A CHIRAL APERIODIC MONOTILE

David Smith[1], Joseph Samuel Myers[2], Craig S. Kaplan[3], and Chaim Goodman-Strauss[4]

[1]*Yorkshire, U.K.*
ds.orangery@gmail.com
[2]*Cambridge, U.K.*
jsm@polyomino.org.uk
[3]*School of Computer Science, University of Waterloo, Waterloo, Ontario, Canada*
csk@uwaterloo.ca
[4]*National Museum of Mathematics, New York, New York, U.S.A.*
chaimgoodmanstrauss@gmail.com

Submitted: May 29, 2023; Accepted: Jun 7, 2024; Published: Sep 30, 2024

Abstract. The recently discovered "hat" aperiodic monotile mixes unreflected and reflected tiles in every tiling it admits, leaving open the question of whether a single shape can tile aperiodically using translations and rotations alone. We show that a close relative of the hat—the equilateral member of the continuum to which it belongs—is a weakly chiral aperiodic monotile: it admits only non-periodic tilings if we forbid reflections by fiat. Furthermore, by modifying this polygon's edges we obtain a family of shapes called Spectres that are strictly chiral aperiodic monotiles: they admit only homochiral non-periodic tilings based on a hierarchical substitution system.

Keywords. Tilings, aperiodic order, polyforms

Mathematics Subject Classifications. 05B45, 52C20, 05B50

1. Introduction

The recently discovered "hat" aperiodic monotile admits tilings of the plane, but none that are periodic [SMKGS24]. This polygon settles the question of whether a single shape—a closed topological disk in the plane—can tile aperiodically without any additional matching rules or other constraints on tile placement.

The hat is asymmetric: it is not equal to its image under any non-trivial isometry of the plane. In particular, a hat cannot be brought into perfect correspondence with its own mirror reflections. Moreover, all tilings formed by copies of the hat must use both unreflected and reflected tiles. Some people have wondered whether the hat and its reflection ought to be considered two distinct shapes, thereby invalidating its status as a monotile. To some extent, this question is about tiles as physical objects rather than mathematical abstractions. A hat cut from paper or plastic can easily be turned over in three dimensions to obtain its reflection, but a glazed ceramic tile cannot. More broadly, a wide range of three-dimensional objects, from organic molecules to shoes, behave very differently in their left- and right-handed forms.

Since Felix Klein introduced the *Erlangen* program [Gre07], each metric geometry has been defined by first choosing a set of isometries, the rigid motions of the geometric space in which we are working. Isometries may be classified as *orientation-preserving*, which map left-handed shapes to left-handed shapes (and right-handed to right-handed), or *orientation-reversing*, which exchange left- and right-handedness. In the plane, the orientation-preserving isometries comprise translations and rotations. We will refer to all orientation-reversing isometries generically as "reflections", and the image of a shape under a reflection as a "reflected" copy of that shape.

From that point of view, the question of whether a single tile admits monohedral tilings of the plane depends upon the set of isometries that may be used to transform tiles. If this set is not specified, one may reasonably assume that the full set of planar isometries is intended. Grünbaum and Shephard explicitly permit reflections in their definition of monohedral tilings [GS16, Section 1.2], so that when they later ask for a tile that "only admits monohedral non-periodic tilings" [GS16, Section 10.7], we can be confident that they considered reflected tiles to be in play. Ammann's aperiodic pairs of tiles were considered congruent under reflections [AGS92], and the disconnected aperiodic Taylor–Socolar tile [ST11, ST12] also requires reflections to tile the plane. Likewise, we regard the hat as an aperiodic monotile. Still, the fact that every tiling by hats mixes unreflected and reflected tiles leaves open the question of whether there might exist a monotile that achieves aperiodicity without the use of reflections.

A *tile* T is a closed topological disk in the plane, and a *monohedral tiling* admitted by it is a countable collection $\mathcal{T} = \{T_1, T_2, \ldots\}$ of congruent copies of T with disjoint interiors, whose union is the entire plane. Each T_i is of the form $g_i T$ for some planar isometry g_i. We say that a monohedral tiling $\mathcal{T}$ is a *homochiral tiling* if for every pair $T_i, T_j \in \mathcal{T}$, there is an orientation-preserving isometry mapping T_i to T_j. (Note that a monohedral tiling whose tile has bilateral symmetry is automatically homochiral according to this definition.) We then define a *weakly chiral aperiodic monotile* to be a tile whose homochiral tilings are all non-periodic (and that admits at least one such tiling), and a *strictly chiral aperiodic monotile* to be a tile that admits *only* homochiral non-periodic tilings. Following Klein, the weak case is aperiodic if we decree reflections to be off limits, even if the tile admits periodic tilings when reflections are allowed. The strict case remains both chiral and aperiodic in the presence of reflections. With these definitions in hand, we ask: *Do there exist any weakly or strictly chiral aperiodic monotiles?*[1]

The discovery of the hat is an effective reminder of how little we understand the possibilities and subtleties of monohedral tilings. Certainly there is no evidence to suggest that the hat (and the continuum of shapes to which it belongs) is somehow unique, and we might therefore hope that a zoo of interesting new monotiles will emerge in its wake. In this context, a chiral aperiodic monotile does not seem particularly unlikely. Nonetheless, we did not expect to find one so close at hand: a solution follows in short order from the hat.

Our previous work [SMKGS24] related the discovery of two separate aperiodic polykites—the hat, and a 10-kite known as the "turtle". We showed that these two shapes are members of

[1]We might call this the "vampire einstein" problem, as we are seeking a shape that is not accompanied by its reflection.

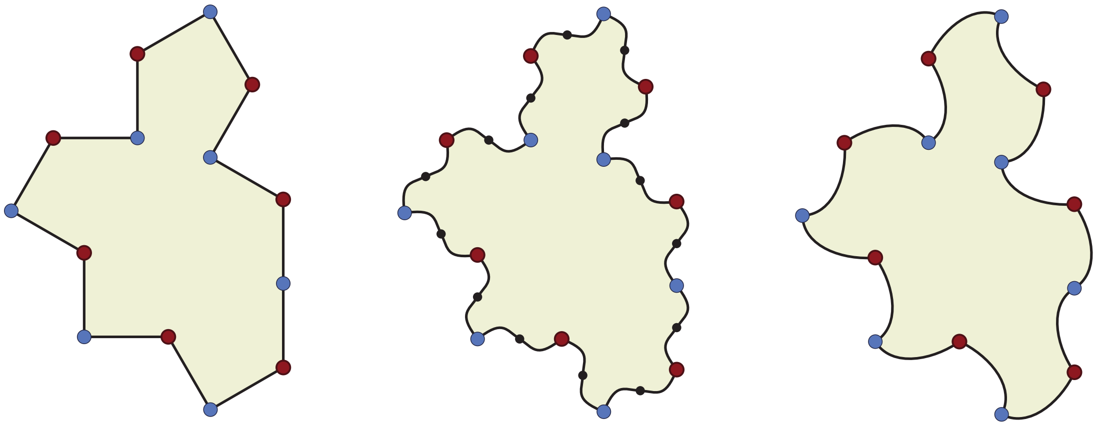

Figure 1.1: The 14-sided polygon $\mathrm{Tile}(1,1)$, shown on the left, is a weakly chiral aperiodic monotile: if by fiat we forbid tilings that mix unreflected and reflected tiles, then it admits only non-periodic tilings. By modifying its edges, as shown in the centre and right for example, we obtain strictly chiral aperiodic monotiles called "Spectres" that admit only non-periodic tilings even when reflections are permitted.

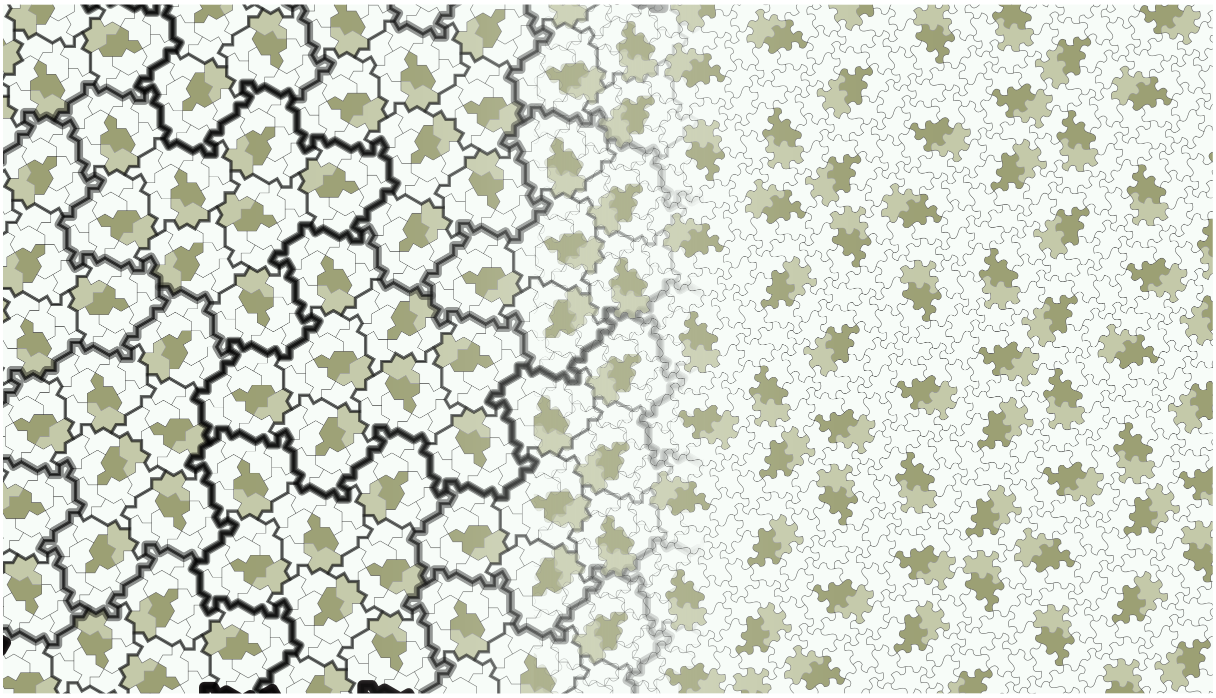

Figure 1.2: A patch from a non-periodic tiling by Spectres. On the left, tiles are drawn as copies of $\mathrm{Tile}(1,1)$, and thickened outlines show the hierarchy of supertiles to which these tiles will be shown to belong. On the right, tile boundaries are modified in a manner similar to Figure 1.1 (right). Tile colours will be explained later in the paper.

a continuum parameterized by choices of two non-negative edge lengths a and b. Each member of that continuum is a 14-sided polygon (with two collinear sides) denoted $\mathrm{Tile}(a,b)$; the hat is $\mathrm{Tile}(1,\sqrt{3})$ and the turtle is $\mathrm{Tile}(\sqrt{3},1)$. All members of this continuum are aperiodic monotiles, with three exceptions: $\mathrm{Tile}(0,1)$ (the "chevron"), $\mathrm{Tile}(1,0)$ (the "comet"), and the equilateral polygon $\mathrm{Tile}(1,1)$ (Figure 1.1, left).

