l(S_{\varphi_{i-1}}) \geq l(R_{i-1})$ with equality only if $\varphi_{i-1} > \dots > \varphi_2$. If $l(S_{\varphi_i}) > l(S_{\varphi_{i-1}})$, then by Proposition 5.9 again, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_i > \varphi_{i-1} > \dots > \varphi_2$. Because equality does indeed hold, we must have $\varphi_n = n$.

Now suppose that $l(S_{\varphi_i}) \leq l(S_{\varphi_{i-1}})$. We will show that this is narrowly possible, but μ will be so specifically determined that we will be able to reduce to the previous case. By Corollary 3.13, Part 3, we have $M_{i-1,i}(\lambda) < \min\{|R_{i-1}|, |R_i|\}$, so we must have $r(S_{\varphi_i}) < r(S_{\varphi_{i-1}})$. Again, by Corollary 3.13, Part 3, we have $M_{t,t+1}(\lambda) > 0$ for $1 \leq t \leq i-2$, and therefore $M_{\varphi_t, \varphi_{t+1}}(\mu) > 0$ and the rows $S_1, S_{\varphi_2}, \dots, S_{\varphi_{i-1}}$ must overlap; to be precise, $S_1 \cup \dots \cup S_{\varphi_{i-1}}$ must be a row. Now because $r(S_1) < r(S_{\varphi_n})$ and $M_{\varphi_t, \varphi_n}(\mu) = 0$ for $2 \leq t \leq i-1$, we must have $r(S_{\varphi_{i-1}}) < l(S_{\varphi_n})$. Because $S_1 \prec S_{\varphi_i}$, we now have

$$r(S_1) \leq r(S_{\varphi_i}) + 1 \leq r(S_{\varphi_{i-1}}) \leq l(S_{\varphi_n}) - 1 \leq r(S_1), \tag{5.12}$$

so we have equality everywhere and in particular, $M_{1,i-1}(\lambda) = \min\{|R_1|, |R_{i-1}|\} > 0$. Because $M_{1,t}(\lambda) = 0$ for $3 \leq t \leq i-1$ by Proposition 5.6, Part 1, this means that in fact $i = 3$. Now by (5.12), we have $M_{1,4}(\lambda) = M_{3,4}(\lambda) = 0$, so $r(R_1) = r(R_3) = l(R_4) - 1$ by Corollary 3.13, Part 3, and $l(R_3) < l(R_1)$ because $R_1 \not\prec R_3$. Because $M_{1,2}(\lambda) = \min\{|R_1|, |R_2|\}$, we either have $R_1 \prec R_2$ or $R_2 \prec R_1$, but because $l(R_2) < l(R_3) < l(R_1)$, we cannot have $R_2 \prec R_1$ by Proposition 5.2, Part 2, so $R_1 \prec R_2$ and $r(R_2) = r(R_1) - 1$ by Corollary 3.13, Part 3. But now $|R_3| - 1 = M_{2,3}(\lambda) = M_{\varphi_2, \varphi_3}(\mu) = |S_{\varphi_2}| - 1 = |R_2| - 1$, so $|R_2| = |R_3|$. Moreover, we have $R_1 \prec R_2$, $R_1 \prec R_3$, and $M_{2,4}(\lambda) = M_{3,4}(\lambda) = 0$ so in fact these two vertices are equivalent in $\Pi(\lambda)$, and we can swap the roles of these two vertices to reduce to the case where $l(S_{\varphi_i}) > l(S_{\varphi_{i-1}})$. This possibility is illustrated below. It is barely possible to

have $l(S_{\varphi_3}) \leq l(S_{\varphi_2})$, but this requires S_{φ_2} and S_{φ_3} to play equivalent roles in $\Pi(\boldsymbol{\mu})$, so by instead considering the identity isomorphism $\tilde{\varphi} : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$, we can reduce to the previous case.



Case 2b: We have $l(R_{i-1}) > \dots > l(R_1)$, $R_t \prec R_n$ for $2 \leq t \leq i-1$, and $l(R_i) < l(R_{i-1})$. Let $k \in \{\varphi_t : 2 \leq t \leq i-1\}$ be such that $l(S_k)$ is maximal. By Proposition 5.13, we have $r(S_k) \geq r(R_{i-1})$ with equality only if $k = i-1$. If $R_{i-1} \not\prec R_i$, then we must have $k = \varphi_{i-1}$ because $S_{\varphi_t} \prec S_{\varphi_i}$ for every $2 \leq t \leq i-1$ except for S_k . Now by Proposition 5.9, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_i > \varphi_{i-1} = k = i-1$. Because equality does indeed hold, we must have $\varphi_n = n$. On the other hand, if $R_{i-1} \prec R_i$, then $S_k \prec S_{\varphi_i}$, and by Corollary 3.13, Part 3, we have $r(S_{\varphi_i}) \geq r(S_k) - 1 \geq r(R_{i-1}) - 1 = r(R_i)$ with equality only if $\varphi_i > k = i-1$. Then by Proposition 5.9 again, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_i$. Because equality does indeed hold, we must have $\varphi_n = n$.

Case 2c: We have $n = 4$, $l(R_2) > l(R_1)$, $l(R_3) < l(R_2)$, and $R_4 \not\prec R_2$. By Proposition 5.11, we have $l(S_{\varphi_2}) > l(S_1)$ and $l(S_{\varphi_4}) > l(S_{\varphi_3})$, so by Proposition 5.9, we have $l(S_{\varphi_4}) \geq l(R_4)$ with equality only if $\varphi_4 > \varphi_3 > \varphi_2$. Because equality does indeed hold, we must have $\varphi_4 = 4$.

Case 3: We have $R_1 \not\prec R_i$ for some minimal $3 \leq i \leq n-2$. We first show that $l(S_{\varphi_i}) \geq l(R_i)$ with equality only if $\varphi_i > \varphi_t$ for all $t < i$. If $i = j = 3$, $l(R_2) > l(R_1)$, $R_n \not\prec R_2$, and $l(R_3) < l(R_2)$, then we have $l(S_{\varphi_2}) > l(S_1)$ by Proposition 5.11 and the result follows as before from Proposition 5.9. Now suppose that $l(R_{i-1}) < \dots < l(R_1)$ and $l(R_i) > l(R_{i-1})$. We show that $l(S_{\varphi_i}) > l(S_{\varphi_{i-1}})$. Suppose that $l(S_{\varphi_i}) \leq l(S_{\varphi_{i-1}})$. By Corollary 3.13, Part 3, we have $M_{i-1,i}(\boldsymbol{\lambda}) < \min\{|R_{i-1}|, |R_i|\}$, so we must have $r(S_{\varphi_n}) \leq r(S_{\varphi_i}) + 1 \leq r(S_{\varphi_{i-1}})$. Because $M_{t,n}(\boldsymbol{\lambda}) = 0$ for $2 \leq t \leq i-1$ and $M_{t,t+1}(\boldsymbol{\lambda}) > 0$ for $1 \leq t \leq i-2$ by Corollary 3.13, Part 3, we must have $r(S_{\varphi_{i-1}}) < l(S_{\varphi_n}) \leq r(S_{\varphi_n})$, a contradiction, so indeed $l(S_{\varphi_i}) > l(S_{\varphi_{i-1}})$. Now by Proposition 5.9, we have $l(S_{\varphi_i}) \geq l(R_i)$ with equality only if $\varphi_i > \varphi_t$ for all $t < i$. Because either $j = i$, or $j = i+1$ and $l(R_j) < l(R_i)$, we in fact have $l(S_{\varphi_j}) \geq l(R_j)$ with equality only if $\varphi_j > \varphi_t$ for all $t < j$. Finally, by rotating and repeating the previous argument, we have $l(S_{\varphi_n}) \geq l(R_n)$ with equality only if $\varphi_n > \varphi_t$ for all $t < n$. Because equality does indeed hold, we must have $\varphi_n = n$. This completes the proof. \square

The payoff of all our work so far is the following Corollary.

Corollary 5.19. *Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a horizontal-strip with $l(R_i) < l(R_{i+1})$ and $R_i \leftrightarrow R_{i+1}$. Let $\boldsymbol{\mu} = (S_1, \dots, S_n)$ and $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$ be such that $S_{\varphi_i} \leftrightarrow S_{\varphi_{i+1}}$. Then there exists a good substitute for $(\boldsymbol{\lambda}, \boldsymbol{\mu})$.*

Proof. By cycling and rotating, we may assume without loss of generality that $i = 1$, $l(S_{\varphi_1}) < l(S_{\varphi_2})$, and $\varphi_1 = 1$. Let $j = \varphi_2$. By applying Lemma 5.5 to μ , either we may replace μ by a similar horizontal-strip to assume that $j = 2$, in which case we have our good substitute and we are done, or there is a minimal noncommuting path $(S_1 = S_{j_1}, \dots, S_{j_k} = S_j)$ in μ from S_1 to S_j . However, in this case, by considering the corresponding rows in λ , we would have $2 = \varphi_j^{-1} = \varphi_{j_k}^{-1} \geq k \geq 3$ by Lemma 5.17, a contradiction. Therefore, there is indeed a good substitute for (λ, μ) . \square

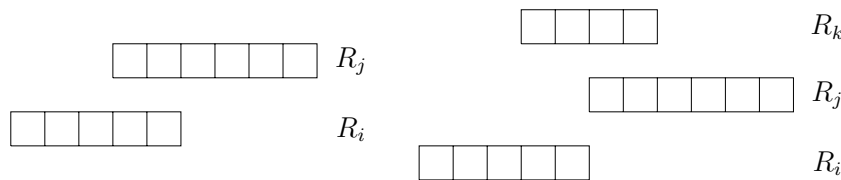
Remark 5.20. The hypothesis that $S_{\varphi_i} \leftrightarrow S_{\varphi_{i+1}}$ is essential to the proof of Corollary 5.19. Note that this hypothesis is not always satisfied, as Figure 5.3 shows an example where $R_1 \leftrightarrow R_2$, but $S_{\varphi_1} \not\leftrightarrow S_{\varphi_2}$. In other words, it is possible for rows not to commute in λ but for the corresponding rows to commute in μ .

Our next goal is to extend Corollary 5.19 by describing properties of the weighted graph $\Pi(\lambda)$ that will force certain rows not to commute.

Definition 5.21. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip. A pair of rows (R_i, R_j) of λ with $i < j$ and $l(R_i) < l(R_j)$ is *strict* if either

1. $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$, or
2. $M_{i,j} = 0$ and $M_{i,k} + M_{j,k} \geq |R_k| + 1$ for some k .

Example 5.22. The two possibilities for strictness are given below. Note that on the right, we have $M_{i,k} + M_{j,k} = 2 + 3 = |R_k| + 1$. Informally, in the second possibility where $M_{i,j} = 0$, the weighted graph normally would not know about the relationship between R_i and R_j . However, the presence of this row R_k glues the rows R_i and R_j together and means that the weighted graph data forces rows R_i and R_j not to commute.



Remark 5.23. Because we define strictness using the weighted graph data, it is preserved under isomorphisms. To be specific, if $\lambda = (R_1, \dots, R_n)$ and $\mu = (S_1, \dots, S_n)$ are horizontal-strips with $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$, then if (R_i, R_j) is strict, we can cycle and rotate to assume that $l(S_{\varphi_i}) < l(S_{\varphi_j})$ and $\varphi_i < \varphi_j$, and then the pair $(S_{\varphi_i}, S_{\varphi_j})$ is strict.

Proposition 5.24. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip and suppose that the pair of rows (R_i, R_j) is strict. Then $R_i \leftrightarrow R_j$.

Proof. If $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$, then $R_i \leftrightarrow R_j$ by Corollary 3.13, Part 1, so suppose that $M_{i,j} = 0$ and $M_{i,k} + M_{j,k} \geq |R_k| + 1$ for some k . Because $l(R_i) < l(R_j)$ and $M_{i,j} = 0$, we have that $l(R_i) \geq r(R_j) + 1$ and because $M_{i,k}, M_{j,k} \leq |R_k|$, we must have $M_{i,k}, M_{j,k} > 0$ so $l(R_k) \leq r(R_i) + 1 \leq l(R_j) \leq r(R_k) + 1$ by Proposition 3.10, Part 1. Now we either have

$M_{i,k} = r(R_i) - l(R_k) + 1 + \chi(i > k)$ or $M_{i,k} = \min\{|R_i|, |R_k|\} \leq r(R_i) - l(R_k) + 1 + \chi(i > k)$ and similarly we have $M_{j,k} \leq r(R_k) - l(R_j) + 1 + \chi(k > j)$. Because $i < j$, we have $\chi(i > k) + \chi(k > j) \leq 1$, so

$$|R_k| + 1 \leq M_{i,k} + M_{j,k} \leq |R_k| + r(R_i) - l(R_j) + 1 + \chi(i > k) + \chi(k > j) \leq |R_k| + 1, \quad (5.13)$$

so we must have equality everywhere and in particular, $r(R_j) - l(R_i) + 1 = 0$, so $R_i \leftrightarrow R_j$. \square

Remark 5.25. This proof shows that if $l(R_i) < l(R_j)$ and $M_{i,j} = 0$, then in fact $M_{i,k} + M_{j,k} \leq |R_k| + 1$ for all k , so we could replace the condition $M_{i,k} + M_{j,k} \geq |R_k| + 1$ with the equivalent condition $M_{i,k} + M_{j,k} = |R_k| + 1$.

