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Decompositions of packed words and self duality of Word Quasisymmetric Functions

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Abstract. By Foissy's work, the bidendriform structure of the Word Quasisymmetric Functions Hopf algebra (WQSym) implies that it is isomorphic to its dual. However, the only known explicit isomorphism due to Vargas does not respect the bidendriform structure. This structure is entirely determined by so-called totally primitive elements (elements such that the two half-coproducts vanish). In this paper, we construct two bases indexed by two new combinatorial families called red (dual side) and blue (primal side) biplane forests in bijection with packed words. In those bases, primitive elements are indexed by biplane trees and totally primitive elements by a certain subset of trees. We carefully combine red and blue forests to get bicolored forests. A simple recoloring of the edges allows us to obtain the first explicit bidendriform automorphism of WQSym.

Keywords. Bidendriform Hopf algebras, Word Quasisymmetric Functions, packed words, permutation, primitive elements, duality, tree, forest, global descents

Mathematics Subject Classifications. 05A05, 05A19, 05E05, 05E18

Introduction

Combinatorial Hopf algebras are a common meeting point of different communities. The operad theory gives a lot of examples, as in numerous cases free algebras over an operad admit a Hopf algebra structure. For many operads, one can make the structure explicit using combinatorics, one of the most basic example being the free dendriform algebra on one generator realized as the Loday–Ronco Hopf algebra of binary trees [LR98].

On the other hand, the theory of symmetric functions often proceeds through non-commutative lifting to better understand the identities. Hence, the community introduced a series of larger and larger Hopf algebras over a large variety of combinatorial structures. One of the first step was the introduction of the dual pair of quasi-symmetric functions and non-commutative symmetric functions [Ges84, GKL$^+$95] to understand the inner product of characters through the descent algebra [MR95]. It leads to the discovery of the Malvenuto–Reutenauer algebra **FQSym** of permutations. Another example was the introduction by Poirier [Poi96] of an algebra of Young tableaux. In [DHT02], it was realized that it can lead to a very simple proof of the Littlewood–Richardson rule.

An early meeting between the symmetric function and the operad communities was the discovery [HNT05] that the same procedure allows to construct both the algebra of tableaux and the algebra of binary trees from the algebra of permutations. One just has to enforce some simple relation (respectively plactic and Sylvester relation) in the variable of the polynomial realization.

Aside from the algebra of permutations, there is another non-commutative lifting of quasi-symmetric functions. Indeed Hivert's action on polynomials whose invariant are quasi-symmetric functions [DHT02, Hiv99] can be lifted to words. Here, its non-commutative invariants spans the Hopf algebra **WQSym** of packed words or, equivalently, surjections or even ordered set partitions. This algebra has various applications in the theory of free Lie algebras is closely related to the Solomon–Tits algebra and twisted descents, the development of which was motivated by the geometry of Coxeter groups, the study of Markov chains on hyperplane arrangements (see [NPT13] and the reference therein).

To better understand these algebra, one has to investigate their structure. For the binary tree algebra it was shown in [HNT05] that it is free as an algebra and isomorphic to its dual. Though those properties are quite obvious for the algebra **FQSym** of permutation, the situation of **WQSym** is much more difficult. Its first study is due to Bergeron–Zabrocki [BZ09]. They showed that it is free and co-free. However, it was only conjectured in [DHT02] that its primitive Lie algebra is free and that it is self dual. It is only by a deep theorem of Foissy [Foi07] that one can show that the second one is too. In particular, until Vargas's work [Var19], no concrete isomorphism was known.

Independently, Novelli–Thibon worked on parking functions which is a super-set of packed words. They endowed the Hopf algebra of parking functions **PQSym** with a bidendriform bialgebra structure [NT07]. Then they describe **WQSym** as a sub-bidendriform bialgebra of **PQSym** [NT06]. Recall that a dendriform algebra is an abstraction of a shuffle algebra where the product is split in two half-products. If the coproduct is also split, and certain compatibilities hold, one gets the notion of bidendriform bialgebra [Foi07].

Building on the work of Chapoton and Ronco [Ron00, Cha02], Foissy [Foi07] showed that the structure of a bidendriform bialgebra is very rigid. In particular, he defined a specific subspace called the space of totally primitive elements, and showed that it characterizes the whole structure. This does not only re-prove the freeness and co-freeness, as well as the freeness of the primitive lie algebra, but also shows that the structure of a bidendriform bialgebra depends only on its Hilbert series (the series of dimensions of its homogeneous components). In particular, any such algebra is isomorphic to its dual. However, Foissy's isomorphism is not fully explicit and depends on a choice of a basis of the totally primitive elements. To this end, one needs an explicit basis of the totally primitive elements. Foissy described such a construction for **FQSym** [Foi11]. In this paper, we construct a far reaching generalization for packed words and **WQSym** so that the basis discribed in [Foi11] is simply a restriction to permutations and **FQSym** is a sub-bidendriform bialgebra of **WQSym**. We provide two explicit bases of totally primitive elements, for **WQSym** and its dual, using a bijection with certain families of trees called biplane.

We begin with a background section presenting Foissy's two rigidity structure theorems that prove, among other things, the self-duality of any bidendriform bialgebra (Theorem 1.2 and Corollary 1.3). We then define the notion of packed words as well as the two specific bases ($\mathbb{Q}$ and $\mathbb{R}$) of $\mathbf{WQSym}$ and its dual, which will be the starting point of our combinatorial analysis.

Section 2 is devoted to the combinatorial construction of biplane forests (Definitions 2.32 and 2.67) which are our first key ingredient. They record a recursive decomposition of packed words according to their global descents (Lemma 1.16) and positions of the maximum letter (Lemma 2.7) or the value of the last letter (Lemma 2.46). We show that the cardinalities of some specific sets of biplane trees match the dimensions of primitive and totally primitive elements (Theorems 2.40 and 2.76).

In Section 3 we construct two new bases ($\mathbb{O}$ and $\mathbb{P}$ Definitions 3.5 and 3.13) of $\mathbf{WQSym}$ and its dual which each contain as a subset a basis for the primitive and totally primitve elements (see Theorems 3.7 and 3.15). To do so we decompose the space of totally primitive elements as a certain direct sum which matchs the combinatorial decomposition of packed words (Lemmas 3.4 and 3.12).

Finally in Section 4 we make explicit how bases $\mathbb{O}$ and $\mathbb{P}$ are sufficient to have an infinite number of bidendriform automorphism of $\mathbf{WQSym}$. Then we give an explicit isomorphism based on an involution on packed words. The definition of the bijection require a new kind of forest mixing red and blue, namely bicolored-packed forests.

1. Background

1.1. Cartier–Milnor–Moore theorems for Bidendriform bialgebras

The goal of this section is to recall the elements of the definition of bidendriform bialgebras which are useful for the comprehension of this paper. We refer to [Foi07] for the full list of axioms.

A bialgebra is a vector space over a commutative field K, endowed with an unitary associative product $\cdot$ and a counitary coassociative coproduct Δ satisfying a compatibility relation called the Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$. In this paper all bialgebras are assumed to be graded and connected (*i.e.* the homogeneous component of degree 0 is K). They are therefore Hopf algebras, as the existence of the antipode is implied.

A **dendriform algebra** (see [Lod01, LR98, Ron00, Ron02]) A is a K-vector space, endowed with two binary bilinear operations $\prec$, $\succ$ satisfying the following axioms, for all $a, b, c \in A$:

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \tag{1.1}$$

$$(a \succ b) \prec c = a \succ (b \prec c), \tag{1.2}$$

$$(a \prec b + a \succ b) \succ c = a \succ (b \succ c). \tag{1.3}$$

Adding together Equations (1.1) to (1.3) show that the product $a \cdot b := a \prec b + a \succ b$ is associative. Adding a subspace of scalars, this defines a unitary algebra structure on $K \oplus A$. In this paper, all the dendriform algebras are graded and have null 0-degree component so that the associated algebra is connected.

Dualizing, one gets a notion of **co-dendriform co-algebra** (see [Foi07]) which is a K-vector space with two binary co-operations (*i.e.*, linear maps $A \to A \otimes A$) denoted by $\Delta_{\prec}$, $\Delta_{\succ}$ satisfying the dual axioms of Equations (1.1) to (1.3):

$$(\Delta_{\prec} \otimes \mathrm{Id}) \circ \Delta_{\prec}(a) = (\mathrm{Id} \otimes \Delta_{\prec} + \mathrm{Id} \otimes \Delta_{\succ}) \circ \Delta_{\prec}(a), \tag{1.4}$$
$$(\Delta_{\succ} \otimes \mathrm{Id}) \circ \Delta_{\prec}(a) = (\mathrm{Id} \otimes \Delta_{\prec}) \circ \Delta_{\succ}(a), \tag{1.5}$$
$$(\Delta_{\prec} \otimes \mathrm{Id} + \Delta_{\succ} \otimes \mathrm{Id}) \circ \Delta_{\succ}(a) = (\mathrm{Id} \otimes \Delta_{\succ}) \circ \Delta_{\succ}(a). \tag{1.6}$$

Adding together Equations (1.4) to (1.6) show that the reduced coproduct $\tilde{\Delta}(a) := \Delta_{\prec}(a) + \Delta_{\succ}(a)$ is co-associative. On $K \oplus A$, setting $\Delta(a) := 1 \otimes a + a \otimes 1 + \tilde{\Delta}(a)$ defines a co-associative and co-unitary coproduct.

A **bidendriform bialgebra** is a K-vector space which is both a dendriform algebra and a co-dendriform co-algebra satisfying a set of four relations relating respectively $\prec$ and $\succ$ with $\Delta_{\prec}$, $\Delta_{\succ}$ (see [Foi07] for more details). In these equations, we use the Sweedler notation where $\tilde{\Delta}(a) = a' \otimes a''$ and $\Delta_{\alpha}(b) = b'_{\alpha} \otimes b''_{\alpha}$ with $\alpha \in \{\prec, \succ\}$.

$$\Delta_{\succ}(a \succ b) = a'b'_{\succ} \otimes a'' \succ b''_{\succ} + b'_{\succ} \otimes a \succ b''_{\succ} + ab'_{\succ} \otimes b''_{\succ} + a' \otimes a'' \succ b + a \otimes b, \tag{1.7}$$
$$\Delta_{\succ}(a \prec b) = a'b'_{\succ} \otimes a'' \prec b''_{\succ} + b'_{\succ} \otimes a \prec b''_{\succ} + a' \otimes a'' \prec b, \tag{1.8}$$
$$\Delta_{\prec}(a \succ b) = a'b'_{\prec} \otimes a'' \succ b''_{\prec} + b'_{\prec} \otimes a \succ b''_{\prec} + ab'_{\prec} \otimes b''_{\prec}, \tag{1.9}$$
$$\Delta_{\prec}(a \prec b) = a'b'_{\prec} \otimes a'' \prec b''_{\prec} + b'_{\prec} \otimes a \prec b''_{\prec} + a'b \otimes a'' + b \otimes a. \tag{1.10}$$

Adding those four relations shows that $\cdot$ and Δ as defined above defines a proper bi-algebra.

We recall here the relevant results of Foissy [Foi07] on the rigidity of bidendriform bialgebras based on the works of Chapoton and Ronco [Ron00, Cha02].

Let A be a bidendriform bialgebra. We denote $\mathrm{Prim}(A) := \mathrm{Ker}(\tilde{\Delta})$ the set of **primitive** elements of A. We also denote by $\mathcal{A}(z)$ and $\mathcal{P}(z)$ the Hilbert series of A and $\mathrm{Prim}(A)$ defined as $\mathcal{A}(z) := \sum_{n=1}^{+\infty} \dim(A_n) z^n$ and $\mathcal{P}(z) := \sum_{n=1}^{+\infty} \dim(\mathrm{Prim}(A_n)) z^n$. The present work is based on two analogues of the Cartier–Milnor–Moore theorems [Foi07] which we present now. The first one is extracted from the proof of [Foi11, Proposition 6]:

Proposition 1.1. *Let A be a bidendriform bialgebra and let $p_1 \ldots p_n \in \mathrm{Prim}(A)$. Then the map*

$$p_1 \otimes p_2 \otimes \ldots \otimes p_n \mapsto p_1 \prec (p_2 \prec (\ldots \prec p_n) \ldots). \tag{1.11}$$

is an isomorphism of co-algebras from $T^+(\mathrm{Prim}(A))$ (the non trivial part of the tensor algebra with deconcatenation as coproduct) to A. As a consequence, taking a basis $(p_i)_{i \in I}$ of $\mathrm{Prim}(A)$, the family $(p_{w_1} \prec (p_{w_2} \prec (\cdots \prec p_{w_n}) \ldots))_w$ where $w = w_1 \ldots w_n$ is a non empty word on I defines a basis of A. This implies the equality of Hilbert series $\mathcal{A} = \mathcal{P}/(1 - \mathcal{P})$.

One can further analyze $\mathrm{Prim}(A)$ using the so-called **totally primitive** elements of A defined as $\mathrm{TPrim}(A) := \mathrm{Ker}(\Delta_{\prec}) \cap \mathrm{Ker}(\Delta_{\succ})$. The associated Hilbert serie is defined as

$$\mathcal{T}(z) := \sum_{n=1}^{+\infty} \dim(\mathrm{TPrim}(A_n)) z^n.$$

Recall that a brace algebra is a K-vector space A together with an n-multilinear operation denoted as $\langle \ldots ; \rangle$ for all $n \geqslant 2$ which satisfies certain relations (see [Ron00] for details).

Theorem 1.2 ([Foi11, Theorem 4 and 5]). *Let A be a bidendriform bialgebra. Then* $\mathrm{Prim}(A)$ *is freely generated as a brace algebra by* $\mathrm{TPrim}(A)$ *with brackets given by*

$$\langle p_1, \dots, p_{n-1}; p_n\rangle := \sum_{i=0}^{n-1} (-1)^{n-1-i} \\ (p_1 \prec (p_2 \prec (\cdots \prec p_i)\cdots)) \succ p_n \prec ((\cdots(p_{i+1} \succ p_{i+2}) \succ \cdots) \succ p_{n-1}).$$

Sometimes also called planar trees, **ordered trees** are define as a root and an ordered list (possibly empty) of ordered trees as children. A basis of $\mathrm{Prim}(A)$ is described by ordered trees that are decorated with elements of $\mathrm{TPrim}(A)$ where p_n is the root and $p_1, \dots, p_{n-1}$ are the children (see [Ron00, Cha02, Foi11]). This is reflected on their Hilbert series as [Foi07, Corollary 37]: $\mathcal{T} = \mathcal{A}/(1+\mathcal{A})^2$ or equivalently $\mathcal{P} = \mathcal{T}(1+\mathcal{A})$.

Using Proposition 1.1 and Theorem 1.2 together with a dimension argument, one can show the two following corollaries:

Corollary 1.3 ([Foi11, Theorem 2]). *Let A be a bidendriform bialgebra. Then A is freely generated as a dendriform algebra by* $\mathrm{TPrim}(A)$.

Corollary 1.4 ([Foi02a, Foi02b]). *A basis of A is described by ordered forests of ordered trees that are decorated with a basis of* $\mathrm{TPrim}(A)$.

On this basis, the product can be described using grafting (see Proposition 28 in [Foi02b]) and the coproduct as the deconcatenation of forests that are word of trees (see Theorem 35 equation 7.(c) in [Foi02a]).

1.2. Packed words

The algebra $\mathbf{WQSym}$ is a Hopf algebra whose bases are indexed by ordered set partitions or equivalently surjections or even packed words. In this paper, we use the latter which we define now.

In this paper we will deal with words over the alphabet of positive integers $\mathbb{N}_{>0}$. We start with basic notations: first, $\max(w)$ is the maximum letter of the word w with the convention that $\max(\epsilon) = 0$. Then $|w|$ is the length (or size) of the word w. The concatenation of the two words u and v is denoted as $u \cdot v$. The shift of a word w of a value i is denoted by $w^{[i]}$. Once that said, $u/v := u^{[\max(v)]} \cdot v$ (resp. $u\backslash v := u \cdot v^{[\max(u)]}$)[1] is the left-shifted (resp. right-shifted) concatenation of the two words where all the letters of the left (resp. right) word are shifted by the maximum of the right (resp. left) word: $1121/3112 = 44543112$ and $1121\backslash 3112 = 11215334$. We also use the notation $u|_{\leqslant i}$ (resp. $u|_{>i}$) for the subword containing all letters smaller (resp. strictly greater) than a value i.

[1]Note that "over"$/$ and "under"$\backslash$ are reversed compared to what can be found in Loday and Ronco [LR02] where it was first introduced: indeed, "over" and "under" need to be consistent with our cartesian representation of packed words whereas Loday and Ronco use a matricial representation.

Definition 1.5. A word over the alphabet $\mathbb{N}_{>0}$ is **packed** if all the letters from 1 to its maximum m appears at least once. By convention, the empty word ϵ is packed. For $n \in \mathbb{N}$, we denote by $\mathbf{PW}_n$ the set of all packed words of length (also called size) n and $\mathbf{PW} = \bigsqcup_{n\in\mathbb{N}} \mathbf{PW}_n$ the set of all packed words.

n	1	2	3	4	5	6	7	8	9	OEIS
$\mathbf{PW}_n$	1	3	13	75	541	4 683	47 293	545 835	7 087 261	A000670

Table 1.1: Number of packed words of size smaller than 9.

Definition 1.6. The packed word $u := \mathrm{pack}(w)$ associated with a word over the alphabet $\mathbb{N}_{>0}$ is obtained by the following process: if $b_1 < b_2 < \cdots < b_r$ are the distinct letters occurring in w, then u is the image of w by the homomorphism $b_i \mapsto i$.

A word u is packed if and only if $\mathrm{pack}(u) = u$.

Example 1.7. The word 4152142 is not packed because the letter 3 does not appear while the maximum letter is $5 > 3$. Meanwhile $\mathrm{pack}(4152142) = 3142132$ is a packed word. Here are all packed words of size 1, 2 and 3 in lexicographic order:

$$1, \quad 11\ 12\ 21, \quad 111\ 112\ 121\ 122\ 123\ 132\ 211\ 212\ 213\ 221\ 231\ 312\ 321$$

The fonction $\mathrm{pack}(w)$ is the analogue of the **standardization** $\mathrm{std}(w)$ that returns a permutation.

Definition 1.8. The standardized word $\mathrm{std}(w)$ associated with a word over the alphabet $\mathbb{N}_{>0}$ is obtained by iteratively scanning w from left to right, and labelling the occurrences of its smallest letter, then labelling the occurrences of the next one, and so on.

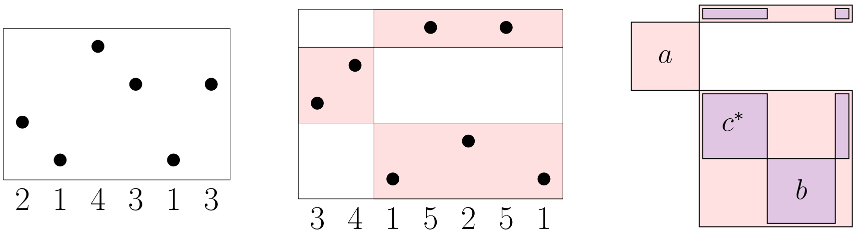
Example 1.9. For example, $\mathrm{std}(4152142) = \mathrm{std}(3142132) = 5173264$.

For the reader familiar with ordered set partitions, there is a classical bijection between packed words and ordered set partitions. The one corresponding to a packed word $w_1 \cdot w_2 \cdots w_n$ is obtained by placing the index i into the w_i-th block.

Example 1.10. The word 121 is associated with $\{\{1, 3\}, \{2\}\}$ and the word 113223 with $\{\{1, 2\}, \{4, 5\}, \{3, 6\}\}$.

To depict some definitions or lemmas, we will use box diagrams with Cartesian coordinates for packed words. On these diagrams, positions are from left to right (as reading direction) and values are from bottom to top. These diagrams will also be used to represent different decompositions with different colors. Transparency will order the decompositions.

Example 1.11. Here we have three examples: the representation of the packed word 214313. Then the word 3415251 decomposed with red-factorization (see Lemma 2.7). Finally the general case of the red-blue-factorization (see Definition 4.8) where it can be seen clearly that the blue-factorization is done after the red-factorization thanks to transparency.



Global descent are defined in [AS05] on permutations, and generalised on packed words in [Var19].

Definition 1.12 ([Var19, Definition 6.10])**.** A **global descent** of a packed word w is a position c such that all the letters before or at position c are strictly greater than all letters after position c.

Example 1.13. The global descents of $w = 54664312$ are the positions 5 and 6. Indeed, all letters of 54664 are greater than the letters of 312 and this is also true for 546643 and 12.

Definition 1.14. A packed word w is **irreducible** if it is non empty and it has no global descent.

n	1	2	3	4	5	6	7	8	9	OEIS
p_n	1	2	8	48	368	3 376	35 824	430 512	5 773 936	A095989

Table 1.2: Number of irreducible packed words of size smaller than 9.

Example 1.15. The word $w' = 21331$ is irreducible.

Lemma 1.16. *Each word w admits a unique factorization as $w = w_1/w_2/\dots/w_k$ such that w_i is irreducible for all i.*

Example 1.17. The global descent decomposition of 54664312 is $21331/1/12$. The word $n \cdot n-1 \cdot \dots \cdot 1$ has $1/1/\dots/1$ as global descent decomposition.

Definition 1.18. $u \sqcup\!\sqcup\, v$ denotes the shuffle product of the two words. It is recursively defined by $u \sqcup\!\sqcup\, \epsilon := \epsilon \sqcup\!\sqcup\, u := u$ and

$$ua \sqcup\!\sqcup\, vb := (u \sqcup\!\sqcup\, vb) \cdot a + (ua \sqcup\!\sqcup\, v) \cdot b \tag{1.12}$$

where u and v are words and a and b are letters. Analogously to the shifted concatenation, one can define the right shifted-shuffle $u \,\overline{\sqcup\!\sqcup}\, v := u \sqcup\!\sqcup\, v^{[\max(u)]}$ where all the letters of the right word v are shifted by the maximum of the left word u.

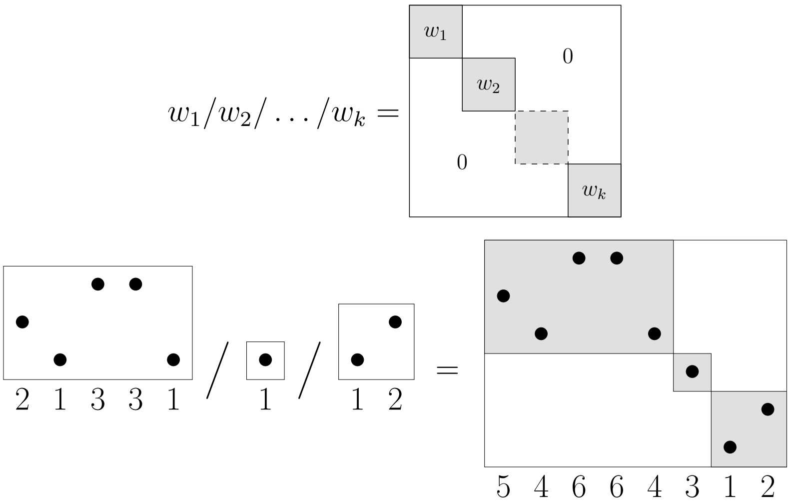
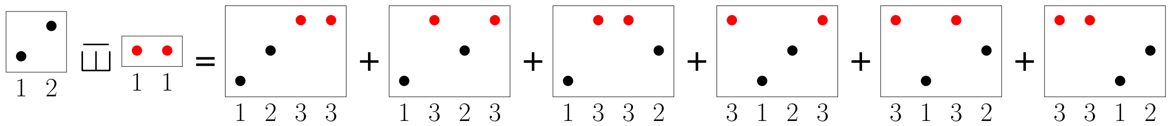


Figure 1.1: Box diagrams: global descent decomposition.

Example 1.19. $12 \,\overline{\sqcup\!\sqcup}\, 11 = 12 \sqcup\!\sqcup 33 = 1233 + 1323 + 1332 + 3123 + 3132 + 3312.$



Definition 1.20 ([Lod01, Example 5.4.(a)])**.** The recursive definition of the shuffle product Equation (1.12) contains two summands. The two half shuffle products on words $\prec$ and $\succ$ are defined respectively by:

$$ua \prec vb := (u \sqcup\!\sqcup vb) \cdot a \qquad \text{and} \qquad ua \succ vb := (ua \sqcup\!\sqcup v) \cdot b. \tag{1.13}$$

Example 1.21. $12 \prec 33 = 1332 + 3132 + 3312 \qquad \text{and} \qquad 12 \succ 33 = 1233 + 1323 + 3123.$

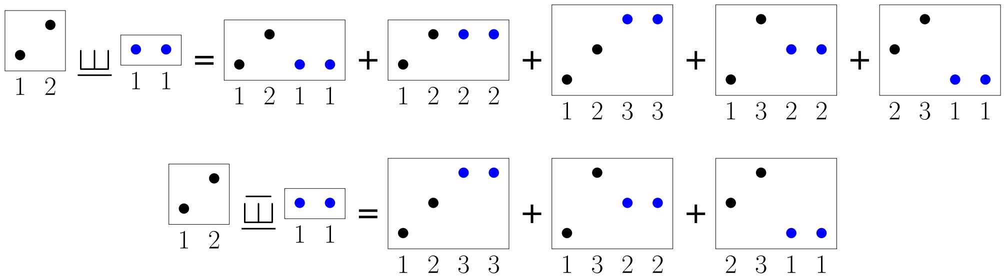
Definition 1.22. $u \underline{\sqcup\!\sqcup} v$ denote the dualisation of the deconcatenation using the function $\mathrm{pack}(w)$ of Definition 1.6.

$$u \underline{\sqcup\!\sqcup} v := \sum_{\substack{u=\mathrm{pack}(u')\\ v=\mathrm{pack}(v')}} u' \cdot v' \tag{1.14}$$

where u, v and $u' \cdot v'$ are packed words. We also use the non-overlapping shuffle product on values by adding the constraint that letters of the two parts are distinct:

$$u \,\overline{\underline{\sqcup\!\sqcup}}\, v := \sum_{\substack{u=\mathrm{pack}(u')\\ v=\mathrm{pack}(v')\\ \forall i,j, u'_i \neq v'_j}} u' \cdot v'. \tag{1.15}$$

Example 1.23. $12 \shuffle 11 = 1211 + 1222 + 1233 + 1322 + 2311, \quad 12\,\overline{\shuffle}\,11 = 1233 + 1322 + 2311.$



Remark 1.24. Using the classical bijection between ordered set partitions and packed words (see Example 1.10), the product $\overline{\shuffle}$ is equivalent to the shifted shuffle on ordered set partitions defined in [BZ09].

Definition 1.25. Analogously to the two half shuffle product of Definition 1.20, we split the two products $\shuffle$ and $\overline{\shuffle}$ in two parts.

$$u \ll v := \sum_{\substack{u=\mathrm{pack}(u')\\ v=\mathrm{pack}(v')\\ \max(u')>\max(v')}} u' \cdot v' \quad \text{and} \quad u \gg v := \sum_{\substack{u=\mathrm{pack}(u')\\ v=\mathrm{pack}(v')\\ \max(u')\leqslant\max(v')}} u' \cdot v'. \tag{1.16}$$

$$u \preceq v := \sum_{\substack{u=\mathrm{pack}(u')\\ v=\mathrm{pack}(v')\\ \forall i,j, u'_i\neq v'_j\\ \max(u')>\max(v')}} u' \cdot v' \quad \text{and} \quad u \succeq v := \sum_{\substack{u=\mathrm{pack}(u')\\ v=\mathrm{pack}(v')\\ \forall i,j, u'_i\neq v'_j\\ \max(u')<\max(v')}} u' \cdot v'. \tag{1.17}$$

Example 1.26.

$$12 \ll 11 = 1211 + 1322 + 2311 \quad \text{and} \quad 12 \gg 11 = 1222 + 1233.$$
$$12 \preceq 11 = 1322 + 2311 \quad \text{and} \quad 12 \succeq 11 = 1233.$$

To sum up in a few words, in $u \prec v$ the last letter is comming from u and the rest is shuffled, in $u \preceq v$ the maximum value is comming from u and the rest is shuffled.

1.3. The Hopf algebra of word-quasisymmetric functions WQSym

We are now in position to define the Hopf algebra of word-quasisymmetric functions **WQSym**. It was first defined as a Hopf algebra in [Jö98] and independently in [Hiv99]. Novelli–Thibon proved later that **WQSym** and its dual are bidendriform bialgebras [NT06, Theorems 2.5 and 2.6]. Their products and coproducts in the monomial basis $(\mathbb{M}_w)_{w\in\mathbf{PW}}$ involve overlapping-shuffle. However, to deal with the bidendriform structure, it will be easier for us to chose, among the various bases known in the literature [Jö98, Hiv99, BZ09, NT06, Var19] a basis where the shuffles are non-overlapping. Therefore, for $\mathbf{WQSym}^*$, we take the basis denoted $(\mathbb{Q}_w)_{w\in\mathbf{PW}}$ of [BZ09, Equation 23] using the classical bijection between ordered set partitions and packed words (see Example 1.10). For the primal **WQSym**, we define the dual basis denoted $(\mathbb{R}_w)_{w\in\mathbf{PW}}$. In this section we transfer the bidendriform structure on the bases $(\mathbb{Q}_w)_{w\in\mathbf{PW}}$ and $(\mathbb{R}_w)_{w\in\mathbf{PW}}$.

Following Novelli–Thibon, we start from the basis $(\mathbb{M}_w)_{w\in\mathbf{PW}}$ and compute expressions of half product Equation (1.22) and half coproduct Equations (1.23) and (1.24) in the basis $(\mathbb{Q}_w)_{w\in\mathbf{PW}}$. Then we dualise these operations Equations (1.25) to (1.27) to define the basis $(\mathbb{R}_w)_{w\in\mathbf{PW}}$. The Hopf algebra product and reduced coproduct are respectively recovered as the sum of the half products and half coproducts.