Because $\mathrm{Tile}(1,1)$ is equilateral, copies of the tile may fit together in more ways than generic members of the continuum, and in fact it admits a simple periodic tiling using equal numbers of left- and right-handed tiles [SMKGS24, Figure 6.2]. But what happens if we rein in that freedom slightly? In this paper we prove that $\mathrm{Tile}(1,1)$ is a weakly chiral aperiodic monotile: if by fiat we restrict ourselves to tilings using only translations and rotations, then $\mathrm{Tile}(1,1)$ admits only non-periodic tilings. More importantly, by modifying the edges of this polygon (Figure 1.1, centre and right), we define a family of strictly chiral aperiodic monotiles that we call "Spectres", which admit only homochiral tilings, even when reflections are permitted (Theorem 2.2). Figure 1.2 shows non-periodic tilings by $\mathrm{Tile}(1,1)$ (left) and a Spectre (right).

2. The Spectre and its tilings

Our main results concern Spectres, the set of shapes whose tilings correspond exactly to the homochiral tilings admitted by $\mathrm{Tile}(1,1)$, thereby boosting us from weakly chiral to strictly chiral aperiodicity. It is helpful to begin by giving a precise definition of this set.

We regard $\mathrm{Tile}(1,1)$ as an equilateral polygon with 14 unit-length edges and 14 vertices, where one of those vertices lies between two collinear edges. A tile X is a *Spectre* if and only if

- X admits only homochiral tilings;
- Every tiling admitted by X corresponds to one by $\mathrm{Tile}(1,1)$: if $\{g_iX\}$ is a tiling for a set of isometries $\{g_i\}$, then $\{g_i\mathrm{Tile}(1,1)\}$ is also a tiling;
- Every homochiral tiling admitted by $\mathrm{Tile}(1,1)$ corresponds to one by X: if $\{g_i\mathrm{Tile}(1,1)\}$ is a homochiral tiling, then $\{g_iX\}$ is also a tiling; and
- If $\{g_iX\}$ is a tiling, then that tiling and $\{g_i\mathrm{Tile}(1,1)\}$ have the same tiling vertices (points shared by three or more tiles).

We do not attempt to characterize the space of all Spectres, but we can assert that the space is non-empty.

Lemma 2.1. *There exists a Spectre.*

Proof. We present two explicit constructions that yield families of Spectres, where the first is a special case of the second.

Note first that any tiling by $\mathrm{Tile}(1,1)$ is *vertex-to-vertex*: no vertex of a tile can lie in the interior of an edge of an adjacent tile (where we explicitly separate the long side of $\mathrm{Tile}(1,1)$ into two short edges, introducing a vertex with an interior angle of $180°$). By considering angles at consecutive vertices of the tile, we can see that any maximal line segment in the union of the boundaries of the tiles in a tiling has length 1 or 2 (and in particular is finite), forcing the edges on either side of that segment to be aligned.

We call a simple planar path an *s-curve* if it is symmetric under $180°$ rotation about its centre. Choose a smooth, non-straight s-curve σ, and construct a new tile S by replacing every edge of $\mathrm{Tile}(1,1)$ by a translated and rotated copy of σ, fit to the edge's vertices. As shown in Figure 1.1 (centre), we can choose σ in such a way that we obtain a topological disk S. Because the path is smooth, an argument similar to the one above guarantees that every tiling by S will be aligned along whole copies of σ, and therefore correspond to a tiling by $\mathrm{Tile}(1,1)$ with the same tiling vertices. Furthermore, the symmetry of σ prevents S and its reflection from being adjacent in a tiling, meaning that the only possible tilings by S correspond to homochiral tilings by $\mathrm{Tile}(1,1)$. Thus S is a Spectre.

The s-curves above are in some sense the minimally invasive way to "chiralize" a polygon that admits vertex-to-vertex tilings, preserving as many of its legal adjacencies as possible while forcing homochirality. However, we can take advantage of additional structure in $\mathrm{Tile}(1,1)$ to offer a more general construction. Observe that the interior angles at the vertices of $\mathrm{Tile}(1,1)$ strictly alternate between multiples of $90°$ and multiples of $120°$ (coloured blue and red, respectively, in Figure 1.1). In any tiling by $\mathrm{Tile}(1,1)$, colours must agree where vertices of neighbouring tiles meet. Thus we can choose *any* (simple, non-straight, smooth) path γ and replace the edges of $\mathrm{Tile}(1,1)$ by copies of γ, in such a way that copies associated with consecutive edges of the tile are related by a rotation around the vertex between them (so that the paths alternately face "in" and "out"). As before, provided that γ is chosen so that this process yields a topological disk, as in Figure 1.1 (right), that shape will be a Spectre. This construction subsumes the first one, because the two possible ways to affix an s-curve to an edge yield identical results. □

From this point on, we will not take the trouble to distinguish between $\mathrm{Tile}(1,1)$ restricted by fiat to homochiral tilings, and the Spectres defined above, which only admit homochiral tilings—we know that we are working with the same universe of tilings in either case. For simplicity we will usually speak of "the" Spectre in reference to any of these shapes, and we will use the polygon $\mathrm{Tile}(1,1)$ in figures showing patches of Spectres. However, if we refer to the Spectre in the context of strictly chiral aperiodicity, it should be understood that we are excluding $\mathrm{Tile}(1,1)$.

2.1. Main result

When a substitution tiling is aperiodic, the proof of aperiodicity often relies in some way on showing that the tiles are nested within an infinite hierarchical superstructure in every tiling they admit. We say that a set of tiles is *hierarchical* if, in every tiling admitted by those tiles, every tile is nested within an infinite hierarchy of ever-larger supertiles [GS99]. If these hierarchies are uniquely determined, then the tilings that contain them must be non-periodic [GS16, Theorem 10.1.1]. (For suppose a hierarchical tiling $\mathcal{T}$ is periodic, meaning that there exists a non-zero vector $\mathbf{v}$ for which $\mathcal{T} + \mathbf{v} = \mathcal{T}$. Then there would exist some supertile $\mathcal{C}$ that is so large that it has a non-empty intersection with $\mathcal{C} + \mathbf{v}$. Every tile T that lies in that intersection belongs to two distinct supertiles at the same level of the hierarchy, contradicting the uniqueness of that hierarchy.) This approach to aperiodicity informs our main result.

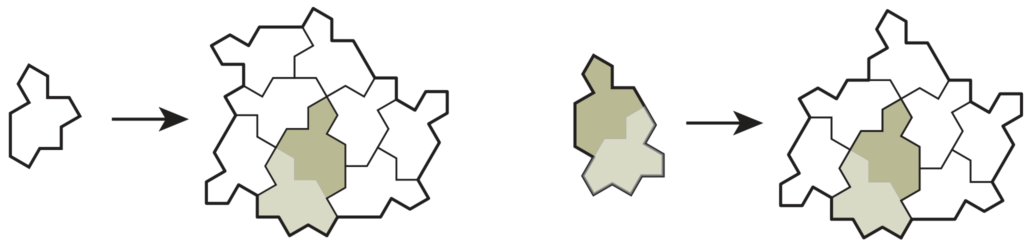

Figure 2.1: Two substitution rules that can be iterated to tile the plane with Spectres. The rules are based on a single Spectre and a two-Spectre compound called a Mystic. The first rule (left) replaces the Spectre by a cluster containing a Mystic and seven Spectres, all reflected; the second (right) replaces a Mystic by a cluster containing a Mystic and six Spectres. In Section 4 we show that every tiling by Spectres can be composed into non-overlapping congruent copies of these clusters.

Theorem 2.2. *The Spectre admits a tiling, and in any tiling it admits, each tile is contained within an infinite, unique hierarchy of larger and larger supertiles. Thus the Spectre is a strictly chiral aperiodic monotile.*

The first step in proving this theorem is to show that in any tiling by Spectres, it is possible to partition the tiles into congruent copies of the clusters of eight and nine Spectres shown in Figure 2.1. In Section 4 we give a computer-assisted proof of this fact. We further show that these clusters belong to nine distinct categories based on the relative positions of neighbouring clusters (Figure 4.1), and that each category may be viewed combinatorially as a regular hexagon marked with matching rules in the form of labelled edges (Figure 4.2). The proof relies on exhaustive enumeration of patches of Spectres that can occur in tilings. At the outset, such an enumeration appears complicated: unlike hats, Spectres are not polyforms, which suggests that generation of patches cannot be reduced to discrete computations on an underlying grid. But as we show in Section 3, any tiling by Spectres is combinatorially equivalent to a tiling by hats and turtles [SMKGS24], allowing us to perform the computations we need with minor modifications to battle-tested software.

In Section 5 we prove that in any tiling by the nine marked hexagons described above, the hexagons can be uniquely composed into supertiles that are combinatorially equivalent to reflections of those hexagons. The marked hexagons are therefore hierarchical. We then observe that we can "reverse" the supertile composition to obtain well-defined substitution rules on the marked hexagons. These rules can be iterated to produce a patch of hexagons of any size. It follows that the hexagons tile the plane, and from any such tiling we may construct a combinatorially equivalent tiling of the plane by Spectres, meaning that the Spectre tiles the plane as well. Because all such tilings were previously shown to be non-periodic, the Spectre is a strictly chiral aperiodic monotile.