Corollary 5.26. *Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip with a pair of adjacent strict rows (R_i, R_{i+1}) . Then λ is good.*

Proof. Let $\mu = (S_1, \dots, S_n)$ and $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$. By definition, we have $M_{i,i+1}(\lambda) < \min\{|R_i|, |R_{i+1}|\}$, so $M_{\varphi_i, \varphi_{i+1}}(\mu) < \min\{|S_{\varphi_i}|, |S_{\varphi_{i+1}}|\}$ and $l(S_{\varphi_i}) \neq l(S_{\varphi_{i+1}})$. Therefore, by cycling and rotating, we may assume without loss of generality that $l(S_{\varphi_i}) < l(S_{\varphi_{i+1}})$ and $\varphi_i < \varphi_{i+1}$, so the pair $(S_{\varphi_i}, S_{\varphi_{i+1}})$ is strict. By Proposition 5.24, we have $S_{\varphi_i} \leftrightarrow S_{\varphi_{i+1}}$, so by Corollary 5.19, there exists a good substitute for (λ, μ) . \square

We now investigate some useful properties of strict pairs.

Proposition 5.27. *Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip with $i < j$, $l(R_i) < l(R_j)$, and $R_i \leftrightarrow R_j$. Suppose that $R_i \leftrightarrow R_k$ and $R_j \leftrightarrow R_k$ for some k with $k < i$ or $k > j$. Then the pair (R_i, R_j) is strict.*

Proof. By Corollary 3.13, Part 3, we cannot have $M_{i,j} = \min\{|R_i|, |R_j|\}$, and if $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$, then we are done, so suppose that $M_{i,j} = 0$ and therefore $l(R_j) = r(R_i) + 1$ by Corollary 3.13, Part 3. Because $R_i \leftrightarrow R_k$, by Proposition 3.10, Part 1 we must have $l(R_k) \leq r(R_i) + 1 = l(R_j)$, and because $R_j \leftrightarrow R_k$, we must have $r(R_k) \geq l(R_j) - 1 = r(R_i)$, so by Corollary 3.13, Part 2, we in fact have $l(R_i) < l(R_k) < l(R_j)$. Now by (3.6), we have

$$M_{i,k} + M_{j,k} = r(R_i) - l(R_k) + 1 + \chi(i > k) + r(R_k) - l(R_j) + 1 + \chi(k > j) = |R_k| + 1, \quad (5.14)$$

so the pair (R_i, R_j) is strict. \square

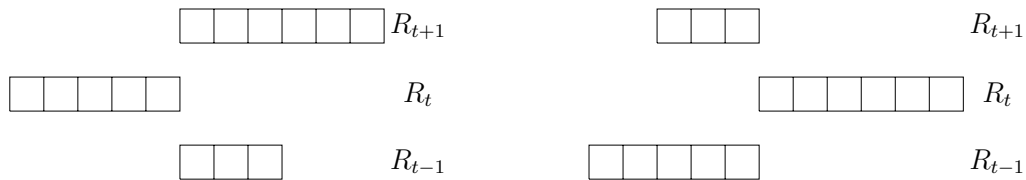
Proposition 5.28. *Let $\lambda = (R_1, \dots, R_n)$ be a minimal noncommuting path and suppose that the pairs $(R_{t'}, R_{t'+1})$ are not strict for $1 \leq t' < n - 1$. Then we have the following.*

1. *If $l(R_{t+1}) > l(R_t)$, then $M_{t,t+1} = 0$ and $l(R_{t+1}) = r(R_t) + 1$.*
2. *If $l(R_{t+1}) > l(R_t)$ and $l(R_t) < l(R_{t-1})$, then $R_{t+1} \overset{\sim}{\prec} R_{t-1}$.*
3. *If $l(R_{t+1}) < l(R_t)$ and $l(R_t) > l(R_{t-1})$, then $R_{t-1} \overset{\sim}{\prec} R_{t+1}$.*

Proof.

1. By Corollary 3.13, Part 3 we cannot have $M_{t,t+1} = \min\{|R_t|, |R_{t+1}|\}$, and if $0 < M_{t,t+1} < \min\{|R_t|, |R_{t+1}|\}$, then the pair (R_t, R_{t+1}) would be strict, so we must have $M_{t,t+1} = 0$ and $l(R_{t+1}) = r(R_t) + 1$.
2. By the previous part, we must have $M_{t,t+1} = 0$. By Proposition 5.6, Part 2, we must have $R_{t-1} \prec R_{t+1}$ or $R_{t+1} \prec R_{t-1}$. However, if $R_{t-1} \prec R_{t+1}$, then because $M_{t-1,t} > 0$ by Corollary 3.13, Part 3, we would have $M_{t-1,t} + M_{t-1,t+1} \geq |R_{t-1}| + 1$ and the pair (R_t, R_{t+1}) would be strict, so we must have $R_{t+1} \succ R_{t-1}$.
3. By rotating, this follows from the previous part. □

Example 5.29. The diagrams below illustrate the contradictions that we deduce in the proofs of Parts 2 and 3 of Proposition 5.28. If $R_{t-1} \prec R_{t+1}$ as on the left, then the pair (R_t, R_{t+1}) would be strict, so we concluded that $R_{t+1} \succ R_{t-1}$. If $R_{t+1} \prec R_{t-1}$ as on the right, then the pair (R_{t-1}, R_t) would be strict, so we concluded that $R_{t-1} \succ R_{t+1}$.



It will also be convenient to make the following definition.

Definition 5.30. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip. A *strict sequence* of λ is a sequence of rows $(R_{j_1}, \dots, R_{j_k})$ such that $k \geq 2$, $j_1 < \dots < j_k$, $M_{j_t, j_{t'}} = 0$ for every $1 \leq t < t' \leq k$, and there is some h with $h < j_1$ or $h > j_k$ for which

$$M_{j_t, h} > 0 \text{ for all } 1 \leq t \leq k \text{ and } M_{j_1, h} + \dots + M_{j_k, h} \geq |R_h| + 1. \tag{5.15}$$

Note that if a pair of rows (R_i, R_j) is a strict sequence, then it meets the second condition of being a strict pair.

Example 5.31. In Figure 5.4, we have $M_{t,t'} = 0$ for every $1 \leq t < t' \leq 6$, $M_{t,7} > 0$ for $1 \leq t \leq 6$, and $M_{1,7} + \dots + M_{6,7} = 2 + 5 + 2 + 4 + 3 + 3 = 19 = |R_7| + 1$, so (R_1, \dots, R_6) is a strict sequence. Informally, our next Proposition will show that every strict sequence looks very much like this example. Because $M_{t,t'} = 0$ for $1 \leq t < t' \leq 6$, the weighted graph normally would not know about the relationship between these rows. However, the presence of the row R_7 glues these rows together and means that the weighted graph data forces the adjacent rows not to commute.

Remark 5.32. Because we define a strict sequence using the weighted graph data, it is preserved under isomorphisms. To be specific, if $\lambda = (R_1, \dots, R_n)$ and $\mu = (S_1, \dots, S_n)$ are horizontal-strips with $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$, then if $(R_{j_1}, \dots, R_{j_k})$ is a strict sequence, we can cycle to

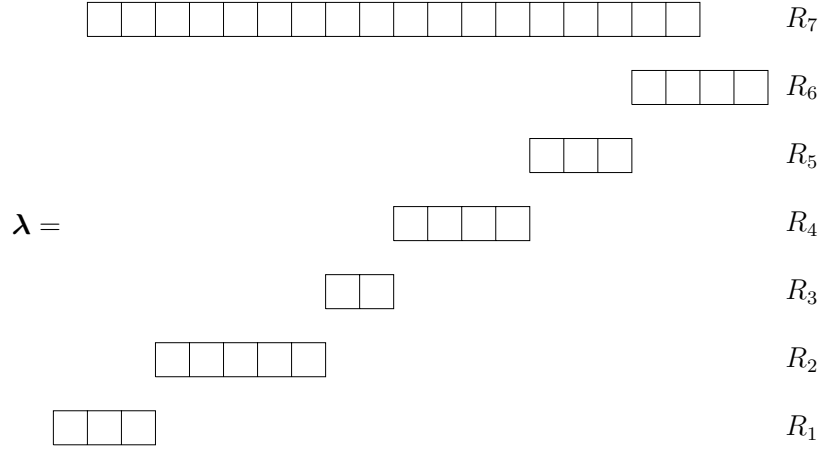


Figure 5.4: An example of a strict sequence.

assume that $\varphi_h > \varphi_{j_t}$ for every $1 \leq t \leq k$. Because $M_{j_t, j_{t'}}(\boldsymbol{\lambda}) = 0$ for $1 \leq t < t' \leq k$, the integers $l(S_{\varphi_{j_t}})$ for $1 \leq t \leq k$ must be distinct, so we can let $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ be the permutation that sorts them in increasing order, in other words

$$l(S_{\varphi_{j_{\sigma_1}}}) < l(S_{\varphi_{j_{\sigma_2}}}) < \dots < l(S_{\varphi_{j_{\sigma_k}}}). \quad (5.16)$$

Then the sequence $(S_{\varphi_{j_{\sigma_1}}}, \dots, S_{\varphi_{j_{\sigma_k}}})$ is strict.

Proposition 5.33. *Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a horizontal-strip with a sequence of rows of the form $(R_{j_1}, \dots, R_{j_k})$ with $k \geq 2$, $j_1 < \dots < j_k$, $M_{j_t, j_{t'}} = 0$ for every $1 \leq t < t' \leq k$, and there is some h with $h < j_1$ or $h > j_k$ for which $M_{j_t, h} > 0$ for every $1 \leq t \leq k$. Then this sequence is strict if and only if $l(R_{j_{t+1}}) = r(R_{j_t}) + 1$ for every $1 \leq t \leq k - 1$ and*

$$l(R_{j_1}) + \chi(j_1 > h) \leq l(R_h) \leq r(R_h) + \chi(h > j_k) \leq r(R_{j_k}). \quad (5.17)$$

Proof. By Proposition 3.10 Part 2, because all of the $M_{j_t, j_{t'}}$ are zero, the integers $l(R_{j_t})$ for $1 \leq t \leq k$ are distinct, so let $\sigma : \{1, \dots, k\} \rightarrow \{j_1, \dots, j_k\}$ sort the rows R_{j_t} so that $l(R_{j_t})$ is increasing. Then because the $M_{j_t, j_{t'}}$ are zero we have

$$l(R_{\sigma_1}) < r(R_{\sigma_1}) + 1 + \chi(\sigma_1 > \sigma_2) \leq l(R_{\sigma_2}) < r(R_{\sigma_2}) + 1 + \chi(\sigma_2 > \sigma_3) \leq \dots \leq l(R_{\sigma_k}). \quad (5.18)$$

Because $M_{\sigma_1, h} > 0$, we must have $l(R_h) \leq r(R_{\sigma_1}) + 1 \leq l(R_{\sigma_2})$ and because $M_{\sigma_k, h} > 0$, we must have $r(R_h) \geq l(R_{\sigma_k}) - 1 \geq r(R_{\sigma_{k-1}})$. We now have that

$$M_{\sigma_1, h} \leq r(R_{\sigma_1}) - l(R_h) + 1 + \chi(\sigma_1 > h) \leq l(R_{\sigma_2}) - l(R_h) + \chi(\sigma_1 > h) - \chi(\sigma_1 > \sigma_2) \quad (5.19)$$

$$M_{\sigma_2, h} = |R_{\sigma_2}| = r(R_{\sigma_2}) - l(R_{\sigma_2}) + 1 \leq l(R_{\sigma_3}) - l(R_{\sigma_2}) - \chi(\sigma_2 > \sigma_3)$$

$\dots =$

$$M_{\sigma_{k-1}, h} = |R_{\sigma_{k-1}}| = r(R_{\sigma_{k-1}}) - l(R_{\sigma_{k-1}}) + 1 \leq l(R_{\sigma_k}) - l(R_{\sigma_{k-1}}) - \chi(\sigma_{k-1} > \sigma_k)$$

$$M_{\sigma_k, h} \leq r(R_h) - l(R_{\sigma_k}) + 1 + \chi(h > \sigma_k).$$

Also note that $\chi(\sigma_1 > h) - \chi(\sigma_1 > \sigma_2) - \dots - \chi(\sigma_{k-1} > \sigma_k) + \chi(h > \sigma_k) \leq 1$ with equality only if $\sigma_1 < \dots < \sigma_k$. Therefore, by summing the above equations, we have

$$\sum_{t=1}^k M_{\sigma_t, h} \leq r(R_h) - l(R_h) + 1 + 1 = |R_h| + 1, \tag{5.20}$$

so the sequence is strict if and only if we have equality everywhere, meaning that $\sigma_1 < \dots < \sigma_k$, $l(R_{j_{t+1}}) = r(R_{j_t}) + 1$ for $1 \leq t \leq k - 1$, and (5.17) holds. \square

Remark 5.34. This proof shows that if $l(R_{j_1}) < \dots < l(R_{j_k})$ and the $M_{j_t, j_{t'}} = 0$, then in fact $M_{j_1, h} + \dots + M_{j_k, h} \leq |R_h| + 1$ for all h , so we could replace the condition $M_{j_1, h} + \dots + M_{j_k, h} \geq |R_h| + 1$ with the equivalent condition $M_{j_1, h} + \dots + M_{j_k, h} = |R_h| + 1$.

