Pretty good state transfer among large sets of vertices

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Abstract. In a continuous-time quantum walk on a network of qubits, pretty good state transfer is the phenomenon of state transfer between two vertices with fidelity arbitrarily close to 1. We construct families of graphs to demonstrate that there is no bound on the size of a set of vertices that admits pretty good state transfer between any two vertices of the set.

Keywords. Continuous-time quantum walks, pretty good state transfer

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1. Introduction

Let X be a simple finite graph, with adjacency matrix A, for some fixed ordering of the vertex set $V(X)$. Let $\mathbb{C}^{V(X)}$ denote the Hilbert space in which the characteristic vectors e_a, $a \in V(X)$, form an orthonormal basis. A continuous-time quantum walk on X, based on the XX-Hamiltonian [Kay11, IV.E], is given by the family $U(t) = \exp(-itA)$ of unitary matrices, for $t \in \mathbb{R}$, operating on $\mathbb{C}^{V(X)}$. Two phenomena of central importance in the theory are *perfect state transfer* (PST) and *pretty good state transfer* (PGST). Let a and b be vertices of X. We say that we have perfect state transfer from a to b at time τ if $|U(\tau)_{b,a}| = 1$. In other words, an initial state e_a concentrated on the vertex a evolves at time τ to one concentrated on b. The concept of pretty good state transfer is an approximate version of perfect state transfer. We say that we have pretty good state transfer from a to b if for every real number $\epsilon > 0$ there exists a $\tau \in \mathbb{R}$ and a unimodular complex number γ such that $\|U(\tau)e_a - \gamma e_b\| < \epsilon$. It is not hard to see that the relation on $V(X)$, whereby vertices a and b are related if we have perfect state transfer from a

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to b at some time, is an equivalence relation. Likewise the relation defined by pretty good state transfer from a to b is another equivalence relation. In these terms, a well known observation of Kay [Kay11, IV.D] states that each PST-equivalence class can have at most two members. This property of perfect state transfer is undesirable for the purpose of routing. By contrast, [PB17, example 4.1] shows that in the cartesian product $P_2\square P_3$ of paths of lengths 2 and 3, the 4 vertices of smallest degree form a PGST-equivalence class. A natural question then is: How large can a PGST-equivalence class be? The main aim of this paper is to construct examples of graphs with arbitrarily large PGST-equivalence classes. Our examples are generalizations of the example above in that they are finite cartesian products of paths of different lengths and the large PGST-equivalence class is the set of vertices of minimum degree (the "corners"). In order for the corners to be PGST-equivalent the path lengths of the cartesian factors have to be selected rather carefully to satisfy certain arithmetic conditions which enter into the problem via Kronecker's approximation theorem, a theorem whose relevance to pretty good state transfer was first noted in [GKSS12] and [VZ12]. Please see [CG21] for more background on continuous-time quantum walk on graphs.

Since pretty good state transfer in a cartesian product implies pretty good state transfer for each cartesian factor, the paths involved must have pretty good state transfer between their end-vertices. Such paths have been classified in [GKSS12]; they are the paths of length $p-1$ or $2p-1$, where p is a prime, or of length 2^e-1 for $e>1$. Thus, we consider only cartesian products of paths of these lengths. In Section 3 we first classify the cartesian products for which all corners are strongly cospectral, as strong cospectrality is necessary for pretty good state transfer. The results in Sections 4 and 5 lead to the following classification which is the main result of this paper.

Theorem 1.1. *Let X be a cartesian product of paths. All corners of X belong to the same PGST-equivalence class if and only if one of the following holds, up to permutation of the cartesian factors:*

1. $X=P_{2^e-1}\square P_{p-1}$, *for some $e\geqslant 2$ and prime $p\geqslant 3$.*
2. $X=P_{2^e-1}\square P_{2p-1}$, *for some $e\geqslant 2$ and prime $p\geqslant 3$.*
3. $X=P_{p_1-1}\square\cdots\square P_{p_h-1}\square P_{2q_1-1}\square P_{2q_k-1}$, *where $h,k\geqslant 0$ and $p_1,\ldots,p_h,q_1,\ldots,q_k$ are distinct primes such that $p_1,\ldots,p_h\equiv 1 \pmod 8$ and $q_1,\ldots,q_k\equiv 1 \pmod 4$.*
4. $X=P_{2^e-1}\square P_{p_1-1}\square\cdots\square P_{p_h-1}\square P_{2q_1-1}\square P_{2q_k-1}$, *where $h,k\geqslant 0$, $e\geqslant 2$ and $p_1,\ldots,p_h$, $q_1,\ldots,q_k$ are distinct primes such that $p_1,\ldots,p_h\equiv 1 \pmod 8$ and $q_1,\ldots,q_k\equiv 1 \pmod 4$.*

We reach the following consequence immediately.

Corollary 1.2. *For integer $n\geqslant 2$, there exists a graph that has pretty good state transfer among n vertices.*

In Section 6, we show on the contrary that there is no cartesian product of paths with pretty good state transfer among all corners when the Laplacian matrix is used as the Hamiltonian of the quantum walk.

2. Notation and background results

Let A be the adjacency matrix of X. We consider the spectral decomposition of A,

$$A = \sum_{r=1}^{k} \theta_r E_r, \tag{2.1}$$

where $\theta_1,\ldots, \theta_k$ are the distinct eigenvalues of A and E_r is the idempotent projector onto the θ_r eigenspace.

Two vertices a and b are said to be *strongly cospectral* if and only if for all r we have $E_r e_a = \pm E_r e_b$. The terminology is justified by the fact that the above condition implies that $(E_r)_{a,a} = (E_r)_{b,b}$ for all r, which is one of several equivalent definitions of *cospectrality* of a and b.

Strong cospectrality is a fundamental notion in the study of quantum state transfer, and is a necessary condition for both perfect state transfer and pretty good state transfer [God12]. The *eigenvalue support* Φ_u of a vertex u is the set of eigenvalues θ_r for which $E_r e_u \neq 0$. If u and v are strongly cospectral then $\Phi_u = \Phi_v$ and this set is the disjoint union of $\Phi^+_{u,v} = \{\theta_r \mid E_r e_u = E_r e_v\}$ and $\Phi^-_{u,v} = \{\theta_r \mid E_r e_u = -E_r e_v\}$.

If an eigenvalue of X is simple, then the corresponding projector is a rank one symmetric matrix. The following lemma is then immediate from the above definition of cospectrality.

Lemma 2.1. *If X has simple eigenvalues then two vertices that are cospectral are strongly cospectral.* □

The following theorem [BCGS17, Theorem 2] (also [KLY17, Lemma 2.2]) is our main tool. It is a direct application of of Kronecker's approximation theorem to quantum walks.

Theorem 2.2. *Let X be a simple graph. Then two vertices u and v are in the same PGST-equivalence class if and only if the following conditions hold.*

(a) u and v are strongly cospectral.

(b) There is no sequence of integers $\{\ell_i\}$ such that all three of the following equations hold:

(i) $\sum_i \ell_i\theta_i = 0$;
(ii) $\sum_i \ell_i = 0$;
(iii) $\sum_{i:\theta_i\in\Phi^-_{u,v}} \ell_i \equiv 1 \pmod 2$. □

Let P_n denote the path of length n. Pretty good state transfer between extremal vertices has been characterized in [GKSS12], and between internal vertices in [vB19]. We shall make use of the extremal case, in which pretty good state transfer occurs if and only if $n + 1 = p$ or $2p$, where p is a prime, or if $n + 1$ is a power of 2. We shall consider the cartesian product

$$X = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_k} \tag{2.2}$$

of k paths, where k is a positive integer.

Lemma 2.3. *If pretty good state transfer occurs between* (x_1, y_1) *and* (x_2, y_2) *in* $X\Box Y$ *then pretty good state transfer occurs between* x_1 *and* x_2 *in* X.

Proof. Let $U_X(t)$ and $U_Y(t)$ denote the transition matrices of X and Y, respectively. Their cartesian product, $X\Box Y$ has transition matrix $U_X(t) \otimes U_Y(t)$. As

$$|U_X(t)_{x_1,x_2}|, |U_Y(t)_{y_1,y_2}| \geqslant \left|\big(U_X(t) \otimes U_Y(t)\big)_{(x_1,y_1),(x_2,y_2)}\right|,$$

pretty good state transfer from (x_1, y_1) to (x_2, y_2) in $X\Box Y$ implies pretty good state transfer from x_1 to x_2 in X and pretty good state transfer from y_1 to y_2 in Y. □

By a corner of $X = P_{n_1}\Box P_{n_2}\Box\cdots\Box P_{n_k}$, we shall mean a vertex $(a_1, \dots, a_k)$ in which every component a_i is an end of P_{n_i}. There are 2^k corners. It follows from Lemma 2.3 that if there is pretty good state transfer between any two corners of X, then for $i = 1, \dots, k$, $n_i + 1 = p_i$ or $n_i + 1 = 2p_i$, for some prime p_i, or $n_i + 1$ is a power of 2.

