

THE SPECTRAL EVEN CYCLE PROBLEM

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This paper is dedicated to the memory of Vladimir Nikiforov.

Abstract. In this paper, we study the maximum adjacency spectral radii of graphs of large order that do not contain an even cycle of given length. For $n > k$, let $S_{n,k}$ be the join of a clique on k vertices with an independent set of $n - k$ vertices and denote by $S_{n,k}^+$ the graph obtained from $S_{n,k}$ by adding one edge. In 2010, Nikiforov conjectured that for n large enough, the C_{2k+2} -free graph of maximum spectral radius is $S_{n,k}^+$ and that the $\{C_{2k+1}, C_{2k+2}\}$ -free graph of maximum spectral radius is $S_{n,k}$. We solve this two-part conjecture.

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Mathematics Subject Classifications. 05C35, 05C50

1. Introduction

The Turán number of a graph F is the maximum number of edges in a graph on n vertices with no subgraph isomorphic to F . We use $ex(n, F)$ to denote the Turán number of F and $EX(n, F)$ for the set of F -free graphs on n vertices with $ex(n, F)$ many edges. In 1941, Turán [Tur41] determined $ex(n, K_{r+1})$, where K_{r+1} is the complete graph on $r + 1$ vertices, showing that the unique extremal graph is the complete r -partite graph with part sizes as balanced as possible

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(called the *Turán graph* and denoted by $T_r(n)$). Another celebrated theorem in extremal combinatorics is the Erdős–Stone–Simonovits theorem [ES65, ES46] which extends Turán’s theorem to all $r + 1$ chromatic graphs, where $r \geq 2$. The statement is as follows,

$$\text{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \frac{n^2}{2}, \quad (1.1)$$

where $\chi(F)$ denotes the chromatic number of the forbidden graph F and the term $o(1)$ goes to zero as n goes to infinity. The Erdős–Stone–Simonovits theorem gives the exact asymptotics of the Turán numbers for any forbidden graph F with chromatic number $\chi(F) > 2$ and essentially says that the extremal graphs cannot do much better than the Turán graphs. However, when the forbidden graphs are bipartite, that is $\chi(F) = 2$, we only get that $\text{ex}(n, F) = o(n^2)$. Very little is known for even the simplest examples of bipartite graphs. Determining the asymptotics of $\text{ex}(n, C_{2k})$ is one of the most famous Turán-type open problem and is notoriously difficult. The order of magnitude is only known for $k \in \{2, 3, 5\}$ [FS13], and determining it for other k is called the even cycle problem.

In this paper, we study the spectral version of the Turán problem for even cycles. Analogous to Turán numbers, among all graphs on n vertices, that do not contain F as a subgraph, let $\text{spex}(n, F)$ denote the maximum value of the spectral radius of their adjacency matrices. Also, let $\text{SPEX}(n, F)$ denote the set of graphs with an adjacency matrix having spectral radius equal to $\text{spex}(n, F)$. Nikiforov [Nik10b] was the first to systematically investigate spectral Turán-type problems, although several sporadic results appeared earlier. In particular, Nikiforov [Nik07] proved that $\text{SPEX}(n, K_{r+1}) = \{T_r(n)\}$. Since the average degree of a graph lower bounds the spectral radius of its adjacency matrices, Nikiforov’s spectral result implies that any K_{r+1} -free graph has at most $e(T_r(n))$ edges. That the spectral Turán theorem implies the edge bound of Turán’s theorem is proved in detail in Proposition 4.20 of [Gui96] and is also explained in equation (3) of [LLF21]. To our knowledge it is not recorded anywhere whether or not any proof of the spectral Turán theorem also implies the uniqueness of the Turán graph as the K_{r+1} -free graph with the maximum number of edges.

Further, Nikiforov [Nik10a], Babai and Guiduli [BG09], independently obtained spectral analogues of the Kővari–Sós–Turán theorem [KTST54], when forbidding a complete bipartite graph $K_{s,t}$. Moreover, using the average degree bound for the spectral radius gives bounds that match the best improvements to the Kővari–Sós–Turán theorem, obtained by Füredi [Für96].

Recently, determining $\text{spex}(n, F)$ for various graphs F has become very popular (see [CDT22, CFTZ20, DKL⁺22, LP22, LLT06, Nik08, SS07, Wil86, YWZ12, ZW12, ZWF20]). This fits into a broader framework of *Brualdi–Solheid problems* [BS86] which investigate the maximum spectral radius over all graphs belonging to a specified family of graphs. Numerous results are known in this area (see [BZ01, BLL18, EZ00, FN10, Nos70, Sta87, SAH08]). In [Nik10b], Nikiforov conjectured the solution to the spectral Turán problem for even cycles. Let $S_{n,k} := K_k \vee \overline{K}_{n-k}$ and $S_{n,k}^+ := K_k \vee (\overline{K}_{n-k-2} \cup K_2)$. The graph $S_{n,k}^+$ has n vertices and does not contain any C_{2k+2} , while, $S_{n,k}$ has n vertices and contains neither any C_{2k+1} nor C_{2k+2} . Nikiforov [Nik10b, Conjecture 15] made the following two-part conjecture.

Conjecture 1.1. Let $k \geq 2$ and G be a graph of sufficiently large n .

- (a) if $\lambda(G) \geq \lambda(S_{n,k})$ then G contains C_{2k+1} or C_{2k+2} unless $G = S_{n,k}$;
- (b) if $\lambda(G) \geq \lambda(S_{n,k}^+)$ then G contains C_{2k+2} unless $G = S_{n,k}^+$.

In this paper, we fully resolve both parts of Conjecture 1.1 in the affirmative.

Theorem 1.2. Let $k \geq 2$ and n be sufficiently large, then $\text{SPEX}(n, C_{2k+2}) = \{S_{n,k}^+\}$.

Theorem 1.2 was proved to be true for $k = 2$ by Zhai and Lin [ZL20]. We settle the remaining cases with $k > 2$. We also prove the following theorem which resolves Part (a) of the conjecture.

Theorem 1.3. Let $k \geq 2$ and n be sufficiently large, then $\text{SPEX}(n, \{C_{2k+1}, C_{2k+2}\}) = \{S_{n,k}\}$.