Proposition 5.35. *Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip with a strict sequence of rows $(R_{j_1}, \dots, R_{j_k})$. Suppose that there is some $j_t < x < j_{t+1}$ with $l(R_x) = l(R_{j_{t+1}})$ and $R_{j_{t+1}} \prec R_x$. Then one of the following holds.*

1. *The sequence $(R_{j_1}, \dots, R_{j_t}, R_x)$ is strict.*
2. *There is a shorter strict sequence of the form $(R_{j_1}, \dots, R_{j_t}, R_x, R_{j_{t'+1}}, \dots, R_{j_k})$ for some $t' \geq t + 2$.*
3. *There is a strict pair $(R_x, R_{j_{t'+1}})$ for some $t' \geq t + 1$.*

Example 5.36. Informally, Proposition 5.35 describes a situation like the one in Figure 5.5. The sequence $(R_{j_1}, \dots, R_{j_6})$ is the strict sequence from Example 5.31. The rows R_{x_1} , R_{x_2} , and R_{x_3} illustrate the three possibilities described in Proposition 5.35. Informally, because R_{x_1} extends past R_{j_t} , the sequence $(R_{j_1}, R_{j_2}, R_{x_1})$ is strict. Because $R_{x_2} = R_{j_3} \cup R_{j_4} \cup R_{j_5}$, it can replace these rows to produce the shorter strict sequence $(R_{j_1}, R_{j_2}, R_{x_2}, R_{j_6})$. Finally, because R_{x_3} ends between the rows R_{j_4} and R_{j_5} , it results in the strict pair (R_{x_3}, R_{j_5}) .

Proof of Proposition 5.35. By the definition of a strict sequence, there is some h with $h < j_1$ or $h > j_k$ for which $M_{j_t, h} > 0$ for all $1 \leq t \leq k$ and $\sum_{t=1}^k M_{j_t, h} \geq |R_h| + 1$, and by cycling we may assume without loss of generality that $h > j_k$, so that $l(R_{j_1}) \leq l(R_h) < r(R_h) + 1 \leq r(R_{j_k})$ by (5.17). Noting that $R_{j_{t+1}} \prec R_x$, let $t + 1 \leq t' \leq k$ be maximal such that $R_{j_{t'}} \prec R_x$. If $t' = k$, then we must have $r(R_x) \geq r(R_{j_k}) \geq r(R_h) + 1$ by Proposition 5.2 and now the sequence $(R_{j_1}, \dots, R_{j_t}, R_x)$ is strict by Proposition 5.33, so the first possibility holds and we may now assume that $t' \leq k - 1$. By maximality of t' , we have that $R_{j_{t'+1}} \not\prec R_x$. If $M_{x, j_{t'+1}} = 0$, then we must have $r(R_x) = r(R_{j_{t'}}) = l(R_{j_{t'+1}}) - 1$ and the sequence $(R_{j_1}, \dots, R_{j_t}, R_x, R_{j_{t'+1}}, \dots, R_{j_k})$ is strict by Proposition 5.33. Also note that because $R_x \not\prec R_{j_{t+1}}$, we must have $t' > t + 1$ so this strict sequence is indeed shorter and the second possibility holds. Finally, if $M_{x, j_{t'+1}} > 0$, then because $l(R_x) = l(R_{j_{t+1}}) < l(R_{j_{t'+1}})$, we have $R_x \not\prec R_{j_{t'+1}}$, and by maximality of t we have $R_{j_{t'+1}} \not\prec R_x$, so $0 < M_{x, j_{t'+1}} < \min\{|R_x|, |R_{j_{t'+1}}|\}$, the pair $(R_x, R_{j_{t'+1}})$ is strict, and the third possibility holds. \square

We now describe the structure of a minimal noncommuting path with no strict pairs.

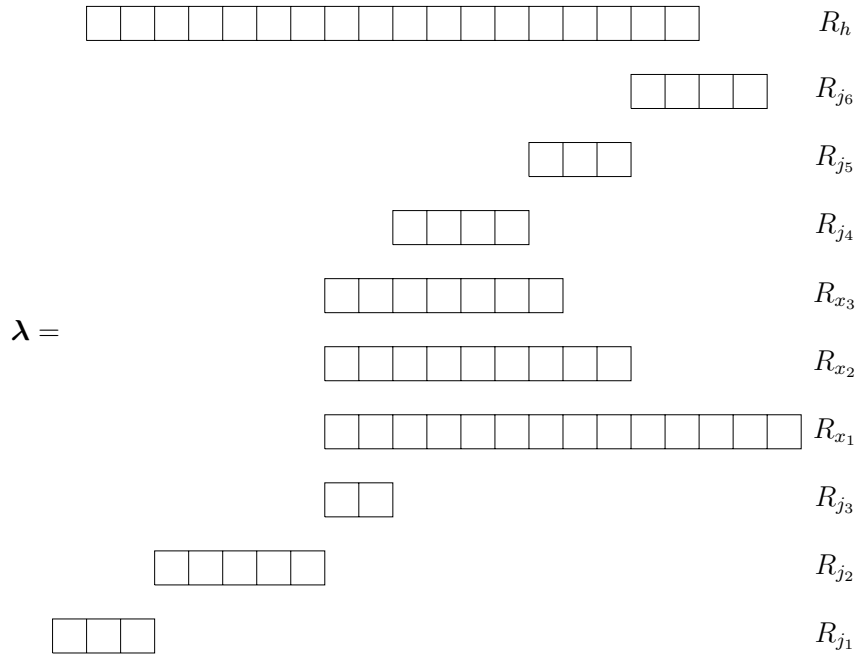
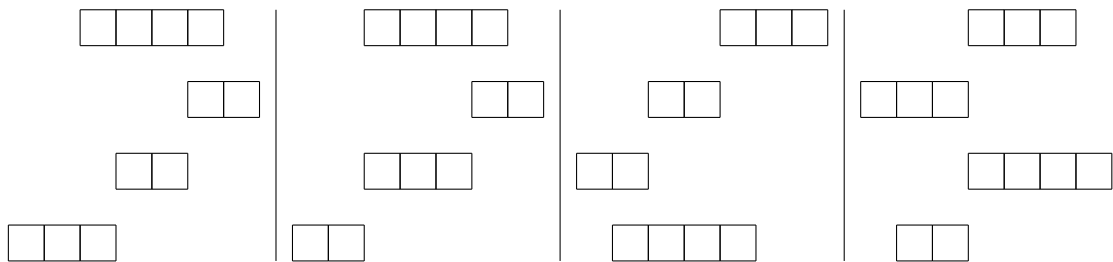


Figure 5.5: A strict sequence with a row R_x as in Proposition 5.35.

Proposition 5.37. *Let $\lambda = (R_1, \dots, R_n)$ be a minimal noncommuting path with $l(R_1) < l(R_n)$ and $R_1 \leftrightarrow R_n$ and suppose that the pairs (R_t, R_{t+1}) are not strict for $1 \leq t \leq n - 1$. Then one of the following holds.*

1. *One of the sequences (R_1, \dots, R_{n-1}) , (R_2, \dots, R_{n-1}) , or (R_2, \dots, R_n) is strict.*
2. *We have $n = 4$, $l(R_2) = l(R_4) = r(R_1) - 1 = r(R_3) - 1$, $R_4 \not\approx R_2$, and $R_1 \not\approx R_3$.*

Example 5.38. Informally, Proposition 5.37 tells us that a minimal noncommuting path with no strict pairs (other than possibly (R_1, R_n)) must look like one of the examples below.



Proof of Proposition 5.37. We consider several cases.

Case 0: We have $n = 3$. In this case, we have $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$. Because $l(R_1) < l(R_3)$, we must have either $l(R_1) < l(R_2)$, in which case the pair (R_1, R_2) is strict by Proposition 5.27, or $l(R_2) < l(R_3)$, in which case the pair (R_2, R_3) is strict by Proposition 5.27, a contradiction.

We may now assume that $n \geq 4$, so it remains to consider the several possibilities outlined in Lemma 5.15.

Case 1: There is no $i \geq 3$ for which $R_1 \not\prec R_i$ and there is no $j \leq n - 2$ for which $R_n \not\prec R_j$.

Case 1a: We have $l(R_2) > l(R_1), l(R_{n-1}) < \dots < l(R_2)$, and $l(R_n) > l(R_{n-1})$. Because $l(R_2) > l(R_1)$ and $l(R_3) < l(R_2)$, we have $R_1 \not\prec R_3$ by Proposition 5.28, contradicting our hypothesis.

Case 1b: We have $l(R_{n-1}) > \dots > l(R_1), R_t \prec R_n$ for $2 \leq t \leq n - 2$, and $l(R_n) < l(R_{n-1})$. By Proposition 5.28, we must have $l(R_{t+1}) = r(R_t) + 1$ for $1 \leq t \leq n - 2$. Now by Proposition 5.33, if $M_{1,n} > 0$ then the sequence (R_1, \dots, R_{n-1}) is strict and if $M_{1,n} = 0$ then the sequence (R_2, \dots, R_{n-1}) is strict, so the first possibility holds.

Case 1c: We have $l(R_2) < l(R_1), l(R_n) > \dots > l(R_2)$, and $R_t \prec R_1$ for $3 \leq t \leq n - 1$. By rotating, the conclusion follows from Case 1b.

We now assume that $R_1 \not\prec R_i$ for some minimal $i \geq 3$ or $R_n \not\prec R_j$ for some maximal $j \leq n - 2$. By rotating, we may assume that $R_1 \not\prec R_i$ for some minimal $i \geq 3$. Note that by Corollary 3.13, Part 3, we cannot have $R_1 \prec R_n$, so we have $i \leq n - 1$. It remains to consider the cases where $i = n - 1$ and where $i \leq n - 2$.

Case 2: We have $R_1 \not\prec R_i$ for some minimal $i \geq 3$, and in fact $i = n - 1$.

Case 2a: We have $l(R_{i-1}) < \dots < l(R_1)$ and $l(R_i) > l(R_{i-1})$. By Proposition 5.7, Part 2, we have $R_{i-2} \prec R_i$, but because $l(R_{i-1}) < l(R_{i-2})$ and $l(R_i) > l(R_{i-1})$, we have $R_i \not\prec R_{i-2}$ by Proposition 5.28, a contradiction.

Case 2b: We have $l(R_{i-1}) > \dots > l(R_1), R_t \prec R_n$ for $2 \leq t \leq i - 1$, and $l(R_i) < l(R_{i-1})$. Because $l(R_i) < l(R_{i-1})$ and $l(R_n) > l(R_i)$, by Proposition 5.28, Part 2 we have $R_n \not\prec R_{i-1}$, contradicting our hypothesis.

Case 2c: We have $n = 4, l(R_2) > l(R_1), l(R_3) < l(R_2)$, and $R_4 \not\prec R_2$. By Proposition 5.2, Proposition 3.10, and Proposition 5.28, Part 1, we must have $r(R_1) + 1 = l(R_2) \leq l(R_4) \leq r(R_1) + 1$, so $l(R_2) = l(R_4)$, and we must have $l(R_4) - 1 = r(R_3) \geq r(R_1) = l(R_2) - 1 = l(R_4) - 1$, so $r(R_3) = r(R_1)$. In particular, we have $M_{1,n} = 0$. By Proposition 5.28, Parts 2 and 3, we must have $R_4 \not\prec R_2$ and $R_1 \not\prec R_3$ and the second possibility holds.

