UNAVOIDABLE ORDER-SIZE PAIRS IN HYPERGRAPHS — POSITIVE FORCING DENSITY

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Abstract. Erdős, Füredi, Rothschild and Sós initiated a study of classes of graphs that forbid every induced subgraph on a given number m of vertices and number f of edges. Extending their notation to r-graphs, we write $(n,e) \to_r (m,f)$ if every r-graph G on n vertices with e edges has an induced subgraph on m vertices and f edges. The *forcing density* of a pair (m,f) is

$$\sigma_r(m,f) = \limsup_{n\to\infty} \frac{|\{e : (n,e) \to_r (m,f)\}|}{\binom{n}{r}} .$$

In the graph setting it is known that there are infinitely many pairs (m,f) with positive forcing density. Weber asked if there is a pair of positive forcing density for $r \geqslant 3$ apart from the trivial ones $(m,0)$ and $(m,\binom{m}{r})$. Answering her question, we show that $(6,10)$ is such a pair for $r=3$ and conjecture that it is the unique such pair. Further, we find necessary conditions for a pair to have positive forcing density, supporting this conjecture.

Keywords. Induced hypergraphs, forcing density

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1. Introduction

The *Turán function* $\mathrm{ex}(n,H)$ is the maximum number of edges in an H-free n-vertex r-graph. The *Turán density* of H, denoted by $\pi(H)$, is defined as follows

$$\pi(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{r}}.$$

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Determining the Turán function for graphs and hypergraphs is a central topic in extremal graph theory with many challenging open problems, trying to identify what graph density forces the occurrence of a specific subgraph. Here, we are concerned with conditions on the graph density that forces the occurrence of an induced subgraph on a given number of vertices and a given number of edges, i.e., a given order-size pair. Erdős, Füredi, Rothschild and Sós [EFRS99] studied the class of graphs that does not contain a vertex subset of a given size m that spans exactly f edges. Given pairs of non-negative integers (n, e) and (m, f) we write

$$(n, e) \rightarrow_r (m, f)$$

if every r-graph G on n vertices and with e edges contains a vertex subset of a given size m that spans exactly f edges. The *forcing density* of a pair (m, f) is

$$\sigma_r(m, f) = \limsup_{n\to\infty} \frac{|\{e : (n, e) \rightarrow_r (m, f)\}|}{\binom{n}{r}}.$$

Erdős, Füredi, Rothschild and Sós [EFRS99] studied $\sigma_2(m, f)$ for different choices of (m, f). They showed that if $(m, f) \in \{(2, 0), (2, 1), (4, 3), (5, 4), (5, 6)\}$, then $\sigma_2(m, f) = 1$; otherwise, $\sigma_2(m, f) \leqslant \frac{2}{3}$. They also gave a construction that shows that for most pairs (m, f) we have $\sigma_2(m, f) = 0$. The upper bound $\frac{2}{3}$ was subsequently improved by He, Ma, and Zhao [HMZ23] to $\frac{1}{2}$. On the other hand, Erdős, Füredi, Rothschild and Sós [EFRS99] showed that there are infinitely many pairs of positive forcing density, in particular there are infinitely many pairs (m, f) with $\sigma_2(m, f) \geqslant \frac{1}{8}$. He, Ma, and Zhao [HMZ23] improved this result, by showing that there are infinitely many pairs (m, f) with $\sigma_2(m, f) \geqslant \frac{1}{2}$. Considering the hypergraph setting, Weber [Web24] showed that for any $r, m \in \mathbb{N}$, $r, m \geqslant 3$, all but at most $m^{\frac{r}{r-1}}$ of all possible $\binom{m}{r}$ pairs (m, f) satisfy $\sigma_r(m, f) = 0$.

Axenovich and Weber [AW24] asked whether there are pairs (m, f) for which not only $\sigma_r(m, f) = 0$, but a stronger statement holds. A pair (m, f) is *absolutely r-avoidable* if there is n_0 such that for each $n > n_0$ and for every $e \in \{0, \dots, \binom{n}{r}\}$, $(n, e) \not\rightarrow_r (m, f)$. In [AW24] it was shown that for $r = 2$ there are infinitely many absolutely avoidable pairs. Moreover, there is an infinite family of absolutely avoidable pairs of the form $(m, \binom{m}{2}/2)$ and for every sufficiently large m, there exists an f such that (m, f) is absolutely avoidable. In [Web24] this result was extended to higher uniformities to show that for every $r \geqslant 3$, there exists m_0 such that for every $m \geqslant m_0$ either $(m, \lfloor\binom{m}{r}/2\rfloor)$ or $(m, \lfloor\binom{m}{r}/2\rfloor - m - 1)$ is absolutely avoidable.

While there are many pairs (m, f) for which $\sigma_r(m, f) = 0$, not a single (non-trivial) pair with positive forcing density was known for r-graphs when $r \geqslant 3$. We denote by K_t^r the r-graph on t vertices where every r-set is an edge. Note that $\sigma_r(r, 1) = \sigma_r(r, 0) = 1$ and for $f = 0$, σ_r corresponds to the Turán density, i.e., $\sigma_r(m, 0) = \sigma_r(m, \binom{m}{r}) = \pi(K_m^r)$, where the best currently known general bounds on the Turán density are

$$1 - \left(\frac{r-1}{m-1}\right)^{r-1} \leqslant \pi(K_m^r) \leqslant 1 - \binom{m-1}{r-1}^{-1},$$

due to Sidorenko [Sid81] and de Caen [dC83]. Weber [Web24] asked whether for $m > r \geqslant 3$, there is any f with $0 < f < \binom{m}{r}$ such that $\sigma_r(m, f) > 0$ and suggested the pair $(6, 10)$ as a candidate when $r = 3$. We answer this question in the affirmative and prove $\sigma_3(6, 10) > 0$.

Given families of r-graphs $\mathcal{F}, \mathcal{G}$, we denote by $\mathrm{ex}(n, {}_{\mathrm{ind}}\mathcal{F}, \mathcal{G})$ the maximum number of edges in an n-vertex r-graph not containing any $F \in \mathcal{F}$ as an induced copy and also not any $G \in \mathcal{G}$ as a copy. Further, denote by $\pi({}_{\mathrm{ind}}\mathcal{F}, \mathcal{G})$ the limit

$$\pi({}_{\mathrm{ind}}\mathcal{F}, \mathcal{G}) = \limsup_{n\to\infty} \frac{\mathrm{ex}(n, {}_{\mathrm{ind}}\mathcal{F}, \mathcal{G})}{\binom{n}{r}}.$$

We mostly consider 3-graphs in this paper. When clear from context, we shall write abc for the set $\{a, b, c\}$ corresponding to an edge in a 3-graph. Denote by $[n] = \{1, 2, \dots, n\}$ the set of the first n integers. The 3-graph on vertex set $[4]$ with edge set $\{123, 124, 134\}$ is denoted by K_4^{3-}. Let $\mathcal{F}_6^{10}$ be the family of 6-vertex 3-graphs containing exactly 10 edges.

Theorem 1.1. *We have that* $\sigma_3(6, 10) = 1 - 2\pi({}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\})$. *Consequently,*

$$0.42622 \leqslant \sigma_3(6, 10) \leqslant 0.47106.$$

We do not know whether other pairs (m, f) with $m > 3$, $0 < f < \binom{m}{3}$ exist, such that $\sigma_3(m, f) > 0$. It seems plausible that for $r = 3$ there are indeed no other pairs with positive forcing density.