2. Organization and Notation

For some fixed $k \geq 2$, let $H_k \in \text{SPEX}(n, C_{2k+2})$ be a spectral extremal graph when forbidding the even cycle of size $2k + 2$ and let $H'_k \in \text{SPEX}(n, \{C_{2k+1}, C_{2k+2}\})$ be a spectral extremal graph when forbidding both C_{2k+1} and C_{2k+2} . All of our arguments in Sections 3 and 4 apply with identical proofs to both H_k and H'_k as we will only be using the fact that the graph is extremal and C_{2k+2} -free, and so for brevity we will only state the results for H_k until the proofs of Theorems 1.2 and 1.3 (Section 5).

For any vertex u and non-negative integer i , let $N_i(u)$ denote the set of vertices at distance i from u , with $d_i(u) := |N_i(u)|$. We use $d(u) = d_1(u)$ to denote the degree of u . For two disjoint subsets $X, Y \subset V(H_k)$, denote by $H_k[X, Y]$ the bipartite subgraph of H_k with vertex set $X \cup Y$ that consists of all the edges with one endpoint in X and the other endpoint in Y . Let $\mathcal{E}(X, Y)$ be the edge-set of $H_k[X, Y]$ and set $e(X, Y) := |\mathcal{E}(X, Y)|$. Denote by $\mathcal{E}(X)$ the set of edges with both endpoints in X and set $e(X) := |\mathcal{E}(X)|$.

For a graph $G = (V, \mathcal{E})$, we denote by $A(G)$ its adjacency matrix and by $\lambda(G)$ the spectral radius of $A(G)$. Associated to the spectral radius of the adjacency matrix of any connected graph is a unique (up to scalar multiples) entrywise positive eigenvector, commonly referred to as the Perron vector of the matrix. Since adding an edge between two disconnected components of a graph does not create any cycles and increases the spectral radius, the graphs H_k and H'_k must be connected. Let \mathbf{v} be the Perron vector of the adjacency matrix of H_k with maximum entry $v_x = 1$, where v_u denotes the coordinate of \mathbf{v} with respect to some vertex u .

We will fix a small constant α below, and we define L to be the following set of vertices of large weight, and denote by S its complement:

$$L := \{u \in V(H_k) \mid v_u > \alpha\} \text{ and } S := V(H_k) \setminus L = \{u \in V(H_k) \mid v_u \leq \alpha\}.$$

Additionally, we will also use the following set in the proof of Lemma 4.1. Let

$$M := \{u \in V(H_k) \mid v_u \geq \alpha/3\}.$$

Finally, we define a subset of L called L' by

$$L' := \{u \mid v_u \geq \eta\},$$

where $\eta > \alpha$ is a constant defined below. For a vertex u , denote

$$L_i(u) := L \cap N_i(u), S_i(u) := S \cap N_i(u), \text{ and } M_i(u) := M \cap N_i(u).$$

If the vertex is unambiguous from context, we will use L_i , S_i , and M_i instead.

With foresight, we choose η , ϵ , and α to be any positive constants satisfying

$$\eta < \left\{ \frac{1}{k+1}, 1 - \frac{1}{16k^3}, \frac{1}{4} - \frac{1}{16k^2} \right\} \quad (2.1)$$

$$\epsilon < \min \left\{ \frac{1}{16k^3}, \frac{\eta}{2}, \frac{\eta}{32k^3 + 2} \right\} \quad (2.2)$$

$$\alpha < \frac{\epsilon^2}{10k}. \quad (2.3)$$

We note that many of the above inequalities are redundant, but we leave them so that it is easier to see exactly what inequalities we are using throughout the proofs. In Sections 3 and 4, we prove lemmas showing the structural properties of H_k and H'_k . We reiterate that every lemma applies to both H_k and H'_k with identical proofs and so the proofs are only written for H_k . In Section 5, we complete the proofs of Theorems 1.2 and 1.3.

The proofs are somewhat technical and so we give a high-level outline of the strategy now. The main goal is to show that H_k (or H'_k) contains a $K_{k,n-k}$. From there it is straightforward to deduce that Conjecture 1.1 holds. Note that we are switching between structural information known for the spectral extremal graph and bounds on its Perron entries, and are recursively improving our estimations for both the structure of the graph and the entries of its Perron vector through a sequence of steps outlined below.

1. We apply Erdős and Gallai's result for Turán numbers of paths on $2k+1$ (or $2k+3$) vertices given in Lemma 3.2, along with the upper bounds appearing in Lemma 3.3 to bound the number of two paths starting at any fixed vertex. This allows us to show that the spectral radius of a spectral extremal graph H_k satisfies $\lambda(H_k) = \Theta(\sqrt{n})$ in Lemma 3.4. We also use the even circuit theorem (see Lemma 3.5) to obtain an upper bound of $o(n)$ for the sizes of L and M given in Lemma 3.6. Using the upper bound on the sizes of L and M we show that every vertex in L has degree $\Theta(n)$ in Lemma 4.1. Then since a result of Nikiforov (see Lemma 3.7) implies that the sum of squares of degrees in any C_{2k+2} -free graph is at most $O(n^2)$, we obtain that there are at most $\Theta(1)$ vertices in L (see Lemma 4.1).
2. Next we refine our calculations from Lemma 4.1. Using the bounds on the size of L coming from Lemma 4.1 along with a combinatorial argument, we show that for any vertex $v \in L'$ with Perron weight $c \geq \eta$, the degree of v is scaling at least linearly with n , so $d(v) \geq cn - \epsilon n$; see Lemma 4.3. The same combinatorial argument, combined with the bounds on the size of L' , coming from Lemma 4.1, when applied on a vertex z with largest Perron weight, gives strong bounds on the number of edges between S_1 and $\{z\} \cup L_1 \cup L_2$; see Lemma 4.5.

3. The structural information gathered from previous lemmas now allows us to further improve our estimations for the Perron entries for vertices corresponding to L' . This in turn gives us better estimates on the size of L' , allowing us to show that $|L'| = k$ and that the common neighborhood of L' , denoted by R , contains all but at most $\frac{n}{8k^2}$ vertices; see Lemma 4.7.
4. The structural information obtained from the previous step allows us to lower bound the sum of all the Perron entries in a neighborhood of a vertex in Lemma 4.8. Since H_k is C_{2k+2} -free, it follows that R has at most one edge. Moreover, let E denote the set of vertices outside L' and R . Then E is adjacent to at most $k - 1$ vertices of L' and 1 vertex of R . Applying the $\Theta(\sqrt{n})$ upper bound for the spectral radius from Lemma 3.4 along with observations from Lemmas 4.8, we can show that every vertex not in L' must be in the common neighborhood of L' . This gives that H_k contains the complete bipartite graph $K_{k,n-k}$; see Lemma 4.9.