The monomial word-quasisymmetric function of a totally ordered alphabet $\mathcal{A}$ associated to the packed word u is the linear combination of words defined by

$$\mathbb{M}_u := \sum_{\substack{w\in\mathcal{A}^*,\\ \mathrm{pack}(w)=u}} w.$$

It turns out that the concatenation of two such elements is a sum of $(\mathbb{M}_u)_{u\in\mathbf{PW}}$ so that $\mathbf{WQSym} := Vect(\mathbb{M}_u \mid u \in \mathbf{PW})$ is an algebra. This can be refined to a bidendriform bialgebra structure. The operations $\ll$, $\gg$, $\Delta_{\ll}$ and $\Delta_{\gg}$ on

$$(\mathbf{WQSym})_+ := Vect(\mathbb{M}_u \mid u \in \mathbf{PW}_n, n \geqslant 1)$$

are defined in the following way: for all $u = u_1 \cdots u_n \in \mathbf{PW}_{n\geqslant 1}$ and $v \in \mathbf{PW}_{m\geqslant 1}$,

$$\mathbb{M}_u \ll \mathbb{M}_v := \sum_{w\in u\ll v} \mathbb{M}_w, \qquad \text{and} \qquad \mathbb{M}_u \gg \mathbb{M}_v := \sum_{w\in u\gg v} \mathbb{M}_w. \tag{1.18}$$

$$\Delta_{\ll}(\mathbb{M}_u) := \sum_{i=u_n}^{\max(u)-1} \mathbb{M}_{u|_{\leqslant i}} \otimes \mathbb{M}_{\mathrm{pack}(u|_{>i})}, \tag{1.19}$$

$$\Delta_{\gg}(\mathbb{M}_u) := \sum_{i=1}^{u_n-1} \mathbb{M}_{u|_{\leqslant i}} \otimes \mathbb{M}_{\mathrm{pack}(u|_{>i})}. \tag{1.20}$$

Example 1.27.

$$\mathbb{M}_{112} \ll \mathbb{M}_{12} = \mathbb{M}_{11312} + \mathbb{M}_{11423} + \mathbb{M}_{22312} + \mathbb{M}_{22413} + \mathbb{M}_{33412},$$
$$\mathbb{M}_{112} \gg \mathbb{M}_{12} = \mathbb{M}_{11212} + \mathbb{M}_{11213} + \mathbb{M}_{11223} + \mathbb{M}_{11234} + \mathbb{M}_{11323} + \mathbb{M}_{11324} + \mathbb{M}_{22313} + \mathbb{M}_{22314},$$

$$\begin{aligned}
\Delta_{\ll}(\mathbb{M}_{212536434}) &= \mathbb{M}_{2123434} \otimes \mathbb{M}_{\mathrm{pack}(56)} + \mathbb{M}_{21253434} \otimes \mathbb{M}_{\mathrm{pack}(6)},\\
&= \mathbb{M}_{2123434} \otimes \mathbb{M}_{12} + \mathbb{M}_{21253434} \otimes \mathbb{M}_{1},\\
\Delta_{\gg}(\mathbb{M}_{212536434}) &= \mathbb{M}_{1} \otimes \mathbb{M}_{\mathrm{pack}(22536434)} + \mathbb{M}_{212} \otimes \mathbb{M}_{\mathrm{pack}(536434)} + \mathbb{M}_{21233} \otimes \mathbb{M}_{\mathrm{pack}(5644)}\\
&= \mathbb{M}_{1} \otimes \mathbb{M}_{11425323} + \mathbb{M}_{212} \otimes \mathbb{M}_{314212} + \mathbb{M}_{21233} \otimes \mathbb{M}_{2311}.
\end{aligned}$$

Theorem 1.28. *[NT06, Theorem 2.5]* $((\mathbf{WQSym})_+, \ll, \gg, \Delta_{\ll}, \Delta_{\gg})$ *is a bidendriform bialgebra.*

As we said, we want a basis without overlapping-shuffle for the product. Following [BZ09], we first define a partial order on packed words then we define the new basis.

Definition 1.29. [BZ09] We say that the packed word u is smaller than v for the relation $\leqslant_*$ if u and v have the same standardization and if $u_i = u_j$ implies $v_i = v_j$ for all i and j.

$$u \leqslant_* v \iff \mathrm{std}(u) = \mathrm{std}(v) \text{ and } (u_i = u_j \implies v_i = v_j).$$

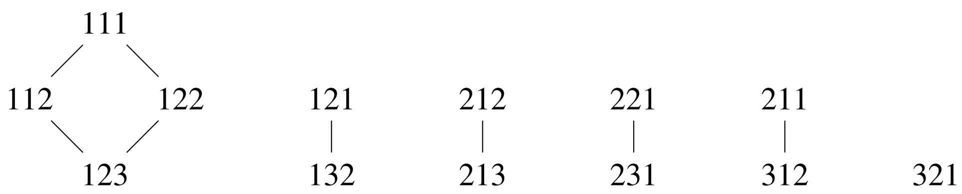


Figure 1.2: The Hasse diagram of $(\mathbf{PW}_3, \leqslant_*)$.

We give two immediate lemmas on this order that are useful.

Lemma 1.30. *For $u \leqslant_* v$, let m_u (resp. m_v) be the set of positions of occurrences of the maximum value letters in u (resp. v). Then m_u is included in m_v and all positions in m_v that are not in m_u are smaller to the minimum of m_u.*

Proof. It is immediate with the definition of $\leqslant_*$. □

Lemma 1.31. *For $u \leqslant_* v$, let i and i' such that $u|_{\leqslant i}$ and $v|_{\leqslant i'}$ are of the same size then $u|_{\leqslant i} \leqslant_* v|_{\leqslant i'}$ and $u|_{>i} \leqslant_* v|_{>i'}$.*

Proof. It is immediate with the definition of $\leqslant_*$. □

Now we can recall ([BZ09, Equation 23]) the definition of the basis $(\mathbb{Q}_w)_{w \in \mathbf{PW}}$

$$\mathbb{Q}_u := \sum_{u \leqslant_* v} \mathbb{M}_v. \tag{1.21}$$

Example 1.32.

$$\mathbb{Q}_{123} = \mathbb{M}_{123} + \mathbb{M}_{122} + \mathbb{M}_{112} + \mathbb{M}_{111} \qquad \mathbb{Q}_{43132} = \mathbb{M}_{43132} + \mathbb{M}_{32121}$$
$$\mathbb{Q}_{412234} = \mathbb{M}_{412234} + \mathbb{M}_{312223} + \mathbb{M}_{311123} + \mathbb{M}_{211112} \qquad \mathbb{Q}_{2131} = \mathbb{M}_{2131} + \mathbb{M}_{2121}$$

It is proved in [BZ09, Theorem 17] that the product in basis $(\mathbb{Q}_w)_{w\in\mathbf{PW}}$ is

$$\mathbb{Q}_u\mathbb{Q}_v = \sum_{w\in u ⧢ v} \mathbb{Q}_w.$$

Thanks to Lemma 1.30, we have the two expressions for the two half products.

$$\mathbb{Q}_u \preceq \mathbb{Q}_v := \sum_{w\in u\preceq v} \mathbb{Q}_w, \qquad \text{and} \qquad \mathbb{Q}_u \succeq \mathbb{Q}_v := \sum_{w\in u\succeq v} \mathbb{Q}_w. \tag{1.22}$$

For the coproduct, we start with the definition of the coproduct in basis $\mathbb{M}$.

$$\begin{aligned}
\Delta(\mathbb{Q}_u) &= \sum_{v\geqslant_* u} \Delta(\mathbb{M}_v) \\
&= \sum_{v\geqslant_* u}\left(\sum_{i=0}^{\max(v)} \mathbb{M}_{v|_{\leqslant i}} \otimes \mathbb{M}_{v|_{>i}}\right) \\
&= \sum_{i=0}^{\max(u)}\left(\sum_{v\geqslant_* u|_{\leqslant i}} \mathbb{M}_v \otimes \sum_{v'\geqslant_* u|_{>i}} \mathbb{M}_{v'}\right) \qquad \text{(by Lemma 1.31)} \\
&= \sum_{i=0}^{\max(u)} \mathbb{Q}_{u|_{\leqslant i}} \otimes \mathbb{Q}_{u|_{>i}}.
\end{aligned}$$

Then, with Lemma 1.30 we have the two expressions for the two half coproducts.

$$\Delta_{\preceq}(\mathbb{Q}_u) := \sum_{i=u_n}^{\max(u)-1} \mathbb{Q}_{u|_{\leqslant i}} \otimes \mathbb{Q}_{\mathrm{pack}(u|_{>i})}, \tag{1.23}$$

$$\Delta_{\succeq}(\mathbb{Q}_u) := \sum_{i=1}^{u_n-1} \mathbb{Q}_{u|_{\leqslant i}} \otimes \mathbb{Q}_{\mathrm{pack}(u|_{>i})}. \tag{1.24}$$

Example 1.33.

$$\begin{aligned}
\mathbb{Q}_{1312} \preceq \mathbb{Q}_{12} &= \mathbb{Q}_{151234} + \mathbb{Q}_{151324} + \mathbb{Q}_{151423} + \mathbb{Q}_{252314} + \mathbb{Q}_{252413} + \mathbb{Q}_{353412}, \\
\mathbb{Q}_{1312} \succeq \mathbb{Q}_{12} &= \mathbb{Q}_{131245} + \mathbb{Q}_{141235} + \mathbb{Q}_{141325} + \mathbb{Q}_{242315}, \\
\Delta_{\preceq}(\mathbb{Q}_{212536434}) &= \mathbb{Q}_{2123434} \otimes \mathbb{Q}_{\mathrm{pack}(56)} + \mathbb{Q}_{21253434} \otimes \mathbb{Q}_{\mathrm{pack}(6)}, \\
&= \mathbb{Q}_{2123434} \otimes \mathbb{Q}_{12} + \mathbb{Q}_{21253434} \otimes \mathbb{Q}_{1}, \\
\Delta_{\succeq}(\mathbb{Q}_{212536434}) &= \mathbb{Q}_{1} \otimes \mathbb{Q}_{\mathrm{pack}(22536434)} + \mathbb{Q}_{212} \otimes \mathbb{Q}_{\mathrm{pack}(536434)} + \mathbb{Q}_{21233} \otimes \mathbb{Q}_{\mathrm{pack}(5644)} \\
&= \mathbb{Q}_{1} \otimes \mathbb{Q}_{11425323} + \mathbb{Q}_{212} \otimes \mathbb{Q}_{314212} + \mathbb{Q}_{21233} \otimes \mathbb{Q}_{2311}.
\end{aligned}$$

Finally we define $\prec$, $\succ$, $\Delta_\prec$ and $\Delta_\succ$ on $(\mathbf{WQSym}^*)_+ := Vect(\mathbb{R}_u \mid u \in \mathbf{PW}_n, n \geqslant 1)$ by dualizing half products and half coproducts of the basis $(\mathbb{Q}_w)_{w\in\mathbf{PW}}$ in the following way: for all $u = u_1 \cdots u_n \in \mathbf{PW}_{n\geqslant 1}$ and $v \in \mathbf{PW}_{m\geqslant 1}$,

$$\mathbb{R}_u \prec \mathbb{R}_v := \sum_{w\in u\prec v^{[\max(u)]}} \mathbb{R}_w, \qquad \text{and} \qquad \mathbb{R}_u \succ \mathbb{R}_v := \sum_{w\in u\succ v^{[\max(u)]}} \mathbb{R}_w. \tag{1.25}$$

$$\Delta_{\prec}(\mathbb{R}_u) := \sum_{\substack{i=k \\ \{u_1,\ldots,u_i\}\cap\{u_{i+1},\ldots,u_n\}=\varnothing \\ u_k=\max(u)}}^{n-1} \mathbb{R}_{\mathrm{pack}(u_1\cdots u_i)} \otimes \mathbb{R}_{\mathrm{pack}(u_{i+1}\cdots u_n)}, \tag{1.26}$$

$$\Delta_{\succ}(\mathbb{R}_u) := \sum_{\substack{i=1 \\ \{u_1,\ldots,u_i\}\cap\{u_{i+1},\ldots,u_n\}=\varnothing \\ u_k=\max(u)}}^{k-1} \mathbb{R}_{\mathrm{pack}(u_1\cdots u_i)} \otimes \mathbb{R}_{\mathrm{pack}(u_{i+1}\cdots u_n)}. \tag{1.27}$$

Note that in Equations (1.26) and (1.27), k is defined by $u_k = \max(u)$. In the case there are several possible k, the condition $\{u_1,\ldots,u_i\} \cap \{u_{i+1},\ldots,u_n\} = \varnothing$ ensure that i varies after (resp. before) all possible k.

Example 1.34.

$$\begin{aligned}
\mathbb{R}_{211} \prec \mathbb{R}_{12} &= \mathbb{R}_{21341} + \mathbb{R}_{23141} + \mathbb{R}_{23411} + \mathbb{R}_{32141} + \mathbb{R}_{32411} + \mathbb{R}_{34211},\\
\mathbb{R}_{221} \succ \mathbb{R}_{12} &= \mathbb{R}_{21134} + \mathbb{R}_{21314} + \mathbb{R}_{23114} + \mathbb{R}_{32114},\\
\Delta_{\prec}(\mathbb{R}_{2125334}) &= \mathbb{R}_{2123} \otimes \mathbb{R}_{112} + \mathbb{R}_{212433} \otimes \mathbb{R}_{1},\\
\Delta_{\succ}(\mathbb{R}_{2125334}) &= \mathbb{R}_{212} \otimes \mathbb{R}_{3112}.
\end{aligned}$$

Theorem 1.35. *$((\mathbf{WQSym})_+, \preceq, \succeq, \Delta_{\preceq}, \Delta_{\succeq})$ and $((\mathbf{WQSym}^*)_+, \prec, \succ, \Delta_{\prec}, \Delta_{\succ})$ are two dual bidendriform bialgebras.*

From now on $\mathrm{Prim}(\mathbf{WQSym})$ and $\mathrm{TPrim}(\mathbf{WQSym})$ are respectively abbreviated to $\mathbf{Prim}$ and $\mathbf{TPrim}$. Moreover, we denote homogeneous components using indices and dualization using a $*$ in exponent as in $\mathbf{Prim}_n^*$. We give the first values of the dimensions, $a_n := \dim(\mathbf{WQSym}_n)$, $p_n := \dim(\mathbf{Prim}_n)$ and $t_n := \dim(\mathbf{TPrim}_n)$:

n	1	2	3	4	5	6	7	8	9	OEIS
a_n	1	3	13	75	541	4 683	47 293	545 835	7 087 261	A000670
p_n	1	2	8	48	368	3 376	35 824	430 512	5 773 936	A095989
t_n	1	1	4	28	240	2 384	26 832	337 168	4 680 272	

Table 1.3: Dimensions of homogeneous components for **WQSym**, **Prim** and **TPrim**.

Though the numbers $(t_n)_n$ are easy to obtain thanks to the relation of Theorem 1.2: $\mathcal{T} = \mathcal{A}/(1+\mathcal{A})^2$, no combinatorial interpretation existed. The first results of this paper are two different subsets of packed words that are counted by these dimensions (red-irreducible and blue-irreducible).

2. Decorated forests

In this paper we will generalize twice the construction of [Foi11], one for $\mathbf{WQSym}$ and one for its dual. This section is devoted to the combinatorial ingredient, that is a notion of biplane forests suitable for indexing the various bases of primitive elements. Each time, we start by decomposing packed words through global descents and removal of specific letters. We then perform those decompositions recursively, encoding the result in a forest. We hence obtain so called biplane forests, which are in bijection with packed words. Later, the recursive structure of forests will be understood as a chaining of brace and dendriform operations generating some elements of $\mathbf{WQSym}$ or its dual. This will allow us to construct two bases of respectively $\mathbf{TPrim}$ and its dual by characterizing a subfamily of biplane trees.

From now on, we associate the color blue to the primal ($\mathbf{WQSym}$) and the color red to the dual ($\mathbf{WQSym}^*$). We start by explaining the construction on $\mathbf{WQSym}^*$ (red) then we dualize the construction to the primal $\mathbf{WQSym}$ (blue).

2.1. Dual (Red)

For the red side $\mathbf{WQSym}^*$, the decomposition of packed words is made through global descents and removal of maximum values. One step of this decomposition is called the red-factorization.

2.1.1 Decomposition of packed words through maximums

In this section, we define two combinatorial operations on packed words (ϕ_I and $\blacktriangleright$) and the red-factorization that uses them. The unary operation ϕ_I inserts new maximums in a word in positions I. A word that cannot be factorized $u \blacktriangleright v$ in a non trivial way is called red-irreducible. Red-irreducible words will index our basis of $\mathbf{TPrim}^*$.

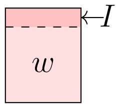
Definition 2.1. Fix $n \in \mathbb{N}$ and $w \in \mathbf{PW}_n$. We write $m' := \max(w) + 1$. For any $p > 0$ and any subset $I \subseteq [1, \dots, n+p]$ of cardinality p, we define $\phi_I(w) := u_1 \dots u_{n+p}$ as the packed word of length $n+p$ obtained by inserting p occurrences of the letter m' in w so that they end up in positions $i \in I$. In other words $u_i = m'$ if $i \in I$ and w is obtained from $\phi_I(w)$ by removing all occurrences of m'. Notice that $\phi_I(w)$ is only defined if $n+p \geqslant i_p$.

Example 2.2. $\phi_{2,4,7}(1232) = 1424324$ and $\phi_{1,2,3}(\epsilon) = 111$.

Note 2.3. For the rest of this paper, $I = [i_1, \dots, i_p]$ will always denote a non-empty ($p > 0$) list of increasing non-zero integers. For any integer k, $I' = I + k$ denote the list $I' = [i_1 + k, \dots, i_p + k]$. Let $\mathbf{PW}_n^I$ denote the set of packed words of size n whose maximums are in positions $i \in I$. This way $\phi_I(w) \in \mathbf{PW}_{n+p}^I$ for any $w \in \mathbf{PW}_n$.

Lemma 2.4. *Let $n \in \mathbb{N}$ and $p > 0$, for any $I = [i_1, \dots, i_p] \subseteq [1, \dots, n+p]$ of size p, ϕ_I is a bijection from $\mathbf{PW}_n$ to $\mathbf{PW}_{n+p}^I$.*

Moreover, for any $W \in \mathbf{PW}_\ell$ where $\ell > 0$ there exists a unique pair (I, w) where $I \subseteq [1 \dots \ell]$ and w is packed, such that $W = \phi_I(w)$.

The box diagram that pictures this lemma is $W =$ 

.

Proof. Let $W \in \mathbf{PW}_\ell$ with $\ell > 0$ and m the value of the maximum letter of W. Let $I = [i_1, \dots, i_p] \subseteq [1, \dots, \ell]$ be the list of the positions of m in W and let w be the word obtain by removing all occurrences of m in W, then $W = \phi_I(w)$. If $\phi_I(u) = \phi_J(v)$ then positions of maximum values are the same so $I = J$ and words obtain by removing these maximum values are also the same so $u = v$. □

Definition 2.5. Let $u, v \in \mathbf{PW}$ with $v \neq \epsilon$. By Lemma 2.4, there is a unique pair (I, v') such that $v = \phi_I(v')$. Let $I' = I + |u|$, we define $u \blacktriangleright v := \phi_{I'}(u/v')$. In other words, we remove the maximum letter of the right word, perform a left shifted concatenation and reinsert the removed letters as new maximums.

Example 2.6. $2123 \blacktriangleright 322312 = 2123 \blacktriangleright \phi_{1,4}(2212) = \phi_{1+4,4+4}(43452212) = 4345622612$.

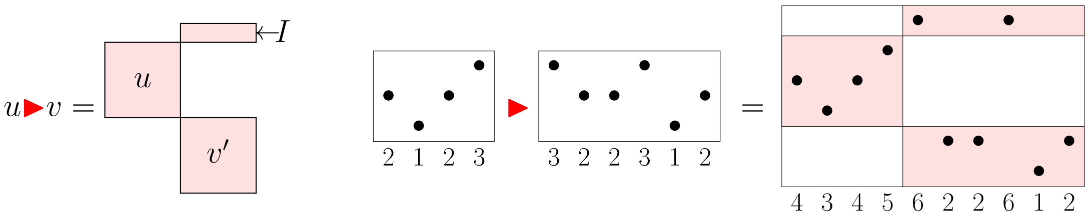


Figure 2.1: Box digrams: the operation $\blacktriangleright$.

Lemma 2.7. *Let w be an irreducible packed word. There exists a unique factorization of the form $w = u \blacktriangleright v$ which maximizes the size of u. In this factorization, let v' and I be such that $v = \phi_I(v')$, then*

- *either $v' = \epsilon$ and $I = [1, \dots, p]$ for some p,*
- *or the global descent decomposition $v' = v_1/\dots/v_r$ of v' satisfies the inequalities $1 \leqslant i_1 \leqslant |v_1|$, and $1 \leqslant (|I| + |v'|) + 1 - i_p \leqslant |v_r|$ with $I = [i_1, \dots, i_p]$.*

We call it the ***red-factorization*** *of a word.*

Example 2.8. Here is a first detailed example of a red-factorization of an irreducible packed word:

Consider the irreducible packed word $w = 543462161$.

- The first step is to remove all the occurrences of the maximal value but keep in memory the positions in the initial word. We get $w' = 5434\_21\_1$ which is a packed word, but is not irreducible.

- The second step is to decompose the new word w' in irreducible factors $w' = 1/212/\_1/1\_1$. We still keep in memory the positions of the removed value. (when we have the choice, we cut to the left of the removed value.)
- We can distinguish two groups of factors, those strictly before the first maximum withdrawn and the others $w' = 1/212 \quad / \quad \_1/1\_1$.
- Finally, by numbering the positions of the maximum removed value in the right factor (positions 1 and 4), we get the following decomposition of w (see Definition 2.1 for ϕ and Definition 2.5 for $\blacktriangleright$):
  $w = 543462161 = (1/212)\blacktriangleright\phi_{1,4}(1/11) = (3212)\blacktriangleright\phi_{1,4}(211) = 3212\blacktriangleright 32131.$

Example 2.9. Here are some other red-factorizations:

$$21331 = 1\blacktriangleright\phi_{2,3}(11) = 1\blacktriangleright 1221 \qquad 1231 = \epsilon\blacktriangleright\phi_3(121) = \epsilon\blacktriangleright 1231$$
$$1233 = 12\blacktriangleright\phi_{1,2}(\epsilon) = 12\blacktriangleright 11 \qquad 111 = \epsilon\blacktriangleright\phi_{1,2,3}(\epsilon) = \epsilon\blacktriangleright 111$$
$$56434126 = 1\blacktriangleright\phi_{1,7}(212/12) = 1\blacktriangleright\phi_{1,7}(43412) = 1\blacktriangleright 5434125$$

Proof. Let w be irreducible and let (I, w') be the unique pair such that $w = \phi_I(w')$ according to Lemma 2.4. By Lemma 1.16, we write $w' = w'_1/w'_2/\dots/w'_k$, the unique decomposition into irreducibles. Let ℓ be such that w'_ℓ is the last factor which is entirely before the first removed maximum, it is the only choice to maximize the size of u. Then with $r = k - \ell$ we can rewrite w' as $(u_1/\dots/u_\ell)/(v_1/\dots/v_r)$. Now we get I' by subtracting $|u_1/\dots/u_\ell|$ to all parts of I ($I' = I - |u_1/\dots/u_\ell|$) and we obtain

$$w = u_1/\dots/u_\ell\blacktriangleright\phi_{I'}(v_1/\dots/v_r)$$

with $i'_1 \leqslant |v_1|$ or $r = 0$.

In the case of $v' \neq \epsilon$, the inequality $(|v'| + |I|) + 1 - i_p \leqslant |v_r|$ is always true otherwise w would not be irreducible. □

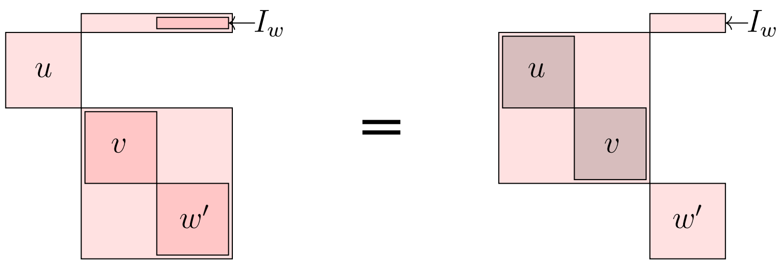
Definition 2.10. A packed word w is said to be **red-irreducible** if w is irreducible and the equality $w = u\blacktriangleright v$ implies that $u = \epsilon$(and $w = v$).

Here are all red-irreducible packed words of size 1, 2, 3 and 4 in lexicographic order:

$$1, \quad 11, \quad 111\ 121\ 132\ 212,$$
$$1111\ 1121\ 1132\ 1211\ 1212\ 1221\ 1231\ 1232\ 1243\ 1312\ 1321\ 1322\ 1323\ 1332$$
$$1342\ 1423\ 1432\ 2112\ 2121\ 2122\ 2132\ 2143\ 2212\ 2312\ 2413\ 3123\ 3132\ 3213.$$

Here are some useful lemmas on the operation $\blacktriangleright$.

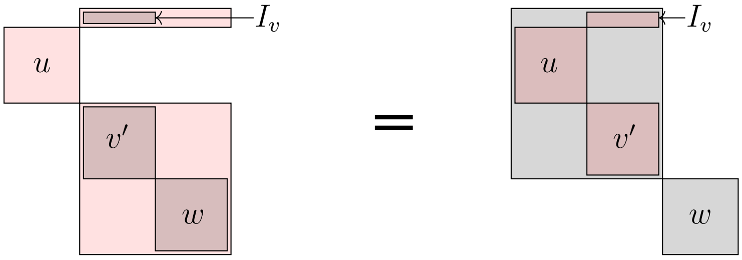
Lemma 2.11. *For any $u, v, w \in \mathbf{PW}$ with $w \neq \epsilon$, we have $u\blacktriangleright(v\blacktriangleright w) = (u/v)\blacktriangleright w$.*



Proof. Let $u, v, w \in \mathbf{PW}$ with $w \neq \epsilon$ and let w' and I_w such that $w = \phi_{I_w}(w')$.

$$\begin{aligned} u \blacktriangleright (v \blacktriangleright w) &= u \blacktriangleright (\phi_{I_w+|v|}(v/w')) \\ &= \phi_{I_w+|v|+|u|}(u/(v/w')) \\ &= \phi_{I_w+|v|+|u|}((u/v)/w') \\ &= (u/v) \blacktriangleright \phi_{I_w}(w') \\ &= (u/v) \blacktriangleright w. \end{aligned}$$
□

Lemma 2.12. *For any $u, v, w \in \mathbf{PW}$ with $v \neq \epsilon$, we have $u \blacktriangleright (v/w) = (u \blacktriangleright v)/w$.*



Proof. Let $u, v, w \in \mathbf{PW}$ with $v \neq \epsilon$ and let v' and I_v such that $v = \phi_{I_v}(v')$.

$$\begin{aligned} u \blacktriangleright (v/w) &= u \blacktriangleright \phi_{I_v}(v'/w) \\ &= \phi_{I_v+|u|}(u/(v'/w)) \\ &= \phi_{I_v+|u|}((u/v')/w) \\ &= \phi_{I_v+|u|}(u/v')/w \\ &= (u \blacktriangleright v)/w. \end{aligned}$$
□

Remark 2.13. Adding the associativity of shifted concatenation $u/(v/w) = (u/v)/w$, the two operations $\blacktriangleright$ and $/$ verify relations of the *skew-duplicial operad* [BDO20].

Corollary 2.14. *For any $u, v \in \mathbf{PW}$, we have that $u \blacktriangleright v$ is irreducible if and only if v is irreducible.*

Proof. By contradiction, if $v = v_1/v_2$ then by Lemma 2.12 $u \blacktriangleright v = (u \blacktriangleright v_1)/v_2$. Now if $u \blacktriangleright v = w_1/w_2$, as the position of the first maximum of $u \blacktriangleright v$ is greater than $|u|$ we have that $w_1 = w_1' \cdot w_1''$ such that $\mathrm{pack}(w_1') = u$. We also have that $\mathrm{pack}(w_1'')/w_2 = v$. □

Proposition 2.15. *For any word w, $w = u \blacktriangleright v$ is the red-factorization of w if and only if v is red-irreducible.*

Proof. Let $w \in \mathbf{PW}$ and let $u \blacktriangleright v$ be the red-factorization of w. Let v_1 and v_2 such that $v = v_1 \blacktriangleright v_2$, then $(u/v_1) \blacktriangleright v_2 = w$ by Lemma 2.11, but in the red-factorization the size of u is maximized so $|(u/v_1)| \leqslant |u|$ and then we have that $v_1 = \epsilon$ so v is red-irreducible.

Let $w \in \mathbf{PW}$ and let u and v such that $w = u \blacktriangleright v$ and v is red-irreducible. By contradiction, suppose that there exists u', v' such that $w = u' \blacktriangleright v'$ with $|u| < |u'|$ and $v' \neq \epsilon$. Then necessarily u is a prefix of u'. Let u'' such that $u' = u \cdot u''$, then $pack(u'') \blacktriangleright v' = v$. But v is red-irreducible. So the size of u is maximal if v is red-irreducible. □

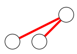
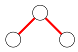
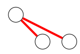
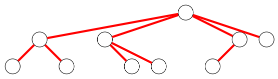
For the reader who is familiar with ordered set partitions, all the definitions in Section 2.1 can be easily written with these. However in Section 2.2 it is easier to do all the definitions on packed words and in Section 4 we must have the same object on both sides to explicit the isomorphism. So we decided to stick to packed words.

2.1.2 Red-forests from decomposed packed words using ϕ

We now apply recursively the red-factorization of the previous section to construct a bijection between packed words and a certain kind of trees that we now define.

Definition 2.16. An unlabeled **biplane tree** is an ordered tree whose children are organized in a pair of two (possibly empty) ordered forests, which we call the left and right forests, a forest being an ordered list of trees.

In the picture, we naturally draw the children of the left (resp. right) forest on the left (resp. right) of their father.

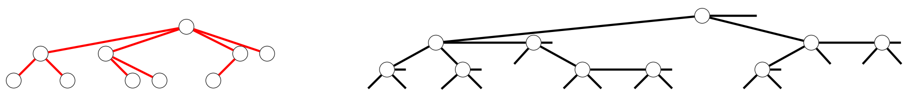
Example 2.17. The biplane trees  ,  and  are different. Indeed in the first case, the left forest contains two trees and the right forest is empty, in the second case both forests contain exactly one tree while in the third case the left forest is empty and the right contains two trees. Here is an example of a bigger biplane tree where the root has two trees in both left and right forests  .

Definition 2.18. A **skeleton biplane tree** is a biplane tree where no node has a right forest.

These skeleton biplane trees can also be seen as planar trees. In [Foi11] we have planar trees recursively labeled by planar trees. Skeleton biplane trees are similar to these planar trees, we prefer to see them as biplane tree with no right forest in order to keep some constistency.

Definition 2.19. The **size** of a biplane tree is the number of node in the tree.

Remark 2.20. Biplane forests $\mathfrak{F}$ (*i.e.* ordered list of biplane trees $\mathfrak{T}$) are counted by the sequence $A001764$ in OEIS [SI20] whose explicit formula is $a(n) = \binom{3n}{n}/(2n+1)$. Biplane forests are in bijection with ternary trees. We give a bijection that is inspired from the well known bijection [FS09, I.5.3] between plane forests and binary trees. The bijection is the following: in a biplane forest a node has a first left child and a first right child and a right brother. A consequence is that unlabeled biplane trees are counted by the sequence $A006013$ in OEIS [SI20] whose explicit formula is $a(n) = \binom{3n+1}{n}/(n+1)$. Indeed, biplane trees are in bijection with pair of ternary trees. Here is an example of the ternary tree in bijection with the biplane forest constituted of one tree, the big biplane tree in Example 2.17 that we show again

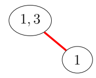
 .

n	0	1	2	3	4	5	6	7	8	OEIS
$\mathfrak{F}_n$	1	1	3	12	55	273	1 428	7 752	43 263	A001764
$\mathfrak{T}_n$	0	1	2	7	30	143	728	3 876	21 318	A006013

Table 2.1: Number of biplane forests and biplane trees.

Remark 2.21. As we can see on OEIS [SI20], sequence $A006013$ which counts unlabeled biplane trees is the dimensions of the free L-algebra on one generator (see [Ler11]). It would be interesting to investigate the link between L-algebras and bidendriform bialgebras using biplane trees.

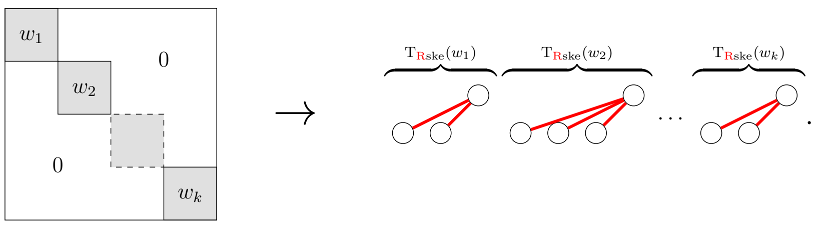
In our construction we will deal with labeled biplane trees with colored edges. For a labeled biplane tree, we denote by $\mathrm{Node}_{\mathrm{R}}(x, f_\ell, f_r)$ the tree whose edges are colored in red, root is labeled by x and whose left (resp. right) forest is given by f_ℓ (resp. f_r). We also denote by $[t_1, \dots t_k]$ a forest of k trees. The edge color (for now, only red) will play a role later in the paper.