Conjecture 1.2. Let m and f be positive integers, $0 < f < \binom{m}{3}$. If $\sigma_3(m, f) > 0$, then $(m, f) = (6, 10)$.

The following result provides evidence for this conjecture to be true.

Theorem 1.3. *Let m and f be positive integers,* $0 < f < \binom{m}{3}$. *If* $\sigma_3(m, f) > 0$, *then there exist* $x_1, x_2, x_3 \in [m-1]$ *such that*

$$f = \binom{x_1}{3} = \binom{m}{3} - \binom{x_2}{3} = \binom{x_3}{3} + \binom{x_3}{2}(m - x_3). \tag{1.1}$$

Thus, in particular if there are no other non-trivial solutions except for $m = 6$, $x_1 = 5$, $x_2 = 5$, $x_3 = 3$, to the above Diophantine equation, then Conjecture 1.2 is true. A computer search for suitable solutions of (1.1) did not give a result for $m \leqslant 10^6$.

This paper is organized as follows: In Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.3. Finally, in Section 4 we make concluding remarks and state open problems.

2. Proof of Theorem 1.1

We say a 3-graph G *induces* $(6, 10)$ if G contains an induced copy of some $F \in \mathcal{F}_6^{10}$. If G does not contain any $F \in \mathcal{F}_6^{10}$ as an induced copy, we say G is $(6, 10)$-free, i.e., a 3-graph is $(6, 10)$-free if no 6-vertex set induces exactly 10 edges.

2.1. Proof idea

Before proving Theorem 1.1, we give a short sketch of the proof. We shall show that for every $\varepsilon > 0$ there is n_0 such that for every $n > n_0$ if G is an n-vertex 3-graph satisfying

$$\frac{e(G)}{\binom{n}{3}} \in \left[\pi({}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\}) + \varepsilon, 1 - \pi({}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\}) - \varepsilon\right], \tag{2.1}$$

then G induces $(6, 10)$. Then we first use a standard Ramsey type argument to partition most of the vertices of G into many large homogeneous sets. First, we rule out the case that there is a large clique and a large independent set that are disjoint. Thus, most of the vertex set of G or its complement G^c can be partitioned into large independent sets. Due to the symmetry of the problem, if we find a $(6, 10)$-set in G^c, we also find a $(6, 10)$-set in G. Thus, without loss of generality, we can assume that most of the vertices of G can be partitioned into many large independent sets. Using a classical supersaturation result and the density assumption on G, we find many copies of K_4^{3-} in G and thus, in particular, four large independent sets spanning many transversal copies of K_4^{3-}. Using a final cleaning argument, we find a $(6, 10)$-set in this substructure.

On the other hand, we fix an arbitrary 3-graph G on $\mathrm{ex}(n, {}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\})$ edges that is $(6, 10)$-free and K_4^{3-}-free. Then every set of 6 vertices spans at most 9 edges, so there is a graph on n vertices and e edges, for any $e \leqslant \mathrm{ex}(n, {}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\})$, that is $(6, 10)$-free. By taking complements, there also is a graph on n vertices and e edges for every $e \geqslant \binom{n}{3} - \mathrm{ex}(n, {}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\})$, that is $(6, 10)$-free.

2.2. Definitions, notations, and construction

An *independent set* in an r-graph is a vertex subset containing no edges. A *clique* in an r-graph is a vertex subset in which every r-set is an edge. A *homogeneous set* in an r-graph is a clique or an independent set.

Let G be a 3-graph and let $X, Y, Z \subseteq V(G)$, not necessarily disjoint from each other. Then, let $E_G(X, Y, Z) = \{(x, y, z) : \{x, y, z\} \in E(G), x \in X, y \in Y, z \in Z, x, y, z$ pairwise distinct$\}$. We say $E_G(X, Y, Z)$ is *complete* if $E_G(X, Y, Z) = \{(x, y, z) : x \in X, y \in Y, z \in Z, x, y, z$ pairwise distinct$\}$, and $E_G(X, Y, Z)$ is *empty* if $E_G(X, Y, Z) = \varnothing$. If the 3-graph G is clear from the context, we might omit the index and simply write $E(X, Y, Z)$. Given a set $S \subseteq V(G)$, the *induced subhypergraph* $G[S]$ is the r-graph whose vertex set is S and whose edge set consists of all of the edges in $E(G)$ that have all endpoints in S.

Let H be an r-graph and $t \in \mathbb{N}$. The *t-blow-up* of H, denoted by $H(t)$, is the r-graph with its vertex set partitioned in $|V(H)|$ sets $V_1, V_2, \ldots, V_{|V(H)|}$, each of size t and edge set $\{\{a_1, \ldots, a_r\} : a_j \in V_{i_j}, j = 1, \ldots, r, \{i_1, \ldots, i_r\} \in E(H)\}$. Informally, $H(t)$ is obtained from H by replacing each vertex i with an independent set V_i and each hyperedge e of H with a complete r-partite hypergraph with parts corresponding to the vertices of e.

We say that a 3-graph G is a *weak t-blowup* of H, which we also call *weak* $H(t)$, if the vertex set of G can be partitioned into $|V(H)|$ sets $V_1, V_2, \ldots, V_{|V(H)|}$ each of size t such that if $ijk \in E(H)$ then for every $a \in V_i, b \in V_j, c \in V_k$ we have $abc \in E(G)$, and if $ijk \notin E(H)$ then for every $a \in V_i, b \in V_j, c \in V_k$ we have $abc \notin E(G)$. Moreover, V_i is an independent set for $i = 1, \ldots, |V(H)|$. Note that we do not impose any condition on 3-tuples of vertices with exactly two vertices in some part V_i.

Denote by $r_r(t,t)$ the *Ramsey number* of K_t^r versus K_t^r, i.e., the minimum number of vertices m such that every 2-coloring of the edges of K_m^r contains a monochromatic K_t^r. Erdős, Hajnal and Rado [EHR65] showed that there exists a constant $c > 0$ such that $r_3(t,t) < 2^{2^{ct}}$.

Next, we shall provide a construction of a $(6,10)$-free graph that we shall use to provide an upper bound in Theorem 1.1.

2.2.1 Construction of the 3-graph H_n^{it}

Let H be the 3-graph with vertex set $[6]$ and edges 123, 124, 345, 346, 561, 562, 135, 146, and 236. Note that adding the edge 245 to H results in a 5-regular 3-graph on 6 vertices, which is K_4^{3-}-free and the basis for the construction for the lower bound on $\pi(K_4^{3-})$ by Frankl and Füredi [FF84].

We define the following iterated unbalanced blow-up of this graph. Denote by H_n the 3-graph on n vertices where the vertex set is partitioned into six sets $A_1, A_2, A_3, A_4, A_5, A_6$, where

$$|A_1| = |A_3| = |A_6| = \left\lceil \frac{n}{3\sqrt{3}} \right\rceil, \quad |A_2| = |A_4| = \left\lceil n\left(\frac{1}{3} - \frac{1}{3\sqrt{3}}\right)\right\rceil$$
$$\text{and} \quad |A_5| = n\left(\frac{1}{3} - \frac{1}{3\sqrt{3}}\right) + O(1).$$

The 3-graph H_n consists of all triples xyz, where $x \in A_i, y \in A_j$ and $z \in A_k$ and $ijk \in E(H)$. Now, let H_n^{it} be the 3-graph constructed from H_n by iteratively adding a copy of $H_{|A_i|}$ with vertex set A_i for all $i \in [6]$ if $|A_i| \geqslant 100$.