3. Large classes of strongly cospectral vertices in path products.

It is well known (an unpublished result of G. Coutinho) that by taking cartesian products of paths of suitable lengths, one can obtain arbitrarily large equivalence classes of mutually strongly cospectral vertices. In this section, we include a proof for completeness, and in order to introduce notations that we shall need later on.

Lemma 3.1. *The automorphism group of* X *acts transitively on the corners. Hence the corners are mutually cospectral.* □

In order for the corners of $X = P_{n_1}\Box P_{n_2}\Box\cdots\Box P_{n_k}$ to be mutually strongly cospectral, we will need to choose the paths of lengths n_i more carefully, so that X will have simple eigenvalues, and Lemma 2.1 will apply. The proof of the simplicity of the eigenvalues will make use of some well known properties the eigenvalues of paths and of cyclotomic fields, which we shall now discuss.

The adjacency matrix of P_n has eigenvalues $2\cos\frac{r\pi}{n+1} = e^{\frac{r\pi}{n+1}i} + e^{-\frac{r\pi}{n+1}i}$. The degree of the minimal polynomial of $2\cos\frac{\pi}{n+1}$ is $d := \frac{1}{2}\phi(2(n+1))$ where ϕ is the totient function. For $r = 1, \dots, d$, using T_r to denote the Chebyshev polynomial of the first kind of degree r, we have

$$\cos\frac{r\pi}{n+1} = T_r\left(\cos\frac{\pi}{n+1}\right).$$

Thus

$$\left\{1, 2\cos\frac{\pi}{n+1}, 2\cos\frac{2\pi}{n+1}, \dots, 2\cos\frac{(d-1)\pi}{n+1}\right\} \tag{3.1}$$

is a basis of the field, F_{n+1}, generated by the eigenvalues of P_n over $\mathbb{Q}$.

When $n + 1 = p$ for some prime p, let $\alpha_r := 2\cos\frac{r\pi}{p}$, $(r = 1, \dots p-1)$. It follows from the minimal polynomial of the primitive $2p$-th root of unity that

$$1 + \sum_{j=1}^{\frac{p-1}{2}} (-1)^j \alpha_j = 0. \tag{3.2}$$

Lemma 3.2. *Let $p \geqslant 5$ be a prime and let $\alpha_j = 2\cos\left(\frac{j\pi}{p}\right)$, $(1 \leqslant j \leqslant \frac{p-1}{2})$. Then for every $1 \leqslant r < s \leqslant \frac{p-1}{2}$, the set $\{1, \alpha_r, \alpha_s\}$ is linearly independent over $\mathbb{Q}$.*

Proof. The degree of α_1 is $d = \frac{p-1}{2}$. If $1 \leqslant r < s < d$, the set $\{1, \alpha_r, \alpha_s\}$ is a subset of the basis $\{1, \alpha_1, \ldots, \alpha_{d-1}\}$ of F_p, so it is linearly independent.

It remains to consider α_d. By Equation (3.2), we may replace any element in

$$\{\alpha_j : 1 \leqslant j \leqslant d-1 \text{ and } j \neq r\}$$

by α_d in $\{1, \alpha_1, \ldots, \alpha_{d-1}\}$ to get another basis of F_p containing 1, α_r and α_d. □

The eigenvalues of P_{2p-1} are $\beta_r := 2\cos\left(\frac{r\pi}{2p}\right)$, for $r = 1, \ldots, 2p-1$. The field F_{2p} generated by these eigenvalues over $\mathbb{Q}$ is the intersection of the $4p$-th cyclotomic field with the field of real numbers, so $|F_{2p} : \mathbb{Q}| = p-1$. We also have $F_{2p} = \mathbb{Q}(\beta_1)$. The minimal polynomial of $e^{\frac{\pi}{2p}i}$ yields

$$1 + \sum_{j=1}^{\frac{p-1}{2}} (-1)^j \beta_{2j} = 0. \tag{3.3}$$

Lemma 3.3. *Let $p \geqslant 7$ be a prime and let $\beta_r = 2\cos\left(\frac{r\pi}{2p}\right)$, $(1 \leqslant r \leqslant p-1)$. Then for every $1 \leqslant r < s \leqslant p-1$, the set $\{1, \beta_r, \beta_s\}$ is linearly independent over $\mathbb{Q}$.*

Proof. The degree of β_1 is $p-1$. For $1 \leqslant r < s \leqslant p-2$, $\{1, \beta_r, \beta_s\}$ is a subset of the basis $\{1, \beta_1, \ldots, \beta_{p-2}\}$. Hence it is linearly independent over $\mathbb{Q}$.

Using Equation (3.3), we may replace any element in $\left\{\beta_{2j} : 1 \leqslant j \leqslant \frac{p-3}{2} \text{ and } 2j \neq r\right\}$ by β_{p-1} in $\{1, \beta_1, \ldots, \beta_{p-2}\}$ to get another basis of F_{2p} containing 1, β_r and β_{p-1}. □

Corollary 3.4. *Suppose $n = p-1$ for some prime $p \geqslant 5$, or $n = 2p-1$ for some prime $p \geqslant 7$. For any two non-zero eigenvalues λ and λ' of P_n such that $\lambda \neq \pm\lambda'$, the set $\{1, \lambda, \lambda'\}$ is linearly independent over $\mathbb{Q}$.*

The eigenvalues of X, counting multiplicity, are the $\prod_{i=1}^{k} n_i$ numbers $\lambda_1 + \lambda_2 + \cdots + \lambda_k$, where λ_i is an eigenvalue of P_{n_i}.

Lemma 3.5. *Let $p_1, \ldots, p_k \geqslant 5$ be distinct primes. Suppose $n_i = p_i - 1$ or $n_i = 2p_i - 1$. Then $X = P_{n_1} \Box \cdots \Box P_{n_k}$ has simple eigenvalues.*

Proof. We may assume $k > 1$, since paths have simple eigenvalues. As X is the Cartesian product of the paths P_{n_i}'s, the eigenvalues of X are the values $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ where λ_i is an eigenvalue of P_{n_i}. Suppose

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = \lambda_1' + \lambda_2' + \cdots + \lambda_k'. \tag{3.4}$$

where the λ_i and λ_i' are eigenvalues of P_{n_i}. Suppose for a contradiction that for some i we have $\lambda_i \neq \lambda_i'$. Then there must be another index j with $\lambda_j \neq \lambda_j'$. Without loss of generality we can assume that $i = 1$ and that $p_1 \geqslant 7$ if $n_1 = 2p_1 - 1$. We rewrite (3.4) as

$$(\lambda_1' - \lambda_1) = (\lambda_2 - \lambda_2') \cdots + (\lambda_k - \lambda_k'). \tag{3.5}$$

The left member of (3.5) lies in the cyclotomic field of order $4p_1$ while the right member lies in a cyclotomic field of order $4m$, where m is coprime to p_1. The intersection of these fields is $\mathbb{Q}(i)$, but we also know that the eigenvalues are real, so it follows that the common value of (3.5) must be rational.

If, without loss of generality, $\lambda_1 = 0$ then $\lambda_1' \notin \mathbb{Q}$ contradicting Equation (3.5). Otherwise, as $\lambda_1 \notin \mathbb{Q}$, we cannot have $\lambda_1' = -\lambda_1$. Then by Corollary 3.4, the set $\{1, \lambda_1, \lambda_1'\}$ is linearly independent over $\mathbb{Q}$, contradicting the rationality of $\lambda_1' - \lambda_1$ in (3.5). Hence X has simple eigenvalues. □

Lemma 3.6. *Let X be as in Lemma 3.5 and let $Y = P_{2^e-1}\square X$, $e \geqslant 2$. Then the eigenvalues of Y are simple.*

Proof. The eigenvalues of Y, counting multiplicity are the $(2^e - 1)\prod_{i=1}^{k}(n_i)$ complex numbers $\lambda_0 + \lambda_i + \cdots + \lambda_k$, where λ_0 is an eigenvalue of P_{2^e-1} and for $1 \leqslant i \leqslant k$ λ_i is an eigenvalue of P_{n_i}. Let λ_i', $i = 0,1,\ldots, k$ similarly denote eigenvalues and consider the equation

$$\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_k = \lambda_0' + \lambda_1' + \lambda_2' + \cdots + \lambda_k'. \tag{3.6}$$

We shall show that $\lambda_i = \lambda_i'$ for all i. Suppose $\lambda_0 = \lambda_0'$. Then we can cancel these terms and we have an equation expressing equality of two eigenvalues of X. So Lemma 3.5 gives the desired conclusion. Therefore we may assume $\lambda_0 \neq \lambda_0'$.

Let F_{2^e} denote the field generated by the eigenvalues of P_{2^e-1}. This is the intersection of the real field with the cyclotomic field $Q(\omega)$, where $\omega = e^{\frac{i\pi}{2^e}}$ is a primitive 2^{e+1}-th root of unity. Let F_X denote the field generated by the eigenvalues of the path factors in X. Then F_X lies in a cyclotomic field of order $4m$, where m is odd. It follows that $F_{2^e} \cap F_X = \mathbb{Q}$. Thus, if we rearrange (3.6) by isolating $\lambda_0 - \lambda_0'$, we see that $\lambda_0 - \lambda_0' \in \mathbb{Q}$. The eigenvalues of P_{2^e-1} are $\omega^r + \omega^{-r}$, for $r = 1, \ldots, 2^e - 1$. We assume for a contradiction that $\lambda_0 \neq \lambda_0'$. The only rational eigenvalue is zero, for $r = 2^{e-1}$, so λ_0 and λ_0' must both be irrational.