Example 2.22. $\mathrm{Node}_{\mathrm{R}}((1), [], []) =$  , and $\mathrm{Node}_{\mathrm{R}}((1,3), [], [\mathrm{Node}_{\mathrm{R}}((1), [], [])]) =$  .

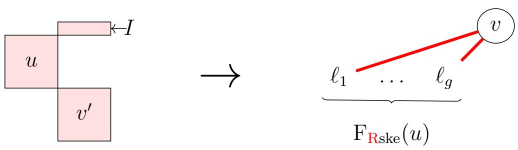
We now apply recursively the global descent decomposition and the red-factorization of Lemmas 1.16 and 2.7. We obtain an algorithm which takes a packed word and returns a biplane forest where nodes are decorated by red-irreducible packed words:

Definition 2.23. We now define two functions $\mathrm{F}_{\mathrm{Rske}}$ and $\mathrm{T}_{\mathrm{Rske}}$. These functions transform respectively a packed word and an irreducible packed word into respectively a skeleton biplane forest and a skeleton biplane tree labeled by red-irreducible words. These functions are defined in a mutual recursive way as follows:

- $\mathrm{F}_{\mathrm{Rske}}(\epsilon) = []$ (empty forest),
- for any packed word w, let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w, then $\mathrm{F}_{\mathrm{Rske}}(w) := [\mathrm{T}_{\mathrm{Rske}}(w_1), \mathrm{T}_{\mathrm{Rske}}(w_2), \dots, \mathrm{T}_{\mathrm{Rske}}(w_k)]$.



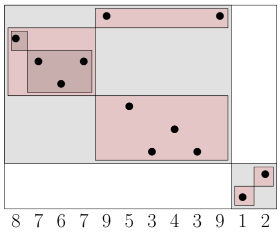
- for any irreducible packed word w, let $w = u \blacktriangleright v$ be the red-factorization of w. We define $\mathrm{T}_{\mathrm{Rske}}(w) := \mathrm{Node}_{\mathrm{R}}(v, \mathrm{F}_{\mathrm{Rske}}(u), [])$.



with $v = \phi_I(v')$.

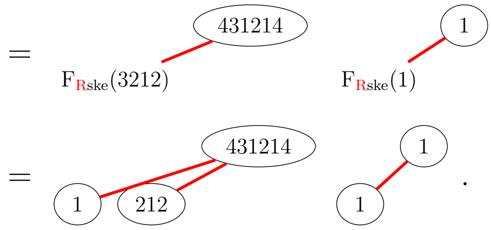
Example 2.24. Let $w = 876795343912$, the global descent decomposition of Lemma 1.16 gives $w = w_1/w_2$ with $w_1 = 6545731217$ and $w_2 = 12$. Now, we have the red-factorization of w_1 and w_2 using Lemma 2.7 as

$$w_1 = 3212 \blacktriangleright \phi_{1,6}(3121) = (1/212) \blacktriangleright 431214 \quad \text{and} \quad w_2 = 1 \blacktriangleright \phi_1(\epsilon) = 1 \blacktriangleright 1.$$



It gives the following forest:

$$\mathrm{F}_{\mathrm{Rske}}(876795343912) = [\mathrm{T}_{\mathrm{Rske}}(6545731217),\ \mathrm{T}_{\mathrm{Rske}}(12)]$$

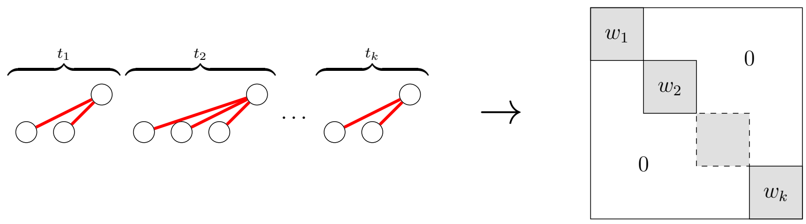


Definition 2.25. A labeled biplane forest (resp. tree) is a **red-skeleton forest** (resp. **tree**) if it is labeled by red-irreducible words and no node has a right child.

We want to prove that the functions $\mathrm{F}_{\mathrm{Rske}}$ and $\mathrm{T}_{\mathrm{Rske}}$ are bijections. To do that we first define two functions that are the inverses.

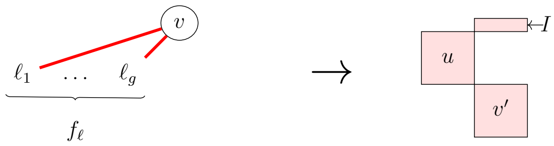
Definition 2.26. We now define two functions $\mathrm{F}^*_{\mathrm{Rske}}$ and $\mathrm{T}^*_{\mathrm{Rske}}$ that transform respectively a red-skeleton forest and tree into packed words. These functions are defined in a mutual recursive way as follows:

- $\mathrm{F}^*_{\mathrm{Rske}}([]) = \epsilon$,
- for any red-skeleton forest $f = [t_1, \dots, t_k]$, we define $\mathrm{F}^*_{\mathrm{Rske}}(f) := \mathrm{T}^*_{\mathrm{Rske}}(t_1)/ \dots / \mathrm{T}^*_{\mathrm{Rske}}(t_k)$.



with $w_i = \mathrm{T}^*_{\mathrm{Rske}}(t_i)$.

- for any red-skeleton tree $t = \mathrm{Node}_{\mathrm{R}}(v, f_\ell, [])$, we define $\mathrm{T}^*_{\mathrm{Rske}}(t) := \mathrm{F}^*_{\mathrm{Rske}}(f_\ell) \blacktriangleright v$.



with $v = \phi_I(v')$ and $u = \mathrm{F}^*_{\mathrm{Rske}}(f_\ell)$.

Lemma 2.27. *The functions* $\mathrm{F}_{\mathrm{Rske}}$ *and* $\mathrm{F}^*_{\mathrm{Rske}}$ *(resp.* $\mathrm{T}_{\mathrm{Rske}}$ *and* $\mathrm{T}^*_{\mathrm{Rske}}$*) are two converse bijections between packed words and red-skeleton forests (resp. irreducible packed words and red-skeleton trees). That is to say* $\mathrm{F}^{-1}_{\mathrm{Rske}} = \mathrm{F}^*_{\mathrm{Rske}}$ *and* $\mathrm{T}^{-1}_{\mathrm{Rske}} = \mathrm{T}^*_{\mathrm{Rske}}$.

Proof. We start to prove that domain and codomain are as announced (see Items (a) and (b) bellow), then we prove that the functions $\mathrm{F}_{\mathrm{Rske}}$ and $\mathrm{F}^*_{\mathrm{Rske}}$ (resp. $\mathrm{T}_{\mathrm{Rske}}$ and $\mathrm{T}^*_{\mathrm{Rske}}$) are inverse to each other (see Items (c) and (d) bellow).

(a) By Definition 2.23, a forest (resp. tree) obtain by $\mathrm{F}_{\mathrm{Rske}}$ (resp. $\mathrm{T}_{\mathrm{Rske}}$) is a red-skeleton forest (resp. tree). Indeed, thanks to Proposition 2.15 nodes are labeled by red-irreducible words because a red-factorization is done and nodes have no right children.

(b) We prove by a mutual induction that $\mathrm{F}^*_{\mathrm{Rske}}$ returns a packed word and that $\mathrm{T}^*_{\mathrm{Rske}}$ returnes an irreducible packed word. Indeed, we do an induction on the size of the forest or the tree. Here is our induction hypothesis for $n \in \mathbb{N}$:

$$\begin{aligned} &\forall t, \text{red-skeleton tree of size } \leqslant n, \mathrm{T}^*_{\mathrm{Rske}}(t) \text{ is an irreducible packed word,} \\ &\forall f, \text{red-skeleton forest of size } \leqslant n, \mathrm{F}^*_{\mathrm{Rske}}(f) \text{ is a packed word.} \end{aligned} \tag{2.1}$$

The base case ($n = 0$) is given by the first item of Definition 2.26 (*i.e.* $\mathrm{F}^*_{\mathrm{Rske}}([]) = \epsilon$ the empty packed word).

Now, let us fix $n \geqslant 1$ and suppose that the hypothesis (2.1) holds. Let $f = [t_1, \dots, t_k]$ be a red-skeleton forest of size $n + 1$.

- If $k = 1$, then f is reduced to a single tree t. We need to prove that $\mathrm{T}^*_{\mathrm{Rske}}(t)$ is an irreducible packed word (which also gives that $\mathrm{F}^*_{\mathrm{Rske}}(f)$ is a packed word as in this case $\mathrm{F}^*_{\mathrm{Rske}}(f) = \mathrm{T}^*_{\mathrm{Rske}}(t)$). Let $t = \mathrm{Node}_{\mathrm{R}}(v, f_\ell, [])$ be a red-skeleton tree of size $n + 1$ (notice that the word v can be of any size). The size of f_ℓ is n, so by induction $\mathrm{F}^*_{\mathrm{Rske}}(f_\ell)$ is a packed word, as v is red-irreducible it is by definition irreducible so by Corollary 2.14 $\mathrm{T}^*_{\mathrm{Rske}}(t) = \mathrm{F}^*_{\mathrm{Rske}}(f_\ell) \blacktriangleright v$ is irreducible.
- If $k \geqslant 2$, *i.e.*, the forest contains at least two trees, since all trees are of size at least one, $t_1, \dots, t_k$ are at must of size n, so we have by induction that $\mathrm{T}^*_{\mathrm{Rske}}(t_1), \dots \mathrm{T}^*_{\mathrm{Rske}}(t_k)$ are irreducible packed words. $\mathrm{F}^*_{\mathrm{Rske}}(f)$ is the shifted concatenation of $\mathrm{T}^*_{\mathrm{Rske}}(t_1), \dots \mathrm{T}^*_{\mathrm{Rske}}(t_k)$ and thus it is a packed word.

(c) We now prove by a mutual induction on the size of the forest or the tree that, for $n \in \mathbb{N}$:

$$\begin{aligned}&\forall t, \text{red-skeleton tree of size } \leqslant n, \mathrm{T}_{\mathrm{Rske}}(\mathrm{T}^*_{\mathrm{Rske}}(t)) = t,\\ &\forall f, \text{red-skeleton forest of size } \leqslant n, \mathrm{F}_{\mathrm{Rske}}(\mathrm{F}^*_{\mathrm{Rske}}(f)) = f.\end{aligned} \tag{2.2}$$

The base case ($n = 0$) is given by the first item of Definitions 2.23 and 2.26 as $\mathrm{F}_{\mathrm{Rske}}(\mathrm{F}^*_{\mathrm{Rske}}([])) = \mathrm{F}_{\mathrm{Rske}}(\epsilon) = []$.

Now let us fix $n \geqslant 1$ and suppose that the hypothesis (2.2) holds. Let $f = [t_1, \ldots, t_k]$ be a red-skeleton forest of size $n + 1$.

- If $k = 1$, then the forest f is reduced to a single tree t, then it is sufficient to prove $\mathrm{T}_{\mathrm{Rske}}(\mathrm{T}^*_{\mathrm{Rske}}(t)) = t$ (as in this case $\mathrm{F}^*_{\mathrm{Rske}}(f) = \mathrm{T}^*_{\mathrm{Rske}}(t)$). Let $t = \mathrm{Node}_{\mathrm{R}}(v, f_\ell, [])$ a red-skeleton tree of size $n + 1$. As the label v is a red-irreducible packed word, with the induction hypothesis on $\mathrm{F}^*_{\mathrm{Rske}}(f_\ell)$ and with Proposition 2.15, $\mathrm{F}^*_{\mathrm{Rske}}(f_\ell) \blacktriangleright v$ is the red-factorization so:

$$\begin{aligned}\mathrm{T}_{\mathrm{Rske}}(\mathrm{T}^*_{\mathrm{Rske}}(t)) &= \mathrm{T}_{\mathrm{Rske}}(\mathrm{F}^*_{\mathrm{Rske}}(f_\ell) \blacktriangleright v)\\ &= \mathrm{Node}_{\mathrm{R}}(v, \mathrm{F}_{\mathrm{Rske}}(\mathrm{F}^*_{\mathrm{Rske}}(f_\ell), []))\\ &= \mathrm{Node}_{\mathrm{R}}(v, f_\ell, []) = t.\end{aligned}$$

- If $k \geqslant 2$, since all trees are of size at least one, they are at most of size n, so we have by induction that:

$$\begin{aligned}\mathrm{F}_{\mathrm{Rske}}(\mathrm{F}^*_{\mathrm{Rske}}(f)) &= \mathrm{F}_{\mathrm{Rske}}(\mathrm{T}^*_{\mathrm{Rske}}(t_1) / \ldots / \mathrm{T}^*_{\mathrm{Rske}}(t_k))\\ &\qquad \text{as } \mathrm{T}^*_{\mathrm{Rske}}(t_i) \text{ are irreducible packed words}\\ &= [\mathrm{T}_{\mathrm{Rske}}(\mathrm{T}^*_{\mathrm{Rske}}(t_1)), \ldots, \mathrm{T}_{\mathrm{Rske}}(\mathrm{T}^*_{\mathrm{Rske}}(t_k))]\\ &= [t_1, \ldots, t_k] = f.\end{aligned}$$

(d) Finally we prove by a mutual induction on the size of the word w that, for $n \in \mathbb{N}$:

$$\begin{aligned}&\forall v \in \mathbf{PW}, \text{irreducible packed word of size } \leqslant n, \mathrm{T}^*_{\mathrm{Rske}}(\mathrm{T}_{\mathrm{Rske}}(v)) = v,\\ &\forall w \in \mathbf{PW}, \text{packed word of size } \leqslant n, \mathrm{F}^*_{\mathrm{Rske}}(\mathrm{F}_{\mathrm{Rske}}(w)) = w.\end{aligned} \tag{2.3}$$

The base case ($n = 0$) is given by the first item of Definitions 2.23 and 2.26 as $\mathrm{F}^*_{\mathrm{Rske}}(\mathrm{F}_{\mathrm{Rske}}(\epsilon)) = \mathrm{F}^*_{\mathrm{Rske}}([]) = \epsilon$.

Now let us fix $n \geqslant 1$ and suppose that the hypothesis (2.3) holds. Let $w \in \mathbf{PW}_{n+1}$ a packed word of size $n + 1$. Let $w = w_1 / w_2 / \ldots / w_k$ be the global descent decomposition of w.

- If $k = 1$, the packed word w is irreducible then $\mathrm{F}_{\mathrm{Rske}}(w) = [\mathrm{T}_{\mathrm{Rske}}(w)]$ so we need to prove that $\mathrm{T}^*_{\mathrm{Rske}}(\mathrm{T}_{\mathrm{Rske}}(w)) = w$. Let $w = u \blacktriangleright v$ be the red-factorization of w, then we can use the induction hypothesis on u, indeed as v is not empty the size of u is smaller than n:

$$\begin{aligned}\mathrm{T}^*_{\mathrm{Rske}}(\mathrm{T}_{\mathrm{Rske}}(w)) &= \mathrm{T}^*_{\mathrm{Rske}}(\mathrm{Node}_{\mathrm{R}}(v, \mathrm{F}_{\mathrm{Rske}}(u), [])) \\ &= \mathrm{F}^*_{\mathrm{Rske}}(\mathrm{F}_{\mathrm{Rske}}(u)) \blacktriangleright v \\ &= u \blacktriangleright v = w.\end{aligned}$$

- If $k \geqslant 2$, then we use the induction hypothesis on each factors, so we have:

$$\begin{aligned}\mathrm{F}^*_{\mathrm{Rske}}(\mathrm{F}_{\mathrm{Rske}}(w)) &= \mathrm{F}^*_{\mathrm{Rske}}([\mathrm{T}_{\mathrm{Rske}}(w_1), \dots, \mathrm{T}_{\mathrm{Rske}}(w_k)]) \\ &= \mathrm{T}^*_{\mathrm{Rske}}(\mathrm{T}_{\mathrm{Rske}}(w_1)) / \dots / \mathrm{T}^*_{\mathrm{Rske}}(\mathrm{T}_{\mathrm{Rske}}(w_k)) \\ &= w_1 / \dots / w_k = w.\end{aligned}$$

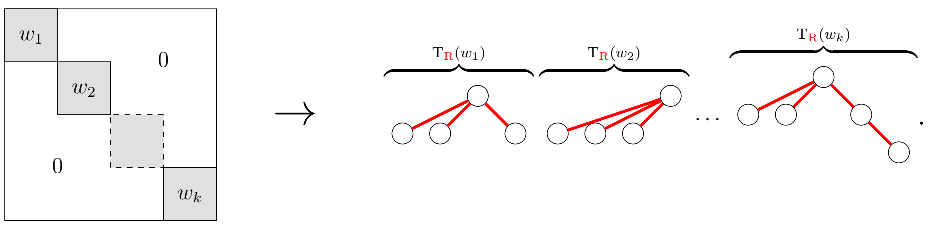
□

Now that we have the red-skeleton, we will add right forests to every nodes to obtain biplane trees. For every node, if v is the red-irreducible word in label, with $\phi_I(v') = v$, then I is the new label and $\mathrm{F}_{\mathrm{R}}(v')$ is the new right forest.

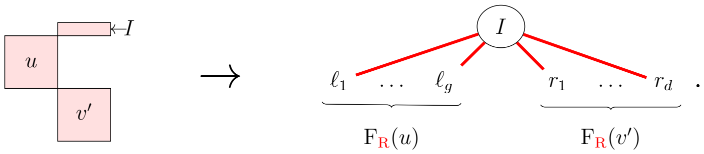
Here is the formal definition of $\mathrm{F}_{\mathrm{R}}(w)$ and $\mathrm{T}_{\mathrm{R}}(w)$ which are very similar to Definition 2.23, only the third item is different. The labels are now lists of integers.

Definition 2.28. The forest $\mathrm{F}_{\mathrm{R}}(w)$ (resp. tree $\mathrm{T}_{\mathrm{R}}(w)$) associated to a packed word (resp. irreducible packed word) w are defined in a mutual recursive way as follows:

- $\mathrm{F}_{\mathrm{R}}(\epsilon) = []$ (empty forest),
- for any packed word w, let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w, then $\mathrm{F}_{\mathrm{R}}(w) := [\mathrm{T}_{\mathrm{R}}(w_1), \mathrm{T}_{\mathrm{R}}(w_2), \dots, \mathrm{T}_{\mathrm{R}}(w_k)]$.

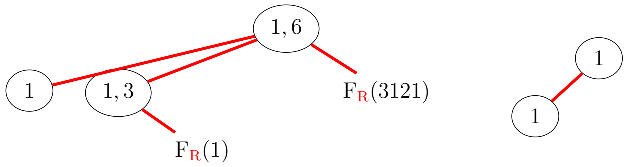


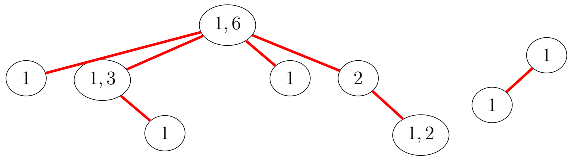
- for any irreducible packed word w, we define $\mathrm{T}_{\mathrm{R}}(w) := \mathrm{Node}_{\mathrm{R}}(I, \mathrm{F}_{\mathrm{R}}(u), \mathrm{F}_{\mathrm{R}}(v'))$ where $w = u \blacktriangleright \phi_I(v')$ is the red-factorization of w.



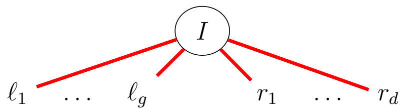
Example 2.29. Consider again $w = 876795343912$. We start from the red-skeleton forest from Example 2.24.

$\mathrm{F}_{\mathrm{Rske}}(876795343912) =$ 

$\mathrm{F}_{\mathrm{R}}(876795343912) =$ 

$\mathrm{F}_{\mathrm{R}}(876795343912) =$ 

Definition 2.30. Let t be a labeled biplane tree. We write $t = \mathrm{Node}_{\mathrm{R}}(I, f_\ell, f_r)$ where $I = [i_1, \dots, i_p]$, $p > 0$, $(1 \leqslant i_1 < \cdots < i_p)$, $f_\ell = [\ell_1, \dots, \ell_g]$ and $f_r = [r_1, \dots, r_d]$, which is depicted as follows:

$t =$  .

The **weight** of t is recursively defined by $\omega(t) = p + \sum_{i=0}^{g} \omega(\ell_i) + \sum_{j=0}^{d} \omega(r_j)$. In particular, if t is a single node then $\omega(t) = p$. By extension, the **weight** of a forest is the sum of the weight of its trees.

Lemma 2.31. *The weight of a forest (resp. a tree) obtained by the functions* F_{R} *(resp.* T_{R}*) is equal to the size of the word, i.e. For all* $w \in \mathbf{PW}$ *then* $\omega(\mathrm{F}_{\mathrm{R}}(w)) = |w|$ *and for all* $w \in \mathbf{PW}$ *with* w *irreducible then* $\omega(\mathrm{T}_{\mathrm{R}}(w)) = |w|$.

Proof. We prove by induction with the following hypothesis, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall w \in \mathbf{PW}_n, \quad & \omega(\mathrm{F}_{\mathrm{R}}(w)) = |w|, \\ \forall w \in \mathbf{PW}_n \text{ with } w \text{ irreducible}, \quad & \omega(\mathrm{T}_{\mathrm{R}}(w)) = |w|. \end{aligned} \tag{2.4}$$

The base case is given by the first item of Definition 2.28 as $\mathrm{F}_{\mathrm{R}}(\epsilon) = []$ and $\omega([]) = |\epsilon| = 0$.

Let us fix $n \geqslant 1$ and suppose that the hypothesis (2.4) holds. Let $w \in \mathbf{PW}_{n+1}$ and $w = w_1/w_2/\dots/w_k$ be the global descent decomposition of w.

- If $k = 1$, we have $\mathrm{F}_{\mathrm{R}}(w) = [\mathrm{T}_{\mathrm{R}}(w)]$. Let $w = u \blacktriangleright \phi_I(v)$ with $I = [i_1, \dots, i_p], p > 0$ be the red-factorization of w, then

$$\omega(\mathrm{T}_{\mathrm{R}}(w)) = \omega(\mathrm{Node}_{\mathrm{R}}(I, \mathrm{F}_{\mathrm{R}}(u), \mathrm{F}_{\mathrm{R}}(v))) = p + \omega(\mathrm{F}_{\mathrm{R}}(u)) + \omega(\mathrm{F}_{\mathrm{R}}(v)).$$

  As $p > 0$, the sizes of u and v are at most n, by induction $\omega(\mathrm{T}_{\mathrm{R}}(w)) = p + |u| + |v| = |w|$.

- If $k \geqslant 2$, by induction on each factors, we have that

$$\omega(\mathrm{F}_{\mathrm{R}}(w)) = \omega(\mathrm{T}_{\mathrm{R}}(w_1)) + \cdots + \omega(\mathrm{T}_{\mathrm{R}}(w_k)) = |w_1| + \cdots + |w_k| = |w|. \qquad \square$$

Definition 2.32. Using the same notations as in previous Definition 2.30, we say that t is a **red-packed tree** if it satisfies:

$$\begin{cases} d = 0, \\ i_k = k \text{ for all } k \leqslant p, \\ \ell_1, \ldots, \ell_g \text{ are red-packed trees.} \end{cases} \quad \text{or} \quad \begin{cases} d \geqslant 1, \\ 1 \leqslant \quad i_1 \quad \leqslant \omega(r_1), \\ 1 \leqslant \quad p + \omega(f_r) + 1 - i_p \quad \leqslant \omega(r_d), \\ \ell_1, \ldots, \ell_g \text{ and } r_1, \ldots, r_d \text{ are red-packed trees.} \end{cases}$$

An ordered list of red-packed trees is a **red-packed forest**.

Remark 2.33. Red-skeleton trees can be interpreted as flattened representations of red-packed trees. Symmetrically, red-packed trees can be interpreted as unfolded representations of red-skeleton trees. We use the operation ϕ_I to change between red-packed and red-skeleton trees.

Note 2.34. From now on, we use these notations:

- $\mathfrak{F}_{Rn}$ the set of red-packed forests of weight n, ($\mathfrak{F}_{Rn} := \{\mathrm{F}_{\mathrm{R}}(w)\}_{w \in \mathbf{PW}_n}$),
- $\mathfrak{T}_{Rn}$ the set of red-packed trees of weight n, ($\mathfrak{T}_{Rn} := \{\mathrm{T}_{\mathrm{R}}(w)\}_{w \in \mathbf{PW}_n}$ with w irreducible),
- $\mathfrak{N}_{Rn}$ the set of red-packed trees of weight n such that the left forest of the root is empty, ($\mathfrak{N}_{Rn} := \{\mathrm{T}_{\mathrm{R}}(w)\}_{w \in \mathbf{PW}_n}$ with w red-irreducible). In particular, the red-skeleton of a tree of $\mathfrak{N}_{Rn}$ consist of a single node labeled by a red-irreducible word.

Remark 2.35. The set $\mathfrak{N}_{Rn}$ can be described as a disjointed union of sets depending on $I = [i_1, \ldots, i_p]$. Let $\mathfrak{F}^I_{Rn}$ denote the set of red-packed forests of weight n that can be right children of a node labeled by I (see Definition 2.32 for conditions), we have the following description:

$$\mathfrak{N}_{Rn} = \bigsqcup_I \{\mathrm{Node}_{\mathrm{R}}(I, [], f_r) \mid f_r \in \mathfrak{F}^I_{Rn-p}\}. \tag{2.5}$$

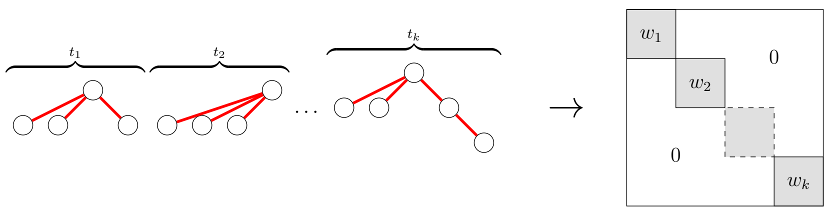
Analogously, we use $\mathfrak{F}_{Rske}$, $\mathfrak{T}_{Rske}$ and $\mathfrak{N}_{Rske}$ for red-skeleton forests, trees and trees with only one node.

We can remark that for $n = 1$ we have $\mathfrak{N}_{R1} = \mathfrak{T}_{R1} = \mathfrak{F}_{R1}$ and $\forall n > 1, \mathfrak{N}_{Rn} \subsetneq \mathfrak{T}_{Rn} \subsetneq \mathfrak{F}_{Rn}$.

As with Definition 2.26, we want to prove that the functions F_{R} and T_{R} are bijections. To do that we first define the two inverse functions.

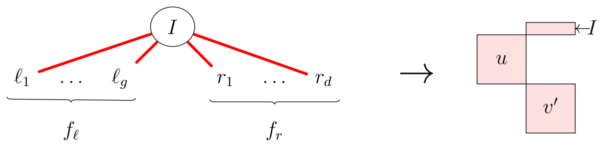
Definition 2.36. We define here the functions $\mathrm{F}^*_{\mathrm{R}}$ (resp. $\mathrm{T}^*_{\mathrm{R}}$) that transform a red-packed forest f (resp. tree t) into a packed word. We reverse all instructions of Definition 2.28 as follows:

- $\mathrm{F}^*_{\mathrm{R}}([]) = \epsilon$,
- for any non empty red-packed forest $f = [t_1, t_2 \ldots, t_k]$, then $\mathrm{F}^*_{\mathrm{R}}(f) = \mathrm{T}^*_{\mathrm{R}}(t_1) / \mathrm{T}^*_{\mathrm{R}}(t_2) / \ldots / \mathrm{T}^*_{\mathrm{R}}(t_k)$.



with $w_i = \mathrm{T}^*_{\mathrm{Rske}}(t_i)$.

- for any non empty red-packed tree $t = \mathrm{Node}_{\mathrm{R}}(I, f_\ell, f_r)$, then $\mathrm{T}^*_{\mathrm{R}}(t) = \mathrm{F}^*_{\mathrm{R}}(f_\ell) \blacktriangleright \phi_I(\mathrm{F}^*_{\mathrm{R}}(f_r))$.



with $v' = \mathrm{F}^*_{\mathrm{R}}(f_r)$ and $u = \mathrm{F}^*_{\mathrm{Rske}}(f_\ell)$.

There might be a problem with this definition since $\phi_I(\mathrm{F}^*_{\mathrm{R}}(f_r))$ is only defined if $i_p \leqslant |\mathrm{F}^*_{\mathrm{R}}(f_r)| + p$ (see Definition 2.1). We prove in the following Lemma 2.37 that the inequality holds if $t \in \mathfrak{T}_R$.

Lemma 2.37. *For any red-packed forest f, $\mathrm{F}^*_{\mathrm{R}}(f)$ is a well defined word of size $\omega(f)$. For any red-packed tree t, $\mathrm{T}^*_{\mathrm{R}}(t)$ is a well defined word of size $\omega(t)$.*

Proof. We prove by induction with the following hypothesis, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall f \in \mathfrak{F}_{R\leqslant n}, \quad & \mathrm{F}^*_{\mathrm{R}}(f) \text{ is well defined and } |\mathrm{F}^*_{\mathrm{R}}(f)| = \omega(f), \\ \forall t \in \mathfrak{T}_{R\leqslant n}, \quad & \mathrm{T}^*_{\mathrm{R}}(t) \text{ is well defined and } |\mathrm{T}^*_{\mathrm{R}}(t)| = \omega(t). \end{aligned} \tag{2.6}$$

The base case is given by the first item of Definition 2.36 as $\mathrm{F}^*_{\mathrm{R}}([]) = \epsilon$ and $\omega([]) = |\epsilon| = 0$.

Let us fix $n \geqslant 1$ and suppose that the hypothesis (2.6) holds. Let $f = [t_1, \dots, t_k] \in \mathfrak{F}_{Rn+1}$.