Case 3: We have $R_i \not\prec R_i$ for some minimal $3 \leq i \leq n - 2$. If $l(R_{i-1}) < l(R_{i-2})$ and $l(R_i) > l(R_{i-1})$, then by Proposition 5.7, Part 2, we have $R_{i-2} \prec R_i$, but by Proposition 5.28, Part 2 we have $R_i \not\prec R_{i-2}$, a contradiction. Therefore, by Lemma 5.15, we must have $i = j = 3, l(R_2) > l(R_1), R_n \not\prec R_2$, and $l(R_3) < l(R_2)$. Similarly, by rotating, we must have $i = j = n - 2$, so $n = 5, l(R_5) > l(R_4), R_1 \not\prec R_4$, and $l(R_4) < l(R_3)$. However, by Proposition 5.7, Parts 1 and 2, we have $R_2 \prec R_4$ and $R_4 \prec R_2$, which is impossible because $n \geq 5$. □

We now describe another operation that we can perform on a horizontal-strip while preserving similarity. We can think of it as a *local rotation*.

Lemma 5.39. *Let $\lambda = (R_1, \dots, R_n)$ and suppose that $l(R_i) = r(R_{i-1}) + 1$ for some $2 \leq i \leq n$. Assume that λ satisfies the inductive hypothesis (3.16) in Lemma 3.23. Let $I = \{1, \dots, i-2, i+1, \dots, n\}$ and define the four disjoint subsets of I*

$$\begin{aligned} A &= \{t \in I : M_{i-1,t} = M_{i,t} = 0\}, \\ B &= \{t \in I : R_{i-1} \prec R_t, R_i \prec R_t\}, \\ C_{i-1} &= \{t \in I : R_t \prec R_{i-1}, M_{i,t} = 0, M_{a,t} = 0 \text{ for all } a \in A, R_t \prec R_b \text{ for all } b \in B\}, \\ C_i &= \{t \in I : R_t \prec R_i, M_{i-1,t} = 0, M_{a,t} = 0 \text{ for all } a \in A, R_t \prec R_b \text{ for all } b \in B\}. \end{aligned}$$

Let $C = C_{i-1} \cup C_i \cup \{i-1, i\}$ and suppose that $A \cup B \cup C = \{1, \dots, n\}$, in other words every row of λ falls into one of these categories. Then there is a horizontal-strip $\mu = (S_1, \dots, S_n) \in \mathcal{S}(\lambda)$ and $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$ with $l(S_{\varphi_t}) = l(R_t)$ for all $t \in A \cup B$ and $l(S_{\varphi_i}) = l(R_{i-1})$.

Example 5.40. Figure 5.6 illustrates a situation where we can apply Lemma 5.39. The rows below λ with crosses signify rows that cannot be present because the condition $A \cup B \cup C = \{1, \dots, n\}$ requires that every row of λ be either disjoint from R_{i-1} and R_i , contained in R_{i-1} , contained in R_i , or containing both. Lemma 5.39 allows us to locally rotate the six rows of C to produce the similar horizontal-strip μ . Although our proof constructs this specific μ , we will only need that $l(S_{\varphi_t}) = l(R_t)$ for all $t \in A \cup B$ and $l(S_{\varphi_i}) = l(R_{i-1})$.

Proof of Lemma 5.39. Informally, we will first use commuting and cycling to bring the rows of C together. To be specific, we first claim that if $t \in C_i$ and $t' \notin C_i$ with either $t > t' > i$, then $R_t \leftrightarrow R_{t'}$. Because $t \in C_i$, we have $R_t \prec R_i$ and $M_{i-1,t} = 0$, so by Proposition 5.2 we have $l(R_i) \leq l(R_t) \leq r(R_t) \leq r(R_i)$. Now if $t' \in A$, then $M_{i-1,t'} = M_{i,t} = M_{t,t'} = 0$, so either $r(R_{t'}) < l(R_{i-1}) - 1 < l(R_t) - 1$ and $R_t \leftrightarrow R_{t'}$ by Proposition 3.10, Parts 2 and 3, or $l(R_{t'}) > r(R_i) \geq r(R_t) \geq l(R_t)$, but now we cannot have $R_t \leftrightarrow R_{t'}$ by Corollary 3.13, Part 3. If $t' \in B$, then $R_{t'} \prec R_{i-1}$, so by Proposition 5.2 we have $l(R_{t'}) \leq l(R_{i-1}) - 1 < l(R_t)$, but now we cannot have $R_t \leftrightarrow R_{t'}$ by Corollary 3.13, Part 3. If $t' \in C_{i-1}$, then because $R_{t'} \prec R_{i-1}$ and $M_{i,t'} = 0$, we have $r(R_{t'}) < l(R_i) - 1 \leq l(R_t) - 1$, so $R_t \leftrightarrow R_{t'}$ by Proposition 3.10, Part 1. This establishes our claim.

Therefore, by cycling and commuting, we may assume that $C_i = \{i+1, \dots, y\}$ for some y and similarly, by considering a rotation of λ , we may assume that $C_{i-1} = \{x, \dots, i-2\}$ for some x . In particular, we have $l(R_{i-1}) \leq l(R_t) \leq r(R_t) \leq r(R_{i-1})$ for every $t \in C_{i-1}$ and $l(R_i) \leq l(R_t) \leq r(R_t) \leq r(R_i)$ for every $t \in C_i$. To summarize, we have

$$\lambda = (R_1, \dots, R_{x-1}, R_x, \dots, R_{i-2}, R_{i-1}, R_i, R_{i+1}, \dots, R_y, R_{y+1}, \dots, R_n), \quad (5.21)$$

where $A \cup B = \{1, \dots, x-1\} \cup \{y+1, \dots, n\}$, $C_{i-1} = \{x, \dots, i-2\}$, and $C_i = \{i+1, \dots, y\}$.

Now let $N = l(R_{i-1}) + r(R_i)$ and define the horizontal-strip $\mu = (S_1, \dots, S_n)$ by $S_t = R_t$ if $t < x$ or $t > y$, and $S_t = N - R_{x+y-t}$ otherwise, that is

$$\mu = (R_1, \dots, R_{x-1}, N - R_y, \dots, N - R_i, N - R_{i-1}, \dots, N - R_x, R_{y+1}, \dots, R_n), \quad (5.22)$$

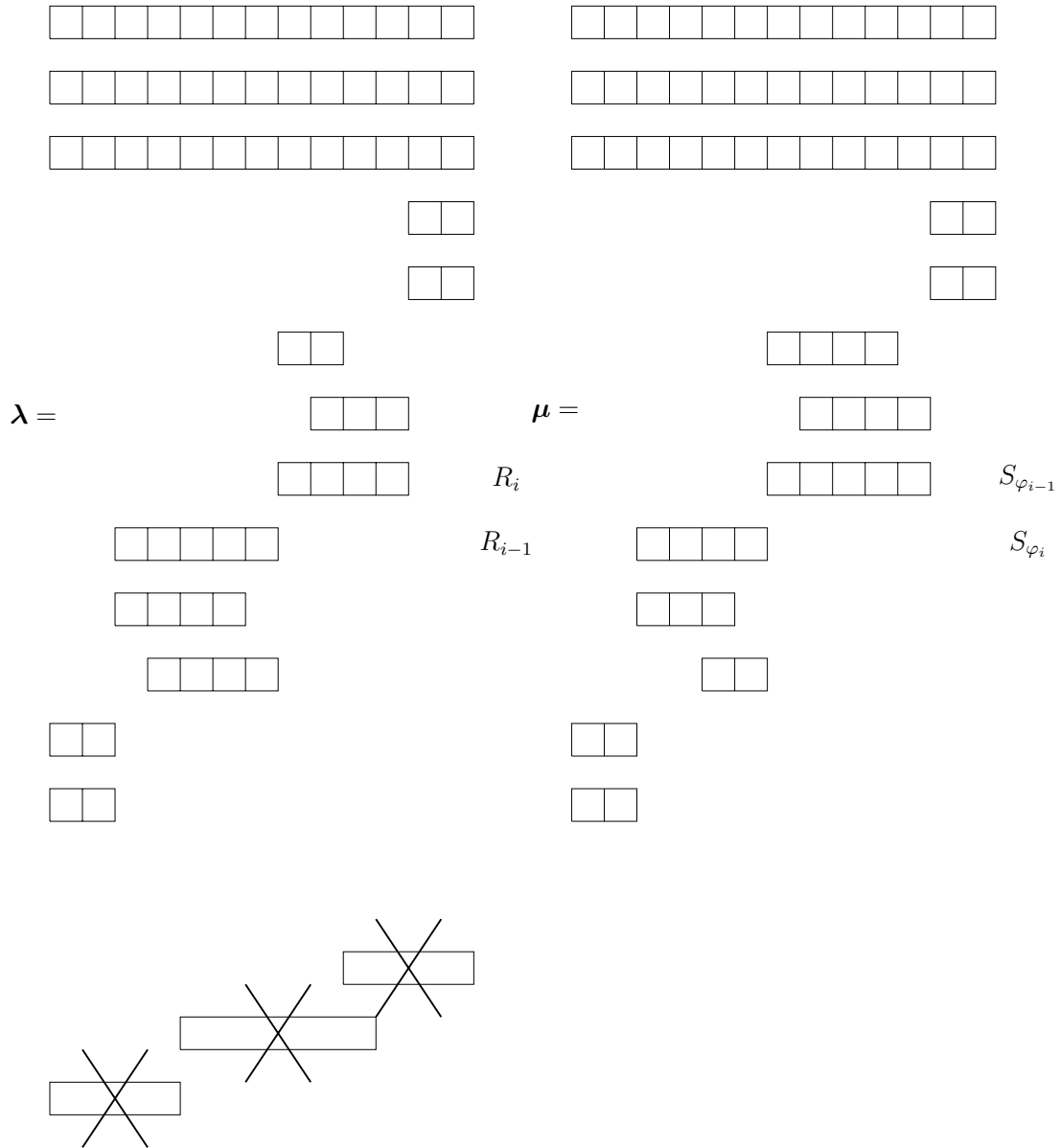


Figure 5.6: A horizontal-strip λ and a local rotation μ .

and define $\varphi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ by $\varphi_t = t$ if $t < x$ or $t > y$ and $\varphi_t = x + y - t$ otherwise. Indeed, we have $l(S_{\varphi_t}) = l(R_t)$ for all $t \in A \cup B$ and $l(S_{\varphi_i}) = \ell(N - R_i) = l(R_{i-1}) + r(R_i) - r(R_i) = l(R_{i-1})$. We claim that $\varphi : \Pi(\boldsymbol{\lambda}) \xrightarrow{\sim} \Pi(\boldsymbol{\mu})$. We have $|R_t| = |S_{\varphi_t}|$ for every $1 \leq t \leq n$, so it remains to check that the edge weights are preserved. If $t, t' \in A \cup B$, then the relative positions of R_t and $R_{t'}$ have not changed, so indeed $M_{t,t'}(\boldsymbol{\lambda}) = M_{\varphi_t, \varphi_{t'}}(\boldsymbol{\mu})$. If $t, t' \in C$, then this follows because $M(R_t, R_{t'}) = M(N - R_{t'}, N - R_t)$. Now suppose that $t \in A \cup B$ and $t' \in C$. We have either $t' < i - 1$ and $R_{t'} \prec R_{i-1}$ or $t' > i$ and $R_{t'} \prec R_i$, so in either case we have

$$l(R_{i-1}) \leq l(R_{t'}) \leq r(R_{t'}) \leq r(R_i), \text{ so } l(R_{i-1}) \leq \ell(N - R_{t'}) \leq r(N - R_{t'}) \leq r(R_i). \quad (5.23)$$

Also note that $\chi(t > i) = \chi(t > i - 1) = \chi(t > t') = \chi(t > x + y - t')$. Now if $t \in A$, we have either

$$l(R_t) > r(R_i) + \chi(i > t) \geq r(N - R_{t'}) + \chi(i > x + y - t') \text{ or} \quad (5.24)$$

$$r(R_t) < l(R_{i-1}) - \chi(t > i) \leq \ell(N - R_{t'}) - \chi(t > x + y - t'), \quad (5.25)$$

so in either case, we have $M_{\varphi_t, \varphi_{t'}}(\boldsymbol{\mu}) = 0$. Similarly, if $t \in B$, we have

$$l(R_t) \leq l(R_{i-1}) + \chi(i - 1 > t) \leq \ell(N - R_{t'}) + \chi(x + y - t' > t) \text{ and} \quad (5.26)$$

$$r(R_t) \geq r(R_i) - \chi(t > i) \geq r(N - R_{t'}) - \chi(t > x + y - t'), \quad (5.27)$$

so we have $N - R_{t'} \prec R_t$ and $M_{\varphi_t, \varphi_{t'}}(\boldsymbol{\mu}) = |S_{\varphi_{t'}}|$.