The Galois group of $Q(\omega)$ over $\mathbb{Q}$ consists of the 2^e automorphisms of the form $\omega \mapsto \omega^a$, where $a \in \mathbb{Z}/2^{e+1}\mathbb{Z}$ is odd. The Galois automorphisms of $F_{2^e} = \mathbb{Q}(\omega + \omega^{-1})$ are obtained by restriction of those for $Q(\omega)$, and form a cyclic group of order 2^{e-1}. By Galois theory, the subfields of F_{2^e} correspond bijectively with the subgroups of the Galois group. Thus, for each d with $1 \leqslant d \leqslant e$ there is a unique subfield of degree 2^{d-1}, and this subfield must be the field $F_{2^d} = \mathbb{Q}(\omega^{2^{e-d}}) \cap \mathbb{R}$, as this field has the right degree. An eigenvalue $\omega^r + \omega^{-r}$ is Galois conjugate to $\omega^{2^d} + \omega^{-2^d}$, where 2^d divides r exactly, so $Q(\omega^r + \omega^{-r}) = F_{2^d}$. Suppose $\mathbb{Q}(\lambda_0) \neq \mathbb{Q}(\lambda_0')$. Then $\{1, \lambda_0, \lambda_0'\}$ is linearly independent over $\mathbb{Q}$, a contradiction. Therefore we may assume $\mathbb{Q}(\lambda_0) = \mathbb{Q}(\lambda_0')$. Thus, $\lambda_0 = \omega^r + \omega^{-r}$ and $\lambda_0' = \omega^s + \omega^{-s}$, where $1 \leqslant r, s \leqslant 2^e - 1$ and r and s are exactly divisible by the same power of 2, say 2^d. In other words ω^r and ω^s are both odd powers of $\omega^{2^{e-d}}$, a primitive 2^{d+1}-th root of unity. Now $2^d + 1$ is odd and the $(2^d + 1)$-th power of $\omega^{2^{e-d}}$ is $-\omega^{2^{e-d}}$. It follows that there is Galois automorphism of $Q(\lambda_0)$ sending λ_0 to $-\lambda_0$ and λ_0' to $-\lambda_0'$. Since $\lambda_0 - \lambda_0'$ is a nonzero rational number, we have our final contradiction. This proves that Y has simple eigenvalues. □

From Lemmas 3.5, 3.6 and 2.1, we draw the following conclusion.

Corollary 3.7. *Let $p_1, \dots, p_k \geqslant 5$ be distinct primes. Suppose $n_i = p_i - 1$ or $n_i = 2p_i - 1$. Further, let $n_0 = 2^e - 1$, $e \geqslant 2$. Then in $X = P_{n_1} \square \cdots \square P_{n_k}$ and $Y = P_{n_0} \square X$, all corners are mutually strongly cospectral.*

□

The following results imply that if the primes in the above corollary are not distinct, then the corners of $X = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_k}$ are not mutually strongly cospectral.

Lemma 3.8. *Let y_1 and y_2 be vertices in a graph Y and E be an idempotent projector for some eigenspace of Y. Let z be a vertex in a graph Z and $\mu \in \Phi_z$ with corresponding idempotent projector F.*

If W is an idempotent projector for an eigenspace of $Y \square Z$ such that $(E \otimes F)W \neq 0$ and $We_{(y_1,z)} = \alpha We_{(y_2,z)}$ for some $\alpha \in \mathbb{C}$, then $Ee_{y_1} = \alpha Ee_{y_2}$.

Proof. Since $(E \otimes F)W \neq 0$, we can write

$$W = E \otimes F + \sum_j E_{r_j} \otimes F_{s_j}$$

where the E_{r_j}'s and the F_{s_j}'s are idempotent projectors for Y and Z, respectively, different from E and F.

Multiplying $E \otimes F$ on the left to both sides of $We_{(y_1,z)} = \alpha We_{(y_2,z)}$ gives

$$Ee_{y_1} \otimes Fe_z = \alpha Ee_{y_2} \otimes Fe_z,$$

and $Ee_{y_1} = \alpha Ee_{y_2}$. □

Corollary 3.9. *Let y_1 and y_2 be vertices in a graph Y and let z be a vertex in a graph Z. If (y_1, z) is strongly cospectral to (y_2, z) in $Y \square Z$, then y_1 is strongly cospectral to y_2 in Y.*

Lemma 3.10. *If $\gcd(n+1, m+1) \geqslant 3$, then $(1, 1)$ and $(n, 1)$ are not strongly cospectral vertices in $P_n \square P_m$.*

Proof. We use E_r to denote the idempotent projector onto the $\left(2\cos\frac{r\pi}{n+1}\right)$-eigenspace of $A(P_n)$, and F_s to denote the idempotent projector onto the $\left(2\cos\frac{s\pi}{m+1}\right)$-eigenspace of $A(P_m)$.

Let $g = \gcd(n+1, m+1)$, $h_n = \frac{n+1}{g}$ and $h_m = \frac{m+1}{g}$. Without loss of generality, we assume h_n is odd. Then

$$\theta = 2\cos\left(\frac{h_n\pi}{n+1}\right) + 2\cos\left(\frac{2h_m\pi}{m+1}\right) = 2\cos\left(\frac{2h_n\pi}{n+1}\right) + 2\cos\left(\frac{h_m\pi}{m+1}\right)$$

is an eigenvalue of $P_n \square P_m$. If W is the idempotent projector of the θ-eigenspace of $P_n \square P_m$, then both $(E_{h_n} \otimes F_{2h_m})W$ and $(E_{2h_n} \otimes F_{h_m})W$ are non-zero. As $E_{h_n}e_1 = E_{h_n}e_n$ and $E_{2h_n}e_1 = -E_{2h_n}e_n$, Lemma 3.8 implies $We_{(1,1)} \neq \pm We_{(n,1)}$. □

Corollary 3.11. *If $\gcd(n_i + 1, n_j + 1) \geqslant 3$, for some $1 \leqslant i < j \leqslant n$, then the corners of $P_{n_1} \square P_{n_2} \square \cdots \square P_{n_k}$ are not mutually strongly cospectral.*

4. No pretty good state transfer among all corners

We shall identify pairs of paths whose cartesian product does not have all four corners in the same PGST-equivalence class. If any of these pairs appears as factors in $P_{n_1}\square\dots\square P_{n-k}$, then Lemma 2.3 implies that the corners of $P_{n_1}\square\dots\square P_{n_k}$ are not in the same PGST-equivalence class.

The following lemma covers $P_{p-1}\square P_{p-1}$, $P_{p-1}\square P_{2p-1}$, $P_{2p-1}\square P_{2p-1}$, and $P_{2^e-1}\square P_{2^f-1}$, for $e, f \geqslant 2$.

Lemma 4.1. *If* $\gcd(n+1, m+1) \geqslant 3$*, then the four corners of* $P_n\square P_m$ *do not form a PGST-equivalence class.*

Proof. It follows immediately from Corollary 3.11 and Theorem 2.2. □

For each pair of paths considered in this section, we give a sequence of integer $\{\ell_{rs}\}$ that satisfies the three conditions in Theorem 2.2(b) using the 2×2 matrices

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}. \tag{4.1}$$

4.1. $\mathbf{P_{p_1-1}\square P_{p_2-1}}$

For $i = 1, 2$, let p_i be an odd prime and $\alpha_r^{(i)} = 2\cos\frac{r\pi}{p_i}$, it follows from Equation (3.2) and $\alpha_r^{(i)} = -\alpha_{p_i-r}^{(i)}$ that

$$2 + \sum_{r=1}^{p_i-1} (-1)^r \alpha_r^{(i)} = 0. \tag{4.2}$$

Lemma 4.2. *If* $p_1 \equiv 3 \pmod 4$*, then there is no pretty good state transfer from* $(1,1)$ *to* $(p_1-1, 1)$ *in* $P_{p_1-1}\square P_{p_2-1}$.

Proof. Define the $(p_1-1)\times(p_2-1)$ matrix

$$L := \begin{bmatrix} A & B & \cdots & B \\ C & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C & 0 & \cdots & 0 \end{bmatrix}$$

containing $\left(\frac{p_2-1}{2}-1\right)$ copies of B and $\left(\frac{p_1-1}{2}-1\right)$ copies of C. Note that the entries of the r-th row of L sum to $(-1)^{r+1}$ and the entries of the s-th column of L sum to $(-1)^s$.