- If $k=1$, it is sufficient to prove the second item of (2.6). Let $t = \mathrm{Node}_{\mathrm{R}}(I, f_\ell, f_r) \in \mathfrak{T}_{Rn+1}$. According to Definition 2.32 with notations of Definition 2.30 there are two cases:
  - $d = 0$ and $I = [1, \dots, p]$. We have that $i_p = p$ and $|\mathrm{F}^*_{\mathrm{R}}(f_r)| = |\epsilon| = 0$ so $i_p \leqslant 0 + p$ and $\phi_I(\epsilon) = 11\dots11$ of size p. Now by induction on f_ℓ, we have that $\mathrm{F}^*_{\mathrm{R}}(f_\ell)$ is a well defined word of size $\omega(f_\ell)$. Finally $\mathrm{T}^*_{\mathrm{R}}(t) = \mathrm{F}^*_{\mathrm{R}}(f_\ell) \blacktriangleright \phi_I(\epsilon) = \mathrm{F}^*_{\mathrm{R}}(f_\ell) \backslash \phi_I(\epsilon)$ is a well defined word of size $|\mathrm{F}^*_{\mathrm{R}}(f_\ell)| + p = \omega(t)$.
  - $d \geqslant 1$. As $p > 0$ we can apply the hypothesis (2.6) on f_r and f_ℓ. According to Definition 2.32 we have that

$$\begin{aligned} 1 &\leqslant p + \omega(f_r) + 1 - i_p, \\ i_p &\leqslant p + |\mathrm{F}^*_{\mathrm{R}}(f_r)|. \end{aligned}$$

    So $\mathrm{T}^*_{\mathrm{R}}(t)$ is well defined. Moreover

$$\begin{aligned} |\mathrm{T}^*_{\mathrm{R}}(t)| &= |\mathrm{F}^*_{\mathrm{R}}(f_\ell) \blacktriangleright \phi_I \, \mathrm{F}^*_{\mathrm{R}}(f_r)| \\ &= |\mathrm{F}^*_{\mathrm{R}}(f_\ell)| + p + |\mathrm{F}^*_{\mathrm{R}}(f_r)| \\ &= \omega(f_\ell) + p + \omega(f_r) = \omega(t). \end{aligned}$$

- If $k \geqslant 2$, the weight of trees are at least 1 so we can apply (2.6) on trees of f. □

Theorem 2.38. *The functions* F_{R} *and* $\mathrm{F}^*_{\mathrm{R}}$ *(resp.* T_{R} *and* $\mathrm{T}^*_{\mathrm{R}}$*) are two converse bijections between packed words of size* n *and red-packed forests (resp. irreducible packed words and red-packed trees) of weight* n*. That is to say* $\mathrm{F}^{-1}_{\mathrm{R}} = \mathrm{F}^*_{\mathrm{R}}$ *and* $\mathrm{T}^{-1}_{\mathrm{R}} = \mathrm{T}^*_{\mathrm{R}}$*.*

Proof. The proof is very similar to the one of Lemma 2.27. Indeed, we start to prove that domain and codomain are as announced (see Items (a) and (b) bellow), then we prove that the functions F_{R} and $\mathrm{F}^*_{\mathrm{R}}$ (resp. T_{R} and $\mathrm{T}^*_{\mathrm{R}}$) are inverse to each other (see Items (c) and (d) bellow).

We now give the differences with the proof of Lemma 2.27 and we advise the reader to read the two proofs in parallel. While for Item (a) in Lemma 2.27 it was simple, we need to do an induction here to prove that conditions on labels are respected. For Items (b), (c) and (d), the same inductions are done with one additional argument, so only the different argument of the induction is explicited here.

(a) We prove by a mutual induction that F_{R} returns a red-packed forest and that T_{R} returns a red-packed tree. Indeed, we do an induction on the size of the word w. Here is our induction hypothesis for $n \in \mathbb{N}$:

$$\begin{aligned} \forall v \in \mathbf{PW}, \text{irreducible packed word of size} \leqslant n, \quad & \mathrm{T}_{\mathrm{R}}(v) \text{ is a red-packed tree,} \\ \forall w, \text{packed word of size} \leqslant n, \quad & \mathrm{F}_{\mathrm{R}}(w) \text{ is a red-packed forest.} \end{aligned} \tag{2.7}$$

The base case $(n = 0)$ is given by the first item of Definition 2.28.

Now let us fix $n \geqslant 1$ and suppose that the hypothesis (2.7) holds. Let $w \in \mathbf{PW}$ be a packed word of size $n+1$ and let $w_1/\cdots/w_k$ be the global descent decomposition of w.

- If $k = 1$ (w is irreducible), then $\mathrm{F}_{\mathrm{R}}(w)$ is reduced to a single tree $\mathrm{T}_{\mathrm{R}}(w)$. We need to prove that $\mathrm{T}_{\mathrm{R}}(w)$ is a red-packed tree (which also gives that $\mathrm{F}_{\mathrm{R}}(w)$ is a red-packed forest). Let $w = u \blacktriangleright v$ be the red-factorization of w. With $\phi_I(v') = v$, $f_\ell = \mathrm{F}_{\mathrm{R}}(u)$ and $f_r = \mathrm{F}_{\mathrm{R}}(v')$ we have that $\mathrm{T}_{\mathrm{R}}(w) = \mathrm{Node}_{\mathrm{R}}(I, f_\ell, f_r)$. The inequalities on I and v' in Lemma 2.7 are the same as the inequalities on I and f_r in Definition 2.32. Therefore by Lemma 2.31 and (2.7) on f_r, we have that $\mathrm{T}_{\mathrm{R}}(w)$ belongs to $\mathfrak{T}_R$.
- If $k \geqslant 2$, the hypothesis (2.7) can be applied to each factors.

(b) Compared to the proof of Lemma 2.27 we use the same general arguments to prove that $\mathrm{F}^*_{\mathrm{R}}$ and $\mathrm{F}^*_{\mathrm{Rske}}$ return a packed word. First of all the base case and the case were the size of the forest is $k \geqslant 2$ are dealt with by a similar argumentation. It remains to prove that $\mathrm{T}^*_{\mathrm{R}}$ returns an irreducible packed word. We thus suppose that the induction hypothesis (2.8) holds for a given $n \in \mathbb{N}$:

$$\begin{aligned} \forall t, \text{red-packed tree of size} \leqslant n, \quad & \mathrm{T}^*_{\mathrm{R}}(t) \text{ is an irreducible packed word,} \\ \forall f, \text{red-packed forest of size} \leqslant n, \quad & \mathrm{F}^*_{\mathrm{R}}(f) \text{ is a packed word.} \end{aligned} \tag{2.8}$$

Let $t = \mathrm{Node}_{\mathrm{R}}(I, f_\ell, f_r)$ be a red-packed tree of size $n+1$. By induction we have that $\mathrm{F}^*_{\mathrm{R}}(f_\ell)$ and $\mathrm{F}^*_{\mathrm{R}}(f_r)$ are packed words. Moreover either $f_r = \epsilon$ and $I = [1, \ldots, p]$ or $1 \leqslant p + \omega(f_r) + 1 - i_p \leqslant \omega(r_d)$ with $I = [i_1, \ldots, i_p]$ and $f_r = [r_1, \ldots, r_d]$. In both cases, $\phi_I(\mathrm{F}_{\mathrm{R}}^*(f_r))$ is an red-irreducible packed word. Indeed, we recognize the two cases of Definition 2.10 with the same inequalities. Finally $\mathrm{T}_{\mathrm{R}}^*(t) = \mathrm{F}_{\mathrm{R}}^*(f_\ell) \blacktriangleright \phi_I(\mathrm{F}_{\mathrm{R}}^*(f_r))$ is an irreducible packed word according to Corollary 2.14.

(c) We now want to prove that for any forest f (resp. any tree t), $\mathrm{F}_{\mathrm{R}}(\mathrm{F}_{\mathrm{R}}^*(f)) = f$ (resp. $\mathrm{T}_{\mathrm{R}}(\mathrm{T}_{\mathrm{R}}^*(t)) = t$). As in Item (b), the arguments are the same as is the proof of Lemma 2.27 for F_{R} and $\mathrm{F}_{\mathrm{Rske}}$. In the case of T_{R} the new arguments are the same as in Item (b) (*i.e.* $\phi_I(\mathrm{F}_{\mathrm{R}}^*(f_r))$ is a red-irreducible packed word).

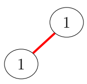
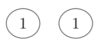
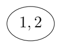
(d) Finally, we want to prove that for any packed word w (resp. irreducible packed word v), $\mathrm{F}_{\mathrm{R}}^*(\mathrm{F}_{\mathrm{R}}(w)) = w$ (resp. $\mathrm{T}_{\mathrm{R}}^*(\mathrm{T}_{\mathrm{R}}(v)) = v$). Once again, the only difference with Lemma 2.27 is the former second case. It remains to prove that point and we thus suppose that the induction hypothesis (2.9) holds for $n \in \mathbb{N}$:

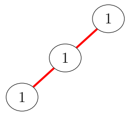
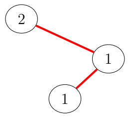
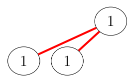
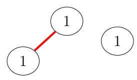
$$\begin{aligned} \forall v \in \mathbf{PW}, \text{irreducible packed word of size } \leqslant n, \quad & \mathrm{T}_{\mathrm{R}}^*(\mathrm{T}_{\mathrm{R}}(v)) = v, \\ \forall w \in \mathbf{PW}, \text{packed word of size } \leqslant n, \quad & \mathrm{F}_{\mathrm{R}}^*(\mathrm{F}_{\mathrm{R}}(w)) = w. \end{aligned} \tag{2.9}$$

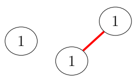
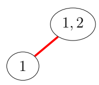
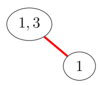
Let v be an irreducible packed word of size $n+1$. Let $v = v' \blacktriangleright \phi_I(v'')$ be the red-factorization of v. We have by Definition 2.28 that $\mathrm{T}_{\mathrm{R}}(v) = \mathrm{Node}_{\mathrm{R}}(I, \mathrm{F}_{\mathrm{R}}(v'), \mathrm{F}_{\mathrm{R}}(v''))$. As $|I| > 0$, the sizes of v' and v'' are smaller than n so we can apply (2.9). We have:

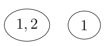
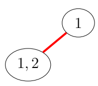
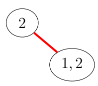
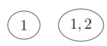
$$\begin{aligned} \mathrm{T}_{\mathrm{R}}^*(\mathrm{T}_{\mathrm{R}}(v)) &= \mathrm{T}_{\mathrm{R}}^*(\mathrm{Node}_{\mathrm{R}}(I, \mathrm{F}_{\mathrm{R}}(v'), \mathrm{F}_{\mathrm{R}}(v''))) \\ &= \mathrm{F}_{\mathrm{R}}^*(\mathrm{F}_{\mathrm{R}}(v')) \blacktriangleright \phi_I(\mathrm{F}_{\mathrm{R}}^*(\mathrm{F}_{\mathrm{R}}(v''))) \\ &= v' \blacktriangleright \phi_I(v'') = v. \end{aligned}$$

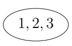
□

Example 2.39. There is a unique forest in $\mathfrak{F}_{R1}$, namely , here are the 3 forests of $\mathfrak{F}_{R2}$ with the associated packed word: $\mathrm{F}_{\mathrm{R}}(12) =$ , $\mathrm{F}_{\mathrm{R}}(21) =$ , $\mathrm{F}_{\mathrm{R}}(11) =$ . We show below the 13 forests of $\mathfrak{F}_{R3}$ with the corresponding packed word:

$\mathrm{F}_{\mathrm{R}}(123) =$ , $\mathrm{F}_{\mathrm{R}}(132) =$ , $\mathrm{F}_{\mathrm{R}}(213) =$ , $\mathrm{F}_{\mathrm{R}}(231) =$ ,

$\mathrm{F}_{\mathrm{R}}(312) =$ , $\mathrm{F}_{\mathrm{R}}(321) =$ , $\mathrm{F}_{\mathrm{R}}(122) =$ , $\mathrm{F}_{\mathrm{R}}(212) =$ ,

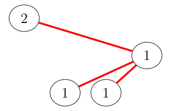
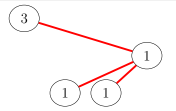
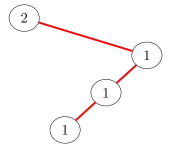
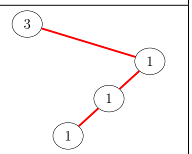
$\mathrm{F}_{\mathrm{R}}(221) =$ , $\mathrm{F}_{\mathrm{R}}(112) =$ , $\mathrm{F}_{\mathrm{R}}(121) =$ , $\mathrm{F}_{\mathrm{R}}(211) =$ ,

$\mathrm{F}_{\mathrm{R}}(111) =$ .

More examples can be found in the annexes section with Tables 4.2 to 4.7.

We conclude by the main theorem of this subsection. It is a generalization of the construction of [Foi11] for **FQSym** and permutations to **WQSym** and packed words. Indeed, if we restrict the construction on permutations and we consider right children of a node as label of this node, we have the same construction as in [Foi11] with a shift of 1 for labels. In Table 2.2 we have some examples of trees in [Foi11] and the equivalent red-packed tree.

All the constructions with red-packed forests have been done in order to have this theorem.

$\bullet_{\mathrm{T}}$ with $T = (\vee, 1)$	
$\bullet_{\mathrm{T}}$ with $T = (\vee, 2)$	
$\bullet_{\mathrm{T}}$ with $T = (\vdots, 1)$	
$\bullet_{\mathrm{T}}$ with $T = (\vdots, 2)$	

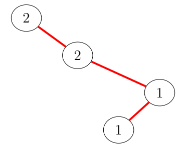
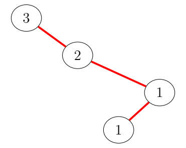
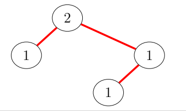
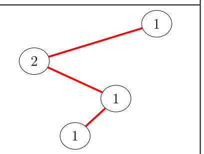
$\bullet_{\mathrm{T}}$ with $T = (\bullet_{\mathrm{T'}}, 1)$ with $T' = (:, 1)$	
$\bullet_{\mathrm{T}}$ with $T = (\bullet_{\mathrm{T'}}, 2)$ with $T' = (:, 1)$	
$:^{\mathrm{T}}$ with $T = (:, 1)$	
$:_{\mathrm{T}}$ with $T = (:, 1)$	

Table 2.2: Equivalence between trees of [Foi11] and red-packed trees.

Theorem 2.40. *For all $n \in \mathbb{N}$ we have the three following equalities :*

$$\dim(\mathbf{WQSym}_n^*) = \#\mathfrak{F}_{Rn} \quad \textit{and} \quad \dim(\mathbf{Prim}_n^*) = \#\mathfrak{T}_{Rn} \quad \textit{and} \quad \dim(\mathbf{TPrim}_n^*) = \#\mathfrak{N}_{Rn}.$$

Proof. Theorem 2.38 proves the first equality. It also gives a relation between $\#\mathfrak{F}_R$ and $\#\mathfrak{T}_R$. Indeed a red-packed forest of weight n is an ordered sequence of red-packed trees of weight (n_k) such that $\sum_k (n_k) = n$. This relation is the same between $\dim(\mathbf{WQSym}_n^*)$ and $\dim(\mathbf{Prim}_n^*)$ according to Proposition 1.1 (*i.e.* $\mathcal{A} = \mathcal{P}/(1-\mathcal{P})$).

Red-skeleton trees are equivalent to ordered trees decorated by red-irreducible words as said in Remark 2.33. Recall that a basis of primitive elements is given by Theorem 1.2 as ordered trees decorated by totally primitive elements. Elements of $\mathfrak{N}_R$ are by definition in bijections with red-irreducible words, labels of red-skeleton trees. □

2.2. Primal (Blue)

Now we do the same work for the primal side: $\mathbf{WQSym}$. This subsection follows the same structure of statements as the previous one. Recall that in Section 2.1 we constructed a bijection between packed words and red-forests by recursively decomposing packed words using global descent and removal of maximums. In this section we follow the same path: incerting the last letter using ψ_{i^α} (lowercase i designates the integer value) instead of new maximums using ϕ_I (uppercase I designates the list of their positions). We define a blue-factorization of packed words. When used recursively, blue-factorization and global descent decomposition construct a bijection between PW and so-called blue-packed forest. Since the general structure of proofs are the same as in the previous section, we will mostly focus on the differences between combinatorials arguments.

2.2.1 Decomposition of packed words through last letter

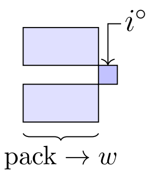
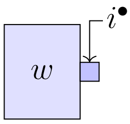
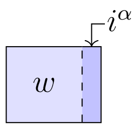
In this section, we define two combinatorial operations on packed words (ψ_{i^α} and $\blacktriangle$) and the blue-factorization that use them. The unary operation ψ_{i^α} insert the new value i at the end of a given word. A word that cannot be factorized $u\blacktriangle v$ in a non trivial way is called blue-irreducible. Blue-irreducible words will index a new basis of $\mathbf{TPrim}$.

Definition 2.41. Fix $n \in \mathbb{N}$ and $w \in \mathbf{PW}_n$. For any $1 \leqslant i \leqslant \max(w) + 1$ (with the convention $\max(\epsilon) = 0$), we denote by $\psi_{i^\circ}(w) = u_1 \cdots u_n \cdot i$ the packed word defined by $u_k = w_k$ if $w_k < i$ and $u_k = w_k + 1$ otherwise. We also define $\psi_{i^\bullet}(w) = w \cdot i$ for any $1 \leqslant i \leqslant \max(w)$.

Example 2.42. $\psi_{2^\circ}(1232) = 13432$, $\psi_{2^\bullet}(1232) = 12322$, $\psi_{4^\circ}(1232) = 12324$ and $\psi_{1^\circ}(\epsilon) = 1$.

Lemma 2.43. *For any $W \in \mathbf{PW}_\ell$ where $\ell > 0$ there exists a unique triplet (i, α, w) where $i \in [1 \ldots \ell + 1]$, $\alpha \in \{\circ, \bullet\}$ and w is a packed word, such that $W = \psi_{i^\alpha}(w)$.*

Depending on α, the box diagram can be represented as

$W =$  or $W =$  . In the general case, we will note $W =$  .

Proof. Let $W \in \mathbf{PW}_\ell$ with $\ell > 0$ and i the value of the last letter of W.

- If i appears multiple times in W, then let $w = W_1 \ldots W_{\ell-1}$, we only remove the last letter i of W. We have $W = \psi_{i^\bullet}(w)$.
- Otherwise, i appears only as the last letter, then let $w = \mathrm{pack}(W_1 \ldots W_{\ell-1})$, we remove the last letter i of W and pack the word. We have $W = \psi_{i^\circ}(w)$.

If $\psi_{i^\alpha}(u) = \psi_{j^\beta}(v)$ then the last letter is the same so $i = j$, the multiplicity of this letter is the same so $\alpha = \beta$ and the prefix are the same $u = v$. $\square$

Definition 2.44. Let $u, v \in \mathbf{PW}$ with $v \neq \epsilon$. By Lemma 2.43, there is a unique triplet (i, α, v') such that $v = \psi_{i^\alpha}(v')$. Let $i' = i + \max(u)$, we define $u\blacktriangle v := \psi_{i'^\alpha}(v'/u)$. In other words, we remove the last letter of the right word, perform a reversed left shifted concatenation and adding back the last letter also shifted.

Example 2.45. $2123\blacktriangle 312312 = 2123\blacktriangle\psi_{(2)^\bullet}(31231) = \psi_{(2+3)^\bullet}(645642123) = 6456421235$.

Lemma 2.46. *Let w be an irreducible packed word. There exists a unique factorization of the form $w = u\blacktriangle v$ which maximizes the size of u. In this factorization, let v' and i^α such that $v = \psi_{i^\alpha}(v')$,*

- *either $v' = \epsilon$ and $i^\alpha = 1^\circ$,*
- *or v' is irreducible and $1 \leqslant i \leqslant \max(v')$.*

*We call it the **blue-factorization** of a word.*

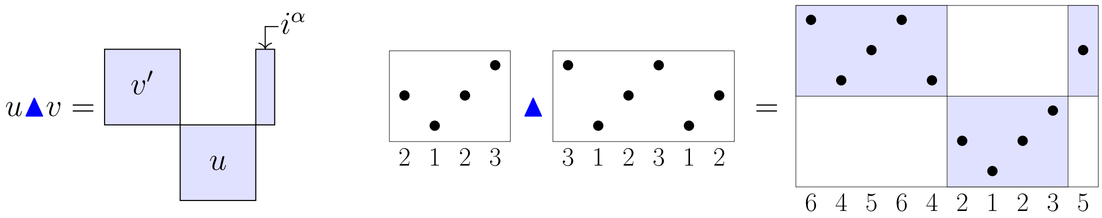


Figure 2.2: Box digrams: the operation ▲.

Example 2.47. Here is a first detailled example of a blue-factorization of an irreducible packed word:

Consider the irreducible packed word $w = 654623314$.

- The first step is to remove the last letter $i = 4$. Here there are multiple occurences of the last letter in w, then $\alpha = \bullet$, we get $w' = 65462331$ which is a packed word, but is not irreducible.
- The second step is to set w'_1 as the first irreducible factor of w' and u the rest of w'. This way $w' = w'_1/u$ and the size of u is maximized. Here $w'_1 = 3213$ and $u = 2331$. Let $i' = i - \max(u) = 4 - 3 = 1$.
- Finally, we get the following decomposition of w (see Definition 2.41 for ψ and Definition 2.44 for ▲):
$w = 654623314 = u▲\psi_{i'^\alpha}(w'_1) = (2331)▲\psi_{1^\bullet}(3213) = 2331▲32131$.

Example 2.48. Here are some other blue-factorizations:

$$234313 = 1▲\psi_{2^\bullet}(1232) = 1▲12322 \qquad 245413 = 1▲\psi_{2^\circ}(1232) = 1▲13432$$
$$11 = \epsilon▲\psi_{1^\bullet}(1) = \epsilon▲11 \qquad 112 = 11▲\psi_{1^\circ}(\epsilon) = 11▲1$$

Proof. Let w be irreducible and let (i, α, w') be the unique triplet such that $w = \psi_{i^\alpha}(w')$ according to Lemma 2.43.

If $i = \max(w)$ and it appears only one time (*i.e.* $\alpha = \circ$ and $i = \max(w') + 1$) then the blue-factorizations is $w = w'▲\psi_{1^\circ}(\epsilon)$.

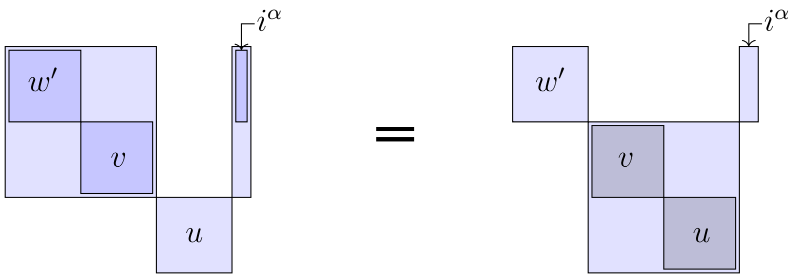
In any other case, we write $w' = w'_1/w'_2/\ldots/w'_k$, the decomposition into irreducibles. Let $u = w'_2/\ldots/w'_k$ and $i' = i - \max(u)$. We have that $i' \leqslant \max(w'_1)$ otherwise w wouldn't be packed and $1 \leqslant i'$ otherwise w wouldn't be irreducible. If $\alpha = \circ$ then $1 \neq i'$ otherwise w wouldn't be irreducible. Then we have $w = u▲\psi_{i'^\alpha}(w'_1)$ where the size of u is maximized. □

Remark 2.49. When restricted to permutations, blue-factorization is equal to a red-factorization applies to the inverse. Let σ be a permutation and $\sigma = \mu▶\nu$ be the red-factorization of σ, then $\sigma^{-1} = \mu^{-1}▲\nu^{-1}$ is the blue-factorization of σ^{-1}.

Definition 2.50. A packed word w is **blue-irreducible** if w is irreducible and $w = u▲v$ implies that $u = \epsilon$(and $w = v$).

Here are some useful lemmas on the operation $\blacktriangle$. There are some similarities with Lemmas 2.11 and 2.12, Corollary 2.14, and Proposition 2.15.

Lemma 2.51. *For any* $u, v, w \in \mathbf{PW}$ *with* $w \neq \epsilon$*, we have* $u\blacktriangle(v\blacktriangle w) = (v/u)\blacktriangle w$.

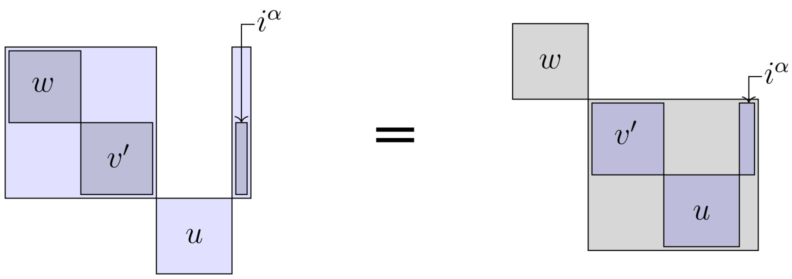


Proof. Let $u, v, w \in \mathbf{PW}$ with $w \neq \epsilon$, and let w' and i^α such that $w = \psi_{i^\alpha}(w')$.

$$\begin{aligned}
u\blacktriangle(v\blacktriangle w) &= u\blacktriangle(\psi_{(i+\max(v))^\alpha}(w'/v)) \\
&= \psi_{(i+\max(v)+\max(u))^\alpha}((w'/v)/u) \\
&= \psi_{(i+\max(v)+\max(u))^\alpha}(w'/(v/u)) \\
&= (v/u)\blacktriangle\psi_{i^\alpha}(w') \\
&= (v/u)\blacktriangle w.
\end{aligned}$$

□

Lemma 2.52. *For any* $u, v, w \in \mathbf{PW}$ *with* $v \neq \epsilon$*, we have* $u\blacktriangle(w/v) = w/(u\blacktriangle v)$.



Proof. Let $u, v, w \in \mathbf{PW}$ with $v \neq \epsilon$ and let v' and i^α such that $v = \psi_{i^\alpha}(v')$.

$$\begin{aligned}
u\blacktriangle(w/v) &= u\blacktriangle(w/\psi_{i^\alpha}(v')) \\
&= u\blacktriangle\psi_{i^\alpha}(w/v') \\
&= \psi_{(i+\max(u))^\alpha}((w/v')/u) \\
&= \psi_{(i+\max(u))^\alpha}(w/(v'/u)) \\
&= w/\psi_{(i+\max(u))^\alpha}(v'/u) \\
&= w/(u\blacktriangle v).
\end{aligned}$$

□

Remark 2.53. Theses relations are the same up to symmetry as the one with $\blacktriangleright$ (Lemmas 2.11 and 2.12). So adding the associativity of shifted concatenation $u/(v/w) = (u/v)/w$, the two operations $\blacktriangle$ and $/$ verify relations of the *skew-duplicial operad* [BDO20].

Corollary 2.54. *For any $u, v \in \mathbf{PW}$, we have that $u\blacktriangle v$ is irreducible if and only if v is irreducible.*

Proof. By contradiction, if $v = v_1/v_2$ then by Lemma 2.52 $u\blacktriangle v = v_1/(u\blacktriangle v_2)$. Now if $u\blacktriangle v = w_1/w_2$ as the value of the last letter of $u\blacktriangle v$ is greater than $\max(u)$ we have that $w_2 = w_2' \cdot w_2'' \cdot i$ such that $\operatorname{pack}(w_2'') = u$. We also have that $w_1/\operatorname{pack}(w_2' \cdot i) = v$. ☐

Proposition 2.55. *For any word w, $w = u\blacktriangle v$ is the blue-factorization of w if and only if v is blue-irreducible.*

Proof. Let $w \in \mathbf{PW}$ and let $u\blacktriangle v$ be the blue-factorization of w. Let v_1 and v_2 such that $v = v_1\blacktriangle v_2$, then $(v_1/u)\blacktriangle v_2 = w$ by Lemma 2.51, but in the blue-factorization the size of u is maximized so $|(v_1/u)| \leqslant |u|$ and then we have that $v_1 = \epsilon$ so v is blue-irreducible.

Let $w \in \mathbf{PW}$ and let u and v such that $w = u\blacktriangle v$ and v is blue-irreducible. By contradiction, suppose that there exists u', v' such that $w = u'\blacktriangle v'$ with $|u| < |u'|$ and $v' \neq \epsilon$. Then necessarily u is a suffix of u'. Let u'' such that $u' = u'' \cdot u$, then $pack(u'')\blacktriangle v' = v$. But v is blue-irreducible. So the size of u is maximal if v is blue-irreducible. ☐

Thanks Remark 2.49 the following proposition is immediate.

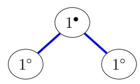
Proposition 2.56. *A permutation σ is blue-irreducible if and only if σ^{-1} is red-irreducible.*

2.2.2 Blue-forests from decomposed packed words using ψ

As in Section 2.1.2 we will apply recursively the blue-factorization of the former section to construct a bijection between packed words and a certain kind of labeled biplane trees.

In this construction, the labels can be a blue-irreducible word for skeleton, or an integer with a sign $\alpha \in \{\circ, \bullet\}$. In order to differentiate the trees from the one of the previous section, we will draw them in blue. As before, for a labeled biplane tree, we denote the trees by $\mathrm{Node}_{\mathrm{B}}(x, f_\ell, f_r)$.

Example 2.57. $\mathrm{Node}_{\mathrm{B}}(1^\circ, [], []) =$  and

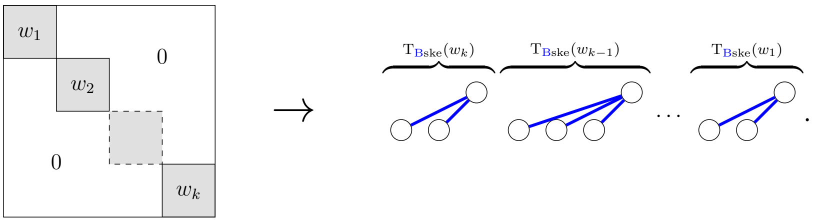
$\mathrm{Node}_{\mathrm{B}}(1^\bullet, [\mathrm{Node}_{\mathrm{B}}(1^\circ, [], [])], [\mathrm{Node}_{\mathrm{B}}(1^\circ, [], [])]) =$  .

We apply recursively the global descent decomposition and the blue-factorization of Lemma 2.46. We obtain an algorithm which takes a packed word and returns a biplane forest where nodes are decorated by blue-irreducible words:

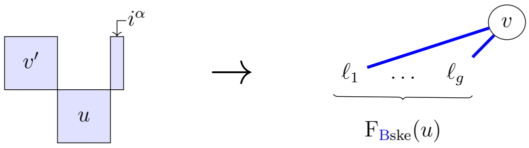
Definition 2.58. Exactly as Definition 2.23 of $\mathrm{F}_{\mathrm{Rske}}$ and $\mathrm{T}_{\mathrm{Rske}}$, we now define two functions $\mathrm{F}_{\mathrm{Bske}}$ and $\mathrm{T}_{\mathrm{Bske}}$. These functions transform respectively a packed word and an irreducible packed word into respectively a biplane forest and a biplane tree. These functions are defined in a mutual recursive way as follow:

- $\mathrm{F}_{\mathrm{Bske}}(\epsilon) = []$ (empty forest),

- for any packed word w, let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w, then $\mathrm{F}_{\mathrm{Bske}}(w) := [\mathrm{T}_{\mathrm{Bske}}(w_k), \mathrm{T}_{\mathrm{Bske}}(w_{k-1}), \dots, \mathrm{T}_{\mathrm{Bske}}(w_1)]$ (notice the inversion compared to Definition 2.23).



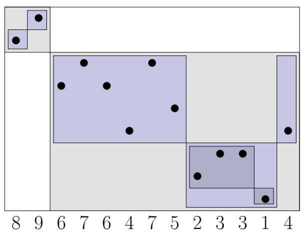
- for any irreducible packed word w, we define $\mathrm{T}_{\mathrm{Bske}}(w) := \mathrm{Node}_{\mathrm{B}}(v, \mathrm{F}_{\mathrm{Bske}}(u), [])$ where $w = u\blacktriangle v$ is the blue-factorization of w.



with $v = \psi_{i^\alpha}(v')$.