Lemma 2.1. *The graph H_n^{it} is an n-vertex 3-graph with $\frac{4}{3+7\sqrt{3}}\binom{n}{3} + o(n^3)$ edges such that every 6 vertices in H_n^{it} induce at most 9 edges. In particular, H_n^{it} is $(6,10)$-free.*

We present the proof of this lemma in the appendix.

2.3. Lemmas

The following lemma shows that every sufficiently large 3-graph can be partitioned into many large homogeneous sets.

Lemma 2.2. *Let $t > 0$. Then there exists $n_0 = n_0(t)$ such that for every $n \geqslant n_0$, if G is an n-vertex 3-graph, then G or G^c contains at least $n/t - \sqrt{n}$ pairwise disjoint homogeneous sets of size t.*

Proof. Let $t > 0$ be fixed. Set $n_0 = (\lceil 2^{2^{ct}} \rceil)^2$ and let $n \geqslant n_0$. Let $G = G_0$ be an n-vertex 3-graph. Since $n \geqslant r_3(t,t)$, there exists a homogeneous set of size t in G. Call it D_0 and define $G_1 = G_0 \setminus D_0$. We iteratively repeat this process. Define $G_{i+1} := G_i \setminus D_i$, where D_i is a homogeneous set of size t in G_i. We can proceed as long as $|V(G_i)| > r_3(t,t)$. Since $r_3(t,t) \leqslant \lceil 2^{2^{ct}} \rceil \leqslant \sqrt{n_0} \leqslant \sqrt{n}$, we have found at least $(n - \sqrt{n})/t \geqslant n/t - \sqrt{n}$ pairwise disjoint homogeneous sets of size t each. □

The following Lemma analyses the structure "between" two large vertex sets. This is partly motivated by a result by Fox and Sudakov [FS08] for 2-graphs.

Lemma 2.3. *Let $t \geqslant 0$. Then there exists n_0 such that for all $n \geqslant n_0$ the following holds. Let G be a 3-graph with vertex set $V(G) = A \cup B$ with $A \cap B = \varnothing$, $|A| = |B| = n$. Then there exist sets $A' \subseteq A$, $B' \subseteq B$ with $|A'| = |B'| = t$ such that each of the edge sets $E(A', A', B')$ and $E(A', B', B')$ is either empty or complete.*

Proof. Let $m = \underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4^t}}}}}}_{2t}$, let $n_0 = \underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4^{2t-1}}}}}}}_{m}$. Let A and B be sets of size $n \geqslant n_0$. For $a \in A, X \subseteq B$ we define an auxiliary 2-graph $G_a^X = (X, \binom{X}{2})$ and an edge-coloring $c_a^X : E(G_a^X) \to \{r, b\}$ with $c_a^X(\{b_1, b_2\}) = \begin{cases} r, & \{a, b_1, b_2\} \in E(G), \\ b, & \text{else}. \end{cases}$
Note that by the standard bound on the diagonal Ramsey number $r_2(s, s) \leqslant 4^s$, each 2-colored 2-clique on k vertices contains a monochromatic clique of size $\log_4(k)$.

Let $A = \{a_1, \ldots, a_n\}$, let $B_1 \subseteq B$ be the vertex set of a monochromatic clique in $G_{a_1}^B$ of size $\log_4(|B|)$. Now assume B_i, $i \geqslant 1$, has been chosen. Let $B_{i+1} \subseteq B_i$ be a monochromatic clique in $G_{a_{i+1}}^{B_i}$ of size $\log_4(|B_i|)$. Thus, after m iterations we obtain a set B_m of size $|B_m| = \underbrace{\log_4 \cdots \log_4}_{m}(n) \geqslant 2t - 1$, such that for each a_i, $i \in [m]$, the set $E(\{a_i\}, B_m, B_m)$ is either empty or complete. Thus, there exists a subset $A'' \subset A$, $|A''| = \lceil \frac{m}{2} \rceil \geqslant \underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4^t}}}}}}_{2t-1}$, such that the set $E(A'', B_m, B_m)$ is either empty or complete.

Now we repeat this process with vertices in $B'' = B_m$, to obtain a subset $A' \subseteq A''$, $|A'| = \underbrace{\log_4 \cdots \log_4}_{|B''|}(|A''|) \geqslant t$, such that for each vertex $b \in B''$, the set $E(A', A', \{b\})$ is either empty or complete. Thus, there exists a subset $B' \subseteq B''$, $|B'| \geqslant \left\lceil \frac{|B''|}{2} \right\rceil = t$ such that the set $E(A', A', B')$ is either empty or complete. The sets A', B' satisfy the conditions of the lemma, completing the proof. □

The next lemma shows that in a $(6, 10)$-free 3-graph there cannot be a large independent set and a large clique that are disjoint.

Lemma 2.4. *There exists $t_0 > 0$ such that for all $t \geqslant t_0$ the following holds. Let G be a $2t$-vertex 3-graph with vertex set $V(G) = A \cup B$ where $A \cap B = \varnothing$, $|A| = |B| = t$, $G[A]$ is a clique and $G[B]$ is an independent set. Then G induces $(6, 10)$.*

Proof. By Lemma 2.3, for sufficiently large t, we can find subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'| = |B'| = 5$ such that the two sets $E(A', A', B')$ and $E(A', B', B')$ are either empty or complete.

If $E(A', B', B')$ is complete, then any vertex from A' together with the 5 vertices from B' induces $(6, 10)$. If $E(A', A', B')$ is empty, then any vertex from B' together with the five vertices from A' induces $(6, 10)$. Hence, we may assume that $E(A', B', B')$ is empty and $E(A', A', B')$ is complete. But then three arbitrary vertices from A' together with three arbitrary vertices from B' induce $(6, 10)$. ☐

Lemma 2.2 together with Lemma 2.4 immediately implies the following lemma.