Let $\ell_{rs} = L_{r,s}$, for $1 \leqslant r \leqslant p_1-1$ and $1 \leqslant s \leqslant p_2-1$. For Condition (b)(i) of Theorem 2.2, we have

$$\begin{aligned} \sum_{r=1}^{p_1-1}\sum_{s=1}^{p_2-1} \ell_{rs}(\alpha_r^{(1)} + \alpha_s^{(2)}) &= \sum_{r=1}^{p_1-1}\left(\sum_{s=1}^{p_2-1}\ell_{rs}\right)\alpha_r^{(1)} + \sum_{s=1}^{p_2-1}\left(\sum_{r=1}^{p_1-1}\ell_{rs}\right)\alpha_s^{(2)} \\ &= \sum_{r=1}^{p_1-1}(-1)^{r+1}\alpha_r^{(1)} + \sum_{s=1}^{p_2-1}(-1)^s\alpha_s^{(2)}, \end{aligned}$$

which is 0 as a result of Equation (4.2).

Since the entries of L sum to zero, Condition (b)(ii) holds.

Now $\Phi^{-}_{(1,1),(p_1-1,1)} = \left\{\alpha_r^{(1)} + \alpha_s^{(2)} \ : \ r \text{ is even}\right\}$, we have

$$\sum_{r,\,s\,:\,r\text{ even}} \ell_{rs} = \sum_{r\text{ even}} (-1)^{r+1} = -\left(\frac{p_1-1}{2}\right).$$

If $p_1 \equiv 3 \pmod 4$, then the sequence $\{\ell_{rs}\}$ satisfies Conditions (b)(i) to (b)(iii) of Theorem 2.2, so there is no pretty good state transfer between $(1,1)$ and $(p_1-1,1)$ in $P_{p_1-1}\square P_{p_2-1}$. □

Lemma 4.3. *If* $p_1 \equiv 5 \pmod 8$ *and* $p_2 \equiv 1 \pmod 4$*, then there is no pretty good state transfer from* $(1,1)$ *to* $(p_1-1,1)$ *in* $P_{p_1-1}\square P_{p_2-1}$.

Proof. Using the 2×2 matrices in Equation (4.1), we define the $\frac{p_1-1}{2} \times \frac{p_2-1}{2}$ matrix

$$L := \begin{bmatrix} A & B & \cdots & B \\ C & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C & 0 & \cdots & 0 \end{bmatrix}$$

containing $\left(\frac{p_2-1}{4}-1\right)$ copies of B and $\left(\frac{p_1-1}{4}-1\right)$ copies of C. Let

$$\ell_{rs} = \begin{cases} L_{r,s} & \text{if } 1 \leqslant r \leqslant \frac{p_1-1}{2}, \text{ and } 1 \leqslant s \leqslant \frac{p_2-1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to the proof of Lemma 4.2, we have

$$\sum_{r=1}^{p_1-1}\sum_{s=1}^{p_2-1} \ell_{rs}(\alpha_r^{(1)} + \alpha_s^{(2)}) = \sum_{r=1}^{\frac{p_1-1}{2}} (-1)^{r+1}\alpha_r^{(1)} + \sum_{s=1}^{\frac{p_2-1}{2}} (-1)^s \alpha_s^{(2)}.$$

It follows from Equation (3.2) that this sum is equal to zero, and Condition (b)(i) of Theorem 2.2 holds. The sequence $\{\ell_{rs}\}$ also satisfies Condition (b)(ii) and

$$\sum_{r,\,s\,:\,r\text{ even}} \ell_{rs} = -\left(\frac{p_1-1}{4}\right).$$

If $p_1 \equiv 5 \pmod 8$, then the sequence $\{\ell_{rs}\}$ satisfies Conditions (b)(i) to (b) (iii) of Theorem 2.2 and there is no pretty good state transfer from $(1,1)$ to $(n_1,1)$. □

4.2. $\mathbf{P_{2p_1-1}\square P_{2p_2-1}}$

For $i = 1,2$, let $\beta_r^{(i)} = 2\cos\frac{r\pi}{2p_i}$. Note that $\beta_{p_i}^{(i)} = 0$.

Lemma 4.4. *If* $p_1 \equiv 3 \pmod 4$, *then there is no pretty good state transfer from* $(1,1)$ *to* $(2p_1-1,1)$ *in* $P_{2p_1-1}\square P_{2p_2-1}$.

Proof. We first consider the case where $p_2 \equiv 1 \pmod 4$. Define the $\frac{p_1+1}{2} \times \frac{p_2-1}{2}$ matrix L as

$$\begin{array}{c} \beta_2^{(1)} \\ \beta_4^{(1)} \\ \vdots \\ \vdots \\ \beta_{p_1-5}^{(1)} \\ \beta_{p_1-3}^{(1)} \\ \beta_{p_1-1}^{(1)} \\ \beta_{p_1}^{(1)} \end{array}
\overset{\begin{array}{cccccc} \beta_2^{(2)} & \beta_4^{(2)} & \cdots & \cdots & \beta_{p_2-3}^{(2)} & \beta_{p_2-1}^{(2)} \end{array}}{\begin{bmatrix} A & B & \cdots & B \\ C & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C & 0 & \cdots & 0 \\ C & 0 & \cdots & 0 \end{bmatrix}}$$

containing $\left(\frac{p_2-1}{4}-1\right)$ copies of B and $\left(\frac{p_1+1}{4}-1\right)$ copies of C. Let

$$\ell_{rs} = \begin{cases} L_{\frac{r}{2},\frac{s}{2}} & \text{if } r \text{ is even with } 2 \leqslant r \leqslant p_1-1, \text{ and } s \text{ is even with } 2 \leqslant s \leqslant p_2-1, \\ L_{\frac{p_1+1}{2},\frac{s}{2}} & \text{if } r = p_1, \text{ and } s \text{ is even with } 2 \leqslant s \leqslant p_2-1, \\ 0 & \text{otherwise.} \end{cases}$$

(We list on the right of L the eigenvalues of P_{2p_1-1} associated with each row of L, and above L the eigenvalues of P_{2p_2-1} associated with each column of L.)

For Conditon (b)(i) in Theorem 2.2, we have

$$\begin{aligned} & \sum_{r=1}^{2p_1-1}\sum_{s=1}^{2p_2-1} \ell_{rs}\left(\beta_r^{(1)}+\beta_s^{(2)}\right) \\ = & \sum_{j=1}^{\frac{p_1-1}{2}}\left(\sum_{k=1}^{\frac{p_2-1}{2}} L_{j,k}\right)\beta_{2j}^{(1)} + \left(\sum_{k=1}^{\frac{p_2-1}{2}} L_{\frac{p_1+1}{2},k}\right)\beta_{p_1}^{(1)} + \sum_{k=1}^{\frac{p_2-1}{2}}\left(\sum_{j=1}^{\frac{p_1+1}{2}} L_{j,k}\right)\beta_{2k}^{(2)} \\ = & \sum_{j=1}^{\frac{p_1-1}{2}}(-1)^{j+1}\beta_{2j}^{(1)} + 0 + \sum_{k=1}^{\frac{p_2-1}{2}}(-1)^k\beta_{2k}^{(2)}. \end{aligned}$$

which is equal to 0 by Equation (3.3).

It is straightforward to check that $\sum_{r,s} \ell_{rs} = 0$. Since

$$\Phi^-_{(1,1),(2p_1-1,1)} = \{\beta^{(1)}_{2j} + \beta^{(2)}_s : 1 \leqslant j \leqslant p_1 - 1, 1 \leqslant s \leqslant 2p_2 - 1\},$$

we have

$$\sum_{j=1}^{p_1-1} \sum_{s=1}^{2p_2-1} l_{2j,s} = \sum_{j=1}^{\frac{p_1-1}{2}} \sum_{k=1}^{\frac{p_2-1}{2}} L_{j,k} = \sum_{j=1}^{\frac{p_1-1}{2}} (-1)^{j+1} \equiv 1 \pmod 2.$$

It follows from Theorem 2.2 that there is no pretty good state transfer from $(1,1)$ to $(2p_1-1,1)$.