3. Lemmas from spectral and extremal graph theory

In this section, we record several lemmas that we will use. Some calculations may only apply for n large enough without being explicitly stated. We start with a standard result from linear algebra which serves as a tool to bound the spectral radius of non-negative matrices.

Lemma 3.1. *For a non-negative symmetric matrix M , a non-negative non-zero vector y and a positive constant c , if $My \geq cy$ entrywise, then $\lambda(M) \geq c$.*

Proof. Assume that $My \geq cy$ entrywise, with the same assumptions for M, y and c as in the statement of the theorem. Then $y^T My \geq y^T cy$ and $\lambda(M) \geq \frac{y^T My}{y^T y} \geq c$. □

Lemma 3.2 (Erdős-Gallai [EG59]). *Any graph on n vertices with no subgraph isomorphic to a path on ℓ vertices has at most $\frac{(\ell - 2)n}{2}$ edges.*

We will be using Part B of the following lemma which appears in [Nik09, Lemma 1].

Lemma 3.3 (Nikiforov [Nik09]). *Suppose that $k \geq 1$ and let the vertices of a graph G be partitioned into two sets U and W .*

(A) *If*

$$2e(U) + e(U, W) > (2k - 2)|U| + k|W|, \tag{3.1}$$

then there exists a path of order $2k$ or $2k + 1$ with both ends in U .

(B) *If*

$$2e(U) + e(U, W) > (2k - 1)|U| + k|W|, \tag{3.2}$$

then there exists a path of order $2k + 1$ with both ends in U .

Let u be any vertex in H_k . Since our graph is C_{2k+2} -free, $N_1(u)$ may not contain a P_{2k+1} . Hence, by Lemma 3.2 we have

$$e(N_1(u)) \leq \frac{2k-1}{2}d(u) < kn. \quad (3.3)$$

Similarly the bipartite subgraph between $N_1(u)$ and $N_2(u)$ may not contain a P_{2k+3} , otherwise there is a P_{2k+1} with both endpoints in $N_1(u)$ and hence a C_{2k+2} . Therefore, by Lemma 3.2 (forbidding P_{2k+3} in the bipartite subgraph) and Lemma 3.3 (forbidding P_{2k+1} with both endpoints in $N_1(u)$), we have

$$e(N_1(u), N_2(u)) \leq \min \left\{ \frac{2k+1}{2}n, (2k-1)d(u) + k(n-d(u)-1) \right\}. \quad (3.4)$$

The spectral radius of $S_{n,k}$ gives a lower bound for $\lambda(H_k)$. We will modify an argument of Nikiforov (proof of [Nik10b, Theorem 3]) to obtain an upper bound for $\lambda(H_k)$.

Lemma 3.4. $\sqrt{kn} \leq \frac{k-1+\sqrt{(k-1)^2+4k(n-k)}}{2} \leq \lambda(H_k) \leq \sqrt{2k(n-1)}$.

Proof. Here the inner lower bound is precisely $\lambda(S_{n,k})$ and the first inequality on the left follows from a straightforward calculation. To prove the upper bound, let $u \in V(H_k)$. We use Lemma 3.3 over the graph $H_k[N_1(u) \cup N_2(u)]$ with $U = N_1(u)$ and $W = N_2(u)$. We know that $|N_1(u)| = d(u)$. Also, there cannot be any path on $2k+1$ vertices in $H_k[N_1(u) \cup N_2(u)]$ with both end points in $N_1(u)$. So (3.2) implies that

$$\begin{aligned} 2e(N_1(u)) + e(N_1(u), N_2(u)) &\leq (2k-1)d(u) + kd_2(u) \\ &\leq (2k-1)d(u) + k(n-d(u)-1) \\ &= (k-1)d(u) + k(n-1). \end{aligned} \quad (3.5)$$

The spectral radius of a non-negative matrix is at most the maximum of the row-sums of the matrix. Applying this result for $A^2(H_k)$ and its spectral radius λ^2 and using (3.5), we obtain that

$$\begin{aligned} \lambda^2 &\leq \max_{u \in V(H_k)} \left\{ \sum_{w \in V(H_k)} A_{u,w}^2 \right\} = \max_{u \in V(H_k)} \left\{ \sum_{v \in N(u)} d(v) \right\} \\ &= \max_{u \in V(H_k)} \left\{ d(u) + 2e(N_1(u)) + e(N_1(u), N_2(u)) \right\} \\ &\leq kd(u) + k(n-1) \leq 2k(n-1). \end{aligned} \quad (3.6)$$

Thus, $\lambda \leq \sqrt{2k(n-1)}$. □

Next we determine an upper bound for the number of vertices in L . We use the same technique as in the proof of [TT17, Lemma 8]. For this work, we use the even-circuit theorem. Note that the best current bounds for $\text{ex}(n, C_{2k})$ are given by He [He21] (see also Bukh and Jiang [BJ17]), but for our purposes, the dependence of the multiplicative constant on k is not important. We use the following version because it makes the calculations slightly easier.

Lemma 3.5 (Even Circuit Theorem [Ver00]). *For $k \geq 1$ and n a natural number,*

$$\text{ex}(n, C_{2k+2}) \leq 8kn^{(k+2)/(k+1)}.$$

Lemma 3.6. $|L| \leq \frac{16k^{1/2}n^{(k+3)/(2k+2)}}{\alpha}$ and $|M| \leq \frac{48k^{1/2}n^{(k+3)/(2k+2)}}{\alpha}$.