Example 2.59. Let $w = 8967647523314$, here is the global descent decomposition $w = w_1/w_2$ with $w_1 = 12$ and $w_1 = 67647523314$. Now, we have the blue-factorization of w_1 and w_2 using Lemma 2.46 as

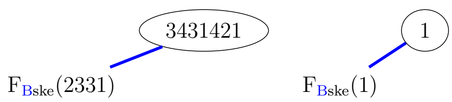
$$w_1 = 1\blacktriangle\psi_{1^\circ}(\epsilon). \quad \text{and} \quad w_2 = 2331\blacktriangle\psi_{1^\bullet}(343142) = (122/1)\blacktriangle 3431421,$$



It gives the following forest:

$$\mathrm{F}_{\mathrm{Bske}}(8967647523314) = [\mathrm{T}_{\mathrm{Bske}}(12),\ \mathrm{T}_{\mathrm{Bske}}(67647523314)]$$

=



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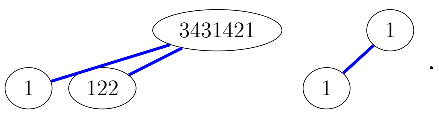


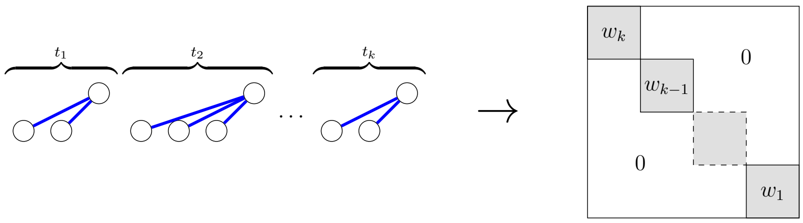
.

Definition 2.60. A labeled biplane forest (resp. tree) is a **blue-skeleton forest** (resp. **tree**) if and only if it is labeled by blue-irreducible words and no node has a right child.

We want to prove that the functions $\mathrm{F}_{\mathrm{Bske}}$ and $\mathrm{T}_{\mathrm{Bske}}$ are bijections. To do that, as for $\mathrm{F}_{\mathrm{Rske}}$ and $\mathrm{T}_{\mathrm{Rske}}$, we first define the two inverse functions.

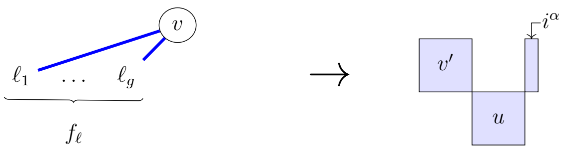
Definition 2.61. We now define two functions $\mathrm{F}^{*}_{\mathrm{Bske}}$ and $\mathrm{T}^{*}_{\mathrm{Bske}}$ that transform respectively a blue-skeleton forest and tree into packed words. These functions are defined in a mutual recursive way as follow:

- $\mathrm{F}^{*}_{\mathrm{Bske}}([]) = \epsilon$,
- for any blue-skeleton forest $f = [t_1, \dots, t_k]$, we define
  $\mathrm{F}^{*}_{\mathrm{Bske}}(f) := \mathrm{T}^{*}_{\mathrm{Bske}}(t_k) / \dots / \mathrm{T}^{*}_{\mathrm{Bske}}(t_1)$.
  (notice the inversion compared to Definition 2.26)



with $w_i = \mathrm{T}^{*}_{\mathrm{Bske}}(t_i)$.

- for any blue-skeleton tree $t = \mathrm{Node}_{\mathrm{B}}(v, f_r, [])$, we define
  $\mathrm{T}^{*}_{\mathrm{Bske}}(t) := \mathrm{F}^{*}_{\mathrm{Bske}}(f_r) \blacktriangle v$.



with $v = \psi_{i^\alpha}(v')$ and $u = \mathrm{F}^{*}_{\mathrm{Bske}}(f_\ell)$.

Lemma 2.62. *The functions $\mathrm{F}_{\mathrm{Bske}}$ and $\mathrm{F}^{*}_{\mathrm{Bske}}$ (resp. $\mathrm{T}_{\mathrm{Bske}}$ and $\mathrm{T}^{*}_{\mathrm{Bske}}$) are two converse bijections between packed words and blue-skeleton forests (resp. irreducible packed words and blue-skeleton trees). That is to say $\mathrm{F}^{-1}_{\mathrm{Bske}} = \mathrm{F}^{*}_{\mathrm{Bske}}$ and $\mathrm{T}^{-1}_{\mathrm{Bske}} = \mathrm{T}^{*}_{\mathrm{Bske}}$.*

Proof. The proof structure is the same as the one of Lemma 2.27 with use of statements comming from this subsection. We can see in this table some of the main statements that are exchanged for this dual part:

Lemma 2.7	Lemma 2.46	red-factorization and blue-factorization.
Definition 2.10	Definition 2.50	red-irreducible words and blue-irreducible words.
Corollary 2.14	Corollary 2.54	$u \blacktriangleright v$ irreducible $\iff$ v irreducible $u \blacktriangle v$ irreducible $\iff$ v irreducible.
Proposition 2.15	Proposition 2.55	$u \blacktriangleright v$ red-factorization $\iff$ v red-irreducible $u \blacktriangle v$ blue-factorization $\iff$ v blue-irreducible.
Definition 2.23	Definition 2.58	$\mathrm{F}_{\mathrm{Rske}}$, $\mathrm{T}_{\mathrm{Rske}}$ and $\mathrm{F}_{\mathrm{Bske}}$, $\mathrm{T}_{\mathrm{Bske}}$.
Definition 2.25	Definition 2.60	red-skeleton forest and blue-skeleton forest.
Definition 2.26	Definition 2.61	$\mathrm{F}^{*}_{\mathrm{Rske}}$, $\mathrm{T}^{*}_{\mathrm{Rske}}$ and $\mathrm{F}^{*}_{\mathrm{Bske}}$, $\mathrm{T}^{*}_{\mathrm{Bske}}$.

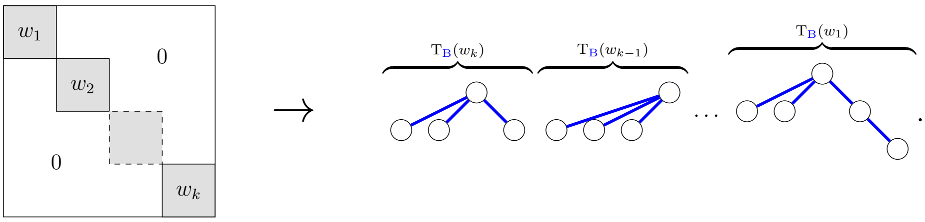
□

Now that we have the blue-skeleton, we will add right forests to every nodes to have biplane trees. For every nodes, if v is the blue-irreducible word in the label, then with Lemma 2.43 $v = \psi_{i^\alpha}(v')$, i^α is the new label and $\mathrm{F}_{\mathrm{B}}(v')$ is the new right forest.

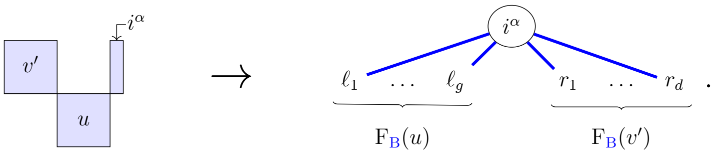
Here is the formal definition of $\mathrm{F}_{\mathrm{B}}(w)$ and $\mathrm{T}_{\mathrm{B}}(w)$ which is very similar to Definition 2.58, only the third item is different. The labels are now pairs of an integer and a sign $\alpha \in \{\circ, \bullet\}$.

Definition 2.63. The forest $\mathrm{F}_{\mathrm{B}}(w)$ (resp. tree $\mathrm{T}_{\mathrm{B}}(w)$) associated to a packed word (resp. irreducible packed word) w are defined in a mutual recursive way as follows:

- $\mathrm{F}_{\mathrm{B}}(\epsilon) = []$ (empty forest),
- for any packed word w, let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w, then $\mathrm{F}_{\mathrm{B}}(w) := [\mathrm{T}_{\mathrm{B}}(w_k), \mathrm{T}_{\mathrm{B}}(w_{k-1}), \dots, \mathrm{T}_{\mathrm{B}}(w_1)]$ (notice the inversion compared to Definition 2.28).

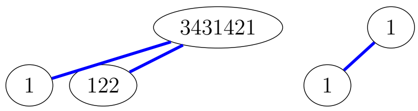


- for any irreducible packed word w, we define $\mathrm{T}_{\mathrm{B}}(w) := \mathrm{Node}_{\mathrm{B}}(i^{\alpha}, \mathrm{F}_{\mathrm{B}}(u), \mathrm{F}_{\mathrm{B}}(v'))$ where $w = u \blacktriangle \psi_{i^{\alpha}}(v')$ is the blue-factorization of w.

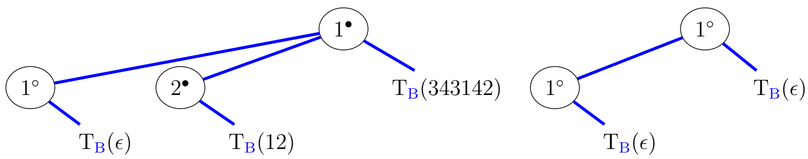


Example 2.64. Consider again $w = 8967647523314$. We start from the blue-skeleton forest from Example 2.59.

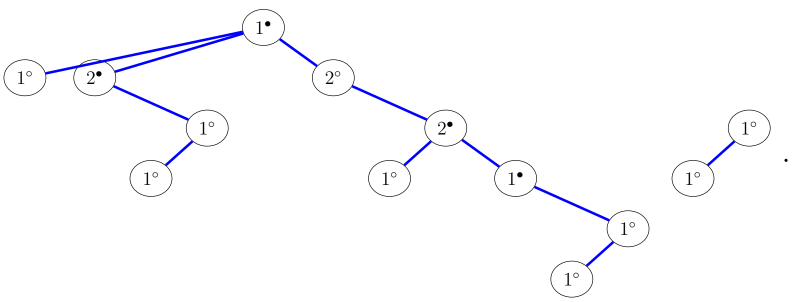
$\mathrm{F}_{\mathrm{Bske}}(8967647523314) =$



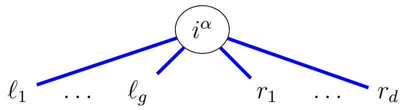
$\mathrm{F}_{\mathrm{B}}(8967647523314) =$



$\mathrm{F}_{\mathrm{B}}(8967647523314) =$



Definition 2.65. Let t be a labeled biplane tree. We write $t = \mathrm{Node}_{\mathrm{B}}(i^\alpha, f_\ell, f_r)$ where $i \in \mathbb{N}_{>0}$, $\alpha \in \{\circ, \bullet\}$, $f_\ell = [\ell_1, \dots, \ell_g]$, and $f_r = [r_1, \dots, r_d]$, which is depicted as follows:

$t =$  .

The **weight** of t ($\omega(t)$) is the number of nodes with $\circ$ in t. By extension, the **weight** of a forest is the sum of the weight of its trees.

Lemma 2.66. *The weight of a forest (resp. a tree) obtain by the functions* F_{B} *(resp.* T_{B}*) is equal to the maximum value of the word. i.e.* $\forall w \in \mathbf{PW}, \omega(\mathrm{F}_{\mathrm{B}}(w)) = \max(w)$, $\forall w \in \mathbf{PW}$ *with* w *irreducible,* $\omega(\mathrm{T}_{\mathrm{B}}(w)) = \max(w)$.

Proof. We prove by induction with the following hypothesis, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall w \in \mathbf{PW}_n, \quad & \omega(\mathrm{F}_{\mathrm{B}}(w)) = \max(w), \\ \forall w \in \mathbf{PW}_n \text{ with } w \text{ irreducible}, \quad & \omega(\mathrm{T}_{\mathrm{B}}(w)) = \max(w). \end{aligned} \tag{2.10}$$

The base case is given by the first item of Definition 2.63 as $\mathrm{F}_{\mathrm{B}}(\epsilon) = []$ and $\omega([]) = \max(\epsilon) = 0$ by convention.

Let us fix $n \geqslant 1$ and suppose that the hypothesis (2.10) holds. Let $w \in \mathbf{PW}_{n+1}$ and $w = w_1/w_2/\dots/w_k$ be the global descent decomposition of w.

- If $k = 1$, we have $\mathrm{F}_{\mathrm{B}}(w) = [\mathrm{T}_{\mathrm{B}}(w)]$. Let $w = u\blacktriangle\psi_{i^\alpha}(v)$ with $i \in \mathbb{N}_{>0}$ and $\alpha \in \{\circ, \bullet\}$ be the blue-factorization of w, then, depending of α the node is counted or not: $\omega(\mathrm{T}_{\mathrm{B}}(w)) = \omega(\mathrm{Node}_{\mathrm{B}}(i^\alpha, \mathrm{F}_{\mathrm{B}}(u), \mathrm{F}_{\mathrm{B}}(v))) = \overbrace{(1+)}^{\text{if } \alpha=\circ} \omega(\mathrm{F}_{\mathrm{B}}(u)) + \omega(\mathrm{F}_{\mathrm{B}}(v))$. The sizes of u and v are at most n so by induction $\omega(\mathrm{T}_{\mathrm{B}}(w)) = (1+)\max(u) + \max(v) = \max(w)$.

- If $k \geqslant 2$, by induction on each factors, we have that

$$\omega(\mathrm{F}_{\mathrm{B}}(w)) = \omega(\mathrm{T}_{\mathrm{B}}(w_1)) + \dots + \omega(\mathrm{T}_{\mathrm{B}}(w_k)) = \max(w_1) + \dots + \max(w_k) = \max(w). \quad \square$$

Definition 2.67. Using the same notations as in previous Definition 2.65, we say that t is a **blue-packed tree** if it satisfies:

$$\begin{cases} d = 0, \\ i^\alpha = 1^\circ, \\ \ell_1, \dots, \ell_g \text{ are blue-packed trees.} \end{cases} \quad \text{or} \quad \begin{cases} d = 1, \\ i^\alpha \neq 1^\circ, \\ 1 \leqslant i \leqslant \omega(r_1), \\ \ell_1, \dots, \ell_g \text{ and } r_1 \text{ are blue-packed trees.} \end{cases} \tag{2.11}$$

An ordered list of blue-packed trees is a **blue-packed forest**.

Remark 2.68. The same remark as Remark 2.33 can be done with blue-skeleton trees that can be interpreted as flattened representations of blue-packed trees. Symmetrically, blue-packed trees can be interpreted as unfolded representations of blue-skeleton trees. We use the operation ψ_{i^α} to change between blue-packed and blue-skeleton trees.

Note 2.69. In the same way as Note 2.34 we add the following notations:

- $\mathfrak{F}_{Bn}$ the set of blue-packed forests of size n, ($\mathfrak{F}_{Bn} := \{\mathrm{F_B}(w)\}_{w\in\mathbf{PW}_n}$),
- $\mathfrak{T}_{Bn}$ the set of blue-packed trees of size n, ($\mathfrak{T}_{Bn} := \{\mathrm{T_B}(w)\}_{w\in\mathbf{PW}_n}$ with w irreducible),
- $\mathfrak{N}_{Bn}$ the set of blue-packed trees of size n such that the left forest of the root is empty, ($\mathfrak{N}_{Bn} := \{\mathrm{T_B}(w)\}_{w\in\mathbf{PW}_n}$ with w blue-irreducible). In particular, the blue-skeleton of a tree of $\mathfrak{N}_{Bn}$ consist of a single node labeled by a blue-irreducible word.

Remark 2.70. The set $\mathfrak{N}_{Bn}$ can be described as a disjointed union of sets depending on i and α. Let $\mathfrak{F}^{i^\alpha}_{Bn}$ denote the set of blue-packed forests of weight n that can be right children of a node labeled by i^α (see Definition 2.67 for conditions), we have the following description:

$$\mathfrak{N}_{Bn} = \bigsqcup_{i,\alpha} \{\mathrm{Node_B}(i^\alpha, [], f_r) \mid f_r \in \mathfrak{F}^{i^\alpha}_{Bn-p}\} \tag{2.12}$$

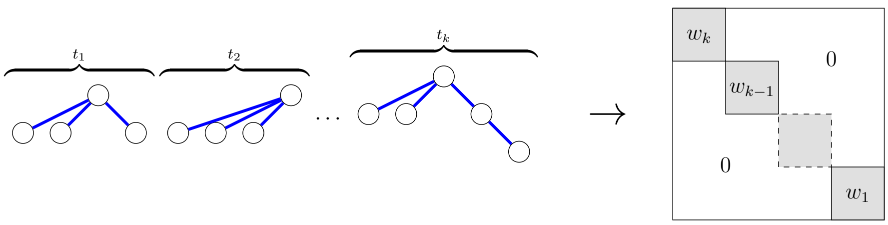
Analogously, we use $\mathfrak{F}_{Bske}$, $\mathfrak{T}_{Bske}$ and $\mathfrak{N}_{Bske}$ for blue-skeleton forests, trees and trees with only one node.

We can remark that for $n = 1$ we have $\mathfrak{N}_{B1} = \mathfrak{T}_{B1} = \mathfrak{F}_{B1}$ and $\forall n > 1, \mathfrak{N}_{Bn} \subsetneq \mathfrak{T}_{Bn} \subsetneq \mathfrak{F}_{Bn}$.

Once again we define two functions in order to prove that $\mathrm{F_B}$ and $\mathrm{T_B}$ are bijections.

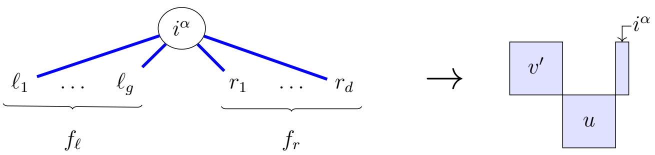
Definition 2.71. We define here the functions $\mathrm{F_B^*}$ (resp. $\mathrm{T_B^*}$) that transform blue-packed forest f (resp. tree t) into a packed word. We reverse all instructions of Definition 2.63 as follows:

- $\mathrm{F_B^*}([]) = \epsilon$,
- for any non empty blue-packed forest $f = [t_1, t_2 \dots, t_k]$, then $\mathrm{F_B^*}(f) = \mathrm{T_B^*}(t_k) / \mathrm{T_B^*}(t_{k-1}) / \dots / \mathrm{T_B^*}(t_1)$ (notice the inversion compared to Definition 2.36).



with $w_i = \mathrm{T_B^*}(t_i)$.

- for any non empty blue-packed tree $t = \mathrm{Node_B}(i^\alpha, f_\ell, f_r)$, then $\mathrm{T_B^*}(t) = \mathrm{F_B^*}(f_\ell) \blacktriangle \psi_{i^\alpha}(\mathrm{F_B^*}(f_r))$.



with $v' = \mathrm{F_B^*}(f_r)$ and $u = \mathrm{F_B^*}(f_\ell)$.

As $\psi_{i^\alpha}(\mathrm{F}_{\mathrm{B}}^*(f_r))$ is only defined if $i \leqslant \max(\mathrm{F}_{\mathrm{B}}^*(f_r))\overbrace{(+1)}^{if \alpha=\circ}$ (see Definition 2.41), there might be a problem with this definition. We prove in the following Lemma 2.72 that this is the case if $t \in \mathfrak{T}_B$.

Lemma 2.72. *For any blue-packed forest f, $\mathrm{F}_{\mathrm{B}}^*(f)$ is a well defined word and its maximum value is $\omega(f)$. For any blue-packed tree t, $\mathrm{T}_{\mathrm{B}}^*(t)$ is a well defined word and its maximum is $\omega(t)$.*

Proof. We prove by induction with the following hypothesis, for $n \in \mathbb{N}$:

$$\begin{array}{ll} \forall f \in \mathfrak{F}_{B\leqslant n}, & \mathrm{F}_{\mathrm{B}}^*(f) \text{ is well defined and } \max(\mathrm{F}_{\mathrm{B}}^*(f)) = \omega(f), \\ \forall t \in \mathfrak{T}_{B\leqslant n}, & \mathrm{T}_{\mathrm{B}}^*(t) \text{ is well defined and } \max(\mathrm{T}_{\mathrm{B}}^*(t)) = \omega(t). \end{array} \tag{2.13}$$

The base case is given by the first item of Definition 2.71 as $\mathrm{F}_{\mathrm{B}}^*([]) = \epsilon$ and $\omega([]) = \max(\epsilon) = 0$.

Let us fix $n \geqslant 1$ and suppose that the hypothesis (2.13) holds. Let $f = [t_1, \ldots, t_k] \in \mathfrak{F}_{Bn+1}$.

- If $k=1$, it is sufficient to prove the second item of (2.13). Let $t = \mathrm{Node}_{\mathrm{B}}(i^\alpha, f_\ell, f_r) \in \mathfrak{T}_{Bn+1}$. According to Definition 2.67 with notations of Definition 2.65 there are two cases:
  - $d = 0$ and $i^\alpha = 1^\circ$. We have that $\max(\mathrm{F}_{\mathrm{B}}^*(f_r)) = \max(\epsilon) = 0$ so $i \leqslant 0+1$ and $\psi_{i^\alpha}(\epsilon) = 1$. Now by induction on f_ℓ, we have that $\mathrm{F}_{\mathrm{B}}^*(f_\ell)$ is a well defined word and its maximum value is $\omega(f_\ell)$. Finally $\mathrm{T}_{\mathrm{B}}^*(t) = \mathrm{F}_{\mathrm{B}}^*(f_\ell) \blacktriangle \psi_{i^\alpha}(\epsilon) = \mathrm{F}_{\mathrm{B}}^*(f_\ell) \backslash \psi_{i^\alpha}(\epsilon)$ is a well defined word and its maximum is the last value: $\max(\mathrm{F}_{\mathrm{B}}^*(f_\ell)) + 1 = \omega(t)$.
  - $d = 1$. In this case, we can directly apply the hypothesis (2.13) on f_r and f_ℓ. According to Definition 2.67 we have that

    $$\begin{array}{lll} i & \leqslant & \omega(r_1), \\ i & \leqslant & \max(\mathrm{F}_{\mathrm{B}}^*(f_r)). \end{array}$$

    So $\mathrm{T}_{\mathrm{B}}^*(t)$ is well defined. Moreover

    $$\begin{array}{rcl} \max(\mathrm{T}_{\mathrm{B}}^*(t)) & = & \max(\mathrm{F}_{\mathrm{B}}^*(f_\ell) \blacktriangle \psi_{i^\alpha}\, \mathrm{F}_{\mathrm{B}}^*(f_r)) \\ & = & \max(\mathrm{F}_{\mathrm{B}}^*(f_\ell)) + \max(\mathrm{F}_{\mathrm{B}}^*(f_r))(+1) \\ & = & \omega(f_\ell) + \omega(f_r)(+1) = \omega(t). \end{array}$$

- If $k \geqslant 2$, the weight of trees are at least 1 so we can apply (2.13) on trees of f. □

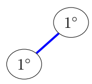
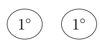
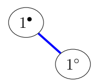
Theorem 2.73. *The functions F_{B} and $\mathrm{F}_{\mathrm{B}}^*$ (resp. T_{B} and $\mathrm{T}_{\mathrm{B}}^*$) are two converse bijections between packed words of size n and blue-packed forests (resp. irreducible packed words and blue-packed trees) of size n. That is to say $\mathrm{F}_{\mathrm{B}}^{-1} = \mathrm{F}_{\mathrm{B}}^*$ and $\mathrm{T}_{\mathrm{B}}^{-1} = \mathrm{T}_{\mathrm{B}}^*$.*

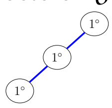
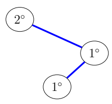
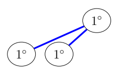
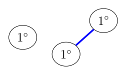
Proof. The proof structure is the same as the one of Theorem 2.38 which is similar to the one of Lemmas 2.27 and 2.62. But with use of statements comming from this subsection. We can see in this table some of the main statements that are exchanged with their counterpart:

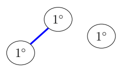
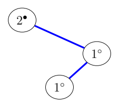
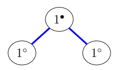
Lemma 2.7	Lemma 2.46	red-factorization and blue-factorization.
Definition 2.10	Definition 2.50	red-irreducible words and blue-irreducible words.
Corollary 2.14	Corollary 2.54	$u \blacktriangleright v$ irreducible $\iff$ v irreducible $u \blacktriangle v$ irreducible $\iff$ v irreducible.
Proposition 2.15	Proposition 2.55	$u \blacktriangleright v$ red-factorization $\iff$ v red-irreducible $u \blacktriangle v$ blue-factorization $\iff$ v blue-irreducible.
Definition 2.28	Definition 2.63	$\mathrm{F_R}$, $\mathrm{T_R}$ and $\mathrm{F_B}$, $\mathrm{T_B}$.
Definition 2.30	Definition 2.65	weight of red-forests ($\sum$ size of labels) weight of blue-forests ($\sum$ nodes with $\circ$).
Lemma 2.31	Lemma 2.66	$\omega(\mathrm{F_R}(w)) = \lvert w \rvert$ and $\omega(\mathrm{F_B}(w)) = \max(w)$.
Definition 2.32	Definition 2.67	red-packed forest and blue-packed forest.
Definition 2.36	Definition 2.71	$\mathrm{F_R^*}$, $\mathrm{T_R^*}$ and $\mathrm{F_B^*}$, $\mathrm{T_B^*}$.

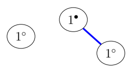
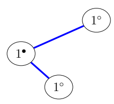
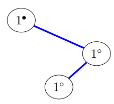
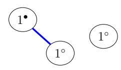
□

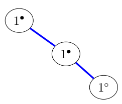
Remark 2.74. As we can see in Sections 2.1 and 2.2, the role of size and weight are exchanged for red and blue-forests. For red forests, the size (number of nodes) is equal to the maximum letter of the word associated while the weight (Definition 2.30) is the number of letter of the associated word. For blue-forests, it is the opposite, the number of letters of the associated word is equal to the size of the forest while le maximum letter is equal to the weight (Definition 2.65) of the forest. That is why we denote the set of red packed forests of **weight** n by $\mathfrak{F}_{Rn}$ and the set of blue-packed forests of **size** n by $\mathfrak{F}_{Bn}$.

Example 2.75. There is a unique forest in $\mathfrak{F}_{B1}$, namely  , here are the 3 forests of $\mathfrak{F}_{B2}$with the associated packed word:$\mathrm{F_B}(12) =$  , $\mathrm{F_B}(21) =$  , $\mathrm{F_B}(11) =$  . We show below the forests of $\mathfrak{F}_{B3}$:

$\mathrm{F_B}(123) =$  , $\mathrm{F_B}(132) =$  , $\mathrm{F_B}(213) =$  , $\mathrm{F_B}(231) =$  ,

$\mathrm{F_B}(312) =$  , $\mathrm{F_B}(321) =$  , $\mathrm{F_B}(122) =$  , $\mathrm{F_B}(212) =$  ,

$\mathrm{F_B}(221) =$  , $\mathrm{F_B}(112) =$  , $\mathrm{F_B}(121) =$  , $\mathrm{F_B}(211) =$  ,

$\mathrm{F_B}(111) =$  .

More examples can be found in the annexes section with Tables 4.2 to 4.7.

We conclude by the main theorem of this subsection. It is the dual of Theorem 2.40.

Theorem 2.76. *For all $n \in \mathbb{N}$ we have the three following equalities :*

$$\dim(\mathbf{WQSym}_n) = \#\mathfrak{F}_{Bn} \quad \textit{and} \quad \dim(\mathbf{Prim}_n) = \#\mathfrak{T}_{Bn} \quad \textit{and} \quad \dim(\mathbf{TPrim}_n) = \#\mathfrak{N}_{Bn}.$$

Proof. The proof is similar to the one of Theorem 2.40 thanks to Theorem 2.73 instead of Theorem 2.38. □

3. Bases for totally primitive elements

In this section we construct two bases of primitive and totally primitive elements of $\mathbf{WQSym}$ and $\mathbf{WQSym}^*$. Thanks to Theorems 2.40 and 2.76 we now have the combinatorial objects to index those bases and we know that their numbers agree with the dimensions. We therefore only need to show that they are linearly independent. We will proceed by showing that the decompositions through maximum and through last letter preserve the total primitivity.

As in Section 2, we start by working on $\mathbf{WQSym}^*$ associated to the color red and do the same work on the primal $\mathbf{WQSym}$ associated to the color blue.

3.1. Dual (Red)

3.1.1 Decomposition through maximums and totally primitive elements

Definition 3.1. Let $I = [i_1, \ldots, i_p]$ with $0 < i_1 < \ldots < i_p$. We define a linear map $\Phi_I : \mathbf{WQSym}^* \to \mathbf{WQSym}^*$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in \mathbf{PW}_n$,

$$\Phi_I(\mathbb{R}_w) := \begin{cases} \mathbb{R}_{\phi_I(w)} & \text{if } i_p \leqslant n+p, \\ 0 & \text{if } i_p > n+p\,. \end{cases} \tag{3.1}$$

Definition 3.2. Let $I = (i_1, \ldots, i_p)$ with $0 < i_1 < \ldots < i_p$. We define a projector $\tau_I : \mathbf{WQSym}^* \to \mathbf{WQSym}^*$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in \mathbf{PW}_n$,

$$\tau_I(\mathbb{R}_w) := \begin{cases} \mathbb{R}_w & \text{if } w_i = \max(w) \text{ if and only if } i \in I, \\ 0 & \text{else.} \end{cases} \tag{3.2}$$

These are orthogonal projectors in the sense that $\tau_I^2 = \tau_I$ and $\tau_I \circ \tau_J = 0$ $(I \neq J)$.

Lemma 3.3. *For any I, we have $\mathrm{Im}(\Phi_I) = \mathrm{Im}(\tau_I)$ where $\mathrm{Im}(f)$ denotes the image of f.*

Proof. For any I, the inclusion $\mathrm{Im}(\Phi_I) \subset \mathrm{Im}(\tau_I)$ is automatic by definition of Φ_I and τ_I. Indeed, for any $w \in \mathbf{PW}_n$ if $i_p \leqslant n+p$ then $\Phi_I(\mathbb{R}_w) = \mathbb{R}_{\phi_I(w)}$ and $\tau_I(\mathbb{R}_{\phi_I(w)}) = \mathbb{R}_{\phi_I(w)}$ and $\Phi_I(\mathbb{R}_w) = 0$ otherwise. By linearity $\mathrm{Im}(\Phi_I) \subset \mathrm{Im}(\tau_I)$.