Finally, we show that $G_{\boldsymbol{\lambda}}(\mathbf{x}; q) = G_{\boldsymbol{\mu}}(\mathbf{x}; q)$. Define the horizontal-strips

$$\boldsymbol{\lambda}' = (R_1, \dots, R_i, R_{i-1}, \dots, R_n), \quad (5.28)$$

$$\boldsymbol{\lambda}'' = (R_1, \dots, R_{i-1} \cup R_i, R_{i-1} \cap R_i, \dots, R_n), \quad (5.29)$$

$$\boldsymbol{\mu}' = (R_1, \dots, N - R_y, \dots, N - R_{i-1}, N - R_i, \dots, N - R_x, \dots, R_n), \text{ and} \quad (5.30)$$

$$\boldsymbol{\mu}'' = (R_1, \dots, (N - R_i) \cup (N - R_{i-1}), (N - R_i) \cap (N - R_{i-1}), \dots, R_n). \quad (5.31)$$

Because Lemma 3.19 describes exactly how to derive the weighted graphs $\Pi(\boldsymbol{\lambda}')$ and $\Pi(\boldsymbol{\lambda}'')$ from $\Pi(\boldsymbol{\lambda})$, we have that $\Pi(\boldsymbol{\lambda}') \cong \Pi(\boldsymbol{\mu}')$ and $\Pi(\boldsymbol{\lambda}'') \cong \Pi(\boldsymbol{\mu}'')$. We also have $n(\boldsymbol{\lambda}'') < n(\boldsymbol{\lambda})$, $n(\boldsymbol{\lambda}') = n(\boldsymbol{\lambda})$, and $M(\boldsymbol{\lambda}') > M(\boldsymbol{\lambda})$, so because $\boldsymbol{\lambda}$ satisfies (3.16) by hypothesis, we have that $G_{\boldsymbol{\lambda}'}(\mathbf{x}; q) = G_{\boldsymbol{\mu}'}(\mathbf{x}; q)$ and $G_{\boldsymbol{\lambda}''}(\mathbf{x}; q) = G_{\boldsymbol{\mu}''}(\mathbf{x}; q)$. Finally, by Lemma 3.18, we have

$$G_{\boldsymbol{\lambda}}(\mathbf{x}; q) = \frac{1}{q}G_{\boldsymbol{\lambda}'}(\mathbf{x}; q) + \frac{q-1}{q}G_{\boldsymbol{\lambda}''}(\mathbf{x}; q) = \frac{1}{q}G_{\boldsymbol{\mu}'}(\mathbf{x}; q) + \frac{q-1}{q}G_{\boldsymbol{\mu}''}(\mathbf{x}; q) = G_{\boldsymbol{\mu}}(\mathbf{x}; q). \quad (5.32)$$

This completes the proof. \square

The hypothesis of Lemma 5.39 that $A \cup B \cup C = \{1, \dots, n\}$ is a little technical so it will be convenient to rephrase it as follows.

Proposition 5.41. *Let $\boldsymbol{\lambda} = (R_1, \dots, R_n)$ be a horizontal-strip with $l(R_i) = r(R_{i-1}) + 1$ and define the sets A, B, C , and I as in Lemma 5.39. If the following hold for every $t \in I$, then we have $A \cup B \cup C = \{1, \dots, n\}$.*

1. If $M_{i-1,t} > 0$ and $M_{i,t} > 0$, then $R_{i-1} \prec R_t$ and $R_i \prec R_t$.
2. If $M_{i-1,t} > 0$ and $M_{i,t} = 0$, then $R_t \prec R_{i-1}$.
3. If $M_{i-1,t} = 0$ and $M_{i,t} > 0$, then $R_t \prec R_i$.
4. If $R_t \prec R_{i-1}$, then $M_{i,t} = 0$, $M_{a,t} = 0$ for all $a \in A$, and $R_t \prec R_b$ for all $b \in B$.
5. If $R_t \prec R_i$, then $M_{i-1,t} = 0$, $M_{a,t} = 0$ for all $a \in A$, and $R_t \prec R_b$ for all $b \in B$.

Proof. Let $t \in I$. We need to show that $t \in A \cup B \cup C_{i-1} \cup C_i$. The integers $M_{i-1,t}$ and $M_{i,t}$ are either zero or nonzero. If $M_{i-1,t} = M_{i,t} = 0$, then $t \in A$. If $M_{i-1,t} > 0$ and $M_{i,t} > 0$, then by (1) we have $t \in B$. If $M_{i-1,t} > 0$ and $M_{i,t} = 0$, then by (2) and (4) we have $t \in C_{i-1}$. If $M_{i-1,t} = 0$ and $M_{i,t} > 0$, then by (3) and (5) we have $t \in C_i$. \square

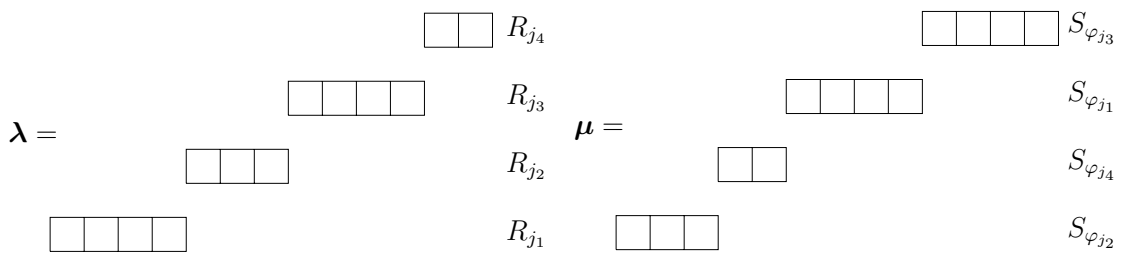
The next Lemma is very technical but it is the key idea that uses local rotation to extend Corollary 5.26 to strict sequences.

Lemma 5.42. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip with a sequence $(R_{j_1}, \dots, R_{j_k})$ with $k \geq 2$, $j_1 < \dots < j_k$, $l(R_{j_{t+1}}) = r(R_{j_t}) + 1$ for $1 \leq t \leq k - 1$, and suppose that there is no noncommuting path in λ from R_{j_t} to $R_{j_{t+1}}$ for any $1 \leq t \leq k - 1$. Assume that λ satisfies (3.16). Let $\mu = (S_1, \dots, S_n)$ and $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$ be such that

$$\text{for some permutation } \sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\} \text{ we have } l(S_{\varphi_{j_{\sigma_{t+1}}}}) = r(S_{\varphi_{j_{\sigma_t}}}) + 1 \quad (5.33)$$

for every $1 \leq t \leq k - 1$. Then there exists a good substitute for (λ, μ) .

Example 5.43. Lemma 5.42 applies in a situation like the one below. Informally, the condition (5.33) asks that these rows in λ still link end to end in μ , although they may be permuted. In this example, we have $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 1$, and $\sigma_4 = 3$.



Remark 5.44. If $(R_{j_1}, \dots, R_{j_k})$ is a strict sequence of λ , then $k \geq 2$, $j_1 < \dots < j_k$, and by Proposition 5.33, $l(R_{j_{t+1}}) = r(R_{j_t}) + 1$ for $1 \leq t \leq k - 1$. Moreover, if $\mu = (S_1, \dots, S_n)$ and $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$, then by Remark 5.32 and by cycling μ if necessary, there will be a permutation $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ with $(S_{\varphi_{j_{\sigma_1}}}, \dots, S_{\varphi_{j_{\sigma_k}}})$ a strict sequence and therefore $l(S_{\varphi_{j_{\sigma_{t+1}}}}) = r(S_{\varphi_{j_{\sigma_t}}}) + 1$ for every $1 \leq t \leq k - 1$. Therefore, a strict sequence satisfies the hypothesis (5.33) of Lemma 5.42.

Remark 5.45. Informally, the strategy will be to apply Lemma 5.39 to perform a series of local rotations to permute the rows of λ to match those of μ . We will be able to perform these local rotations unless some other row R_t of λ violates some condition of Proposition 5.41, forcing certain rows of λ to link end to end. However, in this case, these rows of λ will be a proper subset of rows that satisfies (5.33) and we can use induction to reason about these rows.

Proof of Lemma 5.42. We use induction on k . If $k = 2$, then because $l(R_{j_2}) = r(R_{j_1}) + 1$ and by hypothesis there is no noncommuting path in λ from R_{j_1} to R_{j_2} , we can use Lemma 5.5 to replace λ with a similar horizontal-strip as necessary to assume that $j_2 = j_1 + 1$, and then by (5.33) we have either $l(S_{\varphi_{j_2}}) = r(S_{\varphi_{j_1}}) + 1$ or $l(S_{\varphi_{j_1}}) = r(S_{\varphi_{j_2}}) + 1$, so $S_{\varphi_{j_1}} \leftrightarrow S_{\varphi_{j_2}}$ and the result follows from Corollary 5.19. So we now assume that $k \geq 3$ and that the result holds for $2 \leq k' \leq k - 1$. In particular, if $J \subseteq \{1, \dots, k\}$ is an interval with $2 \leq |J| \leq k - 1$ and such that $\sigma^{-1}(J) = \{1 \leq t' \leq k : \sigma_{t'} \in J\} \subseteq \{1, \dots, k\}$ is an interval, then the sequence of rows $(R_{j_t} : t \in J)$ satisfies (5.33) and we are done by our induction hypothesis on k . In particular, if $\sigma_1 = 1$ or $\sigma_k = 1$, then we can take $J = \{2, \dots, k\}$, and if $\sigma_1 = k$ or $\sigma_k = k$, then we can take $J = \{1, \dots, k - 1\}$, so we may assume that

$$2 \leq \sigma_1, \sigma_k \leq k - 1, \quad (5.34)$$

and in particular, we may assume that $k \geq 4$.

We now continue to use our induction hypothesis to make several additional simplifying assumptions. For $1 \leq t \leq n$ such that $t \neq j_{t'}$ for any $1 \leq t' \leq k$, consider the sets

$$E_t = \{1 \leq t' \leq k : R_{j_{t'}} \prec R_t\} \text{ and } F_t = \{1 \leq t \leq k : M_{j_{t'}, t}(\lambda) > 0\}. \quad (5.35)$$

Note that $E_t \subseteq F_t$. Also, if $t_1 < t_2 < t_3$ and $t_1, t_3 \in F_t$, then by Proposition 3.10, Parts 1 and 3, we have

$$l(R_t) \leq r(R_{j_{t_1}}) + 1 \leq l(R_{j_{t_2}}) \leq r(R_{j_{t_2}}) \leq l(R_{j_{t_3}}) - 1 \leq r(R_t), \quad (5.36)$$

so by Proposition 5.2 we have $R_{t_2} \prec R_t$ and $t_2 \in E_t$, so E_t and F_t are intervals in $\{1, \dots, k\}$ and $|E_t| \geq |F_t| - 2$. Similarly, consider the sets

$$E'_t = \{1 \leq t' \leq k : S_{\varphi_{j_{\sigma_{t'}}}} \prec S_{\varphi_t}\} \text{ and } F'_t = \{1 \leq t' \leq k : M_{\varphi_{j_{\sigma_{t'}}}, \varphi_t}(\mu) > 0\}. \quad (5.37)$$

Note that $E'_t \subseteq F'_t$, $E'_t = \sigma^{-1}(E_t)$, $F'_t = \sigma^{-1}(F_t)$ and as before, if $t_1 < t_2 < t_3$ and $t_1, t_3 \in F'_t$, then by Proposition 3.10, Parts 1 and 3, we have

$$l(S_{\varphi_t}) \leq r(S_{\varphi_{j_{\sigma_{t_1}}}}) + 1 \leq l(S_{\varphi_{j_{\sigma_{t_2}}}}) \leq r(S_{\varphi_{j_{\sigma_{t_2}}}}) \leq l(S_{\varphi_{j_{\sigma_{t_3}}}}) - 1 \leq r(S_{\varphi_t}), \quad (5.38)$$

so by Proposition 5.2, we have $S_{\varphi_{j_{\sigma_{t_2}}}} \prec S_{\varphi_t}$ and $t_2 \in E'_t$, so E'_t and F'_t are intervals in $\{1, \dots, k\}$. Therefore, if $2 \leq |F'_t| \leq k - 1$, then taking $J = F'_t$ above we are done by our induction hypothesis on k . Similarly, if $|F'_t| = k$ and $|E_t| \leq k - 1$, then because $|E_t| \geq k - 2 \geq 2$, taking $J = E_t$ above we are done by our induction hypothesis on k . This means that we may assume that

$$\text{if } |F_t| \geq 2, \text{ then } E_t = F_t = \{1, \dots, k\}. \quad (5.39)$$

Now suppose that $F_t = \{t'\}$. Our goal is to show that we may assume that $R_t \prec R_{j_{t'}}$, that $M_{a,t}(\lambda) = 0$ for every a with $M_{a,j_{t'}}(\lambda) = 0$, and $R_t \prec R_b$ for every b with $R_{j_{t'}} \prec R_b$.