Lemma 2.5. *There exists t_0 such for all $t \geqslant t_0$ the following holds. There exists $n_0 = n_0(t)$ such that for all $n \geqslant n_0$, if G is a $(6, 10)$-free n-vertex 3-graph, then either G or G^c contains at least $n/t - \sqrt{n}$ pairwise disjoint independent sets of size t.*

Lemma 2.6. *Let $t' > 0$. Then there exists $t_0 > 0$ such that for all $t \geqslant t_0$ the following holds. Let G be a $(6, 10)$-free $2t$-vertex 3-graph with vertex set $V(G) = A \cup B$ where $|A| = |B| = t$, $A \cap B = \varnothing$, $G[A]$ and $G[B]$ are independent sets. Then there exists $A' \subseteq A$, $B' \subseteq B$ of sizes $|A'| = |B'| = t'$ such that the two sets $E(A', B', B')$ and $E(A', A', B')$ are empty.*

Proof. We apply Lemma 2.3 for t'. Then there exists t_0 such that for $t \geqslant t_0$, we find $A' \subseteq A$, $B' \subseteq B$, such that the two sets $E(A', A', B')$ and $E(A', B', B')$ are either empty or complete. Assume the set $E(A', A', B')$ is complete. Then we find induced $(6, 10)$ by taking any 5 vertices from A' and 1 vertex from B. By symmetry the same holds for the set $E(A', B', B')$, so in particular, $G[A' \cup B']$ is the empty graph. ☐

Lemma 2.7. *There exists $t_0 > 0$ such that for all $t \geqslant t_0$ a weak $K_4^3(t)$ induces $(6, 10)$, and also a weak $K_4^{3-}(t)$ induces $(6, 10)$.*

Proof. Let G be a weak $K_4^{3-}(t)$ with independent sets V_1, V_2, V_3, V_4. By iteratively applying Lemma 2.6 to all of the tuples (V_i, V_j), $1 \leqslant i < j \leqslant 4$, we obtain an induced copy $H \subseteq G$ of $K_4^{3-}(2)$ with sets X_1, X_2, X_3, X_4, $X_i \subset V_i$, $i \in [4]$, i.e., $H[X_i \cup X_j]$ is empty for all $i \neq j$, the sets $E(X_i, X_j, X_k)$ are complete for $\{i, j, k\} \in \binom{[4]}{3}$ except for $E(X_2, X_3, X_4)$, which is empty. Let $x_1, x_1' \in X_1$, $x_2, x_2' \in X_2$, $x_3 \in X_3$ and $x_4 \in X_4$. Then $\{x_1, x_1', x_2, x_2', x_3, x_4\}$ induces $(6, 10)$.

Now assume there is a weak $K_4^3(t)$ called G with independent sets V_1, V_2, V_3, V_4. By iteratively applying Lemma 2.6 to all of the tuples (V_i, V_j), $1 \leqslant i < j \leqslant 4$, we obtain an induced copy $H \subseteq G$ of $K_4^3(3)$ with sets X_1, X_2, X_3, X_4, $X_i \subset V_i$, $i \in [4]$, i.e., $H[X_i \cup X_j]$ is empty for all $i \neq j$ and the sets $E(X_i, X_j, X_k)$ are complete for all $\{i, j, k\} \in \binom{[4]}{3}$. Let $x_2 \in X_2, x_3 \in X_3, x_4 \in X_4$. Then $H[X_1 \cup \{x_2, x_3, x_4\}]$ is a 6-vertex 3-graph spanning exactly 10 edges. ☐

Lemma 2.8. *Let $t > 0$ be an integer and $\delta > 0$. Then there exists $m_0 = m_0(t, \delta)$ such that for all $m \geqslant m_0$ the following holds. Let G be a 3-graph on $4m$ vertices such that the vertex set of G can be partitioned into four independent sets V_1, V_2, V_3, V_4 of size m each and the number of copies of K_4^{3-} with one endpoint from each of the V_i's is at least δm^4. Then G contains an induced copy of a weak $K_4^3(t)$ or a weak $K_4^{3-}(t)$.*

Proof. Define the auxiliary 4-graph H on $4m$ vertices where a 4-set spans an edge iff the corresponding four vertices in G form a copy of K_4^{3-}. We 5-color the edges of H in the following way: An edge $\{v_1, v_2, v_3, v_4\}$ of H with $v_i \in V_i$ for $i \in [4]$ is colored with $j \in [4]$ if $\{v_1, v_2, v_3, v_4\} \setminus \{v_j\}$ is not an edge in G, and it is colored with color 5 if $\{v_1, v_2, v_3, v_4\}$ induces a K_4^3 in G.

By pigeonhole principle, there exists $(\delta/5)m^4$ edges of the same color. Erdős [Erd64] proved that $\pi(K_4^4(t)) = 0$ and thus, there exists a monochromatic copy of $K_4^4(t)$ in H. Denote by T the vertex set of this monochromatic copy. The 3-graph $G[T]$ is a weak $K_4^3(t)$ or weak $K_4^{3-}(t)$. □

We will use a supersaturation result discovered by Erdős and Simonovits [ES83]. The proof presented below follows a proof given by Keevash (Lemma 2.1. in [Kee11]).

Lemma 2.9. *For $\varepsilon > 0$ and families $\mathcal{F}, \mathcal{G}$ of r-graphs, there exists constants $\delta > 0$ and $n_0 > 0$ so that if G is an r-graph on $n > n_0$ vertices with $e(G) > (\pi(_{\text{ind}}\mathcal{F}, \mathcal{G}) + \varepsilon)\binom{n}{r}$, then G contains at least $\delta\binom{n}{|V(H)|}$ copies of H for some $H \in \mathcal{G}$, or at least $\delta\binom{n}{|V(H)|}$ induced copies of H for some $H \in \mathcal{F}$.*

Proof. Let G be an r-graph on sufficiently many vertices n with $e(G) > (\pi(_{\text{ind}}\mathcal{F}, \mathcal{G}) + \varepsilon)\binom{n}{r}$. Fix an integer $k \geqslant r$, $k \geqslant |V(H)|$ for all $H \in \mathcal{F} \cup \mathcal{G}$ so that $\text{ex}(k, {}_{\text{ind}}\mathcal{F}, \mathcal{G}) \leqslant \left(\pi(_{\text{ind}}\mathcal{F}, \mathcal{G}) + \frac{\varepsilon}{2}\right)\binom{k}{r}$. There are at least $\frac{\varepsilon}{2}\binom{n}{k}$ k-sets $K \subseteq V(G)$ with $e(G[K]) > (\pi(_{\text{ind}}\mathcal{F}, \mathcal{G}) + \frac{\varepsilon}{2})\binom{k}{r}$. Otherwise, we would have

$$\sum_{\substack{K \subseteq V(G) \\ |K| = k}} e(G[K]) \leqslant \binom{n}{k}\left(\pi(_{\text{ind}}\mathcal{F}, \mathcal{G}) + \frac{\varepsilon}{2}\right)\binom{k}{r} + \frac{\varepsilon}{2}\binom{n}{k}\binom{k}{r}$$
$$= (\pi(_{\text{ind}}\mathcal{F}, \mathcal{G}) + \varepsilon)\binom{n}{k}\binom{k}{r},$$

but we also have

$$\sum_{\substack{K \subseteq V(G) \\ |K| = k}} e(G[K]) = \binom{n-r}{k-r} e(G) > \binom{n-r}{k-r}(\pi(_{\text{ind}}\mathcal{F}, \mathcal{G}) + \varepsilon)\binom{n}{r}$$
$$= (\pi(_{\text{ind}}\mathcal{F}, \mathcal{G}) + \varepsilon)\binom{n}{k}\binom{k}{r},$$

a contradiction. By the choice of k, each of these k-sets K contains an induced copy of some $H \in \mathcal{F}$ or a copy of some $H \in \mathcal{G}$. By the pigeonhole principle, there exists $H_1 \in \mathcal{F}$ such that at least $\frac{\varepsilon}{2(|\mathcal{F}|+|\mathcal{G}|)}\binom{n}{k}$ of these k-sets K contain an induced copy of H_1, or there exists $H_2 \in \mathcal{G}$ such that at least $\frac{\varepsilon}{2(|\mathcal{F}|+|\mathcal{G}|)}\binom{n}{k}$ of these k-sets K contain a copy of H_2. Thus, in the first case, the number of induced copies of H_1 is at least

$$\frac{\frac{\varepsilon}{2(|\mathcal{F}|+|\mathcal{G}|)}\binom{n}{k}}{\binom{n-|V(H_1)|}{k-|V(H_1)|}} = \delta\binom{n}{|V(H_1)|}, \quad \text{for} \quad \delta = \frac{\varepsilon}{2(|\mathcal{F}|+|\mathcal{G}|)\binom{k}{|V(H_1)|}}.$$