Proof. For any vertex $u \in V(H_k)$, we have the following equation relating the spectral radius and Perron vector entries,

$$\lambda v_u = \sum_{w \sim u} v_w. \tag{3.7}$$

Because $\sqrt{kn} \leq \lambda$ and $v_w \leq 1$, we get that

$$\sqrt{kn}v_u \leq \lambda v_u \leq d_u,$$

Summing up all these inequalities for $u \in L$, we obtain that

$$\frac{|L|\sqrt{kn}\alpha}{2} \leq \frac{1}{2} \sum_{u \in L} \lambda v_u \leq \frac{1}{2} \sum_{u \in L} d_u \leq \frac{1}{2} \sum_{u \in V(H_k)} d_u \leq \text{ex}(n, C_{2k+2}) \leq 8kn^{(k+2)/(k+1)},$$

which implies that

$$|L| \leq \frac{16k^{1/2}n^{(k+3)/(2k+2)}}{\alpha}. \tag{3.8}$$

The bound for $|M|$ is obtained similarly by replacing α by $\alpha/3$ everywhere above. □

We use the following result of Nikiforov [Nik09, Theorem 2] to get a better upper bound for the size of L in Lemma 4.1.

Lemma 3.7 (Nikiforov [Nik09]). *Let G be a graph with n vertices and m edges. If G does not contain a C_{2k+2} then*

$$\sum_{u \in V(G)} d_G^2(u) \leq 2km + k(n-1)n.$$

Using Lemma 3.4 we obtain the following lower bound for entries in the Perron vector of the extremal graphs by modifying a proof of Tait and Tobin (proof of [TT17, Lemma 10]).

Lemma 3.8. *For any vertex $u \in V(H_k)$, $v_u \geq \frac{1}{\lambda(H_k)} \geq \frac{1}{\sqrt{2k(n-1)}}$.*

Proof. Towards a contradiction, assume that there exists a vertex $u \in V(H_k)$, such that $v_u < \frac{1}{\lambda(H_k)}$. Then by (3.7), u cannot be adjacent to any vertex x such that $v_x = 1$. Let \hat{H}_k be the graph obtained by modifying H_k by removing all the edges adjacent to u and making u adjacent to x . Then using the Rayleigh quotient, we have $\lambda(\hat{H}_k) > \lambda(H_k)$. Because adding a vertex of degree one to a graph cannot create a cycle, \hat{H}_k does not contain any subgraph isomorphic to C_{2k+2} , contradicting that H_k is extremal. □

4. Structural results for extremal graphs

In this section, we will assume that $k \geq 2$ is fixed. We will be working with subgraphs in H_k and due to lack of ambiguity we will drop H_k from some notations now onward. We will continue to use auxiliary constants α , ϵ , and η and we will frequently assume that n is larger than some constant depending only on $\alpha, k, \epsilon, \eta$. Every lemma in this section holds only for n large enough.

Lemma 4.1. *For any vertex $z \in L$, we have $d(z) \geq \frac{\alpha}{20k}n$. Also, $|L| \leq \frac{k+1}{(\alpha/20k)^2}$.*

Proof. For some vertex $z \in L$ such that $v_z = c$, consider the following second degree eigenvalue-eigenvector equations relating λ^2 , v , and entries in the z -th row of A^2 :

$$\begin{aligned} knc &\leq \lambda^2 c = \lambda^2 v_z = \sum_{u \sim z} \sum_{w \sim u} v_w \\ &\leq d(z)c + 2e(N_1(z)) + \sum_{u \sim z} \sum_{\substack{w \sim u \\ w \in N_2(z)}} v_w \\ &\leq 2kd(z) + \sum_{u \sim z} \sum_{\substack{w \sim u \\ w \in N_2(z)}} v_w, \end{aligned}$$

where the last inequality is by (3.3). Now assume to the contrary that there is a vertex $z \in L$ with $d(z) < \frac{\alpha}{20k}n$. Substituting this into the above equation and using $\alpha < c$ since $z \in L$, we have

$$(k - 0.1)nc < \sum_{u \sim z} \sum_{\substack{w \sim u \\ w \in N_2(z)}} v_w.$$

Next we show that many of the terms in the double sum come from vertices in $M_2(z)$ via the following claim.

Claim 4.2. *There are at least $0.9nc$ terms v_w with $w \in M_2$ in the sum*

$$\sum_{u \sim z} \sum_{\substack{w \sim u \\ w \in N_2(z)}} v_w.$$

Proof. Assume to the contrary that there are less than $0.9nc$ terms v_w where $w \in M_2$. As $v_w \leq 1$ for such w ,

$$\begin{aligned} (k - 0.1)nc &< \sum_{u \sim z} \sum_{\substack{w \sim u \\ w \in N_2(z)}} v_w \\ &= \sum_{u \sim z} \sum_{\substack{w \sim u \\ w \in M_2}} v_w + \sum_{u \sim z} \sum_{\substack{w \sim u \\ w \in N_2 \setminus M_2}} v_w \\ &< 0.9nc + e(N_1, N_2 \setminus M_2) \frac{\alpha}{3}, \end{aligned}$$

and so

$$(k - 1)nc < e(N_1, N_2 \setminus M_2) \frac{\alpha}{3}.$$

From $\alpha < c$ and (3.4), we have that

$$(k - 1)n < \frac{2k + 1}{2} \frac{1}{3}n,$$

a contradiction for $k \geq 2$. This proves our claim. □

Therefore, $e(N_1(z), M_2(z)) \geq 0.9nc > 0.9n\alpha$. Because $H_k[N_1(z) \cup M_2(z)]$ contains no P_{2k+3} , by Lemma 3.2, we deduce that

$$0.9n\alpha \leq \frac{2k + 1}{2} |N_1(z) \cup M_2(z)| < \frac{2k + 1}{2} \left(\frac{n\alpha}{20k} + |M_2(z)| \right).$$

Thus,

$$|M_2(z)| > \left(.9\alpha - \frac{(2k + 1)\alpha}{40k} \right) \left(\frac{2}{2k + 1} \right) n.$$

This contradicts the bound in Lemma 3.6 for n sufficiently large. Thus, for n sufficiently large we have that $d(z) \geq \frac{\alpha}{20k}n$ for all $z \in L$. Combined with Lemma 3.7 this gives us that $|L| \leq \frac{k+1}{(\alpha/20k)^2}$. □

We now refine the lower bound on the degrees of vertices in L' .