Suppose that $R_t \not\prec R_{j_{t'}}$. If $2 \leq t' \leq k - 1$, then $M_{j_{t'-1},t}(\lambda) = 0$ and $M_{j_{t'+1},t}(\lambda) = 0$ by definition of F_t and by Proposition 3.10, Parts 1 and 3, we would have

$$l(R_{j_{t'}}) = r(R_{j_{t'-1}}) + 1 \leq l(R_t) \leq r(R_t) \leq l(R_{j_{t'+1}}) - 1 = r(R_{j_{t'}}) \tag{5.40}$$

and therefore $R_t \prec R_{j_{t'}}$ by Proposition 5.2. This means that we must have $t' = 1$ or k and for the same reason, $\sigma_{t'} = 1$ or k , but this contradicts our assumption (5.34). Therefore, we must have $R_t \prec R_{j_{t'}}$.

Now suppose that there is some a with $M_{a,j_{t'}}(\lambda) = 0$ but $M_{a,t}(\lambda) > 0$. Because $R_t \prec R_{j_{t'}}$, we have $M_{j_{t'},t}(\lambda) + M_{a,t}(\lambda) \geq |R_t| + 1$, so either $j_{t'} < a$ and the pair $(R_{j_{t'}}, R_a)$ is strict, or $a < j_{t'}$ and the pair $(R_a, R_{j_{t'}})$ is strict. In either case, by Proposition 5.24 we have $R_{j_{t'}} \leftrightarrow R_a$, and by Corollary 3.13, Part 3, we have either

$$l(R_a) = r(R_{j_{t'}}) + 1 = l(R_{j_{t'+1}}) \text{ or } l(R_{j_{t'}}) = r(R_a) + 1. \tag{5.41}$$

In particular, if $t' = 1$, then either $F_a = \emptyset$ or $\{2\}$, if $2 \leq t' \leq k - 1$, then either $F_a = \{t' - 1\}$ or $\{t' + 1\}$, and if $t' = k$, then either $F_a = \emptyset$ or $\{k - 1\}$. Similarly, F'_a is either empty, in which case $\sigma_{t'} = 1$ or k , or F'_a and F'_t consist of consecutive singletons. Therefore, if $F_a = \emptyset$, then this contradicts (5.34), and otherwise, taking $J = F_t \cup F_a$ we are done by our induction hypothesis on k .

Next, let us suppose that for some b we have $R_{j_{t'}} \prec R_b$ but $R_t \not\prec R_b$. If $2 \leq t' \leq k - 1$, then $M_{j_{t'-1},t}(\lambda) = 0$ and $M_{j_{t'+1},t}(\lambda) = 0$ by definition of F_t and by Proposition 3.10, Parts 2 and 3, we would have

$$l(R_{j_{t'}}) = r(R_{j_{t'-1}}) + 1 \leq l(R_t) \leq r(R_t) \leq l(R_{j_{t'+1}}) - 1 = r(R_{j_{t'}}). \tag{5.42}$$

Now if $R_{j_{t'}} \subseteq R_b$, we would have $R_t \subseteq R_{j_{t'}} \subseteq R_b$ and $R_t \prec R_b$ by Proposition 5.2, so we must have either $R_{j_{t'}} \subseteq R_b^+$ and $l(R_b) \leq l(R_{j_{t'}}) - 1 = r(R_{j_{t'-1}})$, or $R_{j_{t'}} \subseteq R_b^-$ and $r(R_b) \geq r(R_{j_{t'}}) + 1 = l(R_{j_{t'+1}})$. However, this means that either $\{t' - 1, t'\} \subseteq F_b$ or $\{t', t' + 1\} \subseteq F_b$, so by (5.39) we must have $F_b = \{1, \dots, k\}$, so $M_{j_{t'-1},t}(\lambda) > 0$ and $M_{j_{t'+1},t}(\lambda) > 0$, and by Proposition 5.2, we have

$$l(R_b) \leq r(R_{j_{t'-1}}) + 1 = l(R_{j_{t'}}) \leq l(R_t) \leq r(R_t) \leq r(R_{j_{t'}}) = l(R_{j_{t'+1}}) - 1 \leq r(R_b) \tag{5.43}$$

and $R_t \prec R_b$ by Proposition 5.2 after all. Therefore, if $R_{j_{t'}} \prec R_b$ but $R_t \not\prec R_b$, we must have $t' = 1$ or k and for the same reason, $\sigma_{t'} = 1$ or k , but this contradicts our assumption (5.34). To summarize, we may assume that

$$\begin{aligned} &\text{if } |F_t| \geq 2, \text{ then } E_t = F_t = \{1, \dots, k\}, \text{ and if } F_t = \{t'\}, \text{ then } R_t \prec R_{j_{t'}}, \\ &M_{a,t}(\lambda) = 0 \text{ whenever } M_{a,j_{t'}}(\lambda) = 0, \text{ and } R_t \prec R_b \text{ whenever } R_{j_{t'}} \prec R_b. \end{aligned} \tag{5.44}$$

Let t_0 and t_1 be such that $t_0 < t_1$ and $\{t_0, t_1\} = \{\sigma_1, \sigma_2\}$, and let $x = j_{t_0}$ and $y = j_{t_1}$. Note that we have $x < y$, $l(R_y) \geq r(R_x) + 1$, and $S_{\varphi_x} \leftrightarrow S_{\varphi_y}$. We use induction on $l(R_y) - r(R_x)$.

If $l(R_y) - r(R_x) = 1$, then $t_1 = t_0 + 1$ and because there is no noncommuting path in λ from R_x to R_y , we may use Lemma 5.5 to assume that $y = x + 1$. In this case, it now follows from Corollary 5.19 that there exists a good substitute for (λ, μ) . So we now assume that $l(R_y) - r(R_x) \geq 2$, so in particular $t_1 \geq t_0 + 2$. Because there is no noncommuting path in λ from $R_{j_{t_1-1}}$ to R_y , we may use Lemma 5.5 to assume that $y = j_{t_1-1} + 1$.

Our plan now is to apply Lemma 5.39 to the rows R_{y-1} and R_y to replace λ with a similar horizontal-strip for which $l(R_y)$ has decreased and $r(R_x)$ has not changed, so that we will be done by our induction hypothesis on $l(R_y) - r(R_x)$. It remains to check the conditions of Proposition 5.41.

1. If $M_{y-1,t}(\lambda) > 0$ and $M_{y,t}(\lambda) > 0$, then $|F_t| \geq 2$, so by (5.44) we have $E_t = \{1, \dots, k\}$, so $R_{y-1} \prec R_t$ and $R_y \prec R_t$.
2. If $M_{y-1,t}(\lambda) > 0$ and $M_{y,t}(\lambda) = 0$, then $t_1 - 1 \in F_t$ and $t_1 \notin F_t$, so by (5.44) we must have $F_t = \{t_1 - 1\}$ and then $R_t \prec R_{y-1}$.
3. If $M_{y-1,t}(\lambda) = 0$ and $M_{y,t}(\lambda) > 0$, then $t_1 - 1 \notin F_t$ and $t_1 \in F_t$, so by (5.44) we must have $F_t = \{t_1\}$ and $R_t \prec R_y$.
4. If $R_t \prec R_{y-1}$, then $t_1 - 1 \in F_t$. By Proposition 3.10, Part 1, and because $x < y$, we cannot have both $M_{x,t}(\lambda) > 0$ and $M_{y,t}(\lambda) > 0$ because then

$$\begin{aligned} l(R_t) &\leq r(R_x) + \chi(x > t) \text{ and } r(R_t) \geq l(R_y) - \chi(t > y), \text{ so} & (5.45) \\ |R_t| &= r(R_t) - l(R_t) + 1 \geq l(R_y) - r(R_x) + 1 - \chi(t > y) - \chi(x > t) \\ &= r(R_{y-1}) + 1 - l(R_{y-1}) + 1 + 1 - \chi(t > y) - \chi(x > t) > |R_{y-1}|, \end{aligned}$$

contradicting $R_t \prec R_{y-1}$ by Proposition 5.2. Therefore, either $t_0 \notin F_t$ or $t_1 \notin F_t$, so by (5.44) we must have $F_t = \{t_1 - 1\}$ and $M_{y,t}(\lambda) = 0$. Moreover, if $a \in A$, then in particular $M_{a,y}(\lambda) = 0$, so by (5.44) we have $M_{a,t}(\lambda) = 0$, and if $b \in B$, then in particular we have $R_{y-1} \prec R_b$, so by (5.44) we have $R_t \prec R_b$ as well.

5. If $R_t \prec R_y$, then $t_1 \in F_t$. By Proposition 5.2 and because $x < y$, we cannot have $M_{x,t}(\lambda) > 0$ because then

$$\begin{aligned} l(R_t) &\leq r(R_x) + \chi(x > t) \leq l(R_{y-1}) - 1 + \chi(x > t) \leq r(R_{y-1}) - 1 + \chi(x > t) & (5.46) \\ &= l(R_{y-1}) - 1 - 1 + \chi(x > t) \leq l(R_y) - 1 - \chi(t > y) < l(R_y) - \chi(t > y), \end{aligned}$$

contradicting $R_t \prec R_y$ by Proposition 5.2. Therefore, $t_0 \notin F_t$, so by (5.44) we must have $F_t = \{t_1\}$ and $M_{y-1,t}(\lambda) = 0$. Moreover, if $a \in A$, then in particular $M_{a,y-1}(\lambda) = 0$, so by (5.44) we have $M_{a,t}(\lambda) = 0$, and if $b \in B$, then in particular we have $R_y \prec R_b$, so by (5.44) we have $R_t \prec R_b$ as well.

This concludes our verification of the conditions of Proposition 5.41. Therefore, the result follows by Lemma 5.39 and our induction hypothesis on $l(R_y) - r(R_x)$. \square

We now generalize Corollary 5.26 to the case where λ has a pair of strict rows that are not necessarily adjacent.

Corollary 5.46. *Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip that satisfies (3.16). If λ has a strict sequence $(R_{j_1}, \dots, R_{j_k})$ such that the pairs $(R_{i'}, R_{j'})$ are not strict for $j_1 \leq i' < j' \leq j_k$, then λ is good.*

Proof. Because λ has a strict sequence $(R_{j_1}, \dots, R_{j_k})$ such that the pairs $(R_{i'}, R_{j'})$ are not strict for $j_1 \leq i' < j' \leq j_k$, we may assume that $(R_{j_1}, \dots, R_{j_k})$ is such a strict sequence with $j_k - j_1$ minimal, and among such strict sequences, with k minimal. Now if there is a minimal noncommuting path in λ from R_{j_t} to $R_{j_{t+1}}$ for any $1 \leq t \leq k - 1$, then by Proposition 5.37, there is either a strict sequence between R_{j_t} and $R_{j_{t+1}}$, contradicting minimality of $j_k - j_1$, or there is some $j_t < x < j_{t+1}$ with $l(R_x) = l(R_{j_{t+1}})$ and $R_{j_{t+1}} \prec_{\neq} R_x$, but in that case by Proposition 5.35 there is either a strict sequence between nearer rows, contradicting minimality of $j_k - j_1$, there is a shorter strict sequence from R_{j_1} to R_{j_k} , contradicting minimality of k , or there is a strict pair, contradicting our hypothesis. Therefore, there is no minimal noncommuting path in λ from R_{j_t} to $R_{j_{t+1}}$ for any $1 \leq t \leq k - 1$.