Similarly, in the second case, the number of copies of H_2 is at least

$$\delta\binom{n}{|V(H_2)|} \quad \text{for} \quad \delta = \frac{\varepsilon}{2(|\mathcal{F}|+|\mathcal{G}|)\binom{k}{|V(H_2)|}}.$$

□

2.4. Proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\varepsilon > 0$. Fix an integer t whose existence is guaranteed by Lemma 2.7, such that every weak $K_4^3(t)$ and also every weak $K_4^{3-}(t)$ induces $(6,10)$, see the paragraph before Lemma 2.7 for the definition of a weak blow-up. Fix $\delta > 0$ and $n_1 \in \mathbb{N}$, given by Lemma 2.9, such that every $(6,10)$-free 3-graph G on $n \geqslant n_1$ vertices satisfying $e(G) \geqslant (\pi({}_{\text{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\}) + \varepsilon)\binom{n}{3}$ contains at least $2\delta\binom{n}{4}$ copies of K_4^{3-}. Let $m_0 = m_0(t,\delta)$ be given by Lemma 2.8. Fix integers m_1 and n_2 whose existence is guaranteed by Lemma 2.5, such that $m_1 \geqslant m_0$ and for all $n \geqslant n_2$, if G is $(6,10)$-free n-vertex 3-graph, then either G or G^c contains at least $n/m_1 - \sqrt{n}$ pairwise disjoint independent sets of size m_1. Choose $n_0 := \max\{n_1, n_2, m_1^2, \lceil 40000\delta^{-2}\rceil\}$ and let $n \geqslant n_0$.

Let G be a $(6,10)$-free n-vertex 3-graph satisfying the density assumption (2.1):

$$\frac{e(G)}{\binom{n}{3}} \in \left[\pi({}_{\text{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\}) + \varepsilon, 1 - \pi({}_{\text{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\}) - \varepsilon\right].$$

By Lemma 2.5 either G or G^c contains at least $n' := n/m_1 - \sqrt{n}$ pairwise disjoint independent sets, each of size m_1. Since the density assumption is symmetric, and since G induces $(6,10)$ if and only if G^c induces $(6,10)$, we can assume, without loss of generality, that G contains at least n' pairwise disjoint independent sets $V_1, V_2, \ldots, V_{n'}$ of size m_1 each.

By Lemma 2.9, G contains at least $2\delta\binom{n}{4}$ (not necessarily induced) copies of K_4^{3-}. We call a 4-set *transversal* in G if each of the four vertices is in a different V_i. A copy of K_4^{3-} in G is called *transversal* if the vertex set of the copy is transversal in G. The number of 4-sets which are not transversal in G is at most

$$\sqrt{n}n^3 + n'\binom{m_1}{2}n^2 \leqslant n^{\frac{7}{2}} + m_1 n^3 \leqslant 2n^{\frac{7}{2}},$$

for $n \geqslant m_1^2$. The number of transversal copies of K_4^{3-} in G is at least $\frac{3}{2}\delta\binom{n}{4}$, since

$$2\delta\binom{n}{4} - \frac{3}{2}\delta\binom{n}{4} = \frac{\delta}{2}\binom{n}{4} \geqslant \frac{\delta}{2}\frac{n^4}{2\cdot 4!} = \frac{\delta}{96}n^4 > 2n^{7/2},$$

where the last inequality holds for $n \geqslant 40000\delta^{-2}$. By pigeonhole principle there exist $1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant n'$, such that the number of copies of K_4^{3-} with one endpoint in each of $V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}$ is at least

$$\frac{\frac{3}{2}\delta\binom{n}{4}}{\binom{n'}{4}} \geqslant \frac{\delta\frac{n^4}{4!}}{\frac{\left(\frac{n}{m_1}\right)^4}{4!}} = \delta m_1^4.$$

By Lemma 2.8, the 3-graph $G[V_{i_1} \cup V_{i_2} \cup V_{i_3} \cup V_{i_4}]$ contains a weak $K_4^{3-}(t)$ or a weak $K_4^3(t)$ as an induced subhypergraph. This contradicts Lemma 2.7.

We conclude $\sigma_3(6,10) \geqslant 1 - 2\pi({}_{\text{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\})$. In fact, by the following argument, $\sigma_3(6,10) = 1 - 2\pi({}_{\text{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\})$ holds: Let G be an n-vertex K_4^{3-}-free and $(6,10)$-free

3-graph with exactly $\mathrm{ex}(n, {}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\})$ many edges. Since G is K_4^{3-}-free, every four vertices span at most 2 edges, so using double counting, we see that every 6 vertices span at most $\binom{6}{4} \cdot 2/3 = 10$ edges. Since G is also $(6, 10)$-free, every 6 vertices span only at most 9 edges. We conclude that every subgraph $G' \subseteq G$ is $(6, 10)$-free. Further, by symmetry, also the complement 3-graph of any $G' \subseteq G$ is $(6, 10)$-free. This proves the first part of the theorem.

To get specific numerical bounds on the forcing density, recall again that if

$$\frac{e(G)}{\binom{n}{3}} \in \left[\pi({}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\}) + \varepsilon, 1 - \pi({}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\}) - \varepsilon\right],$$

then G induces $(6, 10)$. In particular, if $\frac{e(G)}{\binom{n}{3}} \in \left[\pi(K_4^{3-}) + \varepsilon, 1 - \pi(K_4^{3-}) - \varepsilon\right]$, then G induces $(6, 10)$. The Turán density of K_4^{3-} is not known precisely. The best currently known bounds on the Turán density of K_4^{3-} are $0.28571 \approx \frac{2}{7} \leqslant \pi(K_4^{3-}) \leqslant 0.28689$, where the lower bound construction was given by Frankl and Füredi [FF84]. The upper bound was proved by Vaughan [Vau] who applied the flag algebra method, see also the webpage of Lidický [Lid]. Thus $\sigma_3(6, 10) \geqslant 1 - 2 \cdot 0.28689 = 0.42622$. However, from Lemma 2.1, we have that there is a 3-graph on n vertices and $\frac{4}{3+7\sqrt{3}}\binom{n}{3}(1 + o(1))$ hyperedges, such that each of its subgraphs is $(6, 10)$-free. Moreover, the complement of this 3-graph has $\left(1 - \frac{4}{3+7\sqrt{3}}\binom{n}{3}\right)(1 + o(1))$ hyperedges and each of its supergraphs is $(6, 10)$-free. Thus $\sigma_3(6, 10) \leqslant 1 - 2\frac{4}{3+7\sqrt{3}} \leqslant 0.47106$. □

3. Proof of Theorem 1.3

3.1. Constructions and notations

We shall first construct a special class of 3-graphs.