For any I, the inclusion $\mathrm{Im}(\Phi_I) \supset \mathrm{Im}(\tau_I)$ is a consequence of Lemma 2.4 and linearity. Indeed, for any $w \in \mathbf{PW}$, $\tau_I(\mathbb{R}_w) = \mathbb{R}_w \Leftrightarrow (w_i = \max(w) \Leftrightarrow i \in I)$. If $\tau_I(\mathbb{R}_w) = \mathbb{R}_w$ let w' be such that $\phi_I(w') = w$ using Lemma 2.4, then $\Phi_I(\mathbb{R}_{w'}) = \mathbb{R}_w = \tau_I(\mathbb{R}_w)$. By linearity $\mathrm{Im}(\Phi_I) \supset \mathrm{Im}(\tau_I)$. □

Lemma 3.4. *For any I, the projection by τ_I of a totally primitive element is still a totally primitive element, so that $\tau_I(\mathbf{TPrim}^*) = \mathrm{Im}(\tau_I) \bigcap \mathbf{TPrim}^*$. Moreover,*

$$\mathbf{TPrim}^* = \bigoplus_I \mathrm{Im}(\tau_I) \cap \mathbf{TPrim}^*\ . \tag{3.3}$$

Proof. Let w a packed word. We have $\Delta_\prec(\tau_I(\mathbb{R}_w)) = (\tau_I \otimes \mathrm{Id}) \circ \Delta_\prec(\mathbb{R}_w)$ by definition of τ_I and $\Delta_\prec$. Indeed, in $\Delta_\prec(\mathbb{R}_w)$, the deconcatenations cannot be done before the last maximum letter of w. By linearity, for all $p \in \mathbf{TPrim}^*$, we have $\Delta_\prec(\tau_I(p)) = (\tau_I \otimes \mathrm{Id}) \circ \Delta_\prec(p) = 0$. The same argument works on the right so that $\tau_I(p) \in \mathbf{TPrim}^*$. Morevover τ_I are orthogonal projectors so $\mathbf{TPrim}^* = \bigoplus_I \tau_I(\mathbf{TPrim}^*) = \bigoplus_I \mathrm{Im}(\tau_I) \cap \mathbf{TPrim}^*$. □

3.1.2 The new basis $\mathbb{P}$

Definition 3.5. Let $t_1, \ldots, t_k$ be k red-packed trees, $I = [i_1, \ldots, i_p]$, $f_l = [\ell_1, \ldots, \ell_g]$ be a red-packed forest and $f_r \in \mathfrak{F}_R^I$ be a red-packed forest that can be right children of a node labeled by I,

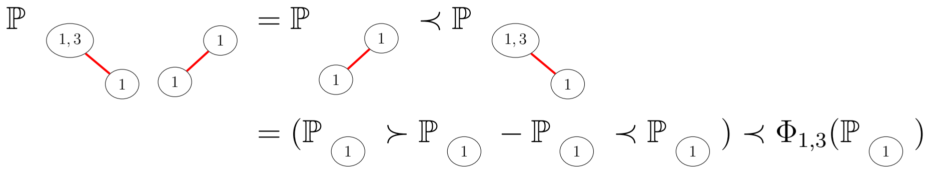
$$\mathbb{P}_{[]} := \mathbb{R}_\epsilon, \tag{3.4}$$

$$\mathbb{P}_{t_1,\ldots,t_k} := \mathbb{P}_{t_k} \prec (\mathbb{P}_{t_{k-1}} \prec (\ldots \prec \mathbb{P}_{t_1})\ldots), \tag{3.5}$$

$$\mathbb{P}_{\mathrm{Node}_{\mathrm{R}}(I, f_l = [\ell_1,\ldots,\ell_g], f_r)} := \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \ldots, \mathbb{P}_{\ell_g}; \mathbb{P}_{\mathrm{Node}_{\mathrm{R}}(I,[],f_r)} \rangle, \tag{3.6}$$

$$\mathbb{P}_{\mathrm{Node}_{\mathrm{R}}(I,[],f_r)} := \Phi_I(\mathbb{P}_{f_r}). \tag{3.7}$$

Example 3.6.



$$= \mathbb{R}_{14342} + \mathbb{R}_{41342} + \mathbb{R}_{43142} + \mathbb{R}_{43412} - \mathbb{R}_{24341} - \mathbb{R}_{42341} - \mathbb{R}_{43241} - \mathbb{R}_{43421}$$

More examples can be found in the annexes section with Table 4.8 and Figures 4.2 and 4.4.

Theorem 3.7. *For all $n \in \mathbb{N}_{>0}$:*

1. $(\mathbb{P}_f)_{f \in \mathfrak{F}_{Rn}}$ *is a basis of* $\mathbf{WQSym}_n^*$,
2. $(\mathbb{P}_t)_{t \in \mathfrak{T}_{Rn}}$ *is a basis of* $\mathbf{Prim}_n^*$,
3. $(\mathbb{P}_t)_{t \in \mathfrak{N}_{Rn}}$ *is a basis of* $\mathbf{TPrim}_n^*$.

Proof. We do a mutually recursive induction on n to prove these three items. As $\dim(\mathbf{WQSym}_1^*) = \dim(\mathbf{Prim}_1^*) = \dim(\mathbf{TPrim}_1^*) = 1$ the base case is trivial. By Proposition 1.1, Item 2 up to degree n implies Item 1 up to degree n. Similarly, Theorem 1.2 shows that Item 3 up to degree n implies Item 2 up to degree n. By induction it is sufficient to show that Items 1 and 2 up to degree $n-1$ implies Item 3 for n.

For all $k \in \mathbb{N}$, let π_k be the canonical projector on the homogeneous component of degree k of $\mathbf{WQSym}^*$. We define $\pi_{<k} := \sum_{i=0}^{k-1} \pi_i$. Fix $I = [i_1, \ldots, i_p]$ with $p \leqslant n$ and u a packed word of size $n-p$. Notice that if $p = n$ we immediately have $u = \epsilon$ and $\Delta_\prec(\Phi_I(\mathbb{R}_\epsilon)) = 0$. We suppose now that $p < n$. By Equation (1.26), in the half coproduct $\Delta_\prec(\Phi_I(\mathbb{R}_u))$ all the maximums must be in the left tensor factor, which therefore must be at least of degree i_p. By linearity, for all $x \in \mathbf{WQSym}_{n-p}^*$,

$$\Delta_\prec(\Phi_I(x)) = \left(\sum_{j=i_p}^{n-1} \Phi_I \circ \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(x). \tag{3.8}$$

Thanks to Corollary 1.4 for $f = [r_1, \dots, r_d] \in \mathfrak{F}_{Rn-p}$, the coproduct $\tilde{\Delta}(\mathbb{P}_f)$ is computed by deconcatenation of forests. So if $f \in \mathfrak{F}^I_{Rn-p}$, in particular r_d is of weight at least $n - i_p + 1$ then for $j \geqslant i_p$ we have $n-1-j < \omega(r_d)$ so that all the terms in the previous sum vanishe. A similar reasoning applies to $\Delta_{\succ}(\Phi_I(x))$ using the fact that $f \in \mathfrak{F}^I_{Rn-p}$ imply $1 \leqslant i_1 \leqslant \omega(r_1)$.

So for all $t = \mathrm{Node}_{\mathrm{R}}(I, [], f_r) \in \mathfrak{N}_{Rn}$, we have that $\Delta_{\prec}(\mathbb{P}_t) = \Delta_{\prec}(\Phi_I(\mathbb{P}_{f_r})) = 0$ and $\Delta_{\succ}(\mathbb{P}_t) = \Delta_{\succ}(\Phi_I(\mathbb{P}_{f_r})) = 0$.

Moreover, by induction we have that $\{\mathbb{P}_f \mid f \in \mathfrak{F}^I_{Rn-p}\}$ are linearity independent as $\{\mathbb{P}_f \mid f \in \mathfrak{F}_{Rn-p}\}$ is a basis of $\mathbf{WQSym}^*_{n-p}$. Recall the description of $\mathfrak{N}_{Rn}$ in Remark 2.35 as a disjointed union of sets depending on I:

$$\mathfrak{N}_{Rn} = \bigsqcup_I \{\mathrm{Node}_{\mathrm{R}}(I, [], f_r) \mid f_r \in \mathfrak{F}^I_{Rn-p}\}. \tag{3.9}$$

Since Φ_I is injective on $\mathbf{WQSym}^*_{n-p}$ then $\{\Phi_I(\mathbb{P}_f) \mid f \in \mathfrak{F}^I_{Rn-p}\}$ are linearly independent. According to Lemma 3.3, for all $f \in \mathfrak{F}^I_{Rn-p}$ we have $\Phi_I(\mathbb{P}_f) \in \mathrm{Im}(\tau_I) \cap \mathbf{TPrim}^*_n$. Moreover, thanks to the direct sum of Equation (3.3):

$$\mathbf{TPrim}^*_n = \bigoplus_I \mathrm{Im}(\tau_I) \cap \mathbf{TPrim}^*_n$$

and by definition of $\mathbb{P}$, in particular $\mathbb{P}_{\mathrm{Node}_{\mathrm{R}}(I,[],f_r)} := \Phi_I(\mathbb{P}_{f_r})$, the family $\{\mathbb{P}_t \mid t \in \mathfrak{N}_{Rn}\}$ are linearly independent. Finally, by cardinalities of Theorem 2.40 it is a basis of $\mathbf{TPrim}^*_n$. □

Remark 3.8. The basis $\mathbb{P}$ is indexed by red-packed forests ($\mathfrak{F}_R$). We will also use red-skeletons ($\mathfrak{F}_{Rske}$) or packed words ($\mathbf{PW}$) as index thanks to the bijections of Remark 2.33 and F_{R} of Definition 2.28.

3.2. Primal (Blue)

3.2.1 Decomposition through last letter and totally primitive elements

Definition 3.9. Let $i \in \mathbb{N}_{>0}$ and $\alpha \in \{\circ, \bullet\}$. We define a linear map $\Psi_{i^\alpha} : \mathbf{WQSym} \to \mathbf{WQSym}$ as follows: for all $n \in \mathbb{N}$ and $w \in \mathbf{PW}_n$,

$$\Psi_{i^\alpha}(\mathbb{Q}_w) := \begin{cases} \mathbb{Q}_{\psi_{i^\circ}(w)} & \text{if } \alpha = \circ \text{ and } 1 \leqslant i \leqslant \max(w) + 1, \\ \mathbb{Q}_{\psi_{i^\bullet}(w)} & \text{if } \alpha = \bullet \text{ and } 1 \leqslant i \leqslant \max(w), \\ 0 & \text{else.} \end{cases} \tag{3.10}$$

Definition 3.10. Let $i \in \mathbb{N}_{>0}$ and $\alpha \in \{\circ, \bullet\}$. We define a projector $\tau_{i^\alpha} : \mathbf{WQSym} \to \mathbf{WQSym}$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in \mathbf{PW}_n$,

$$\tau_{i^\alpha}(\mathbb{Q}_w) := \begin{cases} \mathbb{Q}_w & \text{if } w_n = i \text{ and } \alpha = \bullet \text{ and } i \in [w_1, \dots, w_{n-1}], \\ \mathbb{Q}_w & \text{if } w_n = i \text{ and } \alpha = \circ \text{ and } i \notin [w_1, \dots, w_{n-1}], \\ 0 & \text{else.} \end{cases} \tag{3.11}$$

These are orthogonal projectors in the sense that $\tau^2_{i^\alpha} = \tau_{i^\alpha}$ and $\tau_{i^\alpha} \circ \tau_{j^\beta} = 0$ ($i \neq j$ or $\alpha \neq \beta$).

Lemma 3.11. *For any i and α, we have $Im(\Psi_{i^\alpha}) = Im(\tau_{i^\alpha})$ where $Im(f)$ denotes the image of f.*

Proof. For any i and α, the inclusion $Im(\Psi_{i^\alpha}) \subset Im(\tau_{i^\alpha})$ is automatic by definition of Ψ_{i^α} and τ_{i^α} and linearity. Indeed, for any $w \in \mathbf{PW}_n$ $\tau_{i^\alpha}(\Psi_{i^\alpha}(\mathbb{Q}_w)) = \Psi_{i^\alpha}(\mathbb{Q}_w)$.

For any i and any α, the inclusion $Im(\Psi_{i^\alpha}) \supset Im(\tau_{i^\alpha})$ is a consequence of Lemma 2.43 and linearity. Indeed, for any $w \in \mathbf{PW}$, if $\tau_{i^\alpha}(\mathbb{Q}_w) = \mathbb{Q}_w$ then $w_n = i$.

With $w' = \mathrm{pack}(w_1 \dots w_{n-1})$ we have $\Psi_{i^\alpha}(\mathbb{Q}_{w'}) = \mathbb{Q}_w = \tau_{i^\alpha}(\mathbb{Q}_w)$. □

Lemma 3.12. *For any i and α, the projection by τ_{i^α} of a totally primitive element is still a totally primitive element, so that $\tau_{i^\alpha}(\mathbf{TPrim}) = Im(\tau_{i^\alpha}) \bigcap \mathbf{TPrim}$. Moreover,*

$$\mathbf{TPrim} = \bigoplus_{\alpha,i} Im(\tau_{i^\alpha}) \cap \mathbf{TPrim}\ . \tag{3.12}$$

Proof. Let w a packed word. We have $\Delta_{\preceq}(\tau_{i^\alpha}(\mathbb{Q}_w)) = (\tau_{i^\alpha} \otimes Id) \circ \Delta_{\preceq}(\mathbb{Q}_w)$ by definition of τ_{i^α} and $\Delta_{\preceq}$. Indeed, in $\Delta_{\preceq}(\mathbb{Q}_w)$, the decomposition can't be done under the last letter of w. By linearity, for all $p \in \mathbf{TPrim}$, we have $\Delta_{\preceq}(\tau_{i^\alpha}(p)) = (\tau_{i^\alpha} \otimes Id) \circ \Delta_{\preceq}(p) = 0$. The same argument works on the right so that $\tau_{i^\alpha}(p) \in \mathbf{TPrim}$. Morevover τ_{i^α} are orthogonal projectors so $\mathbf{TPrim} = \bigoplus_{\alpha,i} \tau_{i^\alpha}(\mathbf{TPrim}) = \bigoplus_{\alpha,i} Im(\tau_{i^\alpha}) \cap \mathbf{TPrim}$. □

3.2.2 The new basis $\mathbb{O}$

Definition 3.13. Let $t_1, \dots, t_k, r \in \mathfrak{T}_B$, $f_r \in \{[], [r]\}$ and $f_l = [\ell_1, \dots, \ell_g] \in \mathfrak{F}_B$,

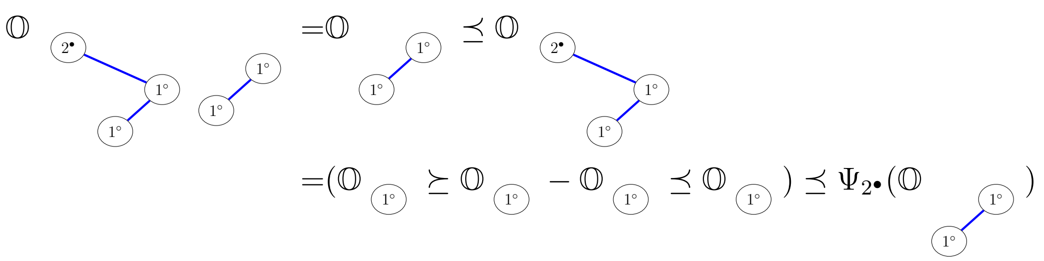
$$\mathbb{O}_{[]} := \mathbb{Q}_\epsilon, \tag{3.13}$$

$$\mathbb{O}_{t_1,\dots,t_k} := \mathbb{O}_{t_k} \preceq (\mathbb{O}_{t_{k-1}} \preceq (\dots \preceq \mathbb{O}_{t_1})\dots), \tag{3.14}$$

$$\mathbb{O}_{\mathrm{Node}_B(i^\alpha, f_l=[\ell_1,\dots,\ell_g], f_r)} := \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{\mathrm{Node}_B(i^\alpha,[],f_r)} \rangle, \tag{3.15}$$

$$\mathbb{O}_{\mathrm{Node}_B(i^\alpha,[],f_r)} := \Psi_{i^\alpha}(\mathbb{O}_{f_r}). \tag{3.16}$$

Example 3.14.



$$= \mathbb{Q}_{34122} + \mathbb{Q}_{24133} + \mathbb{Q}_{14233} + \mathbb{Q}_{43212} + \mathbb{Q}_{42313} + \mathbb{Q}_{41323}$$
$$- \mathbb{Q}_{34212} - \mathbb{Q}_{24313} - \mathbb{Q}_{14323} - \mathbb{Q}_{43122} - \mathbb{Q}_{42133} - \mathbb{Q}_{41233}.$$

More examples can be found in the annexes section with Table 4.8 and Figure 4.5.

Theorem 3.15. *For all* $n \in \mathbb{N}_{>0}$*:*

1. $(\mathbb{O}_f)_{f\in\mathfrak{F}_{Bn}}$ *is a basis of* $\mathbf{WQSym}_n$*,*
2. $(\mathbb{O}_t)_{t\in\mathfrak{T}_{Bn}}$ *is a basis of* $\mathbf{Prim}_n$*,*
3. $(\mathbb{O}_t)_{t\in\mathfrak{N}_{Bn}}$ *is a basis of* $\mathbf{TPrim}_n$*.*

Proof. The proof structure is the same as the one of Theorem 3.7 except for some statements that are exchanged as we can see in this table:

Equation (1.26)	Equation (1.23)	left corproduct in basis $\mathbb{R}$ and $\mathbb{Q}$.
Remark 2.35	Remark 2.70	$\mathfrak{N}_{Rn} = \bigsqcup_I \{\mathrm{Node}_{\mathrm{R}}(I, [], f_r) \mid f_r \in \mathfrak{F}^I_{Rn-p}\}$, $\mathfrak{N}_{Bn} = \bigsqcup_{i,\alpha} \{\mathrm{Node}_{\mathrm{B}}(i^\alpha, [], f_r) \mid f_r \in \mathfrak{F}^{i^\alpha}_{Bn-p}\}$.
Lemma 3.3	Lemma 3.11	$\mathrm{Im}(\Phi_I) = \mathrm{Im}(\tau_I)$ and $\mathrm{Im}(\Psi_{i^\alpha}) = \mathrm{Im}(\tau_{i^\alpha})$.
Lemma 3.4	Lemma 3.12	$\mathbf{TPrim}^* = \bigoplus_I Im(\tau_I) \cap \mathbf{TPrim}^*$, $\mathbf{TPrim} = \bigoplus_{\alpha,i} Im(\tau_{i^\alpha}) \cap \mathbf{TPrim}$.
Theorem 2.40	Theorem 2.76	$\dim(\mathbf{TPrim}^*_n) = \#\mathfrak{N}_{Rn}$, $\dim(\mathbf{TPrim}_n) = \#\mathfrak{N}_{Bn}$.

□

Remark 3.16. The same remark as Remark 3.8 can be done on the basis $\mathbb{O}$. Indeed, it is defined with blue-packed forests ($\mathfrak{F}_B$), it is nevertheless possible to use the blue-skeletons ($\mathfrak{F}_{Bske}$) or packed words($\mathbf{PW}$) thanks to the bijections of Remark 2.68 and F_{B} of Definition 2.63.

4. Isomorphism between $\mathbf{WQSym}$ and $\mathbf{WQSym}^*$

According to Corollary 1.3 $\mathbf{WQSym}$ (resp. $\mathbf{WQSym}^*$) is freely generated as a dendriform algebra by $\mathbf{TPrim}$ (resp. $\mathbf{TPrim}^*$). Therefore, any linear isomorphism between $\mathbf{TPrim}$ and $\mathbf{TPrim}^*$ would lead to a bidendriform isomorphism between $\mathbf{WQSym}$ and its dual. Thanks to the two bases $\mathbb{P}$ and $\mathbb{O}$ any graded bijection between red-irreducible and blue-irreducible packed words leads to such an isomorphism. We first make explicit how this is done. Then the bijection is actually obtained as the restriction to red-irreducibles of an involution on all packed words. The definition of the bijection requires a new kind of forest mixing red and blue factorizations, namely bicolored-packed forests.

4.1. A combinatorial solution to an algebraic problem

In this Section 4.1, we use the skeleton representation for bases $\mathbb{P}$ and $\mathbb{O}$ as said in Remarks 3.8 and 3.16. Moreover we fix a graded bijection μ between red-irreducible and blue-irreducible packed words.

Definition 4.1. Recall that $(\mathbb{P}_t)_{t\in\mathfrak{N}_{Rn}}$ is a basis of $\mathbf{TPrim}^*_n$ (Theorem 3.7) and $(\mathbb{O}_t)_{t\in\mathfrak{N}_{Bn}}$ is a basis of $\mathbf{TPrim}_n$ (Theorem 3.15). By linearity, setting

$$t' := \bigcirc\!\!\!\!\!\mu(v) \in \mathfrak{N}_B, \qquad \text{and} \qquad M_\mu(\mathbb{P}_t) := \mathbb{O}_{t'}. \tag{4.1}$$

for all $t = \textcircled{v} \in \mathfrak{N}_{R}$, defines a linear isomorphism between the vector spaces $\mathbf{TPrim}_n^*$ and $\mathbf{TPrim}_n$

Definition 4.2. We define σ_μ as the extension of μ from red-skeleton to blue-skeleton forests by:

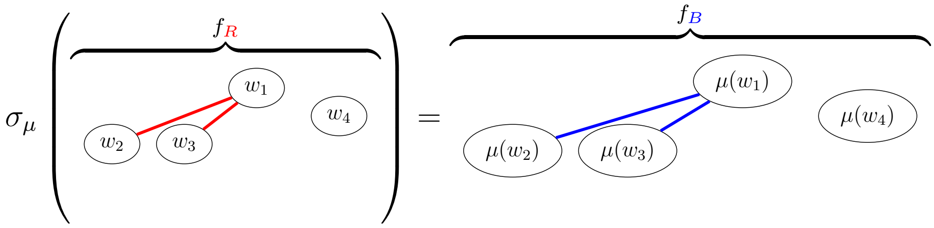
$$\forall f = [t_1, \ldots, t_k] \in \mathfrak{F}_R, \quad \sigma_\mu(f) := [\sigma_\mu(t_1), \ldots, \sigma_\mu(t_k)] \tag{4.2}$$
$$\forall t = \text{Node}_{\text{R}}(v, f_\ell, []) \in \mathfrak{T}_R, \quad \sigma_\mu(t) := \text{Node}_{\text{B}}(\mu(v), \sigma_\mu(f_\ell), []). \tag{4.3}$$

Definition 4.3. We denote Σ_μ the unique bidendriform isomorphism from $\mathbf{WQSym}^*$ to $\mathbf{WQSym}$ which verify for all $f \in \mathfrak{F}_R$:

$$\Sigma_\mu(\mathbb{P}_f) := \mathbb{O}_{\sigma_\mu(f)} \tag{4.4}$$

The existence and unicity are guaranteed by Corollary 1.3.

Example 4.4.



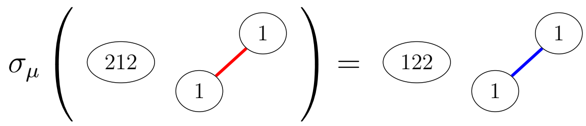
$$\Sigma_\mu\left(\mathbb{P}_{f_R}\right) = \mathbb{O}_{f_B}$$

In the following example, we take $\mu(212) := 122$ and $\mu(w) := w$ for other words with a size less than 3 as they are simultaneously red-irreducible and blue-irreducible packed words.

Here are all red and blue-irreducible packed words of size 1, 2 and 3:

Red : $\quad 1, \quad 11, \quad 132\ 121\ 212\ 111,$ $\qquad$ Blue : $\quad 1, \quad 11, \quad 132\ 121\ 122\ 111.$

Example 4.5.



These two forests are the same as those used in Examples 3.6 and 3.14. So we have here the first example of the isomorphism from the basis $\mathbb{R}$ to the basis $\mathbb{Q}$:

$$\Sigma_\mu\begin{pmatrix} \mathbb{R}_{14342} + \mathbb{R}_{41342} + \mathbb{R}_{43142} + \mathbb{R}_{43412} \\ -\mathbb{R}_{24341} - \mathbb{R}_{42341} - \mathbb{R}_{43241} - \mathbb{R}_{43421} \end{pmatrix} = \begin{pmatrix} \mathbb{Q}_{34122} + \mathbb{Q}_{24133} + \mathbb{Q}_{14233} + \mathbb{Q}_{43212} \\ +\mathbb{Q}_{42313} + \mathbb{Q}_{41323} - \mathbb{Q}_{34212} - \mathbb{Q}_{24313} \\ -\mathbb{Q}_{14323} - \mathbb{Q}_{43122} - \mathbb{Q}_{42133} - \mathbb{Q}_{41233} \end{pmatrix}$$

We now have a construction of a bidendriform isomorphism for any graded bijection μ between red-irreducible and blue-irreducible packed words.

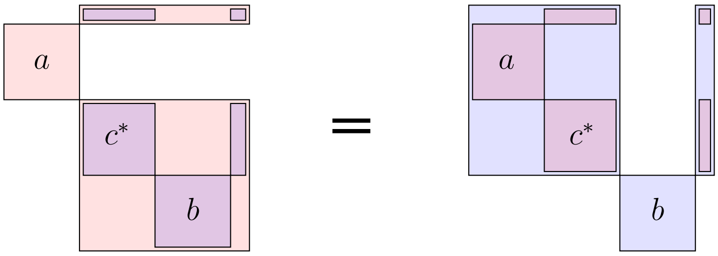
4.2. Full decomposition of packed words into bicolored forests

In order to define a bijection between red-irreducible and blue-irreducible packed words, we need a new kind of forests that mixes up the red and blue factorizations. More precisely, we will recursively alternate these factorizations. We start with an unexpected lemma which implies that starting by red or blue does not matter.

Lemma 4.6. *For all $a, b, c \in \mathbf{PW}$, with $c \neq \epsilon$, the following relations hold:*

$$a{\color{red}\blacktriangleright}(b{\color{blue}\blacktriangle}c) = b{\color{blue}\blacktriangle}(a{\color{red}\blacktriangleright}c) \quad \text{and} \quad a{\color{red}\blacktriangleright}1 = a{\color{blue}\blacktriangle}1, \tag{4.5}$$

where 1 is the packed word of size 1.



Proof. Let $a \in \mathbf{PW}$ and 1 the packed word of size 1, by Definitions 2.5 and 2.44 $a{\color{red}\blacktriangleright}1 = a\backslash 1$ and $a{\color{blue}\blacktriangle}1 = a\backslash 1$.

Let $a, b, c \in \mathbf{PW}$, with $c = c_1 \cdots c_n$ of size $n > 0$. We start by assuming that $c_n \neq \max(c)$ which implies that $c = \phi_I(\psi_{i^\alpha}(c^*)) = \psi_{i^\alpha}(\phi_I(c^*))$ with I, i, α, c^* unique by Lemmas 2.4 and 2.43. With this relation we can deduce:

$$\begin{aligned}
a{\color{red}\blacktriangleright}(b{\color{blue}\blacktriangle}c) &= a{\color{red}\blacktriangleright}(b{\color{blue}\blacktriangle}\psi_{i^\alpha}(\phi_I(c^*))) \\
&= a{\color{red}\blacktriangleright}(\psi_{i+\max(b)^\alpha}(\phi_I(c^*)/b)) \\
&= a{\color{red}\blacktriangleright}(\psi_{i+\max(b)^\alpha}(\phi_I(c^*/b))) \\
&= a{\color{red}\blacktriangleright}(\phi_I(\psi_{i+\max(b)^\alpha}(c^*/b))) \\
&= \phi_{I+|a|}(a/\psi_{i+\max(b)^\alpha}(c^*/b)) \\
&= \phi_{I+|a|}(\psi_{i+\max(b)^\alpha}(a/c^*/b))
\end{aligned}$$

$$\begin{aligned}
b{\color{blue}\blacktriangle}(a{\color{red}\blacktriangleright}c) &= b{\color{blue}\blacktriangle}(a{\color{red}\blacktriangleright}\phi_I(\psi_{i^\alpha}(c^*))) \\
&= b{\color{blue}\blacktriangle}(\phi_{I+|a|}(a/\psi_{i^\alpha}(c^*))) \\
&= b{\color{blue}\blacktriangle}(\phi_{I+|a|}(\psi_{i^\alpha}(a/c^*))) \\
&= b{\color{blue}\blacktriangle}(\psi_{i^\alpha}(\phi_{I+|a|}(a/c^*))) \\
&= \psi_{i+\max(b)^\alpha}(\phi_{I+|a|}(a/c^*)/b) \\
&= \psi_{i+\max(b)^\alpha}(\phi_{I+|a|}(a/c^*/b)) \\
&= \phi_{I+|a|}(\psi_{i+\max(b)^\alpha}(a/c^*/b)).
\end{aligned}$$

The case where $c_n = \max(c)$ can be decomposed into different particular cases. In each of these cases, it is possible to find a relation with two different writings of c that begin with ϕ or ψ just like $c = \phi_I(\psi_{i^\alpha}(c^*)) = \psi_{i^\alpha}(\phi_I(c^*))$. These cases with the associated relation are:

- the case where c is the packed word 1 then $c = \phi_1(\epsilon) = \psi_{1^\circ}(\epsilon)$,
- the case where c is of the form $c'\backslash 1$ then $c = \phi_1(c') = \psi_{1^\circ}(c')$,
- the more general case where there is more than 1 maximum including the one at the end then $c = \phi_{I\cdot c_n}(c^*) = \psi_{c_n^\bullet}(\phi_I(c^*))$.

In each of these cases it is possible to prove with a similar method that $a \blacktriangleright (b \blacktriangle c) = b \blacktriangle (a \blacktriangleright c)$. $\square$

Example 4.7. Here are some examples of this relation:

$$\begin{aligned}
1 \blacktriangleright (1 \blacktriangle 1) &= 213 &&= 1 \blacktriangle (1 \blacktriangleright 1),\\
11 \blacktriangleright (12 \blacktriangle 2111) &= 44533123 &&= 12 \blacktriangle (11 \blacktriangleright 2111),\\
11 \blacktriangleright (21 \blacktriangle 123) &= 5534216 &&= 21 \blacktriangle (11 \blacktriangleright 123),\\
1 \blacktriangleright (112 \blacktriangle 3132) &= 56361124 &&= 112 \blacktriangle (1 \blacktriangleright 3132).
\end{aligned}$$

Definition 4.8. Let w be an irreducible packed word. Let $w = x \blacktriangleright u$ be the red-factorization of w and let $u = y \blacktriangle z$ be the blue-factorization of u. Then $w = x \blacktriangleright (y \blacktriangle z)$ is called the **red-blue-factorization** of w. Symmetrically we define $w = y' \blacktriangle (x' \blacktriangleright z')$ the **blue-red-factorization** of w.

Lemma 4.9. *Let $w = x \blacktriangleright (y \blacktriangle z)$ be the red-blue-factorization and let $w = y' \blacktriangle (x' \blacktriangleright z')$ be the blue-red-factorization of an irreducible packed word w.*

With these two factorizations, we have that $z = z'$ and it is both red-irreducible and blue-irreducible packed word. Moreover,

- *either $z = z' = 1$, $y = x' = \epsilon$ and $x = y'$*
- *or $x = x'$, $y = y'$.*

Example 4.10. Here are some examples of red-blue-factorization and blue-red-factorization:

$$\begin{aligned}
12 \blacktriangleright (\epsilon \blacktriangle 1) &= 213 &&= 12 \blacktriangle (\epsilon \blacktriangleright 1),\\
11 \blacktriangleright (12 \blacktriangle 1211) &= 44353123 &&= 12 \blacktriangle (11 \blacktriangleright 1211),\\
553421 \blacktriangleright (\epsilon \blacktriangle 1) &= 5534216 &&= 553421 \blacktriangle (\epsilon \blacktriangleright 1),\\
1 \blacktriangleright (112 \blacktriangle 3132) &= 56361124 &&= 112 \blacktriangle (1 \blacktriangleright 3132).
\end{aligned}$$

Proof. We start by prooving the case where $z = z' = 1$, $y = x' = \epsilon$ and $x = y'$. Let w' be an irreducible packed word and $w = w'\backslash 1$. We have that $w = w' \blacktriangleright 1$ is the red-factorization of w and $w = w' \blacktriangle 1$ is the blue-factorization of w. In this case we immediately have that $w = w' \blacktriangleright (\epsilon \blacktriangle 1)$ is the red-blue-factorization of w and that $w = w' \blacktriangle (\epsilon \blacktriangleright 1)$ is the blue-red-factorization of w.

Now let w be an irreducible packed word of size n that cannot be written as $w'\backslash 1$. In other words, there is a maximum strictly before the last letter of w ($\exists i < n, w_i = \max(w)$).