The marked hexagons of Figure 4.2, and the substitution rules defined for them in Section 5,

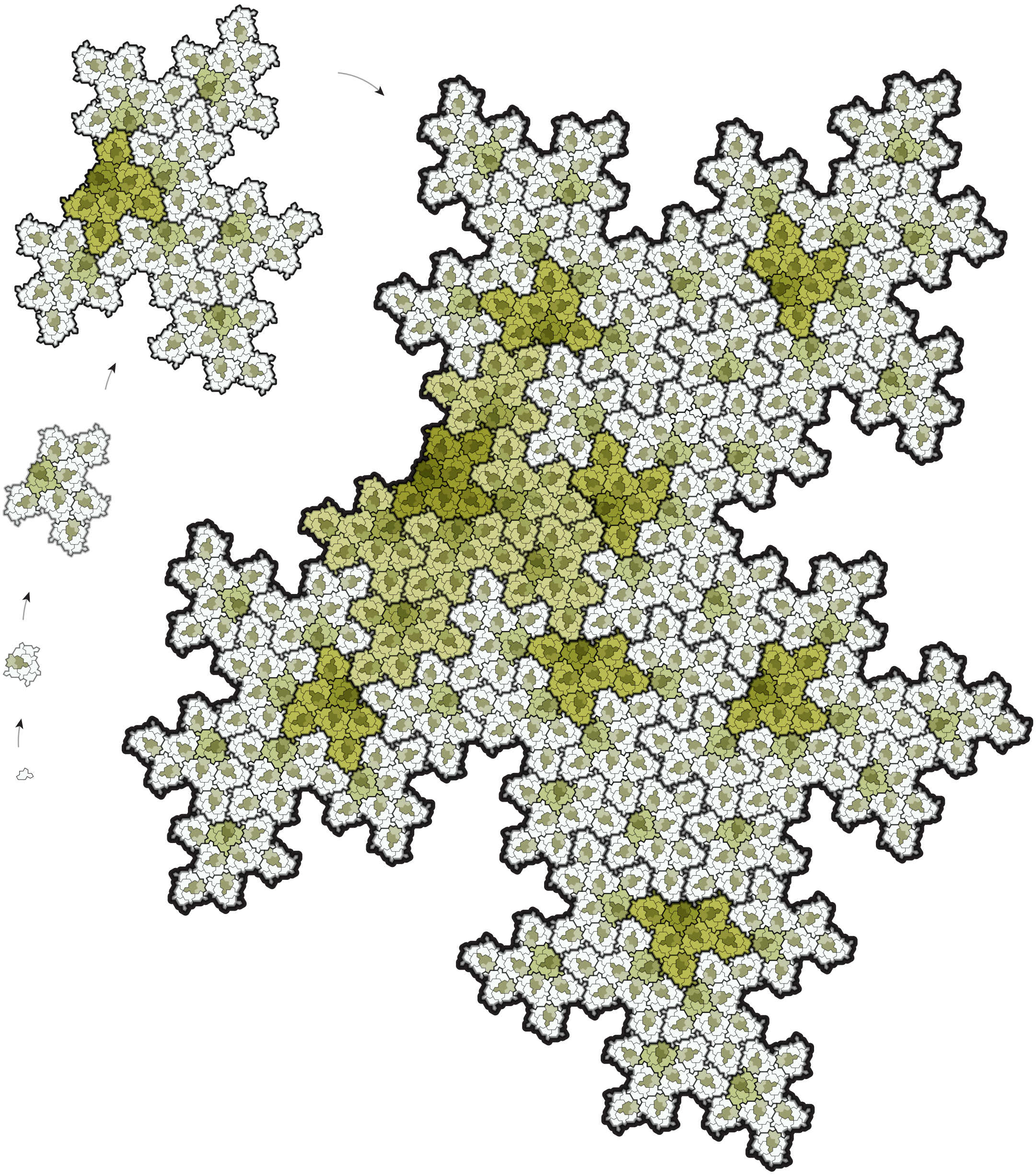

Figure 2.2: Five levels of supertiles created by applying the substitution rules given in Figure 2.1, starting with a single Spectre. The Mystic and its supertiles are shaded in progressive levels of green. Each step uses Spectres of the opposite handedness of the one before it.

suffice to establish the aperiodicity of the Spectre. However, it is also possible to begin this process one step earlier, and define substitution rules at the level of Spectres themselves. These rules are illustrated in Figure 2.1. Here we operate on two base tiles: the first is a single Spectre, and the second is a symmetric two-Spectre compound that we call a "Mystic" because of its resemblance to a Buddha in seated meditation. The first substitution rule replaces a single Spectre with a "Spectre cluster" containing a Mystic and seven Spectres; the second replaces a Mystic by a “Mystic cluster” containing a Mystic and six Spectres. At the end of Section 5 we transcribe the combinatorics of our marked hexagons down onto the Spectre and Mystic, showing that each tile in a tiling by Spectres lies in a unique infinite hierarchy of supertiles.

Based on the substitution rules in Figure 2.1, and the fact that the higher-order rules on hexagons are combinatorially equivalent, we can use an eigenvalue computation to deduce properties of Spectre tilings. We find that in the limit, areas increase by a factor of $4+\sqrt{15}$ with each stage of the substitution, and the ratio of the number of “even” Spectres (white and light green in the figure) to the number of “odd” Spectres (dark green) is also $4+\sqrt{15}$. Even and odd tiles are discussed in greater detail in Section 3. The irrationality of this number immediately implies that any tiling produced using the substitution rules must be non-periodic. Nevertheless, we need the full proof in Sections 4 and 5 to be certain that the substitution rules account for all possible Spectre tilings.

At multiple steps in our proof, we transform a tiling into another tiling by different tiles, and use properties of the new tiling to reason about the old one. These transformations are deterministic and unique. It is also important that they “preserve periodicity”, that is, that the original tiling is periodic if and only if the transformed tiling is. In some situations, such as when grouping tiles into supertiles, this preservation is immediate: the two tilings have the same symmetry group. When working through a less rigid combinatorial equivalence of tilings, we require that when pairs of vertices correspond, then either both pairs are related by translational symmetries of their tilings, or neither is (Section 2.2). This preservation of translational symmetries allows us to conclude that every homochiral tiling by $\mathrm{Tile}(1,1)$, and every tiling by Spectres must be non-periodic.

Remarkably, the rules of Figure 2.1 reverse all tile orientations. Consequently, in any sequence of iterations of these rules, we will necessarily produce homochiral patches of alternating handedness, as illustrated in Figure 2.2. We have never previously encountered this phenomenon when working with substitution tilings. Of course, the alternation can be avoided by defining larger rules that combine two rounds of substitution.

The proofs used in this work are highly combinatorial in nature, which sidesteps the practical question of how to draw patches of Spectres algorithmically. In Appendix A we include a simple algorithm for laying out Spectres, derived from the substitution rules in Figure 2.1. The algorithm can be used to draw large patches like the ones in Figure 2.2. We have created an interactive browser-based visualization tool for drawing patches of Spectres, available at `cs.uwaterloo.ca/~csk/spectre/`.

2.2. Combinatorial equivalence of tilings

At many points in this paper, we deduce information about a tiling by noting its combinatorial equivalence to some other tiling. Before proceeding we consider the notion of combinatorial equivalence in detail. Two patches, or two tilings, are *combinatorially equivalent* if and only if they are homeomorphic as topological complexes. Two sets of tiles are *combinatorially equivalent* if each tiling admitted by one is combinatorially equivalent to a tiling admitted by the other.

The following lemma gives a test for establishing combinatorial equivalence. An *edge patch* is a collection of tiles with disjoint interiors, such that there exists a closed arc e that is a connected component of the intersection of two of the tiles, all the other tiles contain an endpoint of e, and e lies in the interior of the union of the tiles. A *vertex* of an edge patch is a point in the interior of the union of the tiles that is shared by at least three tiles of the patch.

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets of tiles (possibly equipped with matching rules) such that each edge patch of one is combinatorially equivalent to some edge patch admitted by the other. Suppose further that the combinatorial equivalence satisfies the local consistency condition that if two edge patches from one set both have vertices surrounded by the same arrangement of tiles (up to an isometry mapping one vertex onto the other), then the corresponding vertices in the corresponding edge patches from the other set are also surrounded by the same arrangement of tiles. Then $\mathcal{A}$ and $\mathcal{B}$ are combinatorially equivalent.*

Proof. Any tiling admitted by $\mathcal{A}$ may be considered as an abstract complex with labelled vertices, edges, faces and incidences between them, but no further structure. Upon this complex we may chart a global geometry, applying the local geometry provided by the edge patches admitted by $\mathcal{B}$. From this structure we obtain a geometric complex with the same combinatorics as the tiling we started with. This complex is a simply-connected, complete, Riemannian metric manifold of constant zero curvature, and is therefore the Euclidean plane [Hop26]. Thus we have described this complex as a tiling by tiles in $\mathcal{B}$ that is combinatorially equivalent to our initial tiling by tiles in $\mathcal{A}$. □

Note that combinatorial equivalence preserves periodicity. If $\mathcal{T}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{B}}$ are combinatorially equivalent tilings by $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{T}_{\mathcal{A}}$ is periodic if and only if $\mathcal{T}_{\mathcal{B}}$ is. If $\mathcal{T}_{\mathcal{A}}$ has as a symmetry a translation by a vector $\mathbf{v}$, and if two vertices x_1, x_2 separated by $\mathbf{v}$ map to vertices of $\mathcal{T}_{\mathcal{B}}$ separated by a vector $\mathbf{w}$, then any other two vertices y_1, y_2 of $\mathcal{T}_{\mathcal{A}}$ separated by $\mathbf{v}$ also map to vertices separated by $\mathbf{w}$, by considering corresponding sequences of edge patches forming paths from x_1 to y_1 and x_2 to y_2. Thus $\mathbf{w}$ defines a translational symmetry of $\mathcal{T}_{\mathcal{B}}$.

3. From Spectres to hats and turtles

Our computational analysis of the hat [SMKGS24] was simplified by the fact that it is a polyform, specifically a union of eight kites from the Laves tiling $[3.4.6.4]$. Furthermore, the tiles in every tiling by hats must be aligned with the kites of the Laves tiling [SMKGS24, Lemma A.6]. Consequently, patches of hats can be manipulated discretely by associating information with the cells of the underlying kite grid. The aperiodic 10-kite known as the turtle is also compatible with grid-based computations.

$\mathrm{Tile}(1,1)$ is not a polyform, which at the outset appears to rule out such an approach. However, we can regain the ability to perform discrete computations by exploiting a connection between tilings by $\mathrm{Tile}(1,1)$ and tilings by combinations of hats and turtles. We prove the following result.