Now let $\mu = (S_1, \dots, S_n)$ and $\varphi : \Pi(\lambda) \xrightarrow{\sim} \Pi(\mu)$. By cycling, we may assume without loss of generality that $\varphi_h > \varphi_{j_t}$ for every $1 \leq t \leq k$. Because $M_{j_t, j_{t+1}}(\lambda) = 0$ for $1 \leq t \leq k - 1$, the integers $l(S_{\varphi_{j_t}})$ for $1 \leq t \leq k$ must be distinct, so let $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ be the permutation that sorts them in increasing order, in other words

$$l(S_{\varphi_{j_{\sigma_1}}}) < l(S_{\varphi_{j_{\sigma_2}}}) < \dots < l(S_{\varphi_{j_{\sigma_k}}}). \tag{5.47}$$

Now the sequence $(S_{\varphi_{j_{\sigma_1}}}, \dots, S_{\varphi_{j_{\sigma_k}}})$ is strict, so by Lemma 5.42 there exists a good substitute for (λ, μ) . □

Corollary 5.47. *Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip that satisfies (3.16). If λ has a pair of strict rows, then λ is good.*

Proof. Because λ has a pair of strict rows, we may assume that (R_i, R_j) is a strict pair with $j - i$ minimal, in other words the pairs $(R_{i'}, R_{j'})$ are not strict for $i \leq i' < j' \leq j$ with $j' - i' < j - i$. Also note that if there is no minimal noncommuting path in λ from R_i to R_j , then by Lemma 5.5 we may replace λ with a similar horizontal-strip as necessary to assume that $j = i + 1$, in which case we are done by Corollary 5.26. Therefore, we may assume that there is a minimal noncommuting path in λ from R_i to R_j . By Proposition 5.37, there is either a strict sequence $(R_{j_1}, \dots, R_{j_k})$ in λ such that the pairs $(R_{i'}, R_{j'})$ are not strict for $j_1 \leq i' < j' \leq j_k$, in which case we are done by Corollary 5.46, or we have $M_{i,j} = 0$ and there is $i < x < j$ with $l(R_x) = l(R_j)$ and $R_j \prec_{\neq} R_x$. However, in the latter case, because the pair (R_i, R_j) is strict, then since $M_{i,j} = 0$ we must have $M_{i,k} + M_{j,k} = |R_k| + 1$ for some k , but now $M_{i,k} + M_{x,k} = M_{i,k} + M_{j,k} = |R_k| + 1$ so the pair (R_i, R_x) is strict, contradicting minimality of $j - i$. □

Corollary 5.48. *Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip that satisfies (3.16). If λ has a strict sequence $(R_{j_1}, \dots, R_{j_k})$, then λ is good.*

Proof. By Corollary 5.47, we may assume that λ has no strict pairs, after which the result follows by Corollary 5.46. □

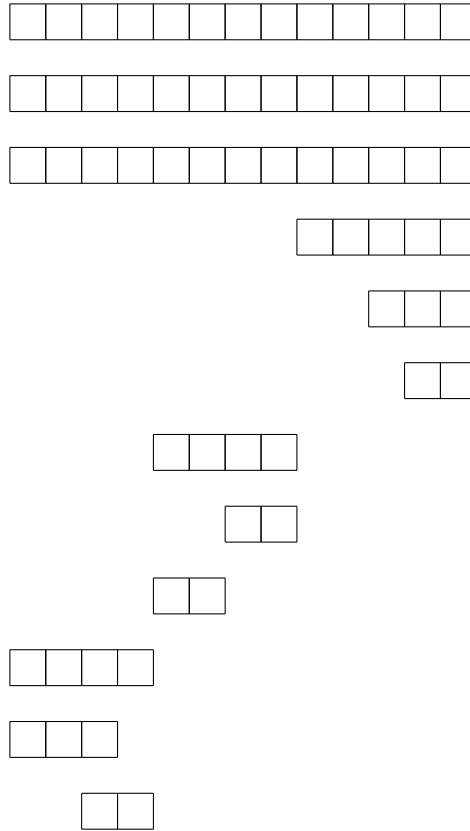


Figure 5.7: A nesting horizontal-strip.

By Corollary 5.47 and Corollary 5.48, we may assume in completing the proof of Lemma 3.23 that there are no strict pairs or strict sequences in λ or any similar horizontal-strip. It will be convenient to make the following definition.

Definition 5.49. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip. We say that λ is *nesting* if for every $1 \leq i < j \leq n$ we have either $M_{i,j} = 0$, $R_i \prec R_j$, or $R_j \prec R_i$, and if $M_{i,j} = 0$, then $M_{i,k} + M_{j,k} \leq |R_k|$ for every k .

Example 5.50. An example of a nesting horizontal-strip is given in Figure 5.7. Informally, every pair of rows is either disjoint or one is contained in the other.

Corollary 5.51. Let $\lambda = (R_1, \dots, R_n)$ be a horizontal-strip that satisfies (3.16). If λ is not nesting, then λ is good.

Proof. If λ is not nesting, then there is some $1 \leq i < j \leq n$ with either $0 < M_{i,j} < \min\{|R_i|, |R_j|\}$ or $M_{i,j} = 0$ and $M_{i,k} + M_{j,k} \geq |R_k| + 1$ for some k . By rotating, we may assume that $l(R_i) < l(R_j)$ so that the pair (R_i, R_j) is strict, and then the result follows from Corollary 5.47. \square

We now explore some properties of a nesting horizontal-strip.

Proposition 5.52. Let $\lambda = (R_1, \dots, R_n)$ be a nesting horizontal-strip.

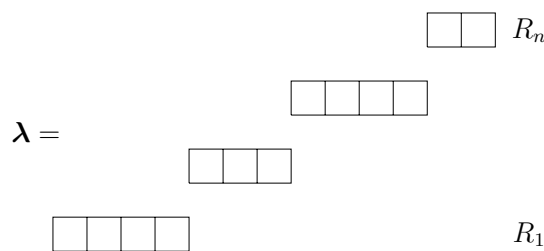
1. The pairs (R_i, R_j) are not strict for $1 \leq i < j \leq n$.
2. If $R_i \prec R_j$ and $R_j \prec R_k$, then $R_i \prec R_k$. In other words, the relation \prec is transitive on the rows of λ .
3. If $R_i \prec R_j$ and $M_{j,k} = 0$, then $M_{i,k} = 0$.
4. We cannot have $i < x < y < j$ with $l(R_x) = l(R_j) = r(R_i) - 1 = r(R_y) - 1$, $R_j \succ R_x$, and $R_1 \succ R_y$.

Proof.

1. This follows directly from the definitions of strictness and nesting.
2. Because $M_{i,j} + M_{j,k} = |R_i| + |R_j| \geq |R_j| + 1$, then by definition of nesting we cannot have $M_{i,k} = 0$, so we must have $R_i \prec R_k$ or $R_k \prec R_i$. If $R_k \prec R_i$, then by Proposition 5.2 we have $|R_k| \leq |R_i| \leq |R_j| \leq |R_k|$, so $|R_i| = |R_k|$ and in fact $R_i \prec R_k$ as well.
3. Because $M_{j,k} = 0$, by definition of nesting we must have $M_{i,j} + M_{i,k} = |R_i| + M_{i,k} \leq |R_i|$, so we must have $M_{i,k} = 0$.
4. Because the conditions $R_j \succ R_x$ and $R_1 \succ R_y$ imply that $|R_x| \geq 2$ and $|R_y| \geq 2$, then we would have $M_{x,y} = 1 < \min\{|R_x|, |R_y|\}$, contradicting that λ is nesting. \square

Proposition 5.53. Let $\lambda = (R_1, \dots, R_n)$ be a nesting minimal noncommuting path with $l(R_1) < l(R_n)$, $M_{1,n} = 0$, and $R_1 \leftrightarrow R_n$. Then we have $l(R_{t+1}) = r(R_t) + 1$ for every $1 \leq t \leq n - 1$.

Example 5.54. Informally, Proposition 5.53 states that if λ is nesting, a minimal noncommuting path must look like the example below.



Proof of Proposition 5.53. If $l(R_{n-1}) > l(R_n)$, then because $l(R_1) < l(R_n)$, we must have $l(R_{t-1}) < l(R_t)$ for some maximal $2 \leq t \leq n - 1$. But then we must have $R_{t-1} \succ R_{t+1}$ by Proposition 5.28 and therefore $R_1 \prec R_{t+1}$ by Proposition 5.7, Part 1, which is impossible by Proposition 5.2 because $l(R_1) < l(R_n) \leq l(R_{t+1})$. Therefore, we must have $l(R_{n-1}) < l(R_n)$, and more specifically, because $M_{n-1,n} < \min\{|R_{n-1}|, |R_n|\}$ by Corollary 3.13, Part 3 and because λ is nesting, we must have $M_{n-1,n} = 0$ and $r(R_{n-1}) = l(R_n) - 1 > r(R_1)$. Similarly,

by rotating, we must have $l(R_2) = r(R_1) + 1 < l(R_n)$. Note that if $n = 3$, then this proves our claim, so we now use induction on $n \geq 4$.

If $l(R_1) < l(R_{n-1})$, then because $R_1 \leftrightarrow R_{n-1}$ by minimality of the noncommuting path and therefore $M_{1,n-1} = 0$ by Proposition 3.10, we have by our induction hypothesis that $l(R_{t+1}) = r(R_t) + 1$ for every $1 \leq t \leq n-2$, which proves our claim. Similarly, by rotating, we are done if $r(R_2) < r(R_n)$. Therefore, the only remaining case to consider is when $l(R_{n-1}) \leq l(R_1)$ and $r(R_2) \geq r(R_n)$. However, we would have $R_1 \prec R_{n-1}$ and $R_n \prec R_2$ by Proposition 5.2, and

$$l(R_{n-1}) \leq l(R_1) \leq r(R_1) + 1 = l(R_2) \leq r(R_{n-1}) = l(R_n) - 1 \leq r(R_n) \leq r(R_2), \quad (5.48)$$

and therefore $M_{2,n-1} > 0$ by Proposition 3.10 Parts 2 and 3. Because λ is nesting, we must have either $R_2 \prec R_{n-1}$ or $R_{n-1} \prec R_2$, but if $R_2 \prec R_{n-1}$, we have $M_{2,n-1} + M_{2,n} = |R_2| + |R_n| \geq |R_2| + 1$, and if $R_{n-1} \prec R_2$, we have $M_{1,n-1} + M_{2,n-1} = |R_1| + |R_{n-1}| \geq |R_{n-1}| + 1$, a contradiction in either case. This completes the proof. \square

We are at long last ready to prove Lemma 3.23, which implies Theorem 2.7. The strategy is as follows. We will use commuting and cycling to rewrite λ so that the bottom two rows are in the desired form. If the corresponding two rows S_i and S_j of μ do not commute, then we are done by Corollary 5.19, and if there is no minimal noncommuting path between them, then we can again use commuting and cycling to bring them closer together until they do not commute. If there is a minimal noncommuting path $(S_i = S_{i_1}, \dots, S_{i_k} = S_j)$, then because we may assume that μ is nesting, Proposition 5.53 specifies the structure of these rows. In this case, we will use an argument similar to that of the proof of Lemma 5.42 to locally rotate pairs of rows $(S_{i_{t-1}}, S_{i_t})$ of μ to again bring S_j toward S_i until they do not commute.

Proof of Lemma 3.23. We first note that by Corollary 5.48 and Corollary 5.51, we may assume that λ is nesting and has no strict pairs or strict sequences. We will first show that there is a similar horizontal-strip $\lambda' = (R'_1, \dots, R'_n) \in \mathcal{S}(\lambda)$ with $l(R'_1) < l(R'_2)$ and $R'_1 \leftrightarrow R'_2$. Because $n(\lambda) - M(\lambda) \geq 1$, by (3.7) we have $M_{i,j}(\lambda) < \min\{|R_i|, |R_j|\}$ for some $1 \leq i < j \leq n$, and because λ is nesting, we must in fact have $M_{i,j}(\lambda) = 0$. By rotating and cycling, we may assume without loss of generality that $i = 1$ and $l(R_1) < l(R_j)$, and then $l(R_j) \geq r(R_1) + 1$ by Proposition 3.10, Parts 2 and 3.

Suppose that $l(R_j) - r(R_1) = 1$, so that $R_i \leftrightarrow R_j$. If there is a minimal noncommuting path in λ from R_1 to R_j , then by Proposition 5.37, either λ has a strict sequence, contradicting our assumption, or there is $1 < x < y < j$ with $l(R_x) = l(R_j) = r(R_i) - 1 = r(R_y) - 1$, $R_j \succ R_x$, and $R_1 \succ R_y$, contradicting that λ is nesting by Proposition 5.52, Part 4. Therefore, there is no minimal noncommuting path in λ from R_1 to R_j , so by Lemma 5.5 we can find our desired horizontal-strip λ' . We now suppose that $l(R_j) - r(R_1) \geq 2$, which means that $R_1 \leftrightarrow R_j$ by Proposition 3.10, Part 1, and we use induction on $l(R_j) - r(R_1)$.