When $p_2 \equiv 3 \pmod 4$, we define the $\frac{p_1+1}{2} \times \frac{p_2+1}{2}$ matrix L as

$$\begin{array}{c} \\ \beta^{(1)}_2\ \beta^{(1)}_4 \\ \vdots \\ \vdots \\ \beta^{(1)}_{p_1-5}\ \beta^{(1)}_{p_1-3} \\ \beta^{(1)}_{p_1-1}\ \beta^{(1)}_{p_1} \end{array} \begin{array}{c} \beta^{(2)}_2\ \beta^{(2)}_4 \quad \cdots \quad \cdots \quad \beta^{(2)}_{p_2-5}\ \beta^{(2)}_{p_2-3} \quad \beta^{(2)}_{p_2-1}\ \beta^{(2)}_{p_2} \\ \begin{bmatrix} A & B & \cdots & B & B \\ C & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C & 0 & \cdots & 0 & 0 \\ C & 0 & \cdots & 0 & 0 \end{bmatrix} \end{array}$$

containing $\left(\frac{p_2+1}{4} - 1\right)$ copies of B and $\left(\frac{p_1+1}{4} - 1\right)$ copies of C. Let

$$\ell_{rs} = \begin{cases} L_{\frac{r}{2},\frac{s}{2}} & \text{if } r \text{ is even with } 2 \leqslant r \leqslant p_1 - 1, \text{ and } s \text{ is even with } 2 \leqslant s \leqslant p_2 - 1, \\ L_{\frac{p_1+1}{2},\frac{s}{2}} & \text{if } r = p_1, \text{ and } s \text{ is even with } 2 \leqslant s \leqslant p_2 - 1, \\ L_{\frac{r}{2},\frac{p_2+1}{2}} & \text{if } s = p_2, \text{ and } r \text{ is even with } 2 \leqslant r \leqslant p_1 - 1, \\ 0 & \text{otherwise.} \end{cases}$$

For Conditon (b)(i) in Theorem 2.2, we have

$$\begin{aligned}
&\sum_{r=1}^{2p_1-1}\sum_{s=1}^{2p_2-1} \ell_{rs}\left(\beta_r^{(1)}+\beta_s^{(2)}\right)\\
&=\sum_{j=1}^{\frac{p_1-1}{2}}\left(\sum_{k=1}^{\frac{p_2+1}{2}} L_{j,k}\right)\beta_{2j}^{(1)}+\left(\sum_{k=1}^{\frac{p_2+1}{2}} L_{\frac{p_1+1}{2},k}\right)\beta_{p_1}^{(1)}+\sum_{k=1}^{\frac{p_2-1}{2}}\left(\sum_{j=1}^{\frac{p_1+1}{2}} L_{j,k}\right)\beta_{2k}^{(2)}+\left(\sum_{j=1}^{\frac{p_1+1}{2}} L_{j,\frac{p_2+1}{2}}\right)\beta_{p_2}^{(2)}\\
&=\sum_{j=1}^{\frac{p_1-1}{2}}(-1)^{j+1}\beta_{2j}^{(1)}+0+\sum_{k=1}^{\frac{p_2-1}{2}}(-1)^k\beta_{2k}^{(2)}+0\\
&=0.
\end{aligned}$$

We also have $\sum_{r,s}\ell_{rs}=\sum_{r,s}L_{r,s}=0$, and

$$\sum_{j=1}^{p_1-1}\sum_{s=1}^{2p_2-1} l_{2j,s}=\sum_{j=1}^{\frac{p_1-1}{2}}\sum_{k=1}^{\frac{p_2+1}{2}} L_{j,k}=\sum_{j=1}^{\frac{p_1-1}{2}}(-1)^{j+1}\equiv 1 \pmod 2.$$

It follows from Theorem 2.2 that pretty good state transfer does not occur between $(1,1)$ and $(2p_1-1,1)$. $\square$

4.3. $\mathbf{P_{2p_1-1}\square P_{p_2-1}}$

For $r=1,\dots,2p_1-1$, let $\beta_r=2\cos\frac{r\pi}{2p_1}$ be the eigenvalues of P_{2p_1-1}. For $s=1,\dots,p_2-1$, let $\alpha_s=2\cos\frac{s\pi}{p_2}$ be the eigenvalues of P_{p_2-1}.

Lemma 4.5. *If* $p_1\equiv 3 \pmod 4$ *and* $p_2\equiv 1 \pmod 4$, *then there is no pretty good state transfer from* $(1,1)$ *to* $(2p_1-1,1)$ *in* $P_{2p_1-1}\square P_{p_2-1}$.

Proof. Define the $\frac{p_1+1}{2}\times\frac{p_2-1}{2}$ matrix L as

$$\begin{array}{c} \\ \beta_2 \\ \beta_4 \\ \vdots \\ \vdots \\ \beta_{(p_1-1)/2} \\ \beta_{p_1}\end{array}
\begin{array}{c} \alpha_1\ \alpha_2 \quad \cdots \quad \cdots \quad \alpha_{\frac{p_2-3}{2}}\ \alpha_{\frac{p_2-1}{2}} \\
\begin{bmatrix} A & B & \cdots & B \\ C & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C & 0 & \cdots & 0 \end{bmatrix}\end{array}$$

containing $\left(\frac{p_2-1}{4}-1\right)$ copies of B and $\left(\frac{p_1+1}{4}-1\right)$ copies of C. Let

$$\ell_{rs}=\begin{cases} L_{\frac{r}{2},s} & \text{if } r \text{ is even with } 2\leqslant r\leqslant p_1-1, \text{ and } 1\leqslant s\leqslant \frac{p_2-1}{2},\\ L_{\frac{p_1+1}{2},s} & \text{if } r=p_1 \text{ and } 1\leqslant s\leqslant \frac{p_2-1}{2},\\ 0 & \text{otherwise.}\end{cases}$$

For Conditon (b)(i) in Theorem 2.2, we have

$$\begin{aligned} &\sum_{r=1}^{2p_1-1}\sum_{s=1}^{p_2-1}\ell_{rs}\left(\beta_r+\alpha_s\right)\\ =\ & \sum_{j=1}^{\frac{p_1-1}{2}}\left(\sum_{s=1}^{\frac{p_2-1}{2}}L_{j,s}\right)\beta_{2j}+\left(\sum_{s=1}^{\frac{p_2-1}{2}}L_{\frac{p_1+1}{2},s}\right)\beta_{p_1}+\sum_{s=1}^{\frac{p_2-1}{2}}\left(\sum_{j=1}^{\frac{p_1+1}{2}}L_{j,s}\right)\alpha_s\\ =\ & \sum_{j=1}^{\frac{p_1-1}{2}}(-1)^{j+1}\beta_{2j}+0+\sum_{s=1}^{\frac{p_2-1}{2}}(-1)^s\alpha_s\\ =\ & 0.\end{aligned}$$

It is straightforward that Condition (b)(ii) holds. For Condition (b)(iii),

$$\Phi^-_{(1,1),(2p_1-1,1)}=\{\beta_r+\alpha_s : r \text{ is even}\},$$

and

$$\sum_{r \text{ is even}}\sum_{s=1}^{p_2-1}\ell_{rs}=\sum_{j=1}^{\frac{p_1-1}{2}}\sum_{s=1}^{p_2-1}L_{j,s}=\sum_{j=1}^{\frac{p_1-1}{2}}(-1)^{j+1}=1 \pmod 2.$$

By Theorem 2.2, there is no pretty good state transfer between $(1,1)$ and $(2p_1-1,1)$ in $P_{2p_1-1}\square P_{p_2-1}$. $\square$

Lemma 4.6. *If* $p_2\equiv 3 \pmod 4$, *then there is no pretty good state transfer from* $(1,1)$ *to* $(1,p_2-1)$ *in* $P_{2p_1-1}\square P_{p_2-1}$.

Proof. Define the $(p_1-1)\times(p_2-1)$ matrix

$$L:=\begin{bmatrix} A & B & \cdots & B\\ C & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ C & 0 & \cdots & 0\end{bmatrix}$$

containing $\left(\frac{p_2-1}{2}-1\right)$ copies of B and $\left(\frac{p_1-1}{2}-1\right)$ copies of C. Let

$$\ell_{rs}=\begin{cases} L_{\frac{r}{2},s} & \text{if } r \text{ is even with } 2\leqslant r\leqslant 2p_1-2, \text{ and } 1\leqslant s\leqslant p_2-1,\\ 0 & \text{otherwise.}\end{cases}$$

For Condition (b)(i) of Theorem 2.2,

$$\begin{aligned}\sum_{r=1}^{2p_1-1}\sum_{s=1}^{p_2-1}\ell_{rs}(\beta_r+\alpha_s) &= \sum_{j=1}^{p_1-1}\left(\sum_{s=1}^{p_2-1}L_{j,s}\right)\beta_{2j}+\sum_{s=1}^{p_2-1}\left(\sum_{j=1}^{p_1-1}L_{j,s}\right)\alpha_s\\ &= \sum_{j=1}^{p_1-1}(-1)^{j+1}\beta_{2j}+\sum_{s=1}^{p_2-1}(-1)^s\alpha_s\\ &= 2\sum_{j=1}^{\frac{p_1-1}{2}}(-1)^{j+1}\beta_{2j}+\sum_{s=1}^{p_2-1}(-1)^s\alpha_s\\ &= 0.\end{aligned}$$

As the entries of L sum to 0, Condition (b)(ii) holds.

For Condition (b)(iii),

$$\Phi^-_{(1,1),(1,p_2-1)} = \{\beta_r+\alpha_s : s \text{ is even}\}$$

and

$$\sum_{s \text{ is even}}\sum_{r=1}^{2p_1-1}\ell_{rs} = \sum_{s \text{ is even}}\sum_{j=1}^{p_1-1}L_{j,s} = \sum_{s \text{ is even}}(-1)^s = \frac{p_2-1}{2} \equiv 1 \pmod 2.$$

By Theorem 2.2, there is no pretty good state transfer between $(1,1)$ and $(1,p_2-1)$ in $P_{2p_1-1}\square P_{p_2-1}$. □

Lemma 4.7. *If* $p_1 \equiv 1 \pmod 4$ *and* $p_2 \equiv 5 \pmod 8$, *then there is no pretty good state transfer from* $(1,1)$ *to* $(1,p_2-1)$ *in* $P_{2p_1-1}\square P_{p_2-1}$.