Let $n, k \in \mathbb{N}$, $k \leqslant n$ and $S \subseteq [2]$. Let $G(S, n, k)$ be the 3-graph with vertex set $A \cup B$, $|A| = k$, $|B| = n-k$, where A and B are disjoint such that A induces a clique, B induces an independent and we have the additional edges $\bigcup_{i \in S} E_i$, where $E_i = \{A' \cup B' : A' \in \binom{A}{i}, B' \in \binom{B}{3-i}\}$. The independent set B we call *base independent set*. The 3-graph $G(\varnothing, n, k)$ is just a clique on k vertices and $n - k$ isolated vertices, and the 3-graph $G([2], n, k)$ is the complete graph on n vertices with a clique of size $n - k$ removed. For an illustration of $G(\{2\}, n, k)$ see Figure 3.1.

Let $f(S, n, k) = |E(G(S, n, k))|$. We call a 3-graph G *m-sparse* if every subset of size at most m vertices in G induces at most m edges. For a fixed choice of m, we say that a 3-graph G is *canonical plus with parameters* (S, n, k), or simply *canonical plus*, if G is a 3-graph obtained as a union of $G(S, n, k)$ and an m-sparse graph whose vertex set is the base independent set B of $G(S, n, k)$. For a fixed choice of m, a 3-graph G is *canonical minus with parameters* (S, n, k), or simply *canonical minus*, if G is a 3-graph obtained from $G(S, n, k)$ by removing the edges of a copy of an m-sparse graph from the clique A of $G(S, n, k)$. We see (letting $\binom{y}{x} = 0$ for $y < x$), that

$$f(S, n, k) = \binom{k}{3} + \sum_{i \in S} \binom{k}{i}\binom{n-k}{3-i}.$$

Moreover, $|f(S, n, x) - f(S, n, x-1)| = O(n^2)$. Note that any induced subgraph of a canonical plus 3-graph with parameters (S, n, k) is a canonical plus 3-graph with parameters (S, n', k'),

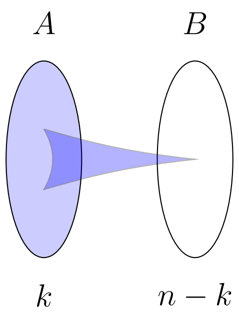

Figure 3.1: Illustration of $G(\{2\}, n, k)$.

for some n' and k'. A similar statement holds for canonical minus graphs. Thus, these two classes of graphs are hereditary.

We see that if an m-vertex 3-graph is canonical plus with parameters (S, m, x), then the number of edges in such a graph is in the interval $[f(S, m, x), f(S, m, x) + m]$. Similarly, the number of edges in a canonical minus graph with parameters (S, m, x) is in the interval $[f(S, m, x) - m, f(S, m, x)]$. Thus, if f is the number of edges of a canonical plus graph and also of a canonical minus graph, with first parameter S and m vertices, then $f \in F(S, m)$, where

$$F(S,m) = \bigcup_{x=0}^{m-1} [f(S,m,x), f(S,m,x)+m] \cap \bigcup_{x=1}^{m} [f(S,m,x)-m, f(S,m,x)] \subseteq \left\{0, 1, \ldots, \binom{m}{3}\right\}.$$

3.2. Proof idea

We are using the following general principle:

Proposition 3.1. *Let m and f be positive integers, $0 < f < \binom{m}{r}$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be classes of r-graphs such that for every $0 < c < 1/2$ there exists $n_0 \in \mathbb{N}$ and for every $n \geqslant n_0$ the following holds:*

- *for every e with $c\binom{n}{r} \leqslant e \leqslant (1-c)\binom{n}{r}$, there is a graph $G_i \in \mathcal{C}_i$ on n vertices and e edges for all $i \in [k]$, and*
- *for some $i \in [k]$, each n-vertex graph in $\mathcal{C}_i$ is (m, f)-free.*

Then $\sigma_r(m, f) = 0$.

Here, we use two classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of 3-graphs that are canonical plus and canonical minus with the same first parameter S. Specifically, the main idea of the proof of Theorem 1.3 is that for any sufficiently large n, any $S \subseteq [2]$, and any e in the interval $[c\binom{n}{3}, (1-c)\binom{n}{3}]$ for $0 < c < 1/2$, there is a canonical plus 3-graph $G_1(S, n, e)$ and a canonical minus 3-graph $G_2(S, n, e)$ with first parameter S, on n vertices and e edges. If, for a pair (m, f), $f \notin F(S, m)$ for some $S \subseteq [2]$, then the pair (m, f) is not representable as a canonical plus or canonical minus graph with first parameter S. Then in particular, $G_1(S, n, e)$ or $G_2(S, n, e)$ is (m, f)-free and $(n, e) \not\to (m, f)$. Letting c be arbitrarily small, we conclude that $\sigma_3(m, f) = 0$ for such a pair (m, f). Finally, we derive number theoretic conditions for a pair (m, f) not being representable by a canonical plus or a canonical minus graph.

3.3. Lemmas

In the following lemmas, n, m, f, e are non-negative integers with $m > 3$, $0 < f < \binom{m}{3}$. In [Web24] it was shown that for any $m \leqslant 15$ and for any $0 < f < \binom{m}{3}$ such that $(m, f) \neq (6, 10)$, $\sigma_3(m, f) = 0$. Thus, we can assume that $m \geqslant 16$. The following result can be obtained by a standard probabilistic argument.

Lemma 3.2. *Let $m > 0$. Then for any sufficiently large n there exists an n-vertex 3-graph with $\Omega(n^{2+\frac{1}{m+1}})$ edges which is m-sparse.*

For a proof of Lemma 3.2 see e.g. [Web24]. The next lemma is a generalization of a similar statement proven in [EFRS99] for graphs.

Lemma 3.3. *Let $S \subseteq [2]$ and c be a constant, $0 < c < 1/2$. For $n \in \mathbb{N}$ sufficiently large and any e where $c\binom{n}{3} < e < (1-c)\binom{n}{3}$, there exist 3-graphs $G_1(S, n, e)$ and $G_2(S, n, e)$ on n vertices and e edges that are canonical plus and canonical minus with first parameter S, respectively.*

Proof. Let n be a given sufficiently large integer. The function $f(S, n, k)$ might not be monotone. Yet, there exists a non-negative integer k such that $f(S, n, k) \leqslant e \leqslant f(S, n, k+1)$ holds, e.g. we can choose $k < n$ to be the largest integer such that $f(S, n, k) \leqslant e$ holds. This is possible as $f(S, n, 0) = 0$ and $f(S, n, n) = \binom{n}{3}$. Note that since $e \leqslant (1-c)\binom{n}{3}$ and $e \geqslant \binom{k}{3}$, we get $\binom{k}{3} \leqslant (1-c)\binom{n}{3}$ and thus, $k \leqslant c'n$, where $c' < 1$ is a constant.

Let G' be an m-sparse 3-graph on $n - k$ vertices with $|E(G')| \geqslant (n-k)^{2+\frac{1}{m+1}}$. The existence of G' is guaranteed by Lemma 3.2. Define G'' to be the 3-graph obtained as a union of $G(S, n, k)$ and a copy of G' on the vertex set that is the base independent set of $G(S, n, k)$. Then $|E(G'')| \geqslant f(S, n, k) + (n-k)^{2+\frac{1}{m+1}} \geqslant f(S, n, k+1) \geqslant e$. Here, the second inequality holds since $f(S, n, k+1) - f(S, n, k) = O(n^2)$. Finally, let $G_1(S, n, e)$ be a subgraph of G'' with e edges, obtained from G'' by removing some edges of G'.