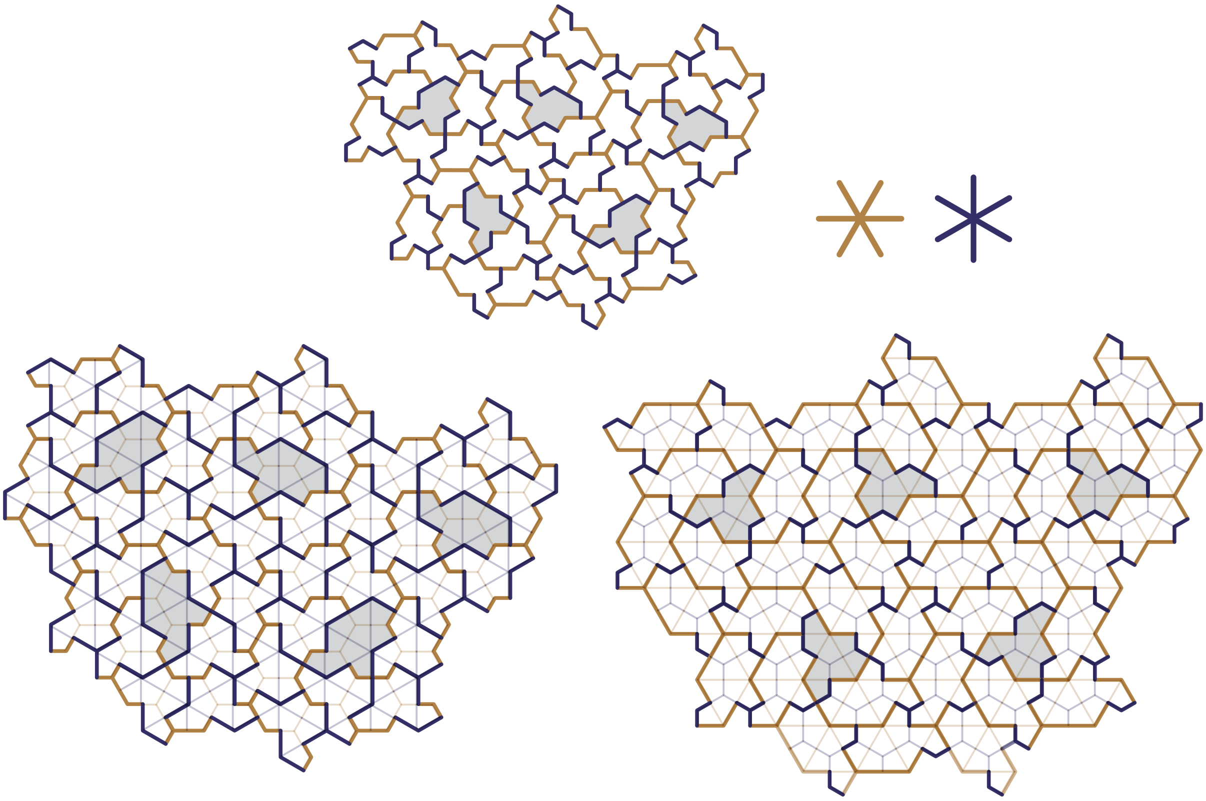

Figure 3.1: Tilings by $\mathrm{Tile}(1, 1)$ are combinatorially equivalent to tilings by hats and turtles, through a change to the lengths of the "even" and "odd" tile edges, oriented at even and odd multiples of $30°$ to the horizontal (top right). All three patches are drawn so that short edges have unit length.

Theorem 3.1. *There is a bijection between combinatorially equivalent tilings by* $\mathrm{Tile}(1, 1)$ *and by the set* $\{hat, turtle\}$*, such that a* $\mathrm{Tile}(1, 1)$ *tiling has a translation as a symmetry if and only if the corresponding hat-turtle tiling has a corresponding translation, and the* $\mathrm{Tile}(1, 1)$ *tiling includes a reflected tile if and only if the hat-turtle tiling does.*

Proof. Note first that in any tiling by a combination of hats and turtles, the tiles must be aligned to an underlying $[3.4.6.4]$ tiling. This fact can be established using an argument similar to the one used by Smith et al. for monohedral tilings by hats alone [SMKGS24, Lemma A.6].

Recall that the hat and the turtle are members of a continuum of tiles parameterized by two non-negative edge lengths a and b. Each such tile is denoted $\mathrm{Tile}(a, b)$; the hat is $\mathrm{Tile}(1, \sqrt{3})$ and the turtle is $\mathrm{Tile}(\sqrt{3}, 1)$.

Let $\mathcal{T}$ be a tiling by $\mathrm{Tile}(1, 1)$. The edges of the tiles of $\mathcal{T}$ lie in six families of parallel lines, spaced evenly by multiples of $30°$. For convenience, we rotate the tiling so that one of those families is horizontal. In that case, the edge vectors of tiles all form angles of $30k°$ with the horizontal; we refer to edges as "even" or "odd" when k is even or odd, respectively. Furthermore, every tile in $\mathcal{T}$ must appear in one of 24 orientations, twelve unreflected and twelve reflected; arbitrarily choosing the tile's two collinear edges as a reference, we label each tile as "even" or "odd" according to the parity of those edges. Figure 3.1 (top) shows a sample patch by $\mathrm{Tile}(1,1)$, with the odd tiles shaded. Next to the patch are two stars showing the even and odd edge directions.

Smith et al. show that the two edge lengths in a tiling by $\mathrm{Tile}(a,b)$ can be modified continuously to produce a family of combinatorially equivalent monotiles [SMKGS24, Theorem 6.1]. Analogous modifications are possible here. Around any tile in $\mathcal{T}$, the edge vectors belonging to any one direction add up to zero. We can therefore change the lengths of all edges in each of those six directions independently of the others, provided that no tile boundary becomes self-intersecting [GS09, Lemma 1.1]. In particular, if we change the lengths of the even edges of $\mathrm{Tile}(1,1)$ to $\sqrt{3}$, then even tiles become turtles and odd tiles become hats, all aligned to a single $[3.4.6.4]$ Laves tiling (Figure 3.1, bottom right). If instead we change the lengths of the odd edges to $\sqrt{3}$, then even tiles become hats and odd tiles become turtles (Figure 3.1, bottom left). Similarly, if we begin with any tiling by hats and turtles and change all edge lengths of $\sqrt{3}$ to 1, we obtain a tiling by $\mathrm{Tile}(1,1)$. We have created an animation that visualizes these continuous edge length modifications on a sample patch of tiles—see `youtu.be/K6wXQvL5KRo`.

Thus there is a direct relationship between a tiling by $\mathrm{Tile}(1,1)$ and a combinatorially equivalent tiling by hats and turtles. Orientations of tiles in the two tilings correspond in the obvious way. Unreflected and reflected copies of $\mathrm{Tile}(1,1)$ correspond to unreflected and reflected hats and turtles. As in Lemma 2.3 the effect of this correspondence on a translation symmetry is given by applying the correspondence to two vertices of one tiling that are related by that translation. □

The next section relies on the enumeration of patches of tiles that can occur in homochiral tilings by $\mathrm{Tile}(1,1)$. While we could in principle perform such calculations directly on a boundary representation of the tile, we instead use the correspondence established here, constructing (and illustrating) patches of unreflected hats and turtles as a proxy for even and odd copies of $\mathrm{Tile}(1,1)$. By doing so, we can harness convenient discrete algorithms similar to the ones we used previously to study the hat [Mye19, Kap22], knowing that any results we obtain apply to $\mathrm{Tile}(1,1)$ as well.

4. From hats and turtles to marked hexagons

We have not yet shown that the Spectre admits any tilings of the plane. However, we can still prove that any such tilings, if they exist, must be non-periodic. This section and the one that follows furnish such a proof. As a by-product we also obtain a substitution system that can produce patches of Spectres of any size.

Any tiling by Spectres is combinatorially equivalent to a homochiral tiling by $\mathrm{Tile}(1,1)$. In turn, that tiling is equivalent to a homochiral tiling by hats and turtles (i.e., all tiles are unreflected, or all tiles are reflected). These equivalences extend to any translational symmetries of the tilings, meaning that the Spectre tiling is periodic if and only if the hat-turtle tiling is. As with the analysis of hat tilings [SMKGS24, Section 4], we show here that in any homochiral tiling by hats and turtles, we can group tiles into non-overlapping clusters. The resulting tiling by the clusters satisfies certain matching rules and has the same symmetries as the original tiling by hats and turtles. Recalling that two tilings are combinatorially equivalent if and only if they are homeomorphic as topological complexes, we arrive at the following result.

Theorem 4.1. *Any tiling by Spectres is equivalent via Theorem 3.1 to a homochiral tiling by hats and turtles. That tiling can be composed into the clusters shown in Figure 4.1, satisfying the matching rules included there. The tiling by those clusters has the same symmetries as the tiling by hats and turtles, and is combinatorially equivalent to a tiling by regular hexagons.*

The first cluster, Γ, consists of eight tiles: one turtle surrounded by seven hats. The other eight clusters all consist of the same arrangement of nine tiles, namely a Γ cluster with one additional hat. These eight nine-tile clusters are distinguished based on the locations marked for their tiling vertices in a tiling by clusters. Edge segments with the same letter and opposite signs (or no sign) must adjoin each other on adjacent tiles. The nine clusters taken together can be regarded abstractly (and concretely, using combinatorial equivalence) as regular hexagons with marked edges, as visualized in Figure 4.2.

Proof of Theorem 4.1. We first assume without loss of generality that the tiling by Spectres and the corresponding tiling by hats and turtles consist entirely of unreflected tiles. If all tiles are reflected, we reflect the whole tiling, perform the analysis below, and reflect the resulting clusters to match the original tiling.

Our proof is based on a computer-assisted case analysis similar to the one used by Smith et al. to establish the aperiodicity of the hat [SMKGS24, Appendix B]. This analysis depends on computing "reduced lists of 1-patches" for sets of tiles, by first generating a large list of candidate 1-patches (defined below), and then "reducing" the list by eliminating those patches that cannot occur in tilings.

Given a set of tiles, define a 0-*patch* to be a single tile. Let $\mathcal{P}$ be a 0-patch and let $\mathcal{S}$ be a set of tiles whose interiors are pairwise disjoint with each other and with $\mathcal{P}$. If every tile of $\mathcal{S}$ is a neighbour of $\mathcal{P}$, and if the union of $\mathcal{P}$ and the tiles in $\mathcal{S}$ is a topological disk with $\mathcal{P}$ in its interior, then we call $\mathcal{S}$ a 1-*corona* of $\mathcal{P}$, and we call $\mathcal{P} \cup \mathcal{S}$ a 1-*patch*. More generally, any k-patch may be completely surrounded by a $(k+1)$-corona to yield a $(k+1)$-patch. We use "corona" as a shorthand for "1-corona".