We define the two sets of triplet of packed words that verify equations of the factorizations for w:

$$\begin{aligned}
S_{RB}(w) &:= \{(a, b, c) \in \mathbf{PW}, c \neq \epsilon, w = a \blacktriangleright (b \blacktriangle c)\},\\
S_{BR}(w) &:= \{(a, b, c) \in \mathbf{PW}, c \neq \epsilon, w = b \blacktriangle (a \blacktriangleright c)\}.
\end{aligned}$$

Thanks to Lemma 4.6 these two sets are equal, we define $S(w) := S_{RB}(w) = S_{BR}(w)$.

In the red-blue-factorization $w = x \blacktriangleright (y \blacktriangle z)$, we maximize the size of x, then we maximize the size of y in the remaining word. In the blue-red-factorization we commute the order of maximizations. We will caracterize $S(w)$ and see the limit of the two maximizations to prove that they can commute.

Let w^* be the packed word comming from w where the last letter and all occurences of the maximum are removed and let $I = [i_1, \dots, i_p], i, \alpha$ such that $w = \psi_{i^\alpha}(\phi_I(w^*))$. By hypothesis, we have that $I \neq \varnothing$. Let $w^* = w_1/\cdots/w_k$ be the global descent decomposition of w^*. Let ℓ be the maximum such that $|w_1/\cdots/w_\ell| \leqslant i_1$, i_1 being the position of the first maximum of w. Let r be the minimum such that $|w_r/\cdots/w_k| \leqslant n - i_p - 1$, i_p being the position of the last maximum of w before the last letter. As $i_1 \leqslant i_p$ by definition, we have that $\ell < r$. We can caracterize the set $S(w)$:

$$\begin{aligned} S(w) = \{(a &= w_1/\cdots/w_{r^0}, \\ b &= w_{\ell^0}/\cdots/w_k, \\ c &= \psi_{(i-\max(b))^\alpha}(\phi_{I-|a|}(w_{r^0+1}/\cdots/w_{\ell^0-1}))), \\ &\text{with } r^0 \leqslant r \text{ and } \ell \leqslant \ell^0\}. \end{aligned}$$

□

Here are all packed words that are both red-irreducible and blue-irreducible of size less than 4:

1, 11, 111 121 132,

1111 1121 1132 1211 1212 1221 1231 1232 1243 1312 1321

1322 1323 1332 1342 1423 1432 2121 2122 2132 2143 3132

n	1	2	3	4	5	6	7	8	9
$\mathfrak{i}_n \in \mathbf{PW}_n$	1	1	3	22	196	2 008	23 184	297 456	4 199 216
$\mathfrak{i}_n \in \mathfrak{S}_n$	1	0	1	5	32	236	1 951	17 827	178 418

Table 4.1: Number of both red-and-blue-irreducible packed words and permutations.

Recall our notations for Hilbert series of an algebra A, $\mathcal{A}(z) := \sum_{n=1}^{+\infty} \dim(A_n) z^n$, $\mathcal{P}(z) := \sum_{n=1}^{+\infty} \dim(\mathrm{Prim}(A_n)) z^n$ and $\mathcal{T}(z) := \sum_{n=1}^{+\infty} \dim(\mathrm{TPrim}(A_n)) z^n$.

Recall the relations between these series:

$$\mathcal{P} = \mathcal{A}/(1+\mathcal{A}) \text{ or equivalently } \mathcal{A} = \mathcal{P}/(1-\mathcal{P}) \text{ (see Proposition 1.1)},$$

$$\mathcal{T} = \mathcal{A}/(1+\mathcal{A})^2 \text{ or equivalently } \mathcal{P} = \mathcal{T}(1+\mathcal{A}) \text{ (see Theorem 1.2)}.$$

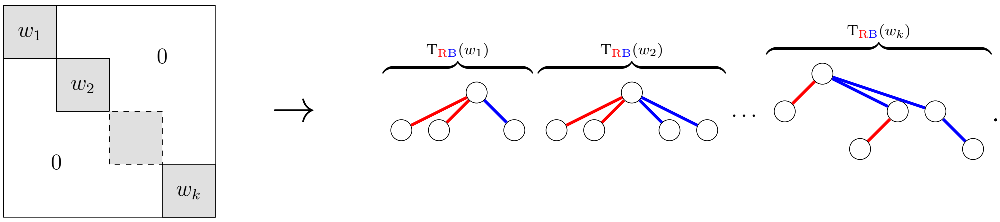
If we define the serie $\mathcal{I} = \sum_{n=1}^{+\infty} \mathfrak{i}_n z^n$ where $\mathfrak{i}_n$ is the number of both red-and-blue-irreducible words of size n, then we have the following relation:

$$\mathcal{I} = \mathcal{A}/(1+\mathcal{A})^3 + z\mathcal{P} \text{ or equivalently } \mathcal{T} = (\mathcal{I} - z)(1+\mathcal{A}) + z.$$

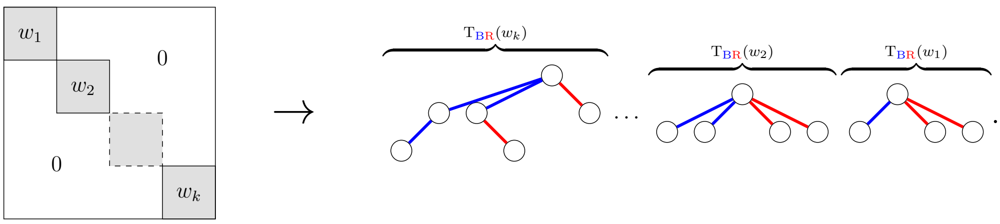
So far we have seen red-biplane trees and blue-biplane trees. In this section we define red-blue-biplane trees and blue-red-biplane trees, the edges of these trees are of two different colors and the labels are red-and-blue-irreducible packed words. We denote by $\mathrm{Node}_{\mathrm{RB}}(x, f_\ell, f_r)$ (resp. $\mathrm{Node}_{\mathrm{BR}}(x, f_\ell, f_r)$) the biplane tree whose edges between the root and the left forest f_ℓ are red (resp. blue) and edges between the root and the right forest f_r are blue (resp. red).

Definition 4.11. The bicolored forests $\mathrm{F}_{\mathrm{RB}}(w)$ and $\mathrm{F}_{\mathrm{BR}}(w)$ (resp. trees $\mathrm{T}_{\mathrm{RB}}(w)$ and $\mathrm{T}_{\mathrm{BR}}(w)$) associated to a packed word (resp. irreducible packed word) w are defined in a mutual recursive way as follows:

- $\mathrm{F}_{\mathrm{RB}}(\epsilon) = \mathrm{F}_{\mathrm{BR}}(\epsilon) = []$ (empty forest),
- for any packed word w, let $w = w_1/w_2/\dots/w_k$ be the global descent decomposition, then $\mathrm{F}_{\mathrm{RB}}(w) := [\mathrm{T}_{\mathrm{RB}}(w_1), \mathrm{T}_{\mathrm{RB}}(w_2), \dots, \mathrm{T}_{\mathrm{RB}}(w_k)]$,

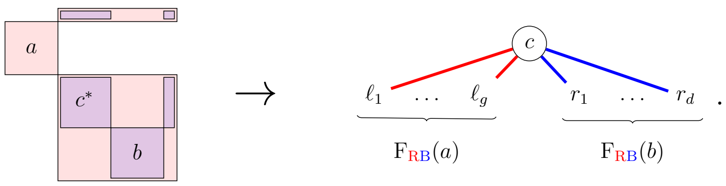


- for any packed word w, let $w = w_1/w_2/\dots/w_k$ be the global descent decomposition, then $\mathrm{F}_{\mathrm{BR}}(w) := [\mathrm{T}_{\mathrm{BR}}(w_k), \mathrm{T}_{\mathrm{BR}}(w_{k-1}), \dots, \mathrm{T}_{\mathrm{BR}}(w_1)]$.

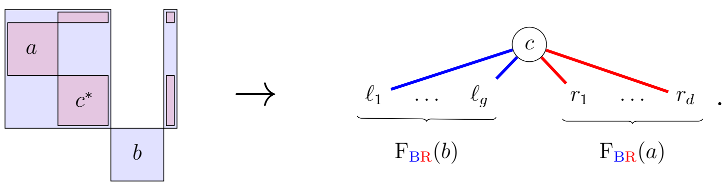


(notice the same inversion as in Definitions 2.58 and 2.63 for F_{BR}.)

- for any irreducible packed word w, let $w = a \blacktriangleright (b \blacktriangle c)$ be the red-blue-factorization, then $\mathrm{T}_{\mathrm{RB}}(w) := \mathrm{Node}_{\mathrm{RB}}(c, \mathrm{F}_{\mathrm{RB}}(a), \mathrm{F}_{\mathrm{RB}}(b))$.



- for any irreducible packed word w, let $w = b \blacktriangle (a \blacktriangleright c)$ be the blue-red-factorization, then $\mathrm{T}_{\mathrm{BR}}(w) := \mathrm{Node}_{\mathrm{BR}}(c, \mathrm{F}_{\mathrm{BR}}(b), \mathrm{F}_{\mathrm{BR}}(a))$.

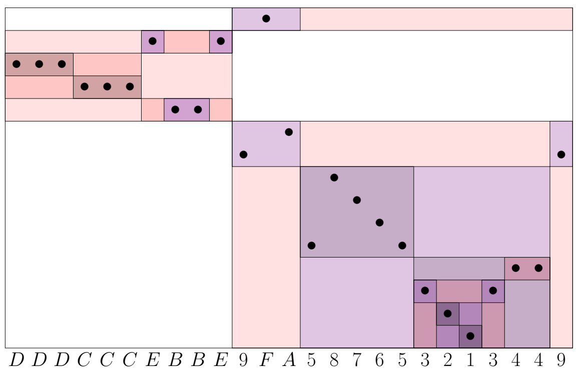


Example 4.12. For this example we write the word w in hexadecimal in order to have a big and clear example, let $w = DDDCCCEBBE9FA587653213449$. The word w is irreducible so there is only one tree in the forest. To have $\mathrm{T_{RB}}(w)$, we start by the blue-red-factorization,

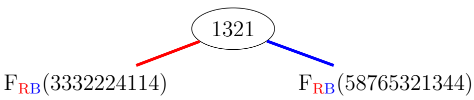
$$w = \underline{DDDCCCEBBE}\,9FA\,\underline{58765321344}\,9 = \underline{3332224114} \blacktriangleright (\underline{58765321344} \blacktriangle 1321).$$

Then we decompose each sub-word according to their global descents and do blue-red-factorizations recursively until we have only both red-irreducible and blue-irreducible packed words:

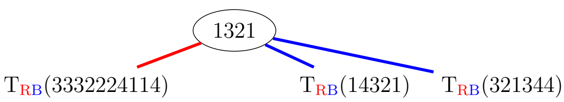
$$\begin{aligned} w &= \underline{3332224114} \blacktriangleright (\underline{\underline{58765}\,\underline{321344}} \blacktriangle 1321), \\ &= \underline{\underline{333222}\,4\,\underline{11}\,4} \blacktriangleright (\underline{(\underline{14321}/\underline{3213}\,\underline{44})} \blacktriangle 1321), \\ &[\ldots] \\ &= ((111/111) \blacktriangleright (11 \blacktriangle 11)) \blacktriangleright ((14321/(((1/1) \blacktriangle 11) \blacktriangleright 11)) \blacktriangle 1321) \end{aligned}$$



$\mathrm{T_{RB}}(w) =$

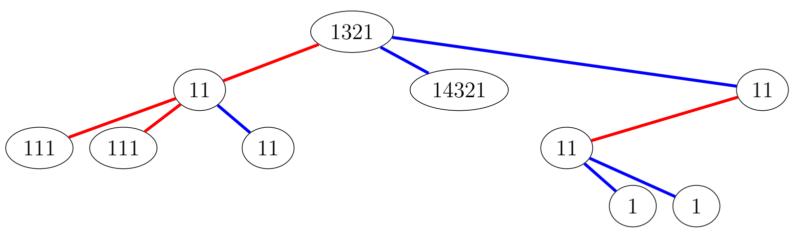


$=$



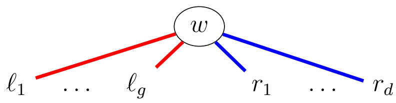
$[\ldots]$

$=$



More examples can be found in the annexes section with Tables 4.2 to 4.7.

Definition 4.13. There are two types of bicolored trees, the only difference is that colors red and blue are inverted. Let t be a labeled biplane tree. We write $t = \mathrm{Node}_{\mathrm{RB}}(w, f_\ell, f_r)$ where $w \in \mathbf{PW}$, $f_\ell = [\ell_1, \dots, \ell_g]$ is the left forest of t and $f_r = [r_1, \dots, r_d]$ is the right forest of t. We depict t as follows:

$t =$  .

We say that t is a **red-blue-packed tree** if it satisfies:

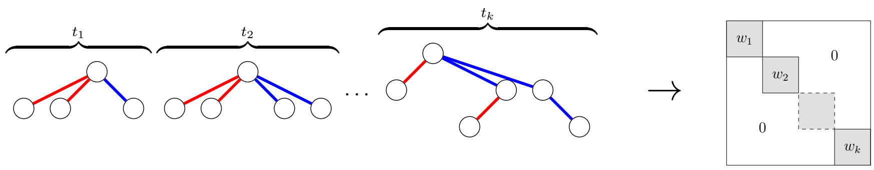
$$\begin{cases} w = 1, \\ f_r = [], \\ f_\ell \text{ is a red-blue-packed forest.} \end{cases} \quad \text{or} \quad \begin{cases} w \neq 1 \text{ is red-irreducible and blue-irreducible,} \\ f_r \text{ and } f_\ell \text{ are red-blue-packed forest.} \end{cases}$$

Definition 4.14. The **weight** of a bicolored-packed tree is the sum of the size of packed words in the nodes.

We have already done it four times (Definitions 2.26, 2.36, 2.61 and 2.71) and we will do it one last time, to prove that $\mathrm{F}_{\mathrm{RB}}, \mathrm{T}_{\mathrm{RB}}, \mathrm{F}_{\mathrm{BR}}, \mathrm{T}_{\mathrm{BR}}$ are bijections, we define $\mathrm{F}^*_{\mathrm{RB}}, \mathrm{T}^*_{\mathrm{RB}}, \mathrm{F}^*_{\mathrm{BR}}, \mathrm{T}^*_{\mathrm{BR}}$ and prove that they are the inverse maps.

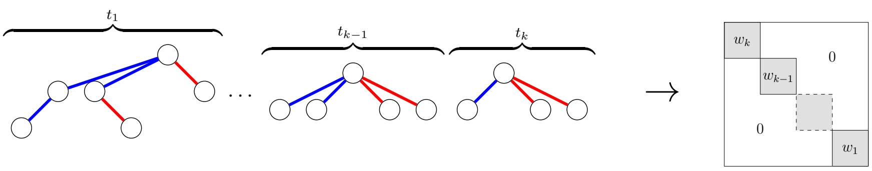
Definition 4.15. We define here the maps $\mathrm{F}^*_{\mathrm{RB}}, \mathrm{T}^*_{\mathrm{RB}}, \mathrm{F}^*_{\mathrm{BR}}, \mathrm{T}^*_{\mathrm{BR}}$ that transforms bicolored-packed forests and trees into packed words. We reverse all instructions of Definition 4.13 as follows:

- $\mathrm{F}^*_{\mathrm{RB}}([]) = \mathrm{F}^*_{\mathrm{BR}}([]) = \epsilon$ (empty packed word),
- for any red-blue-packed forest $f = [t_1, t_2 \dots, t_k]$, we have $\mathrm{F}^*_{\mathrm{RB}}(f) := [\mathrm{T}^*_{\mathrm{RB}}(t_1), \mathrm{T}^*_{\mathrm{RB}}(t_2), \dots, \mathrm{T}^*_{\mathrm{RB}}(t_k)]$.



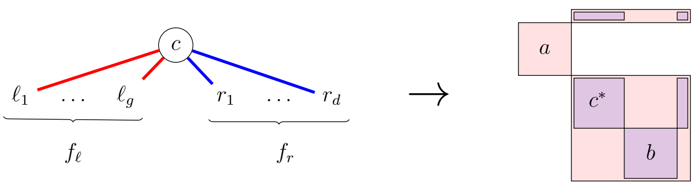
with $w_i = \mathrm{T}^*_{\mathrm{RB}}(t_i)$.

- for any blue-red-packed forest $f = [t_1, t_2 \dots, t_k]$, we have $\mathrm{F}^*_{\mathrm{BR}}(f) := [\mathrm{T}^*_{\mathrm{BR}}(t_k), \mathrm{T}^*_{\mathrm{BR}}(t_{k-1}), \dots, \mathrm{T}^*_{\mathrm{BR}}(t_1)]$.



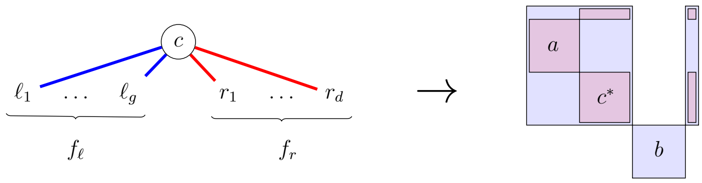
with $w_i = \mathrm{T}^*_{\mathrm{BR}}(t_i)$.

- for any red-blue-packed tree $t = \mathrm{Node}_{\mathrm{RB}}(c, f_\ell, f_r)$, we have $\mathrm{T}^*_{\mathrm{RB}}(t) := \mathrm{F}^*_{\mathrm{RB}}(f_\ell) \blacktriangleright (\mathrm{F}^*_{\mathrm{RB}}(f_r) \blacktriangle c)$.



with $a = \mathrm{F}^*_{\mathrm{RB}}(f_\ell)$
and $b = \mathrm{F}^*_{\mathrm{RB}}(f_r)$.

- for any blue-red-packed tree $t = \mathrm{Node}_{\mathrm{BR}}(c, f_\ell, f_r)$, we have $\mathrm{T}^*_{\mathrm{BR}}(t) := \mathrm{F}^*_{\mathrm{BR}}(f_r) \blacktriangle (\mathrm{F}^*_{\mathrm{BR}}(f_\ell) \blacktriangleright c)$.



with $a = \mathrm{F}^*_{\mathrm{RB}}(f_r)$ and $b = \mathrm{F}^*_{\mathrm{RB}}(f_\ell)$.

Theorem 4.16. *The maps* F_{RB} *and* $\mathrm{F}^*_{\mathrm{RB}}$ *(resp.* T_{RB} *and* $\mathrm{T}^*_{\mathrm{RB}}$*) are two converse bijections between packed words of size* n *and red-blue-packed forests (resp. irreducible packed words and red-blue-packed trees) of weight* n*. That is to say* $\mathrm{F}^{-1}_{\mathrm{RB}} = \mathrm{F}^*_{\mathrm{RB}}$ *and* $\mathrm{T}^{-1}_{\mathrm{RB}} = \mathrm{T}^*_{\mathrm{RB}}$*. We have the same result with inversions of red and blue.*

Proof. It is simple to prove by induction on the size of the trees that domain and codomain are as announced and that the functions are inverse to each other. The proof is similar to the proofs of Lemmas 2.27 and 2.62 and Theorems 2.38 and 2.73 using Definition 4.11 of $\mathrm{F}_{\mathrm{RB}}, \mathrm{T}_{\mathrm{RB}}, \mathrm{F}_{\mathrm{BR}}, \mathrm{T}_{\mathrm{BR}}$, Definition 4.13 of bicolored-packed forests and trees and Definition 4.15 of $\mathrm{F}^*_{\mathrm{RB}}, \mathrm{T}^*_{\mathrm{RB}}, \mathrm{F}^*_{\mathrm{BR}}, \mathrm{T}^*_{\mathrm{BR}}$. □

Remark 4.17. We now have two new families of forests $\mathfrak{F}_{RB}$ and $\mathfrak{F}_{BR}$ that are in bijection with packed words and therefore in bijection with red-packed and blue-packed forests. As in Remarks 3.8 and 3.16, this gives us two other way to index bases $\mathbb{O}$ and $\mathbb{P}$ of **WQSym** and **WQSym***.

4.3. An involution on packed words

We are now in position to define a bijection between red-irreducible and blue-irreducible packed words. This bijection is actually the restriction of an involution defined on all packed words. Precisely, we will define two transformations on bicolored forests

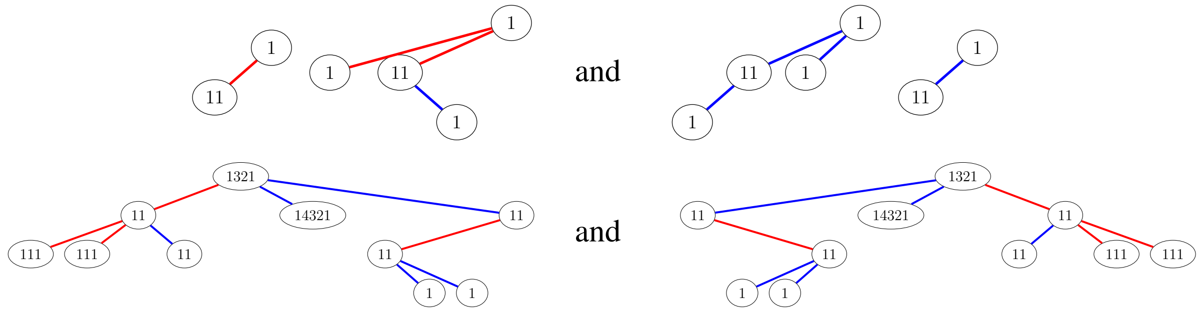
We need to define the notion of mirror transformation of bicolored-packed forests and trees. This transformation is defined from a red-blue to blue-red or from blue-red to red-blue, so in the notations we will use XY instead of RB or BR to point out where the swap is made.

Definition 4.18. The mirror transformation of a bicolored-packed forest $f = [t_1, \dots, t_k]$ is given by $\widetilde{f} := [\widetilde{t_k}, \dots, \widetilde{t_1}]$ where $\widetilde{t_i}$ is the mirror transformation of t_i recursively defined as follows. For any $t = \mathrm{Node}_{\mathrm{XY}}(z, f_\ell, f_r)$ then

$$\widetilde{t} := \begin{cases} \mathrm{Node}_{\mathrm{YX}}(z, \widetilde{f_r}, \widetilde{f_\ell}) & \text{if } z \neq 1, \\ \mathrm{Node}_{\mathrm{YX}}(1, \widetilde{f_\ell}, []) & \text{if } z = 1. \end{cases}$$

Note that when $z \neq 1$, the left and right forests are swapped whereas they are not when $z = 1$. But in the latter case, we have necessarily $f_r = \widetilde{f_r} = []$. These two cases correspond to the two cases of Definition 4.13 so the mirror transformation of a red-blue-packed forest is indeed a blue-red-packed forest.

Example 4.19. Here are two examples of mirror transformations.



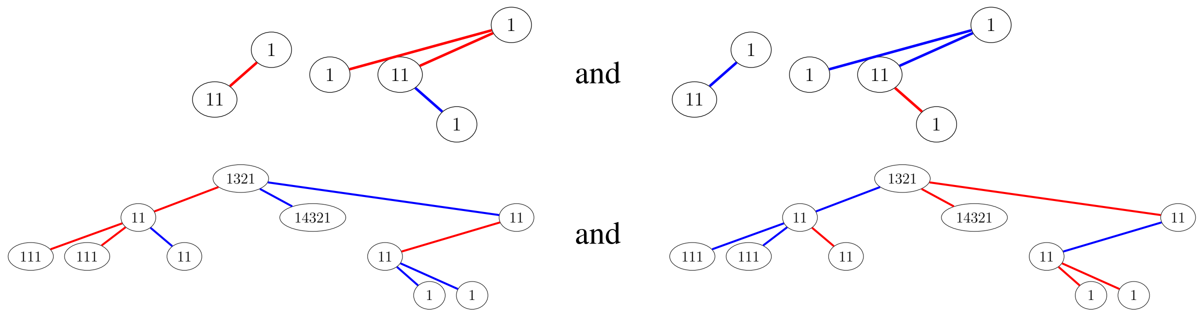
Proposition 4.20. *For all packed words w, both associated bicolored-packed forests $\mathrm{F}_{\mathrm{RB}}(w)$ and $\mathrm{F}_{\mathrm{BR}}(w)$ are mirror image of each other.*

Proof. The proof is a computation of mirror transformation (Definition 4.18) on each items of Definition 4.11 of F_{RB}. For the first three items the computation of $\widetilde{f} := [\widetilde{t_k}, \dots, \widetilde{t_1}]$ is sufficient. Thanks to the relation of Lemma 4.9 ($x \blacktriangleright (y \blacktriangle z) = y \blacktriangle (x \blacktriangleright z)$ in the case $z \neq 1$) the two remaining items are also simple computation of $\widetilde{t}$. □

Definition 4.21. The color swap of a bicolored-packed forest $f = [t_1, \dots, t_k]$ is given by $\widehat{f} := [\widehat{t_k}, \dots, \widehat{t_1}]$ where $\widehat{t_i}$ is the color swap of t_i recursively defined as follows. For any $t = \mathrm{Node}_{\mathrm{XY}}(z, f_\ell, f_r)$ then $\widehat{t} := \mathrm{Node}_{\mathrm{YX}}(z, \widehat{f_\ell}, \widehat{f_r})$.

In other words, it is a recoloration of each edges using the other color. Every blue edges become red and vice versa.

Example 4.22. Here are two examples of color swaps.



More examples can be found in the annexes section with Table 4.8.

When we focus on the packed words associated to these forest, the color swap correspond to the swap of the two operations $\blacktriangleright$ and $\blacktriangle$ in a bicolored-factorization. More precisely, if w is an irreducible packed word and $w = x \blacktriangleright (y \blacktriangle z)$ is the red-blue-factorization of w, then the color swap on the associated forest correspond to $w' = x \blacktriangle (y \blacktriangleright z)$.

Lemma 4.23. *Mirror transformation and color swap commute. It means that for all bicolored-packed forest f, we have $\widehat{\left(\widetilde{f}\right)} = \widetilde{\left(\widehat{f}\right)}$.*

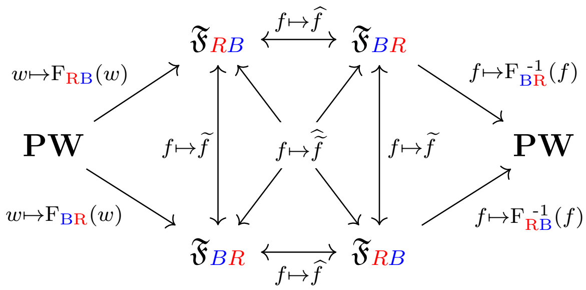


Figure 4.1: Commutative diagram of maps on bicolored-packed forests.

Proof. The proof is immediate. Indeed, the definition of mirror transformation is independant of color swap and symmetrically, the color swap is independant of the tree shape. □

Corollary 4.24. *The diagram on Figure 4.1 is commutative. So* $\widehat{w} := \mathrm{F}_{\mathrm{RB}}^{-1}(\widehat{\mathrm{F}_{\mathrm{BR}}(w)})$ *is an involution on packed words.*

Proof. Thanks to Proposition 4.20 and Lemma 4.23 the diagram is immediately commutative. The mirror transformation and the color swap are independant involutions so the conjunction is an involution. □

Corollary 4.25. *The application* $w \mapsto \widehat{w}$ *send blue (resp. red) irreducibles packed words to red (resp. blue) irreducibles packed words.*

Some examples can be found in th annexes section with Tables 4.9 and 4.10.

Proof. If w is a red-irreducible packed word, then the red-blue-factorization of w is of the form $w = \epsilon \blacktriangleright (y \blacktriangle z)$. Then the color swap correspond to the words $w' = \epsilon \blacktriangle (y \blacktriangleright z)$ which is blue-irreducible. □

4.4. Main theorem

In Section 4.1 we fixed a graded bijection μ between red-irreducible and blue-irreducible packed words. After that, we extend it to all red-skeleton forests as σ_μ. We finished by defining Σ_μ as a bidendriform isomorphism from $\mathbf{WQSym}^*$ to $\mathbf{WQSym}$. Now we can set $\mu : w \mapsto \widehat{w}$ as a graded bijection. The extension σ_μ correspond to the color swap on red-packed forests (*i.e.* $\sigma_\mu : f \mapsto \widehat{f}$). Finally we have the following theorem:

Theorem 4.26. *The linear map* $\Sigma : \mathbf{WQSym}^* \to \mathbf{WQSym}$ *defined as for all packed forest* f,

$$\Sigma(\mathbb{P}_f) := \mathbb{O}_{\widehat{f}}$$

is a bidendriform isomorphism between $\mathbf{WQSym}^*$ *and* $\mathbf{WQSym}$.

Proof. This theorem is a direct consequence of Corollaries 1.3 and 4.25. □

Conclusion

The main contribution of this paper is the combinatorial construction of biplane trees. They are the combinatorial ingredient which completes the algebraic theory of Foissy [Foi11] and allows us to describe the explicit isomorphism. Besides, they are also an innovative combinatorial family and open promising research perspective.

Generalization of the inversion of permutation to packed words

The inherent difficulty of finding an explicit isomorphism between $\mathbf{WQSym}$ and its dual lies in the fact that there is no “inversion” operation on packed words. Indeed, in the case of $\mathbf{FQSym}$, the Hopf algebra indexed by permutations, the isomorphism is given by the inversion of permutations. The solution we offer, using biplane trees, is actually not a generalization of $\mathbf{FQSym}$ in this sense. Indeed, even though permutations are a subset of packed words, the restriction of our involution on packed words to permutation is not the inversion. In particular, if a permutation σ is both red-irreducible and blue-irreducible, its image is itself and not its inverse. This is the case for all packed words which are both red-irreducible and blue-irreducible. Nevertheless, our involution is somehow “compatible” with the inversion in the sense that if we arbitrary decide that the image of σ is σ^{-1} for all σ such that σ is a red-blue-irreducible permutation, then the rest of construction ensures that the image of σ is σ^{-1} for all permutations (not necessarily irreducible anymore). But we don’t know how to define the inversion on red-blue-irreducible elements which are not permutations, which is why to stick with the identity in all case, including permutations.

Stays the open question: is there a generalization of the inversion of permutations on packed words? In other words, one would want an involution on packed words which restricts to the inversion on permutations and gives a bidendriform isomorphism between $\mathbf{WQSym}$ and its dual. A consequence of our work is that it is sufficient to find such an involution on red-blue-irreducible packed words.

Generalization of the biplane trees to parking functions

A long term goal would be to somehow generalize the structure of biplane trees to all bidendriform Hopf algebras. The first step would be to look at the Hopf algebra indexed by parking functions $\mathbf{PQSym}$. Indeed $\mathbf{PQSym}$ is also a bidendriform bialgebra and parking functions are a superset of the packed words. The question of generalizing the structure to parking functions involves both combinatorics and algebra. The first thing is to compute bases of $\mathbf{PQSym}$ in which the shuffle product is not shifted. It can be done with a generalization of Definition 1.29[BZ09].

The lines of research induced by this work are the following:

- How to generalize biplane tree structure to $\mathbf{PQSym}$?
- We will then look for what are the necessary and sufficient ingredients to develop biplane tree structures and obtain bidendriform automorphisms on all bidendriform bialgebras.

Link between bidendriform bialgebras and skew-duplicial operad

As said in Remarks 2.13 and 2.53 the operations ▶ and $/$ (resp. ▲ and $/$) unexpectedly verify relations of the skew-duplicial operad [BDO20]. These relations reveal a new application of the skew-duplical operad applied on packed words.

- Can we find a skew-duplical structure on **WQSym** which is linked to the bidendriform structure?
- More generally, is there a link between bidendriform bialgebra and skew-duplical?