Lemma 4.3. *If z is a vertex of L' with $v_z = c$, then $d(z) \geq cn - \epsilon n$.*

Proof. Given $z \in L$ with $v_z = c$, observe that

$$\begin{aligned} knc &\leq \lambda^2 c = \sum_{u \sim z} \sum_{w \sim u} v_w = d(z)c + \sum_{u \sim z} \sum_{\substack{w \sim u \\ w \neq z}} v_w \\ &\leq d(z)c + \left(\sum_{\substack{u \sim z \\ u \in S_1}} \sum_{\substack{w \sim u \\ w \in L_1 \cup L_2}} v_w \right) + 2e(S_1)\alpha + 2e(L) + e(S_1, L_1)\alpha + e(N_1, S_2)\alpha. \end{aligned}$$

Since N_1 is P_{2k+1} -free and the bipartite graph between N_1 and N_2 is P_{2k+3} -free,

$$\begin{aligned} 2e(S_1) &\leq 2e(N_1) \leq (2k - 1)n \\ e(L_1, S_1) &\leq e(N_1) < kn, \\ e(N_1, S_2) &< 2kn. \end{aligned}$$

Using these and Lemma 4.1, we have

$$2e(S_1)\alpha + 2e(L) \leq 2e(N_1)\alpha + 2 \binom{|L|}{2} \leq (2k - 1)n\alpha + |L|(|L| - 1) < 2kn\alpha,$$

where the last inequality holds for n large enough.

Hence, we have

$$\begin{aligned}
knc &< d(z)c + \left(\sum_{\substack{u \sim z \\ u \in S_1}} \sum_{\substack{w \sim u \\ w \in L_1 \cup L_2}} v_w \right) + 5kn\alpha & (4.1) \\
&\leq d(z)c + e(S_1, L_1 \cup L_2) + 5kn\alpha \\
&< d(z)c + e(S_1, L_1 \cup L_2) + \frac{\epsilon^2 n}{2},
\end{aligned}$$

by the choice of α in (2.3).

If $d(z) \leq (c - \epsilon)n$, then

$$(k - c + \epsilon)nc \leq (kn - d(z))c \leq e(S_1, L_1 \cup L_2) + \frac{\epsilon^2 n}{2}. \quad (4.2)$$

Since $z \in L'$ we have $c \geq \epsilon$. Rearranging and using $\epsilon \leq c \leq 1$, we get that

$$e(S_1, L_1 \cup L_2) \geq (k - 1)nc + \frac{\epsilon^2 n}{2}. \quad (4.3)$$

We will show that $H_k[S_1, L_1 \cup L_2]$ contains a P_{2k+1} with both endpoints in S_1 , thus contradicting the fact that H_k is C_{2k+2} -free. To show this we prove the following claim.

Claim 4.4. *If $\delta := \frac{\epsilon(\alpha/20k)^2}{k+1}$, then there are at least δn vertices inside S_1 with degree at least k in $H_k[S_1, L_1 \cup L_2]$.*

Proof. Assume to the contrary that at most δn vertices in S_1 have degree at least k in $H_k[S_1, L_1 \cup L_2]$. Then

$$e(S_1, L_1 \cup L_2) < (k - 1)|S_1| + |L|\delta n \leq (k - 1)(c - \epsilon)n + \epsilon n,$$

because $|S_1| \leq d(z)$ and by Lemma 4.1. Combining this with (4.3) gives

$$(k - 1)nc - (k - 2)n\epsilon > e(S_1, L_1 \cup L_2) \geq (k - 1)nc + \frac{\epsilon^2 n}{2},$$

a contradiction. \square

Hence, there is some subset of vertices $B \subset S_1$ such that any vertex in B has degree at least k in $H_k[S_1, L_1 \cup L_2]$ and $|B| = \delta n$. Since there are $\binom{|L|}{k}$ options for every vertex in B to choose a set of k neighbors from, we have that there is some set of k vertices in $L_1 \cup L_2$ with at least $\delta n / \binom{|L|}{k}$ common neighbours in B . Therefore, by Lemma 3.2 and Lemma 4.1, for n sufficiently large we have a path on $2k + 1$ vertices with both end points in the common neighbourhood contained in B , a contradiction. \square

Thus, for the vertex x such that $v_x = 1$, we have $d(x) \geq n - \epsilon n$ and $N_1(x)$ contains all but at most ϵn many vertices. Since every vertex in L' has degree more than ϵn (by the definition of L' and Lemma 4.3), this also gives that $L' \setminus \{x\} \subset L_1(x) \cup L_2(x)$. The arguments in the proof of Lemma 4.3 also allow us to show that all vertices of L' have degrees close to n and thus obtain $|L'| = k$.

Lemma 4.5. For any vertex $z \in L'$ with $v_z \geq 1 - \epsilon$, we have $(k - 2\epsilon)n \leq e(S_1, L) \leq (k + \epsilon)n$.

Proof. To obtain the lower bound we refine (4.1). Using $1 - \epsilon \leq v_z \leq 1$ and $d(z) \leq n$, we get

$$kn(1 - \epsilon) < d(z) + e(S_1, L_1 \cup L_2) + \frac{\epsilon^2 n}{2} = e(S_1, L) + \frac{\epsilon^2 n}{2}.$$

Thus $e(S_1, L) > (1 - 2\epsilon)kn$.

To obtain the upper bound, assume to the contrary that $e(S_1, L) > kn + \epsilon n$. We will show that $H_k[S_1, L \setminus \{z\}]$ contains a P_{2k+1} with both endpoints in S_1 , thus contradicting the fact that H_k is C_{2k+2} -free. To show this we prove the following claim.

Claim 4.6. Let $\delta := \frac{\epsilon(\alpha/20k)^2}{k+1}$. Then there are δn vertices inside S_1 with degree at least k in $H_k[S_1, L \setminus \{z\}]$.

Proof. Assume to the contrary that at most δn vertices of S_1 have degree at least k in $H_k[S_1, L \setminus \{z\}]$. Then,

$$e(S_1, L \setminus \{z\}) < (k - 1)|S_1| + |L|\delta n \leq (k - 1)n + |L|\delta n,$$

because $|S_1| \leq d(z)$. This contradicts our assumption that $e(S_1, L) \geq kn + \epsilon n$. □

Hence, there is some subset of vertices $B \subset S_1$ such that any vertex in B has degree at least k in $H_k[S_1, L \setminus \{z\}]$ and $|B| = \delta n$. Since there are only $\binom{|L|}{k}$ options for every vertex in B to choose a set of k neighbors from, we have that there is some set of k vertices in $L \setminus \{x\}$ with at least $\delta n / \binom{|L|}{k}$ common neighbours in B . Therefore, by Lemma 3.2 and Lemma 4.1, for n sufficiently large we have a path on $2k + 1$ vertices with both end points in the common neighbourhood contained in B , a contradiction. □

Lemma 4.7. For all vertices $z \in L'$, we have $d(z) \geq (1 - \frac{1}{8k^3})n$ and $v_z \geq 1 - \frac{1}{16k^3}$. Moreover, $|L'| = k$.