To construct a reduced list of 1-patches for a given a set of tiles, first generate a list of legal pairs of neighbouring tiles, eliminating neighbour relationships that can be determined not to be extendable to a tiling [SMKGS24, Appendix B.1]. If we wish to restrict our attention to homochiral tilings, we also discard all pairs of tiles with opposite handedness. Next, generate a complete list of legal 1-patches of tiles, comprising all ways of surrounding a single central tile by a corona of tiles, yielding a patch in which all neighbours are legal as computed above.

Let $\mathcal{L}$ be a list of 1-patches and $\mathcal{P}$ a patch in $\mathcal{L}$. We say that a tile T in $\mathcal{P}$ is "$\mathcal{L}$-extendable" if we can augment $\mathcal{P}$ with tiles to complete a corona around T that matches one of the patches in $\mathcal{L}$, in such a way that neighbour pairs in the augmented patch are all legal. (It may also be possible to surround T with other coronas not in $\mathcal{L}$, but any such coronas are not relevant here.) The 1-patch $\mathcal{P}$ is $\mathcal{L}$-extendable if all the tiles in its corona are. Now let $\mathcal{L}_1$ be the complete list of legal 1-patches. Any patch in $\mathcal{L}_1$ that is not $\mathcal{L}_1$-extendable cannot appear in a tiling. We remove any non-extendable patches to produce a smaller list $\mathcal{L}_2$. We iterate this process, producing progressively

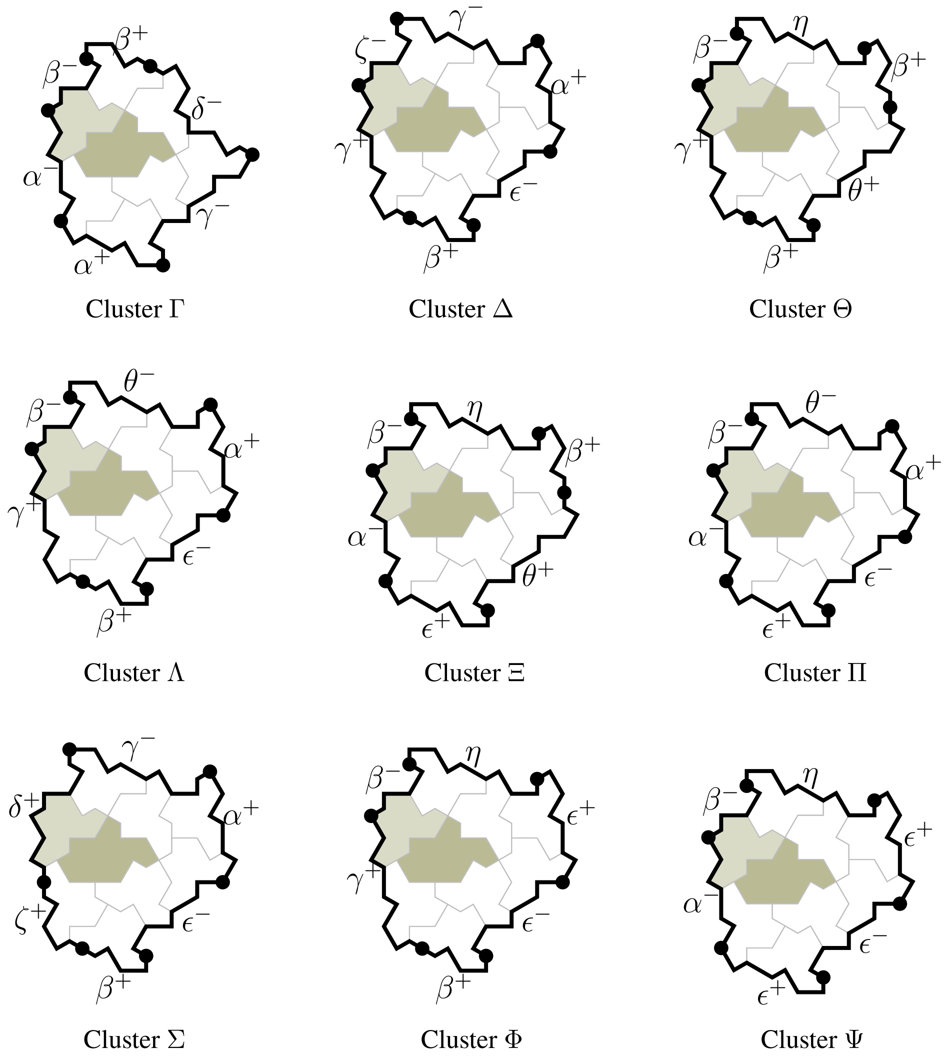

Figure 4.1: The nine marked clusters of hats and turtles. These clusters are combinatorially equivalent to the marked hexagons of Figure 4.2, and to the marked Spectres of Figure 5.3. The markings record the possible ways the unmarked versions of the clusters in Figure 2.1 can fit together in tilings. These clusters define supertiles for substitution rules on the marked Spectres, which may be coloured the same way as the combinatorially equivalent rules of Figure 5.1.

smaller lists of patches, until there are no further removals. The resulting reduced list contains all 1-patches that occur in full tilings, and possibly a few false positives do not interfere with our analysis below. In practice, we test whether T is $\mathcal{L}$-extendable not by attempting to build coronas around it, but by superimposing the central tile of every patch in $\mathcal{L}$ on it and testing for compatibility of the two overlapping patches.

In the next section, we will also use the extended notion of reduced lists of k-patches for $k > 1$. The computation is analogous to that of reduced lists of 1-patches. A list of k-patches may be

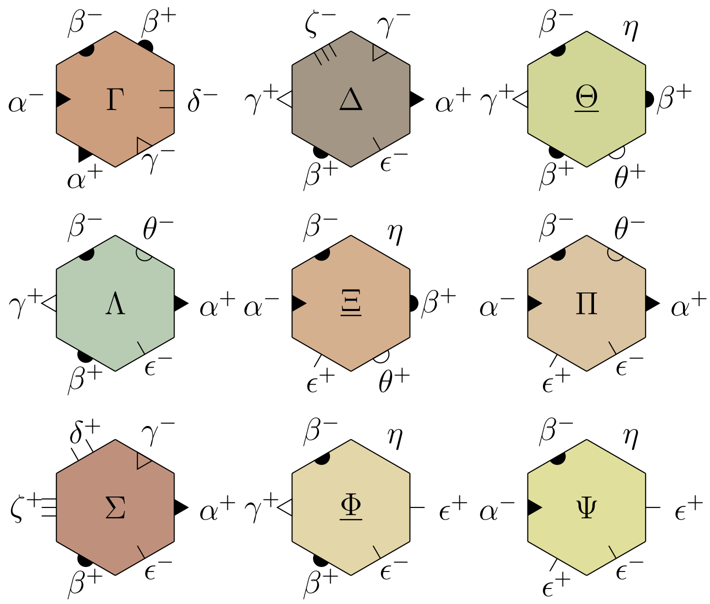

Figure 4.2: Nine marked hexagons, combinatorially equivalent to the marked clusters of Figure 4.1. Hexagon edges are marked with the same lowercase Greek letters as the clusters; edges are also decorated with inward- or outward-facing geometric markings that must agree for adjacent hexagons.

generated from a list of $(k-1)$-patches by considering ways to surround the tiles in the inner 1-corona of each $(k-1)$-patch with one of the $(k-1)$-patches, and then the resulting list may be reduced by considering whether each tile in a patch can be compatibly surrounded by one of the patches in the list.

By computing the reduced 1-patches of a set consisting of an unreflected hat and an unreflected turtle, allowing only neighbours of the same handedness, we obtain (a superset of) the 1-patches that occur in homochiral tilings by hats and turtles. We find that every reduced 1-patch with a turtle at its centre has at most one hat in its corona, with the exception of a single 1-patch where a turtle is surrounded entirely by hats. Likewise, every 1-patch with a hat at centre has at most one turtle in its corona, except for a single 1-patch containing a hat surrounded by turtles.

By considering a connected sequence of tiles from any tile in a tiling to any other, it follows that if any turtle has a turtle neighbour, then every hat is entirely surrounded by turtles that have no other hat neighbour, and if any hat has a hat neighbour, then every turtle is entirely surrounded by hats that have no other turtle neighbour. This behaviour can be seen in the patches of Figure 3.1, where turtles are distributed sparsely among hats or vice versa. Every tiling by the Spectre is equivalent to one tiling of each of these two types, depending on which edges have their lengths changed from 1 to $\sqrt{3}$. Without loss of generality we will work with sparse turtles embedded in a field of hats.

Henceforth we refer to the unique 1-patch consisting of a turtle surrounded by hats, shown in Figure 4.3 (left), as T6H ("turtle and six hats"). Because this patch is unique in the reduced

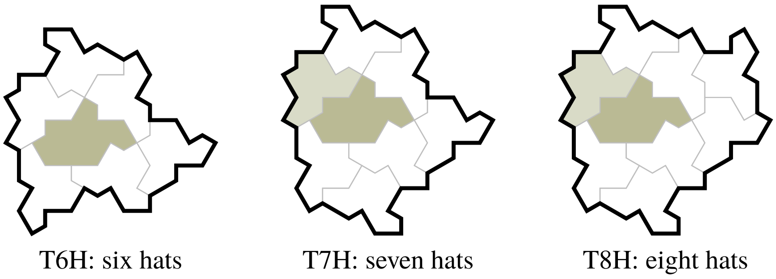

Figure 4.3: Clusters containing a turtle and six, seven or eight hats.

list of 1-patches, and because copies of the patch cannot overlap (as a hat has at most one turtle neighbour), homochiral tilings by hats and turtles correspond to homochiral tilings by two unreflected tiles: hats and copies of T6H.

We now extend this analysis by constructing a reduced list of 1-patches from the set containing the hat and T6H. In this list, we discover that every 1-patch with T6H at its centre must have a hat neighbour in the appropriate position to form the cluster T7H, consisting of a turtle and seven hats, shown in Figure 4.3 (centre). In a homochiral tilings by hats and turtles, we already know that the copies of T6H identified above cannot overlap. So if two copies of T7H overlap, they must do so by sharing a hat.[2] But the only way for them to share a hat would be for the two copies of T7H to be identical. Thus, copies of T7H cannot overlap either. So we continue the grouping process, constructing a reduced list of 1-patches from the hat and T7H. In this list, every 1-patch with the hat at its centre has a T7H neighbour in the appropriate position to form the cluster T8H shown in Figure 4.3 (right). Copies of T7H and T8H cannot overlap in a tiling by hats and turtles. Finally, then, in every homochiral tiling by hats and turtles the tiles can be composed into copies of T7H and T8H. As no arbitrary choices were made in forming these clusters, the tiling has precisely the same symmetries as the original tiling by hats and turtles.