If there is a minimal noncommuting path $(R_1 = R_{i_1}, \dots, R_{i_k} = R_j)$ in λ from R_1 to R_j , then because $R_1 \leftrightarrow R_j$ we have by Proposition 5.53 that $l(R_{i_2}) - r(R_1) = 1$ and we can repeat the above argument with the rows R_1 and R_{i_2} . If there is no minimal noncommuting path in λ from R_1 to R_j , then by commuting and cycling we have $(R_1, \dots, R_{j-1}, R_{j+1}, \dots, R_n, R_j^-) \in \mathcal{S}(\lambda)$ and $l(R_j^-) - r(R_1) < l(R_j) - r(R_1)$, so we are done by our induction hypothesis on $l(R_j) - r(R_1)$.

Therefore, there is indeed a horizontal-strip $\lambda' = (R'_1, \dots, R'_n) \in \mathcal{S}(\lambda)$ with $l(R'_1) < l(R'_2)$ and $R'_1 \leftrightarrow R'_2$.

We will now strengthen the conditions on our choice of λ' . Consider the set

$$\mathcal{S}^*(\lambda) = \{\lambda' = (R'_1, \dots, R'_n) \in \mathcal{S}(\lambda) : l(R'_1) < l(R'_2), R'_1 \leftrightarrow R'_2\} \tag{5.49}$$

and for $\lambda' = (R'_1, \dots, R'_n) \in \mathcal{S}^*(\lambda)$, define the integer

$$h(\lambda') = |\{3 \leq t \leq n : R'_1 \prec R'_t, R'_2 \prec R'_t\}|. \tag{5.50}$$

Because we have shown that the set $\mathcal{S}^*(\lambda)$ is nonempty and because we have a uniform bound $h(\lambda') \leq n - 2$, we may let $\lambda' = (R'_1, \dots, R'_n) \in \mathcal{S}^*(\lambda)$ be such that $h(\lambda')$ is maximal, and among those, with $|R'_2|$ maximal. Let $\mu = (S_1, \dots, S_n)$ and $\varphi : \Pi(\lambda') \xrightarrow{\sim} \Pi(\mu)$, and note that again by Corollary 5.48 and Corollary 5.51 we may assume that μ is nesting and has no strict pairs or strict sequences.

Let $i = \varphi_1$ and $j = \varphi_2$, and note that by cycling and rotating we may assume that $i < j$ and $l(S_i) < l(S_j)$, and then because $M_{i,j}(\mu) = 0$, we have $l(S_j) \geq r(S_i) + 1$ by Proposition 3.10, Parts 2 and 3. If $l(S_j) - r(S_i) = 1$, then $S_i \leftrightarrow S_j$ by Proposition 3.10, Part 3, so there exists a good substitute for (λ, μ) by Corollary 5.19 and we would be done. We now suppose that $l(S_j) - r(S_i) \geq 2$, which means that $S_i \leftrightarrow S_j$ by Proposition 3.10, Part 1, and we use induction on $l(S_j) - r(S_i)$. If there is no minimal noncommuting path in μ from S_i to S_j , then by commuting and cycling we have $(S_i, \dots, S_{j-1}, S_{j+1}, \dots, S_n, S_1^-, \dots, S_j^-) \in \mathcal{S}(\mu)$ and $l(S_j^-) - r(S_i) < l(S_j) - r(S_i)$, so we are done by our induction hypothesis on $l(S_j) - r(S_i)$. Therefore, we may assume that there is a minimal noncommuting path $(S_i = S_{i_1}, \dots, S_{i_k} = S_j)$ in μ from S_i to S_j . Because μ is nesting and $S_i \leftrightarrow S_j$, we have that $l(S_{i_{t+1}}) = r(S_{i_t}) + 1$ for every $1 \leq t \leq k - 1$ by Proposition 5.53. Also note that if there is a minimal noncommuting path in μ from S_{i_t} to $S_{i_{t+1}}$ for any $1 \leq t \leq k - 1$, then by Proposition 5.37 either μ has a strict sequence, contradicting our assumption, or there is $1 < x < y < j$ with $l(S_x) = l(S_j) = r(S_1) - 1 = r(S_y) - 1$, $S_j \succcurlyeq S_x$, and $S_1 \succcurlyeq S_y$, contradicting that μ is nesting by Proposition 5.52, Part 4. Therefore, there is no minimal noncommuting path in μ from S_{i_t} to $S_{i_{t+1}}$ for any $1 \leq t \leq k - 1$,

We now make the following useful observation. For every row R'_t of λ' with $R'_1 \prec R'_t$ and $R'_2 \prec R'_t$, we have $S_i \prec S_{\varphi_t}$ and $S_j \prec S_{\varphi_t}$ and therefore by Proposition 5.2 we have

$$l(S_{\varphi_t}) \leq l(S_1) + 1 \leq r(S_1) + 1 = l(S_{i_2}) \leq r(S_{i_{k-1}}) = l(S_j) - 1 \leq r(S_j) - 1 \leq r(S_{\varphi_t}) \tag{5.51}$$

and therefore $S_{i_{t'}} \prec S_{\varphi_t}$ for every $1 \leq t' \leq k$.

Now suppose that there is some t with $S_{i_{k-1}} \succcurlyeq S_t$ and $M_{j,t}(\mu) = 0$. We must have $r(S_t) \leq l(S_j) - 1 = r(S_{i_{k-1}})$ and by Proposition 5.2, $l(S_t) \leq l(S_{i_{k-1}}) - 1 = r(S_{i_{k-2}})$, so $M_{i_{k-2},t}(\mu) > 0$. Because λ is nesting, this means that either $S_{i_{k-2}} \prec S_t$ or $S_t \prec S_{i_{k-2}}$, but if $S_t \prec S_{i_{k-2}}$ we would have $M_{i_{k-2},t}(\mu) + M_{i_{k-1},t}(\mu) = |S_t| + |S_{i_{k-1}}| \geq |S_t| + 1$ and the pair $(S_{i_{k-2}}, S_{i_{k-1}})$ would be strict, a contradiction by Proposition 5.52, Part 1. Therefore, we must have $S_{i_{k-2}} \prec S_t$. Because there is no minimal noncommuting path in μ from $S_{i_{k-2}}$ to $S_{i_{k-1}}$, we may cycle and use Lemma 5.5 to replace μ with a similar horizontal-strip to assume

that $i_{k-2} = 1$ and $i_{k-1} = 2$, but now we would have $h(\boldsymbol{\mu}) > h(\boldsymbol{\lambda}')$ because S_t is counted only by $h(\boldsymbol{\mu})$, contradicting maximality of $h(\boldsymbol{\lambda}')$. Therefore, we may assume that

$$\text{there is no } t \text{ with } S_{i_{k-1}} \not\prec S_t \text{ and } M_{j,t}(\boldsymbol{\mu}) = 0. \quad (5.52)$$

Similarly, now suppose that there is some t with $S_j \not\prec S_t$ and $M_{i_{k-1},t}(\boldsymbol{\mu}) = 0$. Because there is no minimal noncommuting path in $\boldsymbol{\mu}$ from $S_{i_{k-1}}$ to S_j , we may cycle and use Lemma 5.5 to replace $\boldsymbol{\mu}$ with a similar horizontal-strip to assume that $i_{k-1} = 1$ and $j = 2$. Because $S_j \not\prec S_t$ and $M_{i_{k-1},t}(\boldsymbol{\mu}) = 0$, by Proposition 5.2, we have $l(S_t) = l(S_j)$ and $|S_t| > |S_j|$. Now if there is a minimal noncommuting path in $\boldsymbol{\mu}$ from S_1 to S_t , then as before, by Proposition 5.37 either $\boldsymbol{\mu}$ has a strict sequence, contradicting our assumption, or we contradict that $\boldsymbol{\mu}$ is nesting. Therefore, there is no minimal noncommuting path in $\boldsymbol{\mu}$ from S_1 to S_t , so by Lemma 5.5 we may replace $\boldsymbol{\mu}$ by a similar horizontal-strip to instead assume that $t = 2$. We also note that for every row $S_{t'}$ of $\boldsymbol{\mu}$ with $S_1 \prec S_{t'}$ and $S_j \prec S_{t'}$, we must have $M_{t,t'}(\boldsymbol{\mu}) > 0$ and therefore either $S_t \prec S_{t'}$ or $S_{t'} \prec S_t$. However, if $S_{t'} \prec S_t$, then we would have $M_{1,t'}(\boldsymbol{\mu}) + M_{t,t'}(\boldsymbol{\mu}) = |S_1| + |S_{t'}| \geq |S_{t'}| + 1$ and the pair (S_1, S_t) would be strict, a contradiction, and therefore $S_t \prec S_{t'}$. However, we now have $h(\boldsymbol{\mu}) \geq h(\boldsymbol{\lambda}')$ and $|S_2| > |S_j| = |R'_2|$, contradicting either the maximality of $h(\boldsymbol{\lambda}')$ or the maximality of $|R'_2|$. Therefore, we may assume that

$$\text{there is no } t \text{ with } S_j \not\prec S_t \text{ and } M_{i_{k-1},t}(\boldsymbol{\mu}) = 0. \quad (5.53)$$

Because there is no minimal noncommuting path in $\boldsymbol{\mu}$ from $S_{i_{k-1}}$ to S_j , we may use Lemma 5.5 to replace $\boldsymbol{\mu}$ by a similar horizontal-strip to assume that $j = i_{k-1} + 1$. Our plan is now to apply Lemma 5.39 to the rows S_{j-1} and S_j to replace $\boldsymbol{\mu}$ by a similar horizontal-strip for which $l(S_j)$ has decreased and $r(S_i)$ has not changed, so that we will be done by our induction hypothesis on $l(S_j) - r(S_i)$. It remains to check the conditions of Proposition 5.41.

1. If $M_{j-1,t}(\boldsymbol{\mu}) > 0$ and $M_{j,t}(\boldsymbol{\mu}) > 0$, then because $\boldsymbol{\mu}$ is nesting we must have $S_{j-1} \prec S_t$ or $S_t \prec S_{j-1}$. If $S_t \prec S_{j-1}$, then $M_{j-1,t}(\boldsymbol{\mu}) + M_{j,t}(\boldsymbol{\mu}) = |S_t| + M_{j,t}(\boldsymbol{\mu}) \geq |S_t| + 1$ and the pair (S_{j-1}, S_j) would be strict, contradicting that $\boldsymbol{\mu}$ is nesting. Therefore, we must have $S_{j-1} \prec S_t$ and similarly we must have $S_j \prec S_t$.
2. If $M_{j-1,t}(\boldsymbol{\mu}) > 0$ and $M_{j,t}(\boldsymbol{\mu}) = 0$, then because $\boldsymbol{\mu}$ is nesting we must have $S_{j-1} \prec S_t$ or $S_t \prec S_{j-1}$, but by (5.52) we cannot have $S_{j-1} \not\prec S_t$, so we must have $S_t \prec S_{j-1}$.
3. If $M_{j-1,t}(\boldsymbol{\mu}) = 0$ and $M_{j,t}(\boldsymbol{\mu}) > 0$, then because $\boldsymbol{\mu}$ is nesting we must have $S_j \prec S_t$ or $S_t \prec S_j$, but by (5.53) we cannot have $S_j \not\prec S_t$, so we must have $S_t \prec S_j$.
4. If $S_t \prec S_{j-1}$, then by Proposition 5.52, Part 3, we must have $M_{j,t}(\boldsymbol{\mu}) = 0$ and $M_{a,t}(\boldsymbol{\mu}) = 0$ for every $a \in A$. Moreover, if $b \in B$, then $S_{j-1} \prec S_b$ and therefore $S_t \prec S_b$ because by Proposition 5.52, Part 2, the relation \prec is transitive.
5. If $S_t \prec S_j$, then by Proposition 5.52, Part 3, we must have $M_{j-1,t}(\boldsymbol{\mu}) = 0$ and $M_{a,t}(\boldsymbol{\mu}) = 0$ for every $a \in A$. Moreover, if $b \in B$, then $S_j \prec S_b$ and therefore $S_t \prec S_b$ because by Proposition 5.52, Part 2, the relation \prec is transitive.

This concludes our verification of the conditions of Proposition 5.41. Therefore, the result follows by Lemma 5.39 and our induction hypothesis on $l(S_j) - r(S_i)$. This completes the proof of Lemma 3.23, and therefore our proof of Theorem 2.7. \square

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