Proof. Define the $\frac{p_1-1}{2}\times\frac{p_2-1}{2}$ matrix

$$L := \begin{bmatrix} A & B & \cdots & B\\ C & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ C & 0 & \cdots & 0\end{bmatrix}$$

containing $\left(\frac{p_2-1}{4}-1\right)$ copies of B and $\left(\frac{p_1-1}{4}-1\right)$ copies of C. Let

$$\ell_{rs} = \begin{cases} L_{\frac{r}{2},s} & \text{if } r \text{ is even with } 2\leqslant r\leqslant p_1-1, \text{ and } 1\leqslant s\leqslant \frac{p_2-1}{2},\\ 0 & \text{otherwise.}\end{cases}$$

For Condition (b)(i) of Theorem 2.2,

$$\begin{aligned}\sum_{r=1}^{2p_1-1}\sum_{s=1}^{p_2-1}\ell_{rs}(\beta_r+\alpha_s) &= \sum_{j=1}^{\frac{p_1-1}{2}}\left(\sum_{s=1}^{\frac{p_2-1}{2}}L_{j,s}\right)\beta_{2j}+\sum_{s=1}^{\frac{p_2-1}{2}}\left(\sum_{j=1}^{\frac{p_1-1}{2}}L_{j,s}\right)\alpha_s\\ &= \sum_{j=1}^{\frac{p_1-1}{2}}(-1)^{j+1}\beta_{2j}+\sum_{s=1}^{\frac{p_2-1}{2}}(-1)^s\alpha_s\\ &= 0.\end{aligned}$$

As the entries of L sum to 0, Condition (b)(ii) holds.

For Condition (b)(iii),

$$\Phi^{-}_{(1,1),(1,p_2-1)} = \{\beta_r + \alpha_s : s \text{ is even}\}$$

and

$$\sum_{s \text{ is even}} \sum_{r=1}^{2p_1-1} \ell_{rs} = \sum_{s \text{ is even}} \sum_{j=1}^{\frac{p_1-1}{2}} L_{j,s} = \sum_{s \text{ is even}} (-1)^s = \frac{p_2-1}{4} \equiv 1 \pmod 2.$$

By Theorem 2.2, there is no pretty good state transfer between $(1,1)$ and $(1, p_2 - 1)$ in $P_{2p_1-1} \square P_{p_2-1}$. □

5. Pretty good state transfer among corner vertices

In this section we shall classify the path products where pretty good state transfer occurs among all corners.

By the following lemma, it is enough to fix a corner and show that it has pretty good state transfer to each adjacent corner in a given path product, in order to prove that there is pretty good state transfer between an arbitrary pair of corners.

Lemma 5.1. *Pretty good state transfer defines an equivalence relation on a set of vertices.*

Proof. By Theorem 8.7.2 of [CG21], for $\epsilon > 0$, there exists time τ such that $\|U(\tau) - I\| < \epsilon$, so pretty good state transfer occurs from each vertex to itself in a graph X. Since $U(t)$ is symmetric, pretty good state transfer is a symmetric relation.

Suppose pretty good state transfer occurs from a to b and from b to c in a graph X. For $\epsilon > 0$, there exist times τ_1 and τ_2 and unimodular complex numbers γ_1 and γ_2 such that

$$\|U(\tau_1)e_a - \gamma_1 e_b\| < \frac{\epsilon}{2} \quad \text{and} \quad \|U(\tau_2)e_b - \gamma_2 e_c\| < \frac{\epsilon}{2}.$$

Since $U(\tau_2)^{-1} = U(-\tau_2)$ is unitary, we have $\|\gamma_1 e_b - \gamma_1\gamma_2 U(\tau_2)^{-1} e_c\| < \frac{\epsilon}{2}$. By the triangle inequality

$$\begin{aligned} \|U(\tau_2+\tau_1)e_a - \gamma_1\gamma_2 e_c\| &= \|U(\tau_1)e_a - \gamma_1\gamma_2 U(\tau_2)^{-1} e_c\| \\ &\leqslant \|U(\tau_1)e_a - \gamma_1 e_b\| + \|\gamma_1 e_b - \gamma_1\gamma_2 U(\tau_2)^{-1} e_c\| \\ &< \epsilon. \end{aligned}$$

We see that pretty good state transfer occurs from a to c in X. □

Further, as the automorphism group acts transitively on the set of corners, we may assume that the fixed corner is $(1,1,\dots,1)$, so it is enough to prove pretty good state transfer between $(1,1,\dots,1)$ and $(1,\dots,n_i,\dots,1)$ for all i. For convenience of notation, we may rearrange the cartesian factors so that the factor of interest is the first one.

5.1. $\mathbf{X = P_{p-1} \Box Z}$

Lemma 5.2. *Let* $p \equiv 1 \pmod 8$. *Let* Z *be a finite graph and denote by* F_Z *the field generated by its eigenvalues. Assume that* $F_p \cap F_Z = \mathbb{Q}$. *Then if, for some vertex* z *of* Z *the vertices* $(1, z)$ *and* $(p-1, z)$ *in* $P_{p-1} \Box Z$ *are strongly cospectral, there is pretty good state transfer between* $(1, z)$ *and* $(p-1, z)$.

Proof. We recall that the eigenvalues of P_{p-1} are $\alpha_r = 2 \cos \frac{r\pi}{p}$, $r = 1, \ldots, p-1$. We shall apply Theorem 2.2 to the vertices $(1, z)$ and $(p-1, z)$. By assumption, Condition (a) of Theorem 2.2 holds. Let $\mu_1, \ldots, \mu_m$ be the eigenvalues of Z. Then the eigenvalues of $P_{p-1} \Box Z$ are the values $\alpha_r + \mu_j$ for $1 \leqslant r \leqslant p-1$ and $1 \leqslant j \leqslant m$. We shall assume that there is a sequence of $(p-1)m$ integers ℓ_{rj} that satisfy Conditions (b)(i) and (b)(ii) of Theorem 2.2 and prove that Condition (b)(iii) is impossible.

Condition (b)(i) takes the form

$$\sum_{r,j} \ell_{rj}(\alpha_r + \mu_j) = 0 \tag{5.1}$$

and Condition (b)(ii) is

$$\sum_{r,j} \ell_{rj} = 0. \tag{5.2}$$

The set $\Phi^-_{(1,z),(p-1,z)}$ consists of those eigenvalues $\alpha_r + \mu_j$ for which r is even. Therefore Condition (b)(iii) is

$$\sum_{r,\, j:\ r \text{ even}} \ell_{rj} \equiv 1 \pmod 2.$$

For $r = 1, \ldots, p-1$, let $a_r = \sum_j \ell_{rj}$. Then we can we rewrite (5.1) as

$$\sum_{r=1}^{p-1} a_r \alpha_r = -\sum_{r,j} \ell_{rj} \mu_j. \tag{5.3}$$

The left hand side of (5.3) lies in F_p, while the right hand side lies in F_Z. So by our hypothesis on the intersection of these fields, the common value of (5.3) must be rational and, in fact, an integer, since it is clearly an algebraic integer. We denote this integer by s. Then we have

$$\sum_{r=1}^{p-1} a_r \alpha_r = \sum_{r=1}^{\frac{p-1}{2}} (a_r - a_{p-r}) \alpha_r = s. \tag{5.4}$$

Using Equation (3.2), we can replace 1 in $\{1, \alpha_1, \ldots, \alpha_{\frac{p-3}{2}}\}$ with $\alpha_{\frac{p-1}{2}}$ to form a basis $\{\alpha_1, \ldots, \alpha_{\frac{p-1}{2}}\}$ of F_p. Then Equations (3.2) and (5.4) give

$$\sum_{r=1}^{\frac{p-1}{2}} \left(a_r - a_{p-r} + (-1)^r s\right) \alpha_r = 0,$$

and $a_r - a_{p-r} = (-1)^{r+1}s$, for $r = 1, \dots, \frac{p-1}{2}$.