Next consider a reduced list of 1-patches by T7H and T8H. Classify the central tile in such a 1-patch according to the points on its boundary that it shares with two of its neighbours (i.e., the patch's tiling vertices). This classification produces the nine marked clusters listed in Figure 4.1. The edge labels follow immediately from the shapes of the boundary segments between the marked vertices. Each of these tiles has exactly six vertices of degree 3, the central tile in each 1-patch has exactly six neighbours, and the relation between the orientation of a boundary segment and that of the corresponding edge in Figure 4.2 is consistent. Together, these observations ensure that tilings by T7H and T8H are in bijection with combinatorially equivalent tilings by the marked hexagons of Figure 4.2. □

[2]At this point, the tile T6H is considered as an indivisible unit, not a cluster made up of hats and a turtle. So each T7H contains only a single hat that might potentially overlap with a hat in another T7H.

5. A substitution system for marked hexagons

In this section, we prove that any tiling admitted by the marked hexagons of Figure 4.2 can be uniquely composed into the supertiles of Figure 5.1. This composition yields a unique hierarchy of level-n supertiles for all n, which forces any tiling by these hexagons to be non-periodic. We also observe that the supertiles imply a substitution system that produces patches of hexagons of any size, thus confirming that the hexagons tile the plane.

Theorem 5.1. *The tiles in any tiling by the nine marked hexagons of Figure 4.2 can be composed into the supertiles shown in Figure 5.1. The supertiles are combinatorially equivalent to reflected versions of the hexagons. The tilings by supertiles and marked hexagons have the same symmetries.*

Proof. Each supertile in Figure 5.1 contains a Γ hexagon and either six or seven other hexagons: Supertile Γ contains six other hexagons, while all the other supertiles contain hexagons in those six positions plus a seventh hexagon. Each supertile is annotated with the positions of its vertices and markings on the edges between those vertices. By inspection, the edge markings indicate portions of the supertile boundaries that match if and only if the corresponding marked edges of the original marked hexagons match. Furthermore, exactly three supertiles must meet at each vertex. Thus the supertiles are indeed combinatorially equivalent to reflected versions of the marked hexagons.

To prove Theorem 5.1, it suffices then to give rules to compose the marked hexagons of any tiling into the nine supertiles shown, such that vertices are in the positions marked, with no arbitrary choices involved in those rules (so that the tiling by supertiles necessarily has the same symmetries as the original tiling by marked hexagons). The lack of a translation as a symmetry then follows from the sizes of supertiles after n composition steps going to infinity with n, while the substitution structure implies that the marked hexagons tile arbitrarily large finite regions of the plane, and thus the whole plane.

We accrete one supertile around each Γ hexagon. First assign to that supertile the Γ hexagon itself and the six other hexagons in the positions relative to that Γ that appear in all of the illustrated supertiles. By constructing a reduced list of 5-patches by the marked hexagons, we can confirm that this assignment associates each marked hexagon in any tiling with at most one supertile, but leaves some hexagons unassigned. In the eight supertiles other than Supertile Γ, the remaining hexagon is in a fixed position relative to the Γ hexagon. For each unassigned hexagon, assign it to the supertile where it is in the right position relative to that supertile's Γ hexagon; again, examining a reduced list of 5-patches confirms that every unassigned hexagon is assigned by this rule to exactly one supertile.

It remains to show that the supertiles resulting from these assignment rules are exactly those shown in Figure 5.1 (i.e., that they contain marked hexagons in the positions and orientations illustrated, and have vertices in the indicated positions). For this step, consider those reduced 5-patches with Γ hexagons at their centres. Examining the list of such patches shows that there are exactly the nine possibilities illustrated for the labels and orientations of all the tiles in the supertile of that Γ. Furthermore, for each such possibility, there is a unique division of the hexagons surrounding that supertile into their own supertiles, which determines the locations of

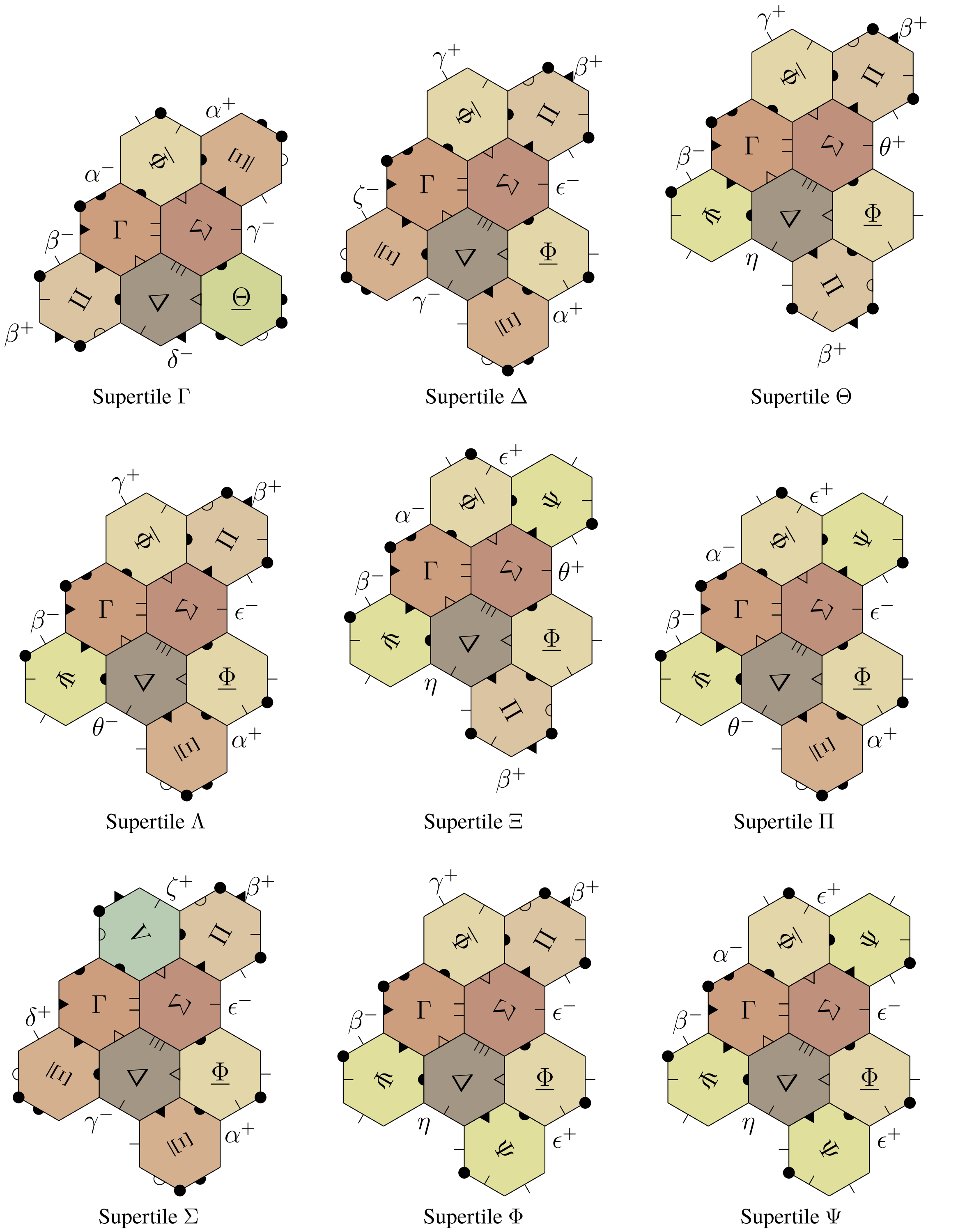

Figure 5.1: Nine supertiles. Each is drawn in the reverse handedness of the corresponding marked hexagon of Figure 4.2, preserving the handedness of the marked hexagons within it.

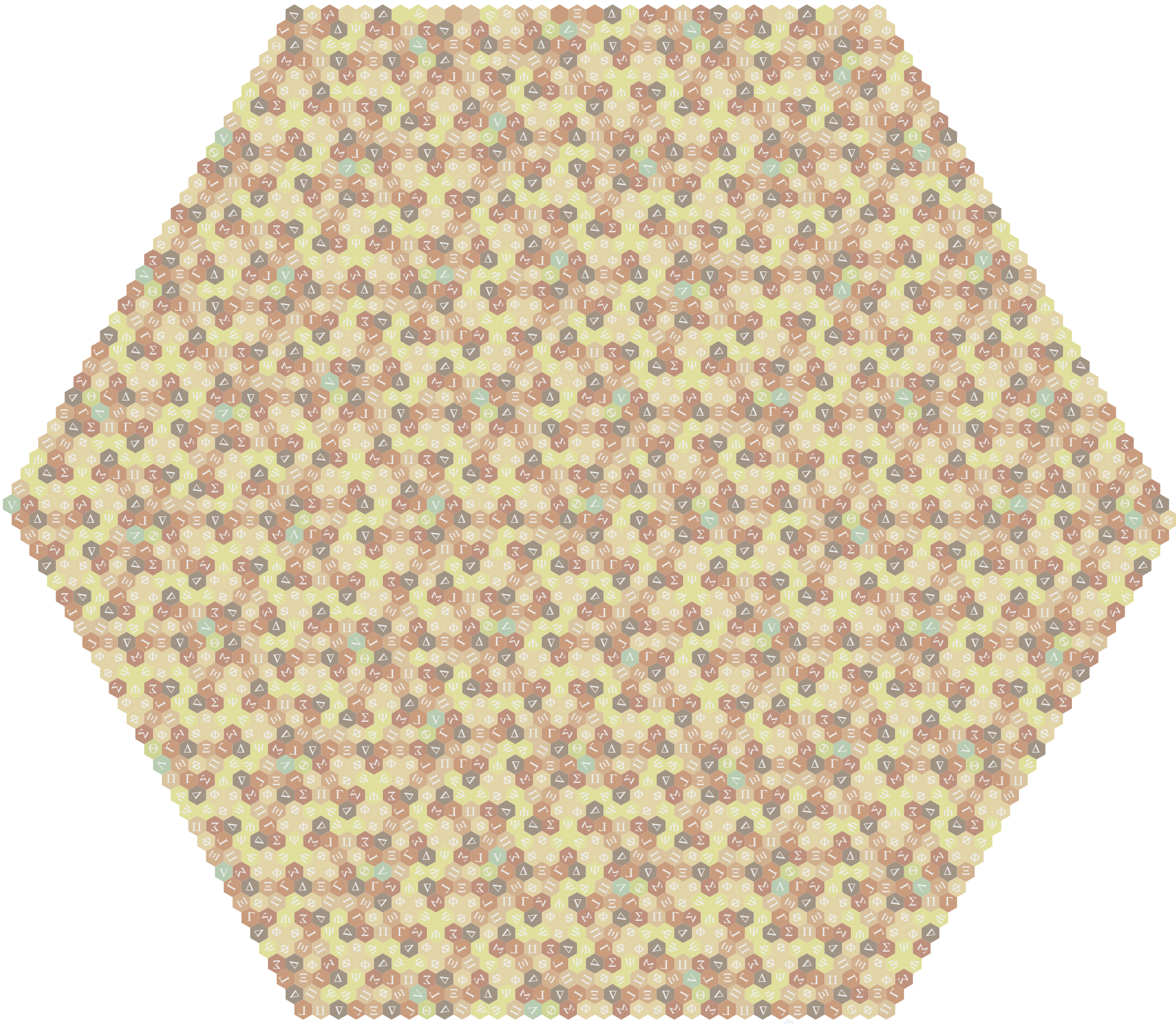

Figure 5.2: A large patch of marked hexagons.