Link between bidendriform bialgebras and L-algebras

As said in Remark 2.21, the sequence that count unlabeled biplane trees is the dimensions of the free L-algebra on one generator (see [Ler11]). It would be interesting to investigate the link between L-algebra and bidendriform bialgebras through the use of biplane trees.

The study of the operad on the three operations $\{$▶, ▲, $/\}$ is a start in order to study the link between bidendriform bialgebras and the skew-duplical operade or L-algebras.

Acknowledgements

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Annexes

In Tables 4.2 to 4.7 we have red-packed forests, blue-packed forest and bicolored-packed forests associated to all packed words of size smaller than 4.

In Table 4.8 we have the isomorphism between **WQSym** (bases $\mathbb{O}$ and $\mathbb{Q}$) and its dual (bases $\mathbb{P}$ and $\mathbb{R}$) for size smaller than 3. Basis $\mathbb{O}$ and $\mathbb{P}$ are indexed by bicolored-packed forests. This illustrates the main theorem of Section 4.4.

In Table 4.9 we have the involution of Corollary 4.25 for all packed words of size 4. They are organized by evaluations. Red-irreducible (resp. blue-irreducible) packed words are underlined in red (resp. blue).

In Table 4.10 we have the involution of Corollary 4.25 for all red-irreducible packed words that are not blue-irreducible. It correspond to words underlined in red in front of a word underlined in blue in Table 4.9.

The matrix of Figure 4.2 is redundant with the column $\mathbb{R}$ and $\mathbb{P}$ of Table 4.8. Note that even though the matrix of Figure 4.3 is symmetric, it is not the case anymore on Figure 4.6. Even if we restrict to permtuations, the matrix is not symmetric for size 5.

	123	132	213	231	312	321	122	212	221	112	121	211	111
123	1	.	.	.	.	.	.	.	.	.	.	.	.
132	.	1	-1	.	1	.	.	.	.	.	.	.	.
213	-1	.	1	.	.	.	.	.	.	.	.	.	.
231	-1	-1	1	1	-1	.	.	.	.	.	.	.	.
312	.	.	-1	.	1	.	.	.	.	.	.	.	.
321	1	.	.	-1	-1	1	.	.	.	.	.	.	.
122	.	.	.	.	.	.	1	.	.	.	.	.	.
212	.	.	.	.	.	.	1	1	.	.	.	.	.
221	.	.	.	.	.	.	.	.	1	-1	.	.	.
112	.	.	.	.	.	.	.	.	.	1	.	.	.
121	.	.	.	.	.	.	-1	.	.	.	1	1	.
211	.	.	.	.	.	.	-1	.	.	.	.	1	.
111	.	.	.	.	.	.	.	.	.	.	.	.	1

Figure 4.2: Change-of-basis matrix from $\mathbb{P}_3$ to $\mathbb{R}_3$.

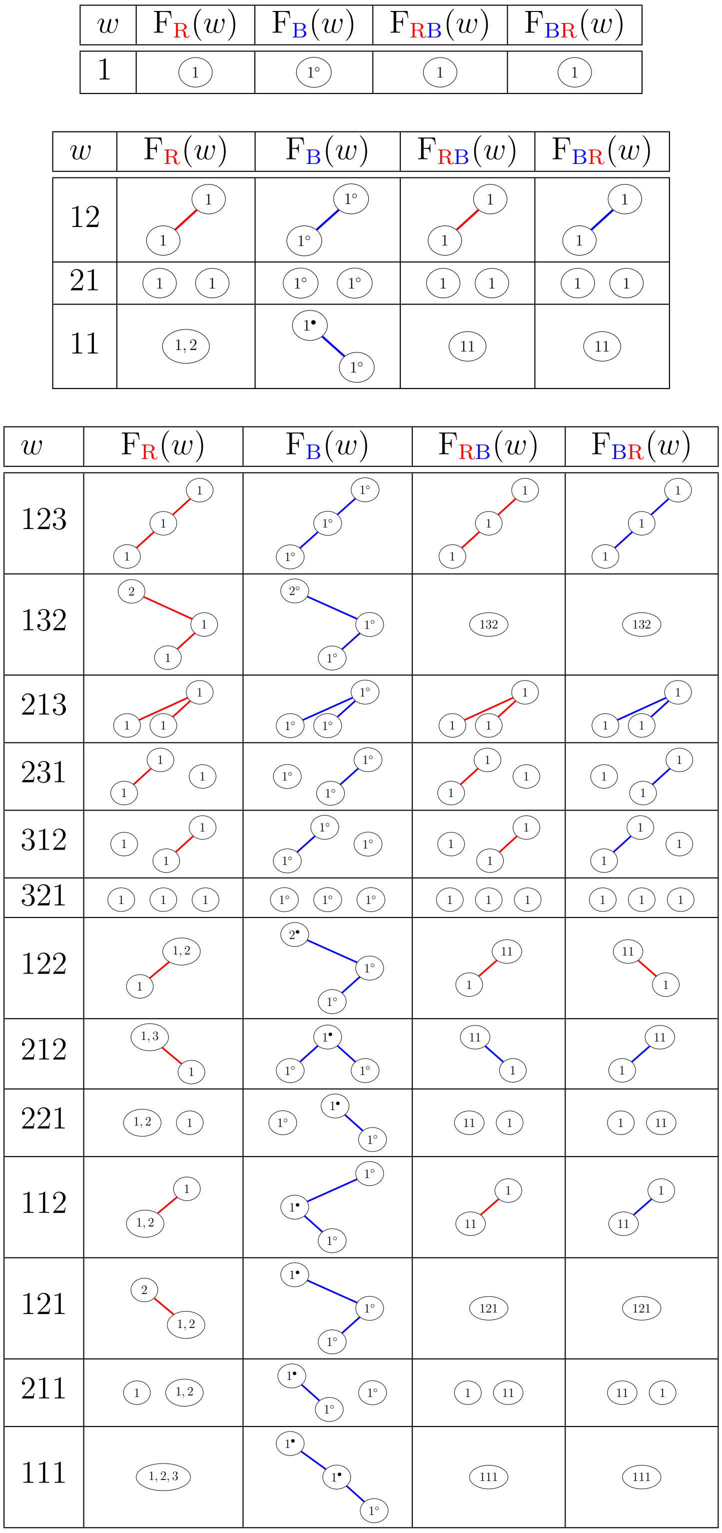


Table 4.2: All packed words of size smaller than 3 and forests associated to it.

	123	132	213	231	312	321	122	212	221	112	121	211	111
123	1	.	.	.	.	.	.	.	.	.	.	.	.
132	.	1	.	.	.	.	.	.	.	.	.	.	.
213	.	.	1	.	.	.	.	.	.	.	.	.	.
231	.	.	.	.	1	.	.	.	.	.	.	.	.
312	.	.	.	1	.	.	.	.	.	.	.	.	.
321	.	.	.	.	.	1	.	.	.	.	.	.	.
122	.	.	.	.	.	.	1	1	.	.	.	.	.
212	.	.	.	.	.	.	1	.	.	.	.	.	.
221	.	.	.	.	.	.	.	.	.	.	1	1	.
112	.	.	.	.	.	.	.	.	.	1	.	.	.
121	.	.	.	.	.	.	.	.	1	.	1	.	.
211	.	.	.	.	.	.	.	.	1	.	.	.	.
111	.	.	.	.	.	.	.	.	.	.	.	.	1

Figure 4.3: Change-of-basis matrix from $\mathbb{Q}_3$ to $\mathbb{R}_3$.

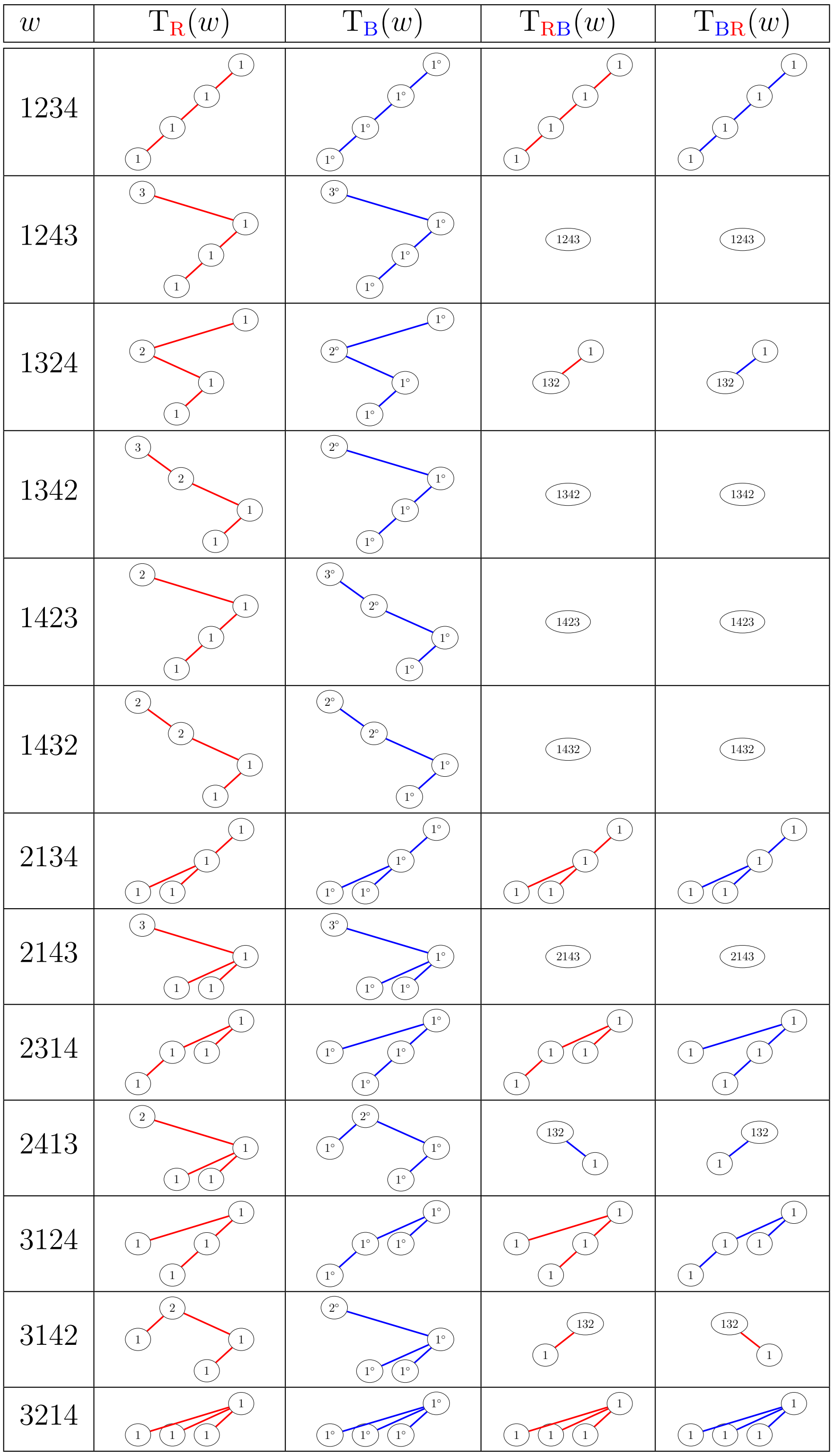


Table 4.3: Packed words of size 4 and associated forests (part 1).

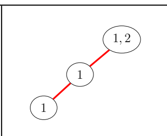
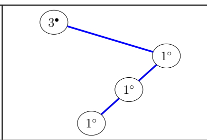
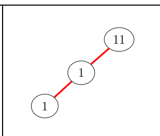
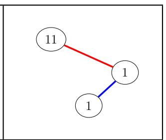
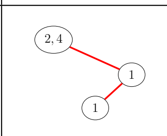
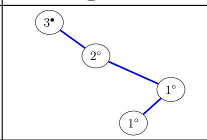
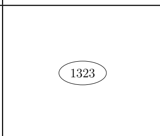
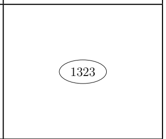
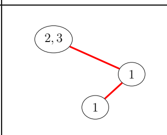
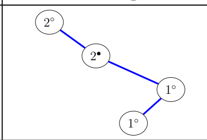
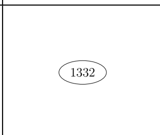
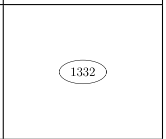
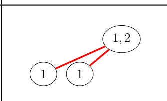
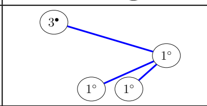
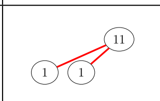
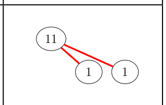
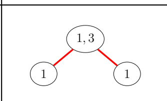
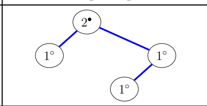
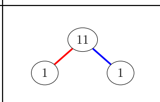
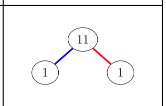
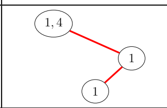
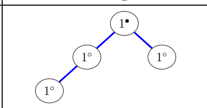
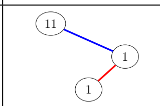
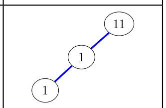
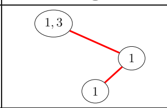
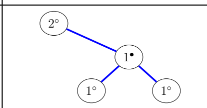
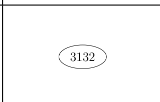
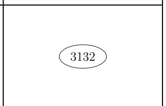
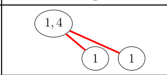
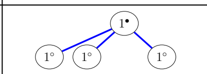
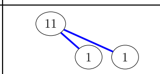
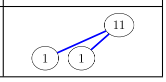
w	$\mathrm{T}_{\mathrm{R}}(w)$	$\mathrm{T}_{\mathrm{B}}(w)$	$\mathrm{T}_{\mathrm{RB}}(w)$	$\mathrm{T}_{\mathrm{BR}}(w)$
1233				
1323				
1332				
2133				
2313				
3123				
3132				
3213				

Table 4.4: Packed words of size 4 and associated forests (part 2).

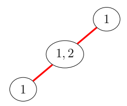
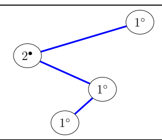
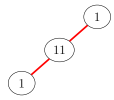
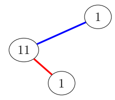
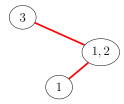
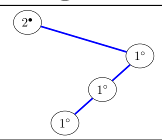
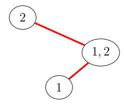
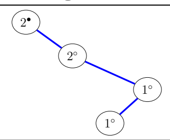
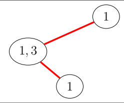
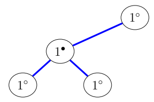
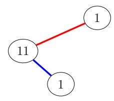
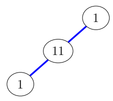
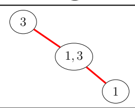
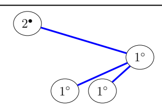
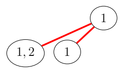
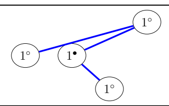
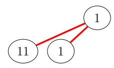
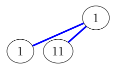
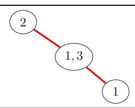
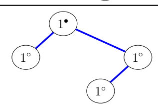
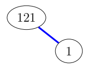
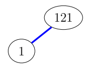
w	$\mathrm{T}_{\mathrm{R}}(w)$	$\mathrm{T}_{\mathrm{B}}(w)$	$\mathrm{T}_{\mathrm{RB}}(w)$	$\mathrm{T}_{\mathrm{BR}}(w)$
1223				
1232			1232	1232
1322			1322	1322
2123				
2132			2132	2132
2213				
2312				

Table 4.5: Packed words of size 4 and associated forests (part 3).

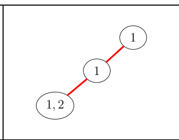
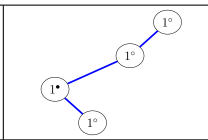
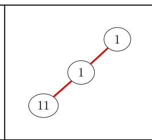
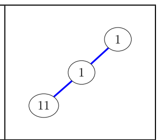
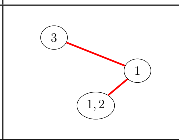
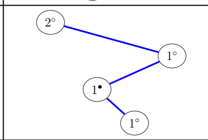
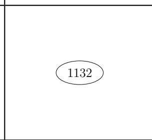
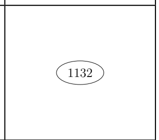
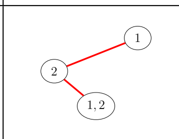
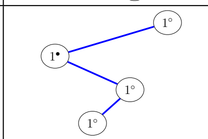
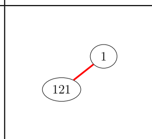
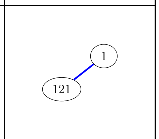
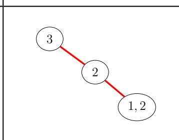
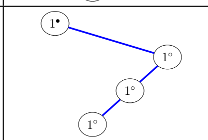
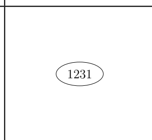
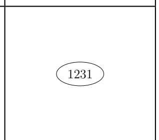
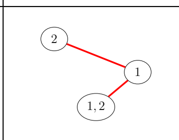
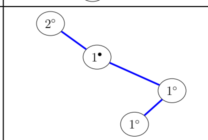
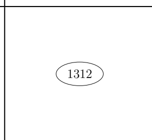
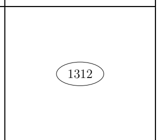
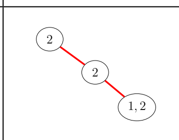
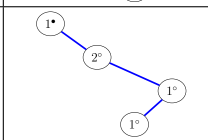
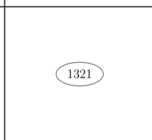
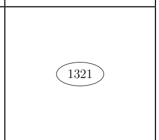
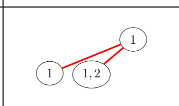
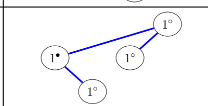
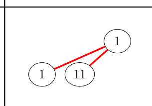
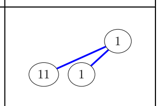
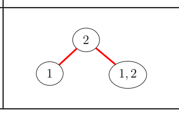
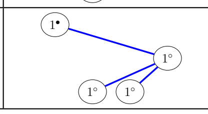
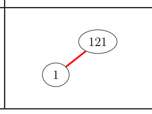
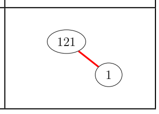
w	$\mathrm{T}_{\mathrm{R}}(w)$	$\mathrm{T}_{\mathrm{B}}(w)$	$\mathrm{T}_{\mathrm{RB}}(w)$	$\mathrm{T}_{\mathrm{BR}}(w)$
1123				
1132				
1213				
1231				
1312				
1321				
2113				
2131				

Table 4.6: Packed words of size 4 and associated forests (part 4).

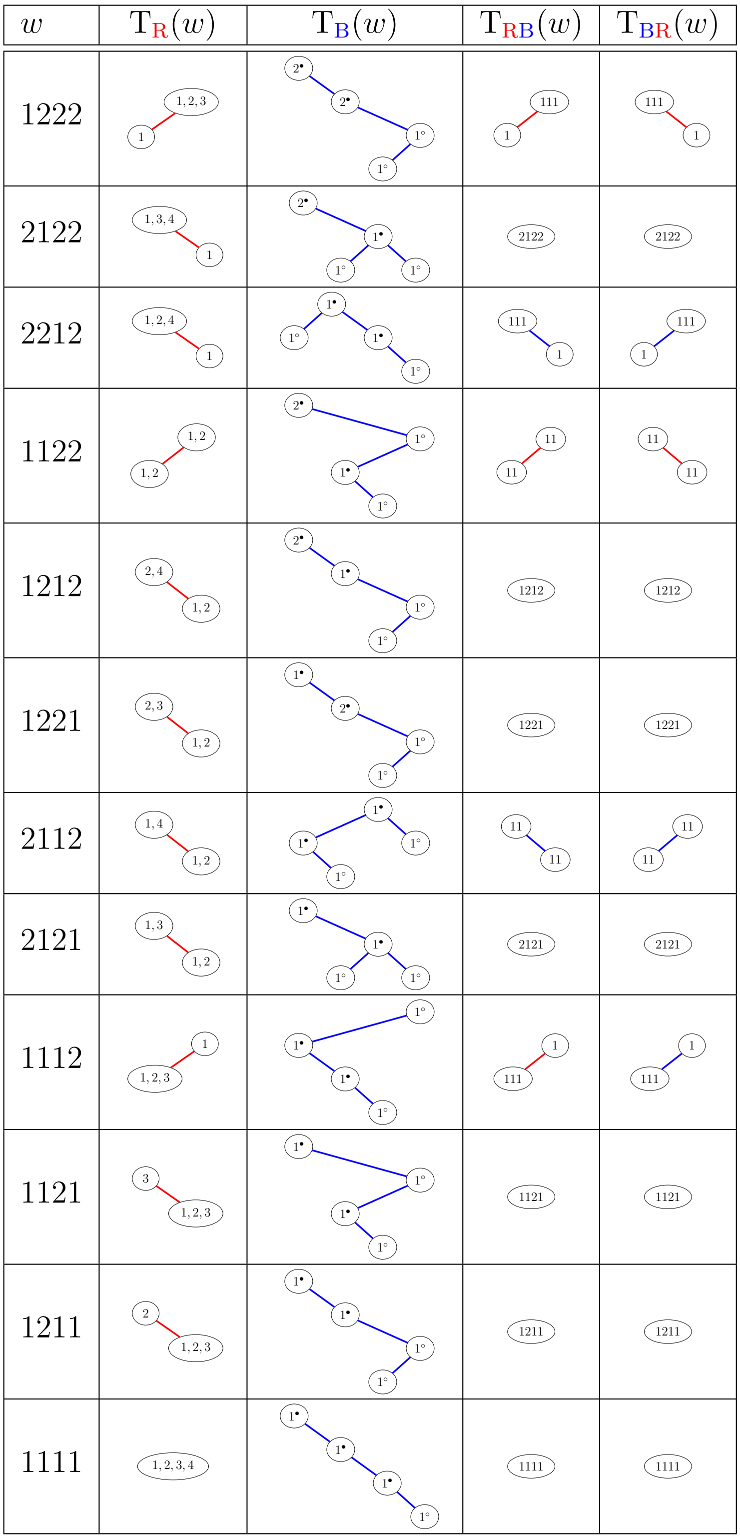
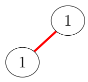
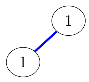
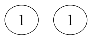
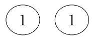


Table 4.7: Packed words of size 4 and associated forests (part 5).

$\mathbb{R}_1$	$\mathbb{P}$ 	$\mathbb{O}$ 	$\mathbb{Q}_1$

$\mathbb{R}_{12} - \mathbb{R}_{21}$	$\mathbb{P}$ 	$\mathbb{O}$ 	$\mathbb{Q}_{12} - \mathbb{Q}_{21}$
$\mathbb{R}_{21}$	$\mathbb{P}$ 	$\mathbb{O}$ 	$\mathbb{Q}_{21}$
$\mathbb{R}_{11}$	$\mathbb{P}$ 	$\mathbb{O}$ 	$\mathbb{Q}_{11}$

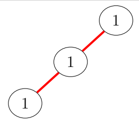
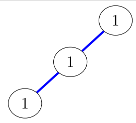
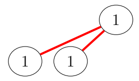
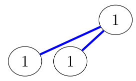
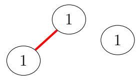
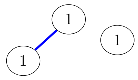
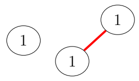
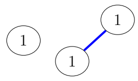
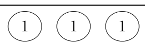
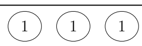
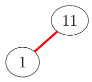
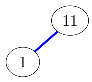
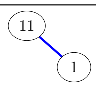
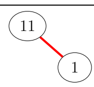
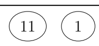
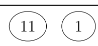
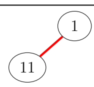
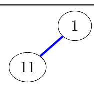
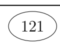
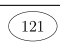
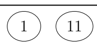
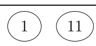
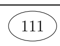
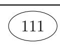
$\mathbb{R}$	$\mathbb{P}$	$\mathbb{O}$	$\mathbb{Q}$
$123 - 213 - 231 + 321$			$123 - 213 - 312 + 321$
$132 - 231$			$132 - 312$
$213 - 312 + 231 - 132$			$213 + 312 - 231 - 132$
$231 - 321$			$312 - 321$
$132 + 312 - 231 - 321$			$231 + 132 - 321 - 312$
321			321
$122 - 121 + 212 - 211$			$122 - 221$
212			$122 - 212$
221			211
$112 - 221$			$112 - 211$
121			$121 - 211$
$121 + 211$			221
111			111

Table 4.8: The automorphism of $\mathbf{WQSym}_{\leqslant 3}$.

1234	1234
1243	1243
1324	1324
1342	1342
1423	1423
1432	1432
2134	2134
2143	2143
2314	3124
2341	4123
2413	3142
2431	4132
3124	2314
3142	2413
3214	3214
3241	4213
3412	3412
3421	4312
4123	2341
4132	2431
4213	3241
4231	4231
4312	3421
4321	4321

1233	3123
1323	1323
1332	1332
2133	3213
2313	2313
2331	3212
3123	1233
3132	3132
3213	2133
3231	3122
3312	2311
3321	3211

1122	2112
1212	1212
1221	1221
2112	1122
2121	2121
2211	2211

1223	2123
1232	1232
1322	1322
2123	1223
2132	2132
2213	2113
2231	3112
2312	2131
2321	3121
3122	3231
3212	2331
3221	3221

1222	2212
2122	2122
2212	1222
2221	2111

1111	1111

1123	1123
1132	1132
1213	1213
1231	1231
1312	1312
1321	1321
2113	2213
2131	2312
2311	3312
3112	2231
3121	2321
3211	3321

1112	1112
1121	1121
1211	1211
2111	2221

Table 4.9: The involution $w \mapsto \widehat{w}$ on packed words of size 4.

Recall that in Table 4.9 packed words are organized by evaluations. Red-irreducible (resp. blue-irreducible) packed words are underlined in red (resp. blue).

23514	41253
24513	41352
25314	41523
25413	41532
32514	42153
35124	34152
35214	43152

24314	31424
24413	31442
41234	12344
41324	13244
42134	21344
42314	31244
42413	34142
43124	23144
43214	32144

23413	31242
24313	31422
32413	32142
34123	23141
34213	32141
32313	23133
33123	12333
33213	21333

22413	31142
23412	31241
24213	31412
24312	31421

23213	21313
23312	21331
31223	21233
32123	12233
32213	21133
32312	23131

22312	21131
23212	21311

31123	11233
31213	12133
32113	22133

22212	12222

23112	22131

21112	11122

24113	33142

22112	11222

Table 4.10: The involution $w \mapsto \widehat{w}$ on red-irreducible packed words that are not blue-irreducible of size 5.

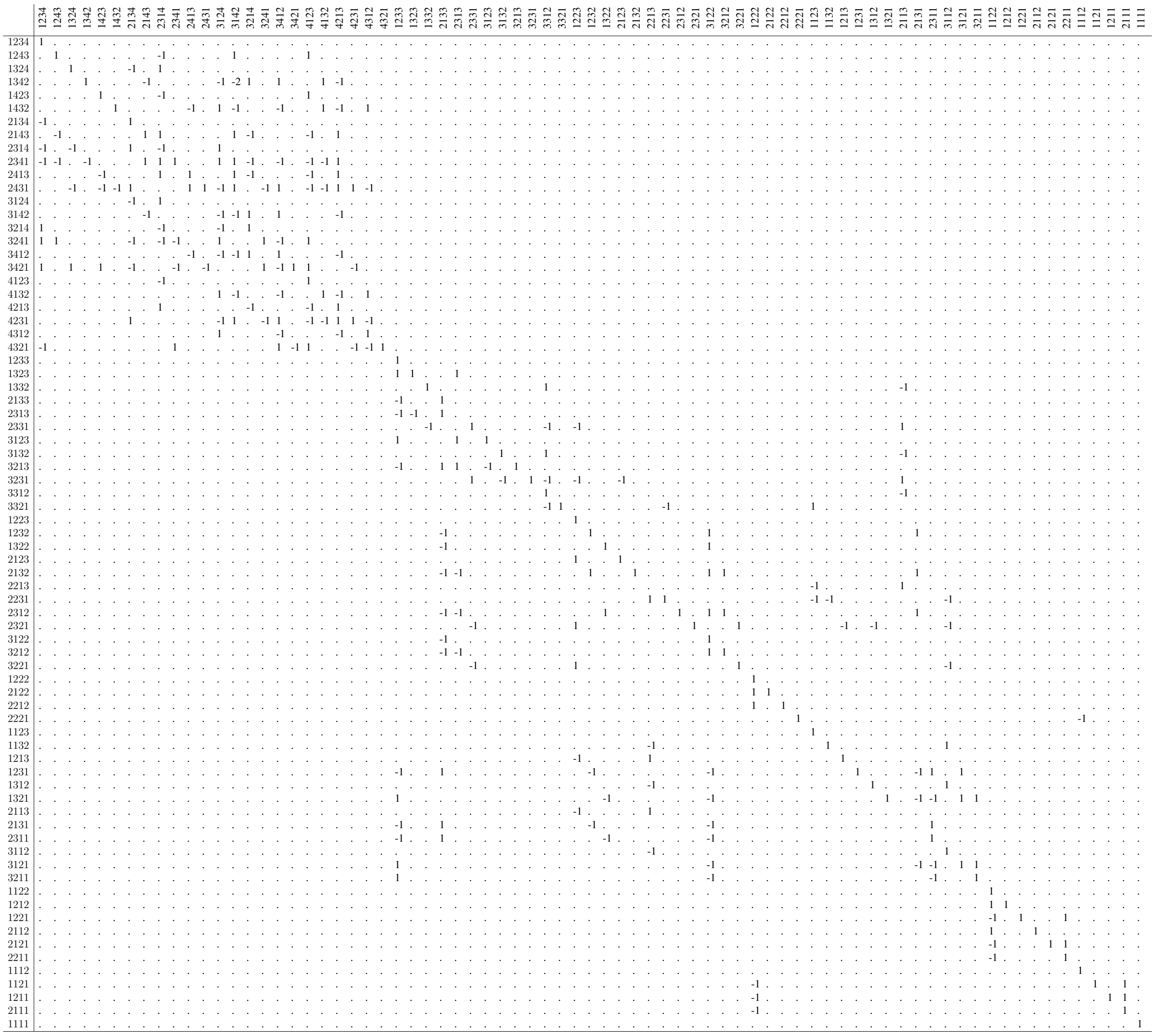


Figure 4.4: Change-of-basis matrix from $\mathbb{P}_4$ to $\mathbb{R}_4$.

	1234	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321	1233	1323	1332	2133	2313	2331	3123	3132	3213	3231	3312	3321	1223	1232	1322	2123	2132	2213	2231	2312	2321	3122	3212	3221	1222	2122	2212	2221	1123	1132	1213	1231	1312	1321	2113	2131	2311	3112	3121	3211	1122	1212	1221	2112	2121	2211	1112	1121	1211	2111	1111
1234	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1243	.	1	.	.	.	.	.	.	.	1	1	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1324	.	.	1	.	.	.	-1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1342	.	.	.	1	.	.	.	.	.	1	.	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1423	.	.	.	.	1	.	.	-1	-1	.	-2	1	.	.	1	-1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1432	.	.	.	.	.	1	.	.	1	.	-1	1	.	-1	.	-1	-1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2134	-1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
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Figure 4.5: Change-of-basis matrix from $\mathbb{O}_4$ to $\mathbb{Q}_4$.

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Figure 4.6: Change-of-basis matrix from $\mathbb{Q}_4$ to $\mathbb{R}_4$.