Proof. If we show that every vertex $z \in L'$ has Perron entry $v_z \geq 1 - \frac{1}{16k^3}$, then it follows from Lemma 4.3 and (2.2) that $d(z) \geq (1 - \frac{1}{8k^3})n$. If all vertices in L' have degree at least $n - \frac{n}{8k^3}$, then $|L'| \leq k$, else there exists a $K_{k+1, k+1}$ in H_k , a contradiction. Also, if $|L'| \leq k - 1$, then by (4.1) and Lemma 4.5, we have

$$\begin{aligned} kn &= knv_x \\ &\leq \lambda^2 \\ &\leq e(S_1(x), L') + e(S_1(x), L \setminus L')\eta + \frac{\epsilon^2 n}{2} \\ &\leq (k - 1)n + (k + \epsilon)n \cdot \eta + \frac{\epsilon^2 n}{2}, \end{aligned}$$

a contradiction by (2.1) and (2.2). Hence, all we need to show is that every vertex in L' has Perron entry at least $1 - \frac{1}{16k^3}$.

By way of contradiction, assume that $z \in L'$ and $v_z < 1 - \frac{1}{16k^3}$. Refining (4.1) applied to the vertex x we have

$$\begin{aligned} kn &< e(S_1(x), L_1(x) \setminus \{z\}) + |N_1(x) \cap N_1(z)|v_z + \frac{\epsilon^2 n}{2} \\ &\leq (k + \epsilon)n - |N_1(x) \cap N_1(z)| + \left(1 - \frac{1}{16k^3}\right) |N_1(x) \cap N_1(z)| + \frac{\epsilon^2 n}{2} \\ &= kn + \epsilon n + \frac{\epsilon^2 n}{2} - \frac{|N_1(x) \cap N_1(z)|}{16k^3}, \end{aligned}$$

where the second inequality is by Lemma 4.5 and the bound on v_z . Therefore, we have that

$$\frac{|N_1(x) \cap N_1(z)|}{16k^3} < 2\epsilon n.$$

On the other hand, since $z \in L'$ we have $v_z \geq \eta$ and so by Lemma 4.3 we have

$$|N_1(x) \cap N_1(z)| \geq (\eta - 2\epsilon)n.$$

Combining the two inequalities is a contradiction by (2.2). \square

Now that we have $|L'| = k$ and every vertex in L' has degree at least $(1 - \frac{1}{8k^3})n$, it follows that the common neighborhood of L' has size at least $(1 - \frac{1}{8k^2})n$. That is, there are at most $\frac{n}{8k^2}$ vertices not adjacent to all of L' . Call this set of “exceptional vertices” E . That is,

$$E := \{v \in V(H_k) \setminus L' : |N_1(v) \cap L'| \leq k - 1\}.$$

Let $R = V(H_k) \setminus (L' \cup E)$ be the remaining vertices. So we have that $V(H_k)$ is the disjoint union of L' , R , and E with $|L'| = k$ and $|E| \leq \frac{n}{8k^2}$. In the next two lemmas we will show that $E = \emptyset$ and this will allow us to prove Theorems 1.2 and 1.3. Note that because R has size larger than $2k + 2$, adding a new vertex adjacent only to the vertices in L' cannot create a C_{2k+2} , otherwise there would have already been one.

Lemma 4.8. *For any vertex $u \in V(H_k)$, the Perron weight in the neighborhood of u satisfies*

$$\sum_{w \sim u} v_w \geq k - \frac{1}{16k^2}.$$

Proof. Assume to the contrary that there exists a vertex u with $\sum_{w \sim u} v_w < k - \frac{1}{16k^2}$. Note that because $\sum_{w \sim u} v_w = \lambda v_u \geq \sqrt{kn}v_u$, we have that $u \notin L'$. Now modify the neighborhood of u by deleting all the edges adjacent to u and joining u to all the vertices of L' . Call the resultant graph, H_k^* . The neighborhood of u in H_k^* has Perron weight at least $k - \frac{1}{16k^2}$ by Lemma 4.7 thus, $\lambda(H_k^*) > \lambda(H_k)$ by the Rayleigh quotient. Moreover, H_k^* does not contain any C_{2k+2} , a contradiction. \square

With this we may show that E is empty.

Lemma 4.9. *The set E is empty and H_k contains the complete bipartite graph $K_{k, n-k}$.*

Proof. Assume to the contrary that $E \neq \emptyset$. Note that $e(R) \leq 1$ and every vertex in E has at most one neighbor in R , else we can embed a C_{2k+2} in H_k . Any vertex $r \in R$, satisfies $v_r < \eta$. Therefore, for any vertex $u \in E$ we have that

$$\sum_{u \sim w} v_w = \lambda v_u = \sum_{\substack{w \sim u \\ w \in L' \cup R}} v_w + \sum_{\substack{w \sim u \\ w \in E}} v_w.$$

By Lemma 4.8 and using that vertices in E have at most $k - 1$ neighbors in L' , we have

$$\sum_{\substack{w \sim u \\ w \in E}} v_w \geq 1 - \frac{1}{16k^2} - \eta, \tag{4.4}$$

Since the Perron weight in E is at least $1 - \frac{1}{16k^2} - \eta > \frac{3}{4}$, the Perron weight outside E is at most $k - 1 + \eta$, and the total Perron weight is λv_u , we have that

$$\sum_{\substack{w \sim u \\ w \in E}} v_w \geq \frac{3}{4k} \lambda v_u$$

Now, applying Lemma 3.1, with $M = A(H_k[E])$, and $y = v|_E$ (the restriction of v to the set E), we have that for any $u \in E$,

$$My_u = \sum_{\substack{w \sim u \\ w \in E}} v_w \geq \frac{3}{4k} \lambda v_u = \frac{3}{4k} \lambda y_u. \tag{4.5}$$

Hence, by Lemma 3.1, $\lambda(M) \geq \frac{3}{4k} \lambda \geq \frac{3}{4} \sqrt{\frac{n}{k}}$. This contradicts Lemma 3.4 because

$$\lambda(M) \leq \sqrt{2k(|E| - 1)} < \sqrt{\frac{n}{4k}}$$

as E induces a C_{2k+2} -free graph. Thus, $E = \emptyset$. Therefore $R = V(H_k) \setminus L'$ and H_k must contain a $K_{k,n-k}$. \square

5. Proofs of Theorems 1.2 and 1.3

With Lemma 4.9 in hand we may complete the proofs of Theorems 1.2 and 1.3. Lemma 4.9 gives us that $K_{k,n-k}$ is a subgraph of both H_k and H'_k and the results follow quickly from this.