For the second part of the lemma, we could apply a similar argument by removing an m-sparse graph from a clique; however, observe that $G_2(S, n, e)$ simply can be constructed as the complement of $G_1(S', n, \binom{n}{3} - e)$, where $S' = \begin{cases} [2] - S, & S \in \{\varnothing, [2]\} \\ S, & \text{else} \end{cases}$, whose existence is guaranteed by the first part of the lemma. □

Lemma 3.4. *Let $S \subseteq [2]$. If $f \notin F(S, m)$, then $\sigma_3(m, f) = 0$.*

Proof. Assume we have integers m, f as above, some $S \subseteq [2]$ and $f \notin F(S, m)$. Let c be a constant, $0 < c < 1/10$, $n \geqslant n_0$, and e be any integer satisfying $c\binom{n}{3} \leqslant e \leqslant (1-c)\binom{n}{3}$. Define graphs $G_1 = G_1(S,n,e)$ and $G_2 = G_2(S,n,e)$ whose existence is guaranteed by Lemma 3.3. Any induced subgraph of G_1 on m vertices is canonical plus with parameters (S,m,x) for some x and thus, its number of edges is in $\bigcup_{x=0}^{m-1}[f(S,m,x), f(S,m,x)+m]$. Any induced subgraph of G_2 on m vertices is canonical minus with parameters (S,m,x) for some x and thus, its number of edges is in $\bigcup_{x=1}^{m}[f(S,m,x)-m, f(S,m,x)]$. Since $f \notin F(S,m)$, we get that G_1 or G_2 is (m,f)-free. Letting c go to zero, we see that $\sigma_3(m,f)=0$. □

In the following lemmas we shall use the set $S=\varnothing$, $S=\{1\}$, or $S=\{2\}$, to claim that for many pairs (m,f), $\sigma_3(m,f)=0$.

Lemma 3.5. *Let* $m \geqslant 7$ *and* $0 < f < \binom{m-1}{2}$. *Then* $\sigma_3(m,f)=0$.

Proof. Let $S=\{1\}$. By Lemma 3.4, it is sufficient to verify that $f \notin F(\{1\},m)$. For that it is sufficient to check that $F(\{1\},m) \cap [1, \binom{m-1}{2}-1] = \varnothing$. Recall that

$$F(\{1\},m) = \bigcup_{x=0}^{m-1}[f(\{1\},m,x), f(\{1\},m,x)+m] \cap \bigcup_{x=1}^{m}[f(\{1\},m,x)-m, f(\{1\},m,x)].$$

Note that $f(\{1\},m,0)=0$, $f(\{1\},m,1)=\binom{m-1}{2}$, and $f(\{1\},m,x) \geqslant \binom{m-1}{2}$, for $x>1$. Thus, we have

$$\begin{aligned}&\bigcup_{x=0}^{m-1}[f(\{1\},m,x), f(\{1\},m,x)+m] \cap [1, \tbinom{m-1}{2}-1]\\ &= [f(\{1\},m,0), f(\{1\},m,0)+m] \cap [1, \tbinom{m-1}{2}-1] = [1,m],\end{aligned}$$

and

$$\begin{aligned}\bigcup_{x=1}^{m}[f(\{1\},m,x)-m, f(\{1\},m,x)] \cap [1, \tbinom{m-1}{2}-1] &= [f(\{1\},m,1)-m, f(\{1\},m,1)-1]\\ &= [\tbinom{m-1}{2}-m, \tbinom{m-1}{2}-1].\end{aligned}$$

In particular, we have

$$F(\{1\},m) \cap [1, \tbinom{m-1}{2}-1] = [1,m] \cap [\tbinom{m-1}{2}-m, \tbinom{m-1}{2}-1] = \varnothing,$$

where in the last step we used that $\binom{m-1}{2} > 2m$. Thus, $\sigma_3(m,f)=0$. □

Lemma 3.6. *Let* $m \geqslant 16$ *and let* f *be an integer such that* $\binom{m-1}{2} \leqslant f < \binom{m}{3}$ *and for any* $x \in [m]$, $f \neq \binom{x}{3}$. *Then* $\sigma_3(m,f)=0$.

Proof. Define f as given in the statement of the lemma and $S=\varnothing$. By Lemma 3.4, it is sufficient to prove that $f \notin F(\varnothing,m)$ and in particular it is sufficient to show that $F(\varnothing,m) \cap [\binom{m-1}{2}, \binom{m}{3}-1] \subseteq \{\binom{x}{3} : x \in [m]\}$. Since $f(\varnothing,m,x)=\binom{x}{3}$, we have

$$F(\varnothing,m) = \bigcup_{x=0}^{m-1}[\tbinom{x}{3}, \tbinom{x}{3}+m] \cap \bigcup_{x=1}^{m}[\tbinom{x}{3}-m, \tbinom{x}{3}],$$

see Figure 3.2 for an illustration of the set $F(\varnothing, m)$. We claim that for $m \geqslant 16$, if $\binom{x}{3} + m \geqslant \binom{m-1}{2}$ for some $x \in [m]$, then $\binom{x+1}{3} - m > \binom{x}{3} + m$. Indeed,

$$(x(x-1))^{\frac{3}{2}} \geqslant x(x-1)(x-2) \geqslant 6\left(\binom{m-1}{2} - m\right) = 3m^2 - 15m + 6$$

and thus $x(x-1) \geqslant (3m^2 - 15m + 6)^{2/3} > 4m$ for $m \geqslant 16$. Therefore $\binom{x}{2} > 2m$, which is equivalent to $\binom{x+1}{3} - m > \binom{x}{3} + m$.

In particular, in this case the interval $[\binom{x}{3}, \binom{x+1}{3}]$ is long enough that we have $[\binom{x}{3}, \binom{x}{3} + m] \cap [\binom{x'}{3} - m, \binom{x'}{3}] = \varnothing$ for $x \neq x'$ and $\max\{\binom{x}{3}, \binom{x'}{3}\} \geqslant \binom{m-1}{2}$. Thus,

$$F(\varnothing, m) \cap [\binom{m-1}{2}, \binom{m}{3}] \subseteq \{\binom{x}{3} : x \in [m]\}.$$

□

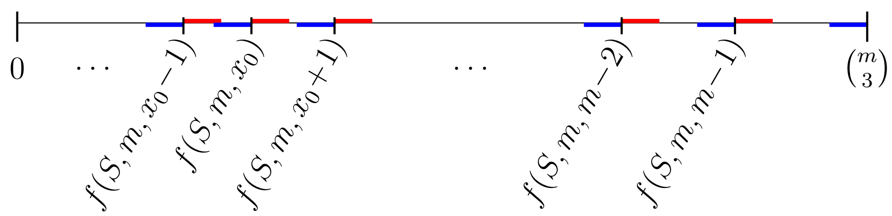

Figure 3.2: This figure displays the set $\bigcup_{x=0}^{m-1} [f(S, m, x), f(S, m, x) + m]$ in red and the set $\bigcup_{x=1}^{m} [f(S, m, x) - m, f(S, m, x)]$ in blue on the number line. Here, x_0 is the smallest integer x such that $f(S, m, x+1) - m > f(S, m, x) + m$.