Thus, for $i = 1, \dots, \frac{(p-1)}{2}$,

$$a_{2i} - a_{p-2i} = -s.$$

Summing over i yields

$$\sum_{r,\,j:\ r\text{ even}} \ell_{rj} - \sum_{r,\,j:\ r\text{ odd}} \ell_{rj} = -\frac{(p-1)}{2}s,$$

and if we add this equation to (5.2) we obtain

$$2\sum_{r,\,j:\ r\text{ even}} \ell_{rj} = -\frac{(p-1)}{2}s.$$

Thus, since $p_1 \equiv 1 \pmod 8$, we have $\sum_{r,\,j:\ r\text{ even}} \ell_{rj} \equiv 0 \pmod 2$. Therefore, we have shown that whenever the first two conditions (b)(i) and (b)(ii) of Theorem 2.2 are true the third condition (b)(iii) is false. □

5.2. $\mathbf{X = P_{2p-1} \square Z}$

Lemma 5.3. *Let $p \equiv 1 \pmod 4$. Let Z be a finite graph and denote by F_Z the field generated by its eigenvalues. Assume that $F_{2p} \cap F_Z = \mathbb{Q}$. Then if, for some vertex z of Z the vertices $(1, z)$ and $(2p-1, z)$ in $P_{2p-1} \square Z$ are strongly cospectral, there is pretty good state transfer between $(1, z)$ and $(2p-1, z)$.*

Proof. We recall that the eigenvalues of P_{2p-1} are $\beta_r = 2\cos\frac{r\pi}{2p}$, $r = 1, \dots, 2p-1$. Let $\mu_1, \dots, \mu_m$ be the eigenvalues of Z. Similar to the proof of Lemma 5.2, we shall assume that there is a sequence of $(2p-1)m$ integers ℓ_{rj} that satisfy Conditions (b)(i) and (b)(ii) of Theorem 2.2 and prove that Condition (b)(iii) is impossible.

Condition (b)(i) takes the form

$$\sum_{r,j} \ell_{rj}(\beta_r + \mu_j) = 0 \tag{5.5}$$

and Condition (b)(ii) is

$$\sum_{r,j} \ell_{rj} = 0.$$

The set $\Phi^-_{(1,z),(2p-1,z)}$ consists of those eigenvalues $\beta_r + \mu_j$ for which r is even. Therefore Condition (b)(iii) is

$$\sum_{r,\,j:\ r\text{ even}} \ell_{rj} \equiv 1 \pmod 2.$$

For $r = 1, \dots, 2p-1$, let $a_r = \sum_j \ell_{rj}$. Then we can we rewrite (5.5) as

$$\sum_{r=1}^{2p-1} a_r\beta_r = -\sum_{r,j} \ell_{rj}\mu_j. \tag{5.6}$$

The left hand side of (5.6) lies in F_{2p}, while the right hand side lies in F_Z. So by our hypothesis on the intersection of these fields, the common value of (5.6) must be an integer, denoted by s. As $\beta_p = 0$, we have

$$\sum_{r=1}^{2p-1} a_r\beta_r = \sum_{r=1}^{p-1}(a_r - a_{2p-r})\beta_r = s. \tag{5.7}$$

Using Equation (3.3), we can replace 1 in $\{1, \beta_1, \ldots, \beta_{p-2}\}$ with β_{p-1} to form a basis $\{\beta_1, \ldots, \beta_{p-1}\}$ of F_{2p}. It follows from Equations (3.3) and (5.7) that

$$\sum_{j=1}^{\frac{p-1}{2}} \left(a_{2j} - a_{2p-2j} + (-1)^j s\right) \beta_{2j} + \sum_{j=1}^{\frac{p-1}{2}}(a_{2j+1} - a_{2p-2j-1})\beta_{2j+1} = 0,$$

and the coefficients of the β_r's are zero. For $i = 1, \ldots, \frac{(p-1)}{2}$, we have

$$a_{2i} - a_{2p-2i} = (-1)^{i+1}s.$$

Since $\frac{p-1}{2}$ is even, summing over i yields

$$\sum_{i=1}^{\frac{p-1}{2}} a_{2i} - \sum_{i=\frac{p+1}{2}}^{p-1} a_{2i} = 0,$$

and

$$\sum_{r,\, j\colon r \text{ even}} \ell_{rj} = \sum_{i=1}^{\frac{p-1}{2}} a_{2i} + \sum_{j=\frac{p+1}{2}}^{p-1} a_{2j} = 2\left(\sum_{j=\frac{p+1}{2}}^{p-1} a_{2j}\right) \equiv 0 \pmod 2.$$

Therefore, we have shown that whenever the first two conditions (b)(i) and (b)(ii) of Theorem 2.2 are true the third condition (b)(iii) is false. □

5.3. $\mathbf{X = P_{2^e-1}\Box Z}$

Let $\gamma_r := 2\cos\frac{r\pi}{2^e}$ $(r = 1\ldots, 2^e-1)$ be the eigenvalues of P_{2^e-1}.

Lemma 5.4. *Let Z be a finite graph and denote by F_Z the field generated by its eigenvalues. Assume that $F_{2^e} \cap F_Z = \mathbb{Q}$. Then if, for some vertex z of Z the vertices $(1, z)$ and $(2^e-1, z)$ in $P_{2^e-1}\Box Z$ are strongly cospectral, there is pretty good state transfer between $(1, z)$ and $(2^e-1, z)$.*

Proof. Let $\mu_1,\ldots,\mu_m$ be the eigenvalues of Z. We apply the criteria of Theorem 2.2. The strong cospectrality condition (a) of that theorem holds by assumption. The eigenvalues in $\Phi^-_{(1,z),(2^e-1,z)}$ are $\gamma_r + \mu_j$ with r even. Suppose Condition (b)(i) of Theorem 2.2 holds. Then, as in previous arguments, we isolate the terms coming from eigenvalues of P_{2^e-1} and use the hypothesis $F_{2^e} \cap F_Z = \mathbb{Q}$, and obtain an equation

$$\sum_{r=1}^{2^e-1} a_r\gamma_r = s,$$

where a_i, $s \in \mathbb{Z}$. Note that $\gamma_{2^{e-1}} = 0$. So we have

$$\sum_{r=1}^{2^{e-1}-1} a_r\gamma_r + \sum_{r=2^{e-1}+1}^{2^e-1} a_r\gamma_r = s.$$

Since $\gamma_{2^e-r} = -\gamma_r$, the above equation becomes

$$\sum_{r=1}^{2^{e-1}-1} (a_r - a_{2^e-r})\gamma_r = s.$$

As γ_1 has degree 2^{e-1}, the set $\{1, \gamma_1, \dots, \gamma_{2^{e-1}-1}\}$ is linearly independent. Hence $a_r = a_{2^e-r}$, for $r = 1, \dots 2^{e-1}$, and $s = 0$. It follows that $\sum_{r \text{ odd}} a_r$ is even. If we assume Condition (b)(ii) in Theorem 2.2, then $\sum_{r=1}^{2^e-1} a_r = 0$. Therefore $\sum_{r \text{ even}} a_r$ is even. Then by Theorem 2.2 we have pretty good state transfer between $(1, z)$ and $(2^e - 1, z)$. □

Lemma 5.5. *For prime $p \geqslant 3$ and $e \geqslant 2$, there is pretty good state transfer between $(1, 1)$ and $(1, p - 1)$ in $P_{2^e-1} \square P_{p-1}$.*

Proof. We shall apply Theorem 2.2. The condition on strong cospectrality is satisfied, by Corollary 3.7. Recall the eigenvalues of P_{p-1} are $\alpha_r = 2\cos(\frac{r\pi}{2p})$, $r = 1,\dots,p-1$. The set $\Phi^-_{(1,1),(1,p-1)}$ consists of the eigenvalues $\gamma_i + \alpha_r$ where r is even.

Suppose Condition (b)(i) of Theorem 2.2 holds. Thus for some integers ℓ_{ir}, we have

$$\sum_{i,r} \ell_{i,r}(\gamma_i + \alpha_r) = 0$$

Then, as in Lemma 5.4 we move the eigenvalues of P_{2^e-1} to the left side and those of P_{p-1} to the right side, resulting in the equation

$$\sum_{i,r} \ell_{ir}\gamma_i = -\sum_{i,r} \ell_{ir}\alpha_r.$$

Next we use use the fact that $F_{2^e} \cap F_p = \mathbb{Q}$. We may argue exactly as in Lemma 5.4 to deduce that the common value of (5.3) is zero. Thus, if we set $a_r = \sum_i \ell_{ir}$, we have

$$\sum_{r=1}^{p-1} a_r\alpha_r = \sum_{r=1}^{\frac{p-1}{2}} (a_r - a_{p-r})\alpha_r = 0.$$

We saw in the proof of Lemma 5.2 that $\{\alpha_1, \dots, \alpha_{\frac{p-1}{2}}\}$ is a basis of F_p. Hence $a_r = a_{p-r}$, for $r = 1, \dots, \frac{p-1}{2}$, and $\sum_{r \text{ even}} a_r = \sum_{r \text{ odd}} a_r$. Taking into account Condition (b)(ii) of Theorem 2.2, we obtain $\sum_{r \text{ even}} a_r = 0$. Therefore, by Theorem 2.2, we have pretty good state transfer from $(1, 1)$ to $(1, p - 1)$. □

Lemma 5.6. *For prime $p \geqslant 3$ and $e \geqslant 2$, there is pretty good state transfer between $(1, 1)$ and $(1, 2p - 1)$ in $P_{2^e-1} \square P_{2p-1}$.*