the vertices of the supertile (this is the step that requires 5-patches, to assign those neighbouring hexagons to their own supertiles, rather than working with k-patches for some $k < 5$). That unique choice of the locations of the vertices is the one shown in Figure 5.1. □

Theorem 5.1 allows us to compose any tiling by marked hexagons into a combinatorially equivalent tiling by supertiles. By induction, we can then iterate this composition process to any number of steps, gathering level-1 supertiles into level-2 super-supertiles, and so on, yielding level-n supertiles for all n. Every marked hexagon in a tiling is therefore nested in a unique way within an infinite hierarchy of level-n supertiles. It follows that any tiling by the marked hexagons is non-periodic, which implies the non-periodicity first of tilings by copies of T7H and T8H, then of homochiral tilings by hats and turtles, and finally of tilings by Spectres, all through combinatorial equivalence.

To complete a proof of aperiodicity, we must also show that any of these families of shapes actually admits tilings. Here we need only "reverse" the composition process to obtain substitution rules that replace each marked hexagon in Figure 4.2 by the corresponding supertile in Figure 5.1. These rules are well-defined, in the sense that they may be iterated to produce level-n supertiles for every $n \geqslant 0$, where the supertiles are considered as combinatorial patches (collections of marked hexagons with associated adjacency relations) rather than geometric ones (marked hexagons in particular positions and orientations in the plane). We can therefore construct combinatorial patches of marked hexagons of any size; Figure 5.2 gives an example of a large patch. From any such patch we can derive a combinatorial patch of Spectres, which in the limit implies the existence of an unbounded combinatorial patch. Because of the consistency of edge labels and shapes of boundary segments in Figure 4.1, and the consistency of the relation between the orientation of a boundary segment and that of the corresponding edge (as noted at the end of Section 4), every combinatorial edge and vertex arrangement that arises in that patch can be realized geometrically, and so we may apply the geometry of the marked hexagons, or the geometry of the Spectre clusters, to any combinatorial patch to construct a geometric tiling of the plane [GS09, Lemma 1.1]. Thus the Spectre tiles the plane, which completes the proof of Theorem 2.2: the Spectre is a strictly chiral aperiodic monotile.

Most of our aperiodicity proof for the Spectre relies not on the Spectres themselves, but on the marked clusters defined in Figure 4.1. It is convenient to position the proof at this level: the clusters have the comparatively simple combinatorics of regular hexagons, and our labelled edges make it easier to verify that supertiles and clusters have combinatorially equivalent matching rules. However, it is also possible to take a step backwards and annotate the Spectres themselves with these same markings. Figure 5.3 shows a Mystic and eight Spectres, with the same tile and edge markings as the associated marked hexagons. The tiling substitution rules on marked hexagons apply equally to these marked Spectres. Erasing the markings restores the Spectre tiling substitution rules of Figure 2.1. Thus the Spectre enforces those substitution rules directly.

6. Conclusion

Many problems in tiling theory depend implicitly or explicitly on an initial decision of when to consider two tiles in a tiling "the same". In the Euclidean plane, if this decision is not otherwise articulated then we assume two tiles are the same if they can be brought into coincidence through any planar isometry. In this paper we first answer the einstein problem in a world where sameness is restricted to orientation-preserving isometries. The polygon $\mathrm{Tile}(1,1)$ is a *weakly chiral aperiodic monotile*: a shape that tiles aperiodically if only translations and rotations are permitted. Then, by modifying the edges of that polygon, we obtain a class of shapes called Spectres, which are *strictly chiral aperiodic monotiles*: they tile aperiodically using tiles of a single handedness, even when reflections are allowed.

Other variations of the einstein problem can be posed in which tiles are restricted to images under any given group of isometries. Each such problem comes in weak and strict forms, as above. In the weak case, we ask whether a shape's tilings are non-periodic if tilings are restricted by fiat to the given isometries, even if it would tile periodically when all isometries are allowed. In the strict case we require that by design, a shape does not admit any tilings that use isometries outside the group.

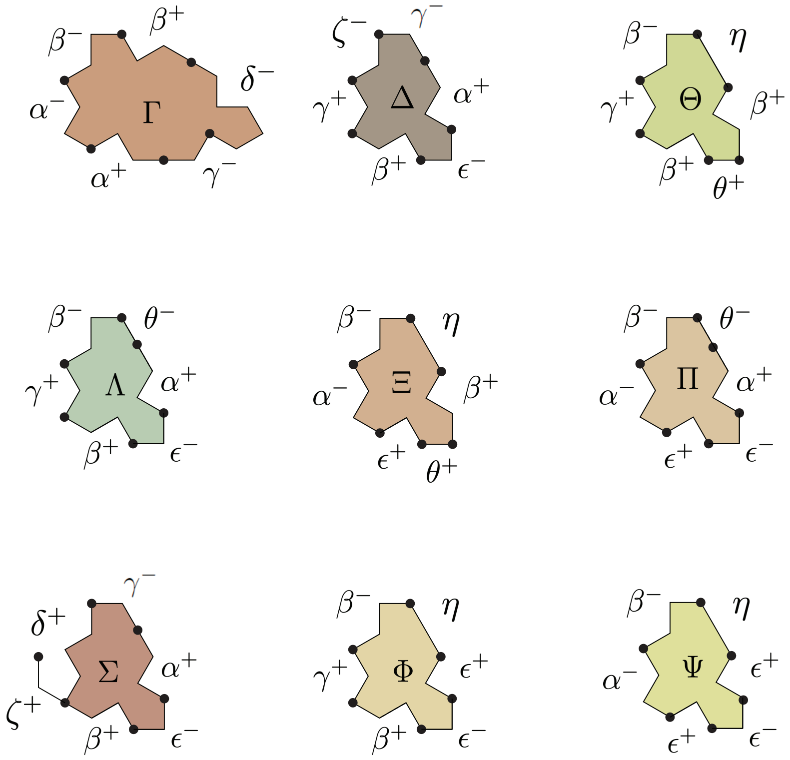

Figure 5.3: Marked Spectres that are combinatorially equivalent to the marked hexagons of Figure 4.2 and to the clusters of Figure 4.1. The Σ tile has additional line segments drawn outside the tile, which should be interpreted as the boundary of part of the tile with zero thickness; the right-hand side of those segments forms part of the δ^+ edge, while the left-hand side is the ζ^+ edge (similar segments arise in boundaries of the clusters forming metatiles for the hat [SMKGS24, Section 4]). The labelled clusters define substitution rules on the marked Spectres that are combinatorially equivalent to the substitution rules on the marked hexagons, shown in Figure 5.1.

If we restrict our attention to translations alone, then the work of Girault-Beauquier and Nivat [GBN91] and Kenyon [Ken92, Ken93, Ken96] shows that a topological disk with a tiling by translation cannot be an aperiodic monotile. However, Greenfeld and Tao [GT23, GT24] and Greenfeld and Kolountzakis [GK23] showed that translational aperiodic monotiles do exist in sufficiently high dimensions. The next simplest case is where $180°$ rotations are allowed along with translations. Schattschneider [SF88, Problem 18.E1] posed the question of whether any tile admitting such a tiling of the plane also satisfies the Conway criterion [Gar75, Sch80], implying that it must be isohedral.

Our work presents a single family of strictly chiral aperiodic monotiles, which are all essentially the same up to trivial modifications of tiling edges. It would be interesting to search for other weakly chiral or strictly chiral einsteins. It would be particularly worthwhile to find (or disprove the existence of) an aperiodic monotile with bilateral reflection symmetry, a shape for which chirality becomes moot.

We showed in Lemma 2.1 that we can replace the 14 edges of $\mathrm{Tile}(1,1)$ by suitably oriented copies of any smooth path to construct a Spectre, provided the replacement results in a tile with a non-self-intersecting boundary. In fact, the construction works generally for C^1 paths, which suffice to force the vertices of the original polygon to meet each other in tilings. However, we leave open the question of whether some other paths (e.g., piecewise-linear paths) might permit different tilings, or whether all non-trivial choices of path produce valid Spectres.

As part of our proof of aperiodicity, we derived the chiral marked hexagons of Figure 4.2. These hexagons are meant to encode the combinatorics of clusters of Spectres, but they display interesting properties of their own that may be worthy of further study. Figure 5.2 hints at emergent patterns in tilings by marked hexagons.

Some of the hexagons of Figure 4.2 have very similar arrangements of markings; removing the distinctions between some of the edge labels can produce a smaller set of marked hexagons without reducing the set to a single trivially marked tile. In particular, combining Θ, Ξ, Φ and Ψ into a single marked hexagon, and combining Λ and Π into another marked hexagon, yields a smaller set of five marked hexagons, which preliminary computations suggest might also be aperiodic. It would be of interest to understand what small aperiodic sets of chiral marked hexagons are possible with this style of edge markings, similar to the work of Jeandel and Rao [JR21] on small aperiodic sets of Wang tiles.

Acknowledgements

Special thanks to Yoshiaki Araki, whose artistic explorations with the equilateral (or nearly equilateral) member of the hat-turtle continuum first inspired us to consider the tiling-theoretic properties of the Spectre in detail. Thanks also to Lucy Birkinshaw for pointing out an error in one of the figures, to Jeff Wilson, Dave Keenan, and Doug Blumeyer for suggesting improvements to the terminology used in the article.