Proof of Theorem 1.2. We have shown that $K_{k,n-k} \subset H_k$, where the part with k vertices is L' and the other part with $n - k$ vertices is R . Thus $\mathcal{E}(L', R) = \{lr | l \in L', r \in R\}$. Now we know that $e(R) \leq 1$ and $H_k[L']$ is isomorphic to some subgraph of K_k . Thus, H_k is a subgraph of $S_{n,k}^+$ and by the monotonicity of the spectral radius over subgraphs, we have that $H_k \cong S_{n,k}^+$. \square

Proof of Theorem 1.3. We have $\mathcal{E}(L', R) = \{lr | l \in L', r \in R\}$. In addition, $e(R) = 0$, otherwise we can embed a C_{2k+1} . Also, $H_k[L']$ is isomorphic to some subgraph of K_k . Thus, H_k is a subgraph of $S_{n,k}$ and by the monotonicity of the spectral radius over subgraphs, we have that $H_k \cong S_{n,k}$. \square

6. Concluding remarks

The proof of Theorems 1.2 and 1.3 additionally allow us to prove the following conjecture of Zhai, Wang, and Fang. Here the fan F_ℓ is the join of a vertex with a path on $\ell - 1$ vertices and the wheel W_ℓ is the join of a vertex with a cycle on $\ell - 1$ vertices.

Conjecture 6.1 (Conjecture 1.4 in [ZWF20]). Let $k \geq 2$ and G be a graph of sufficiently large order n .

1. If G is $\{F_{2k+1}, C_{2k+2}\}$ -free, then $\lambda(G) \leq \lambda(S_{n,k})$ with equality if and only if $G \cong S_{n,k}$.
2. If G is $\{W_{2k+1}, C_{2k+2}\}$ -free, then $\lambda(G) \leq \lambda(S_{n,k}^+)$ with equality if and only if $G \cong S_{n,k}^+$.

The proofs of Theorems 1.2 and 1.3 of our paper can be used to prove the two-part Conjecture 1.4 of Zhai, Wang, and Fang [ZWF20].

Part 2 of Conjecture 1.4 follows trivially from Theorem 1.2 because any graph G that is $\{W_{2k+1}, C_{2k+2}\}$ -free must be C_{2k+2} -free. We have proved that for sufficiently large n , $S_{n,k}^+$ is the unique spectral extremal graph forbidding C_{2k+2} . Moreover, $S_{n,k}^+$ is itself W_{2k+1} -free, thus, it must also be the unique spectral extremal graph forbidding $\{W_{2k+1}, C_{2k+2}\}$, for n sufficiently large.

Part 1 of the conjecture follows by first observing that an $\{F_{2k+1}, C_{2k+2}\}$ -free graph must also be C_{2k+2} free. Thus following the proof of Theorem 1.2 gives that a spectral extremal graph G forbidding $\{F_{2k+1}, C_{2k+2}\}$, must contain $K_{k,n-k}$. The independent set on $n - k$ vertices may induce at most one edge, since G is C_{2k+2} -free. Further, if this set on $n - k$ vertices induces an edge, then the part on k vertices cannot not induce a complete graph K_k . However, one can observe by looking at the Rayleigh quotient that the spectral radius is maximized if the part on k vertices induces a K_k , and the part on $n - k$ vertices induces no edges. Thus, $S_{n,k}$ is the unique spectral extremal graph forbidding $\{F_{2k+1}, C_{2k+2}\}$.

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References

- [BG09] László Babai and Barry Guiduli. Spectral extrema for graphs: the Zarankiewicz problem. *The Electronic Journal of Combinatorics*, 16(1):R123, 2009. doi:10.37236/212.
- [BJ17] Boris Bukh and Zilin Jiang. Erratum for a bound on the number of edges in graphs without an even cycle. *Combinatorics, Probability and Computing*, 26(6):952–953, 2017. doi:10.1017/s0963548317000396.
- [BLL18] Béla Bollobás, Jonathan Lee, and Shoham Letzter. Eigenvalues of subgraphs of the cube. *European Journal of Combinatorics*, 70:125–148, 2018. doi:10.1016/j.ejc.2017.12.007.