Proof. The proof is very similar to that of Lemma 5.5. We shall apply Theorem 2.2. The condition on strong cospectrality is satisfied, by Corollary 3.7. Recall the eigenvalues of P_{2p-1} are $\beta_r = 2\cos(\frac{r\pi}{2p})$, $r = 1,\ldots,2p-1$. Note that $\beta_p = 0$. The set $\Phi^-_{(1,1),(1,2p-1)}$ consists of the eigenvalues $\gamma_i + \beta_r$ where r is even. Suppose Condition (b)(i) of Theorem 2.2 holds. Thus for some integers ℓ_{ir}, we have

$$\sum_{i,r} \ell_{ir}(\gamma_i + \beta_r) = 0$$

Then, as in Lemma 5.4 we move the eigenvalues of P_{2^e-1} to the left side and those of P_{2p-1} to the right side, resulting in the equation

$$\sum_{i,r} \ell_{ir}\gamma_i = -\sum_{i,r} \ell_{ir}\beta_r.$$

Next we use use the fact that $F_{2^e} \cap F_{2p} = \mathbb{Q}$. We may argue exactly as in Lemma 5.4 to deduce that the common value of (5.3) is zero. Thus, if we set $a_r = \sum_i \ell_{ir}$, we have

$$\sum_{r=1}^{2p-1} a_r\beta_r = \sum_{r=1}^{p-1}(a_r - a_{2p-r})\beta_r = 0.$$

Since β_1 has degree $p-1$, $\{\beta_1,\ldots,\beta_{p-1}\}$ is a basis of F_{2p} which implies $a_r = a_{2p-r}$, for $r = 1,\ldots,2p-1$. Thus

$$\sum_{r \text{ even}} a_r = 2\,(a_2 + a_4 + \cdots + a_{p-1})$$

so Condition (b)(iii) of Theorem 2.2 can never hold. Therefore by Theorem 2.2, we have pretty good state transfer between $(1,1)$ and $(1,2p-1)$. □

5.4. Proof of Theorem 1.1

With Lemma 2.3 and the results in Sections 4 and 5 at our disposal we are now ready to prove Theorem 1.1, the classification of path products in which there is pretty good transfer among all corners.

Suppose $X = P_{n_1}\square\ldots\square P_{n_k}$, $k \geqslant 2$, has pretty good state transfer occurring between any two corners. By Lemma 2.3, $n_i + 1$ is either a power of two, p or $2p$ for some prime p, for $i = 1,\ldots,k$. By Lemma 4.1, we can assume that there are distinct primes $p_1,\ldots,p_f$, $q_1,\ldots,q_h$ such that

$$X = P_{p_1-1}\square\ldots\square P_{p_f-1}\square P_{2q_1-1}\square\ldots P_{2q_h-1}$$

where f and h are non-negative integers whose sum is at least two, or

$$X = P_{2^e-1}\square P_{p_1-1}\square\ldots\square P_{p_f-1}\square P_{2q_1-1}\square\ldots P_{2q_h-1},$$

for $e \geqslant 2$ and $f + h \geqslant 1$.

Suppose $f + h \geqslant 2$. It follows from Lemmas 4.2 to 4.6 that $p_1, \ldots, p_f \equiv 1 \pmod 8$ and $q_1, \ldots, q_h \equiv 1 \pmod 4$. Thus, X has the form of parts (3) or (4) in the statement of Theorem 1.1.

Conversely, suppose X has the form of (3) or (4) in Theorem 1.1. We will show that X has pretty good state transfer among all of its corners. By the discussion at the beginning of this section, it suffices to show that there is pretty good state transfer between $(1, 1, \ldots, 1)$ and an adjacent corner, which we can assume to be $(n_1, 1, \ldots, 1)$, where n_1 is one of the path lengths in (3) or (4). First by Lemma 3.7, we have strong cospectrality between these vertices. Then, depending on n_1, we can apply Lemmas 5.2 to 5.4, taking Z to be the product of the other factors in X and z to be the vertex $(1, \ldots, 1)$ of Z, to deduce pretty good state transfer. We must check that the hypotheses on the field intersections in Lemmas 5.2 to 5.4 are satisfied. The eigenvalues of a path P_n lie in the intersection of the cyclotomic field of order $2(n + 1)$ with the real field. It follows that the field $F_{n_1+1} \cap F_Z$ in these Lemmas is a subfield of the threefold intersection of the real numbers, a cyclotomic field of order $2(n_1 + 1)$ and a cyclotomic field of order m, where $\gcd(m, 2(n_1 + 1)) = 4$. Hence $F_{n_1+1} \cap F_Z = \mathbb{Q}$. This completes the proof that the graphs X in (3) and (4) have pretty good state transfer among all corners.

When $f + h = 1$, Lemmas 3.7, 5.4 to 5.6, show that pretty good state transfer occurs among all four corners of $P_{2^e-1} \square P_{p-1}$ and $P_{2^e-1} \square P_{2p-1}$ for $e \geqslant 2$ and prime $p \geqslant 3$. □

Theorem 1.1 gives a construction of graphs that admit pretty good state transfer among n vertices, for $n \geqslant 2$.

6. No Laplacian Pretty good state transfer among corner vertices

For the Heisenberg Hamiltonian, the continuous-time quantum walk on X is given by $\exp(-itL_X)$, where L_X is the Laplacian matrix of X [Kay11, IV.E]. We shall see in this section that, contrary to Theorem 1.1, there is no cartesian product of two or more paths with Laplacian pretty good state transfer between any two corners.

Given two simple finite graphs X and Y on n and m vertices, respectively, the Laplacian matrix of their cartesian product is

$$L_{X \square Y} = L_X \otimes I_m + I_n \otimes L_Y.$$

The transition matrix of $X \square Y$ based on the Heisenberg Hamiltonian is

$$\exp(-itL_X) \otimes \exp(-itL_Y).$$

Hence Lemmas 2.3 and 3.8 apply to the Laplacian matrix of $X \square Y$.

Laplacian pretty good state transfer occurs between extremal vertices of P_n if and only if n is a power of 2 [BCGS17]. By Lemma 2.3, if $P_{n_1} \square \cdots \square P_{n_k}$ has Laplacian pretty good state transfer between any two corners then each n_i is a power of 2. The Laplacian matrix of a path P_{2^e} has eigenvalues 0, and $2 + 2\cos\frac{\pi r}{2^e}$, for $r = 1, \ldots, 2^e - 1$. The idempotent projector of the 0-eigenspace is $\frac{1}{n} J_n$, where J_n denotes the $n \times n$ matrix of all ones. The extremal vertices of P_{2^e} have full eigenvalue support with

$$\Phi^{-}_{1,2^e} = \left\{ 2 + 2\cos\frac{\pi r}{2^e} \;:\; r \text{ odd} \right\}.$$

Lemma 6.1. *Suppose $f \geqslant e \geqslant 1$. The vertices $(1,1)$ and $(2^e,1)$ in $P_{2^e} \square P_{2^f}$ are not strongly cospectral.*

Proof. Let E_r be the idempotent projector of the $\left(2+2\cos\frac{\pi r}{2^e}\right)$-eigenspace of $L(P_{2^e})$, and let F_s be the idempotent projector of the $\left(2+2\cos\frac{\pi s}{2^f}\right)$-eigenspace of $L(P_{2^f})$.

Let W be the idempotent projector of the Laplacian matrix of $P_{2^e} \square P_{2^f}$ corresponding to the eigenvalue $\theta = 2+2\cos\frac{\pi}{2^e}$. As

$$\theta = \left(2+2\cos\frac{\pi}{2^e}\right) + 0 = 0 + \left(2+2\cos\frac{2^{f-e}\pi}{2^f}\right),$$

both $\left(E_1 \otimes \left(\frac{1}{2^f}J_{2^f}\right)\right)W$ and $\left(\left(\frac{1}{2^e}J_{2^e}\right) \otimes F_{2^{f-e}}\right)W$ are non-zero. Since $E_1 e_1 = -E_1 e_{2^e}$ and $\left(\frac{1}{2^e}\right)J_{2^e}e_1 = +\left(\frac{1}{2^e}\right)J_{2^e}e_{2^e}$, it follows from Lemma 3.8 that $We_{(1,1)} \neq \pm We_{(2^e,1)}$. □

Theorem 6.2. *There is no Laplacian pretty good state transfer among the corners of $P_{n_1} \square \cdots \square P_{n_k}$, for all $n_1, \ldots, n_k \geqslant 2$.*

Proof. By Lemma 2.3, we need to consider only the case where $n_1, \ldots, n_k$ are powers of two.

First observe that $P_{n_1} \square \cdots \square P_{n_k}$ is isomorphic to $(P_{n_1} \square P_{n_2}) \square (P_{n_3} \square \ldots \square P_{n_k})$ with an isomorphism given by

$$(v_1, v_2, \ldots, v_n) \mapsto \big((v_1, v_2), (v_3, \ldots, v_k)\big).$$

Lemmas 6.1 and 2.3 rule out Laplacian pretty good state transfer from $((1,1),(1,\ldots,1))$ to $((n_1,1),(1,\ldots,1))$ in $(P_{n_1} \square P_{n_2}) \square (P_{n_3} \square \ldots \square P_{n_k})$ when n_1 and n_2 are powers of two. Hence there is no Laplacian pretty good state transfer from $(1,1,\ldots,1)$ to $(n_1,1,\ldots,1)$ in $P_{n_1} \square \cdots \square P_{n_k}$. □

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