The next lemma follows the same ideas as Lemma 3.6 with red/blue intervals in Figure 3.2 having respective endpoints at $f(S, m, x) = \binom{x}{3} + \binom{x}{2}(m - x)$, where $S = \{2\}$.

Lemma 3.7. *Let* $m \geqslant 13$ *and* f *be an integer, such that* $\binom{m-1}{2} \leqslant f \leqslant \binom{m}{3} - \binom{m-1}{2}$ *and for any* $x \in [m]$, $f \neq \binom{x}{3} + \binom{x}{2}(m - x)$. *Then* $\sigma_3(m, f) = 0$.

Proof. Consider m and f as given in the statement of the lemma and let $S = \{2\}$. By Lemma 3.4, it is sufficient to prove that $f \notin F(S, m)$ and in particular, it is sufficient to show that

$$F(\{2\}, m) \cap [\binom{m-1}{2}, \binom{m}{3} - \binom{m-1}{2}] \subseteq \{\binom{x}{3} + \binom{x}{2}(m - x) : x \in [m]\}.$$

Recall that

$$F(\{2\}, m) = \bigcup_{x=0}^{m-1} [f(\{2\}, m, x), f(\{2\}, m, x) + m] \cap \bigcup_{x=1}^{m} [f(\{2\}, m, x) - m, f(\{2\}, m, x)].$$

From the definition of f, we have that $f(\{2\}, m, x) = \binom{x}{3} + \binom{x}{2}(m - x)$. Note that for $x < 4$ we have $f(\{2\}, m, x) + m < \binom{m-1}{2}$ and for $x > m - 4$, $f(\{2\}, m, x) - m > \binom{m}{3} - \binom{m-1}{2}$.

Therefore it is sufficient to consider only

$$\bigcup_{x=4}^{m-4}[f(\{2\},m,x),f(\{2\},m,x)+m]\cap\bigcup_{x=4}^{m-4}[f(\{2\},m,x)-m,f(\{2\},m,x)].$$

For $m \geqslant 13$ and $4 \leqslant x \leqslant m-4$, $f(\{2\},m,x)-f(\{2\},m,x-1)=m(x-1)+x-x^2>2m$, where the last inequality holds for $x=4,5,6$ by checking separately and for $x \geqslant 7$ because

$$m(x-1)+x-x^2-2m=m(x-3)+x-x^2 \geqslant (x+4)(x-3)+x-x^2=2x-12>0.$$

Thus,

$$\bigcup_{x=4}^{m-4}[f(\{2\},m,x),f(\{2\},m,x)+m]\cap\bigcup_{x=4}^{m-4}[f(\{2\},m,x)-m,f(\{2\},m,x)]$$
$$=\{f(\{2\},m,x):4\leqslant x\leqslant m-4\}.$$

In particular, we have

$$F(\{2\},m)\cap[\tbinom{m-1}{2},\tbinom{m}{3}-\tbinom{m-1}{2}]\subseteq\{\tbinom{x}{3}+\tbinom{x}{2}(m-x):4\leqslant x\leqslant m-4\}.\qquad\square$$

3.4. Proof of Theorem 1.3

Proof. For $m \leqslant 15$ it was already shown in [Web24], that the only possible pair (m,f) with $0<f<\binom{m}{3}$ and $\sigma_3(m,f)>0$ is $(6,10)$, where $10=\binom{5}{3}=\binom{6}{3}-\binom{5}{3}=\binom{3}{3}+\binom{3}{2}(6-3)$. Now let $m>15$, and assume that for some f we have $\sigma_3(m,f)>0$. Then applying Lemma 3.5 to (m,f) and $(m,\binom{m}{3}-f)$, we obtain that $\binom{m-1}{2}\leqslant f\leqslant\binom{m}{3}-\binom{m-1}{2}$. Applying Lemma 3.6 to (m,f) gives us that $f=\binom{x_1}{3}$, for some x_1; applying it again to $(m,\binom{m}{3}-f)$ gives us that $f=\binom{m}{3}-\binom{x_2}{3}$, for some x_2. Lemma 3.7 shows the existence of some x_3, for which we have $f=\binom{x_3}{3}+\binom{x_3}{2}(m-x_3)$. This completes the proof.

Note that applying Lemma 3.7 to $(m,\binom{m}{3}-f)$ will not yield an additional constraint on f, since $\binom{m}{3}-\binom{y}{3}-\binom{y}{2}(m-y)=\binom{x}{3}+\binom{x}{2}(m-x)$ for $y=m-x$. $\square$

4. Concluding Remarks

In this paper we investigate 3-uniform hypergraphs and forcing densities $\sigma_3(m,f)$. We show that $\sigma_3(6,10)>0$ and provide more specific bounds. Apart from the pairs $(m,0)$, $(m,\binom{m}{3})$, the pair $(6,10)$ is the only known non-trivial pair for which the forcing density is positive. We conjecture that $(6,10)$ is the unique pair (m,f) with $0<f<\binom{m}{3}$ for which $\sigma_3(m,f)>0$.

Theorem 1.3 implies that if there is no $m\neq 6$ for which there is a solution (x_1,x_2,x_3), $x_i\in[m-1]$, of the system of Diophantine equations

$$\binom{x_1}{3}=\binom{m}{3}-\binom{x_2}{3}=\binom{x_3}{3}+\binom{x_3}{2}(m-x_3),\tag{4.1}$$

then Conjecture 1.2 is true. However, we do not know much about solutions (x_1, x_2, x_3) to the above system of equations. A computer search for suitable solutions of (4.1) for any given $m \leqslant 10^6$ did not give a result. Considering only the equation $\binom{x_1}{3} = \binom{m}{3} - \binom{x_2}{3}$, Sierpiński [Sie62] found an infinite class of solutions.

It might be possible to find stronger necessary conditions for a pair to have positive forcing density using different constructions than the ones used in the proof of Theorem 1.3. In particular, the reader might wonder why Lemma 3.4 and the corresponding constructions in Lemma 3.3 were not used when $S = \{1\}$. The reason for this is that the respective function $f(\{1\}, m, x) = \binom{x}{3} + x\binom{m-x}{2}$ is not monotone, making it difficult to capture the structure of the set $F(\{1\}, m)$. However, this construction could very well be used to conclude that certain pairs (m, f) have forcing density zero.

Determining the exact value of $\sigma_3(6, 10)$ remains open. We believe that the upper bound from Theorem 1.1, coming from the iterated construction H_n^{it} in Lemma 2.1, is tight.

Conjecture 4.1. We have $\sigma_3(6, 10) = 1 - 2\frac{4}{3+7\sqrt{3}} \approx 0.47105$.

We remark that a standard flag algebra calculation yields $\pi({}_{\mathrm{ind}}\mathcal{F}_6^{10}, \{K_4^{3-}\}) \leqslant 0.275 < 2/7$. Using the first part of Theorem 1.1, this gives $\sigma_3(6, 10) \geqslant 0.45$ which improves the lower bound on $\sigma_3(6, 10)$ given in the second part of Theorem 1.1.

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