- [BS86] Richard A Brualdi and Ernie S Solheid. On the spectral radius of complementary acyclic matrices of zeros and ones. *SIAM Journal on Algebraic Discrete Methods*, 7(2):265–272, 1986. doi:10.1137/0607030.
- [BZ01] Abraham Berman and Xiao-Dong Zhang. On the spectral radius of graphs with cut vertices. *Journal of Combinatorial Theory, Series B*, 83(2):233–240, 2001. doi:10.1016/j.disc.2010.08.011.
- [CDT22] Sebastian Cioabă, Dheer Noal Desai, and Michael Tait. The spectral radius of graphs with no odd wheels. *European Journal of Combinatorics*, 99:103420, 2022. doi:10.1016/j.ejc.2021.103420.
- [CFTZ20] Sebastian Cioabă, Lihua Feng, Michael Tait, and Xiao-Dong Zhang. The maximum spectral radius of graphs without friendship subgraphs. *The Electronic Journal of Combinatorics*, P4.22, 2020. doi:10.37236/9179.
- [DKL⁺22] Dheer Noal Desai, Liying Kang, Yongtao Li, Zhenyu Ni, Michael Tait, and Jing Wang. Spectral extremal graphs for intersecting cliques. *Linear Algebra and its Applications*, 2022. doi:10.1016/j.laa.2022.03.015.
- [EG59] Paul Erdős and Tibor Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.* v10, pages 337–356, 1959. doi:10.1007/bf02024498.
- [ES46] Paul Erdős and Arthur H Stone. On the structure of linear graphs. *Bulletin of the American Mathematical Society*, 52(12):1087–1091, 1946.
- [ES65] Paul Erdős and Miklós Simonovits. A limit theorem in graph theory. In *Studia Sci. Math. Hung.*, 1965.
- [EZ00] Mark N Ellingham and Xiaoya Zha. The spectral radius of graphs on surfaces. *Journal of Combinatorial Theory, Series B*, 78(1):45–56, 2000. doi:10.1006/jctb.1999.1926.
- [FN10] Miroslav Fiedler and Vladimir Nikiforov. Spectral radius and hamiltonicity of graphs. *Linear Algebra and its Applications*, 432(9):2170–2173, 2010. doi:10.1016/j.laa.2009.01.005.
- [FS13] Zoltán Füredi and Miklós Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdős Centennial*, pages 169–264. Springer, 2013. doi:10.1007/978-3-642-39286-3_7.
- [Für96] Zoltán Füredi. An upper bound on Zarankiewicz’s problem. *Combinatorics, Probability and Computing*, 5(1):29–33, 1996. doi:10.1017/s0963548300001814.
- [Gui96] Barry Danzero Guiduli. *Spectral extrema for graphs*. PhD thesis, University of Chicago, Department of Mathematics, 1996.
- [He21] Zhiyang He. A new upper bound on extremal number of even cycles. *The Electronic Journal of Combinatorics*, pages P2–41, 2021. doi:10.37236/9861.
- [KTST54] T Kővári, Vera T Sós, and Pál Turán. On a problem of Zarankiewicz. In *Colloquium Mathematicum*, volume 3, pages 50–57. Polska Akademia Nauk, 1954.
- [LLF21] Yongtao Li, Weijun Liu, and Lihua Feng. A survey on spectral conditions for some extremal graph problems. 2021. arXiv:2111.03309.

- [LLT06] Mei Lu, Huiqing Liu, and Feng Tian. A new upper bound for the spectral radius of graphs with girth at least 5. *Linear Algebra and its Applications*, 414(2-3):512–516, 2006. doi:10.1016/j.laa.2005.10.043.
- [LP22] Yongtao Li and Yuejian Peng. The spectral radius of graphs with no intersecting odd cycles. *Discrete Mathematics*, 345(8):112907, 2022. doi:10.1016/j.disc.2022.112907.
- [Nik07] Vladimir Nikiforov. Bounds on graph eigenvalues II. *Linear Algebra and its Applications*, 427(2):183–189, 2007. doi:10.1016/j.laa.2007.07.010.
- [Nik08] Vladimir Nikiforov. A spectral condition for odd cycles in graphs. *Linear Algebra and its Applications*, 428(7):1492–1498, 2008. doi:10.1016/j.laa.2007.09.029.
- [Nik09] Vladimir Nikiforov. Degree powers in graphs with a forbidden even cycle. *The Electronic Journal of Combinatorics*, 16(1):R107, 2009. doi:10.37236/196.
- [Nik10a] Vladimir Nikiforov. A contribution to the Zarankiewicz problem. *Linear Algebra and its Applications*, 432(6):1405–1411, 2010. doi:10.1016/j.laa.2009.10.040.
- [Nik10b] Vladimir Nikiforov. The spectral radius of graphs without paths and cycles of specified length. *Linear Algebra and its Applications*, 432(9):2243–2256, 2010. doi:10.1016/j.laa.2009.05.023.
- [Nos70] Eva Nosal. Eigenvalues of graphs. Master’s thesis, University of Calgary, 1970.
- [SAH08] Dragan Stevanović, Mustapha Aouchiche, and Pierre Hansen. On the spectral radius of graphs with a given domination number. *Linear Algebra and its Applications*, 428(8-9):1854–1864, 2008. doi:10.1016/j.laa.2007.10.024.
- [SS07] Lingsheng Shi and Zhipeng Song. Upper bounds on the spectral radius of book-free and/or $K_{2,l}$ -free graphs. *Linear Algebra and its Applications*, 420(2-3):526–529, 2007. doi:10.1016/j.laa.2006.08.007.
- [Sta87] Richard P Stanley. A bound on the spectral radius of graphs with e edges. *Linear Algebra and its Applications*, 87:267–269, 1987. doi:10.1016/0024-3795(87)90172-8.
- [TT17] Michael Tait and Josh Tobin. Three conjectures in extremal spectral graph theory. *Journal of Combinatorial Theory, Series B*, 126:137–161, 2017. doi:10.1016/j.jctb.2017.04.006.
- [Tur41] Paul Turán. On an external problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.
- [Ver00] Jacques Verstraëte. On arithmetic progressions of cycle lengths in graphs. *Combinatorics, Probability and Computing*, 9(4):369–373, 2000. doi:10.1017/s0963548300004478.
- [Wil86] Herbert S Wilf. Spectral bounds for the clique and independence numbers of graphs. *Journal of Combinatorial Theory, Series B*, 40(1):113–117, 1986. doi:10.1016/0095-8956(86)90069-9.

- [YWZ12] Wanlian Yuan, Bing Wang, and Mingqing Zhai. On the spectral radii of graphs without given cycles. *The Electronic Journal of Linear Algebra*, 23:599–606, 2012. doi:10.13001/1081-3810.1544.
- [ZL20] Mingqing Zhai and Huiqiu Lin. Spectral extrema of graphs: Forbidden hexagon. *Discrete Mathematics*, 343(10):112028, 2020. doi:10.1016/j.disc.2020.112028.
- [ZW12] Mingqing Zhai and Bing Wang. Proof of a conjecture on the spectral radius of C_4 -free graphs. *Linear Algebra and its Applications*, 437(7):1641–1647, 2012. doi:10.1016/j.laa.2012.05.006.
- [ZWF20] Mingqing Zhai, Bing Wang, and Longfei Fang. The spectral Turán problem about graphs with no 6-cycle. *Linear Algebra and its Applications*, 590:22–31, 2020. doi:10.1016/j.laa.2019.12.032.