A. Constructing Spectre tilings

The methods used in this paper to prove the aperiodicity of $\mathrm{Tile}(1,1)$ and Spectres rely heavily on layers of combinatorial equivalence. Because these methods do not require geometrically rigid information about the locations of tiles, we emerge at the end without a practical algorithm for drawing patches of tiles. Such an algorithm would of course be useful for visualization and artistic experimentation, and so in this appendix we provide one. This algorithm also forms the basis for an interactive browser-based tool for constructing patches of Spectres, available at `cs.uwaterloo.ca/~csk/spectre/`.

In Section 4 we show, through a computer-assisted analysis of cases, that any homochiral tiling by hats and turtles must consist of a sparse arrangement of turtles surrounded by hats, or sparse hats surrounded by turtles. We then show that the sparse turtle tiling can be composed into copies of two clusters, named T7H ("turtle and seven hats") and T8H ("turtle and eight hats"), illustrated in Figure 4.3. Working backwards through the equivalence described in Section 3, from T7H and T8H we can recover corresponding clusters of eight and nine Spectres. Figure 2.1

introduced these clusters, rotated so that each contains a single "odd" Spectre (shaded in dark green) surrounded by "even" Spectres. The odd Spectres play a role similar to that played by reflected hats in the hat tiling (except of course that they are not reflected): they form a sparse subset, and each one is surrounded by a congruent arrangement of neighbours.

These two clusters also form the basis for the substitution rules shown in Figure 2.1. A corollary of the main line of the paper is that these substitution rules can indeed produce patches of Spectres of any size. We can also use these rules to develop a practical drawing algorithm. As shown in Figure 2.1, we combine one even and one odd Spectre into a compound called a "Mystic", and define substitution rules that replace Spectres and Mystics with clusters of reflected tiles.

These rules are combinatorial—although they show the cluster that replaces each Spectre or Mystic in a growing patch, they say nothing about the exact location of that cluster. To make the rules precise, we identify four "key points" on the boundary of the Spectre. We then use those points to determine the translations of the Spectres and Mystics in substituted clusters, as well as next-generation key points on the boundaries of those clusters. These new key points are equivalent to the originals, in that they can drive the same process of snapping clusters together into superclusters and the choice of their key points, and so on through any number of generations.

We determined the locations of these key points through manual experimentation. We chose points on the boundaries of T7H and T8H where three or more clusters meet, in such a way that every cluster in a supercluster is connected to every other through a path of neighbouring clusters with coincident key points (thus fully determining the necessary translations mentioned above). By comparing the arrangement of Spectres in a cluster with the arrangement of clusters in a supercluster, we could then deduce the corresponding locations of key points on individual tiles. In hindsight, the key points used here are subsets of the marked dots in Figures 4.1 and 5.1. They are minimal in the sense that we could have begun by distinguishing just two of the marked dots as key points in each of the nine marked Spectres of Figure 5.3, and propagated those pairs of dots to the nine cluster types and their superclusters. Each of those pairs consists of two of the four key points used here; operating with all four key points at every stage avoids the complexity of tracking nine cluster types.

The construction is illustrated in Figure A.1. We begin by marking four of the Spectre's vertices as key points in Step 1. In the figure we draw the quadrilateral joining the points for visualization purposes only; it is not explicitly needed in the construction. We can then build a Spectre cluster as shown in Step 2a. Note that the sets of key points associated with the Mystic and Spectres form a closed chain: each tile shares exactly two key points with neighbouring tiles. We can therefore construct the cluster by placing one Spectre at any desired location and orientation as a starting point, and then gluing on neighbours by following the chain. At each step we rotate the tile to be placed so it has the correct orientation relative to its predecessor as shown in Step 2a, and compute the translation needed to bring the associated key points of the neighbouring tiles into coincidence. In Step 2a, four key points belonging to different Spectre tiles are indicated with arrows; these become the key points of the Spectre cluster itself in Step 2b. Note that we can also easily define the Mystic cluster by deleting a single even Spectre, as shown above the Spectre cluster.

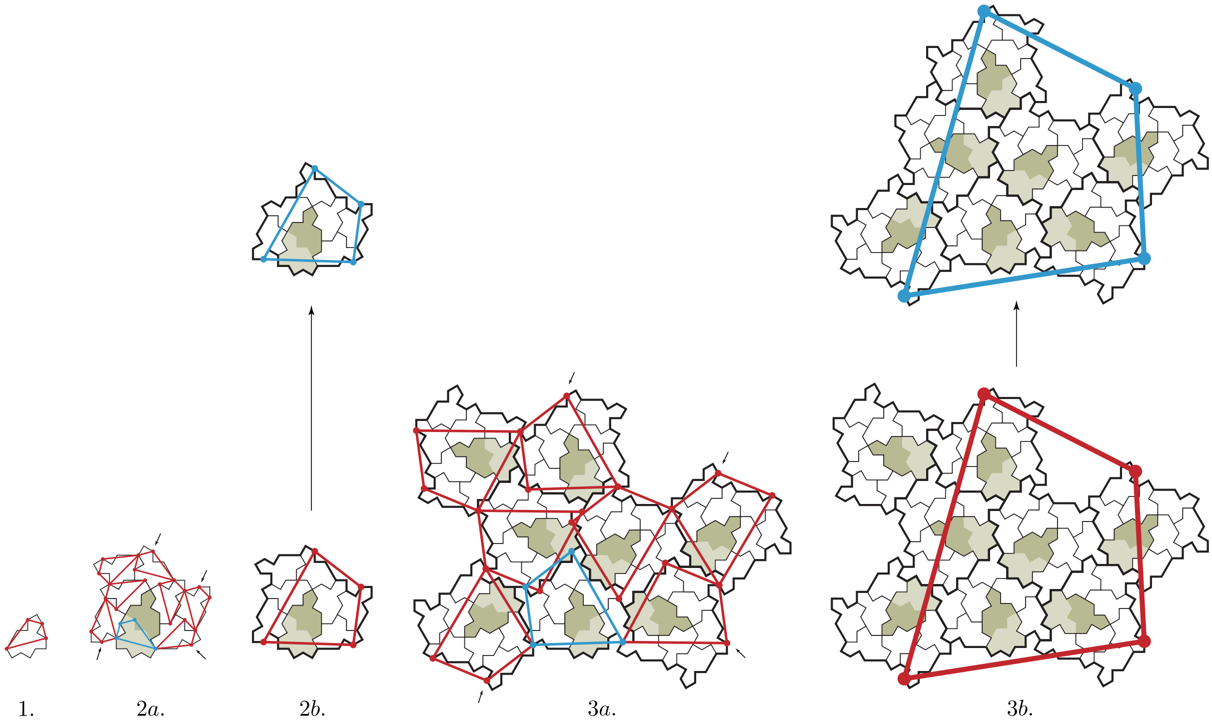

Figure A.1: A visualization of the algorithm for iterating Spectre and Mystic cluster construction. A single Spectre is marked with four "key points" (Step 1), which guide the placement of Spectres and Mystics in the construction of clusters (Step 2a), as well as the choice of key points for those clusters (Step 2b). The process may then be repeated to build superclusters (Steps 3a and 3b), super-superclusters, and so on. Spectre clusters and Mystic clusters are marked with red points and blue points, respectively.

We can then repeat this process, assembling Spectre and Mystic clusters into a Spectre supercluster (Step 3a), and extracting the supercluster's key points (Step 3b) as a subset of the key points of the clusters that make it up, as was done to compute the key points of those clusters. By deleting one of the placed clusters, we also construct the Mystic supercluster. This construction can be iterated as many times as desired to define a patch of Spectres of any size. Figure 2.2 shows five levels of Spectre clusters. Notice that at each level of this construction, individual tiles have the opposite handedness of those in the previous level.

As with the metatiles used in the analysis of the hat [SMKGS24], these substitution rules cannot be expressed as a set of rigid motions and uniform scalings that pack tiles into supertiles. This behaviour is evident from the fact that the quadrilaterals in Figure A.1 are not related by similarities. However, also like metatiles, they quickly converge to limit shapes.

References

[AGS92] Robert Ammann, Branko Grünbaum, and G. C. Shephard. Aperiodic tiles. *Discrete Comput. Geom.*, 8:1–25, 1992. doi:10.1007/BF02293033.

[Gar75] Martin Gardner. More about tiling the plane: the possibilities of polyominoes, polyiamonds, and polyhexes. *Scientific American*, 233(2):112–115, 1975. URL: https://www.jstor.org/stable/24949870.

[GBN91] D. Girault-Beauquier and M. Nivat. Tiling the plane with one tile. In *Topology and category theory in computer science (Oxford, 1989)*, Oxford Sci. Publ., pages 291–333. Oxford Univ. Press, New York, 1991.

[GK23] Rachel Greenfeld and Mihail N. Kolountzakis. Tiling, spectrality and aperiodicity of connected sets. 2023. arXiv:2305.14028.

[Gre07] Marvin J. Greenberg. *Euclidean and Non-Euclidean Geometries: Development and History*. W.H. Freeman, fourth edition, 2007.

[GS99] Chaim Goodman-Strauss. Aperiodic hierarchical tilings. In *Foams and emulsions (Cargèse, 1997)*, volume 354 of *NATO Adv. Sci. Inst. Ser. E: Appl. Sci.*, pages 481–496. Kluwer Acad. Publ., Dordrecht, 1999. doi:10.1007/978-94-015-9157-7_28.

[GS09] Chaim Goodman-Strauss. Regular production systems and triangle tilings. *Theoret. Comput. Sci.*, 410(16):1534–1549, 2009. doi:10.1016/j.tcs.2008.12.012.

[GS16] Branko Grünbaum and G.C. Shephard. *Tilings and Patterns*. Dover, second edition, 2016.

[GT23] Rachel Greenfeld and Terence Tao. Undecidable Translational Tilings with Only Two Tiles, or One Nonabelian Tile. *Discrete Comput. Geom.*, 70(4):1652–1706, 2023. doi:10.1007/s00454-022-00426-4.

[GT24] Rachel Greenfeld and Terence Tao. A counterexample to the periodic tiling conjecture. *Ann. of Math. (2)*, 200(1):301–363, 2024. doi:10.4007/annals.2024.200.1.5.

[Hop26] H. Hopf. Zum Clifford-Kleinschen Raumproblem. *Math. Ann.*, 95:313–339, 1926. doi:10.1007/BF01206614.

[JR21] Emmanuel Jeandel and Michaël Rao. An aperiodic set of 11 Wang tiles. *Adv. Comb.*, (1):1–37, 2021. doi:10.19086/aic.18614.

[Kap22] Craig S. Kaplan. Heesch numbers of unmarked polyforms. *Contributions to Discrete Mathematics*, 17(2):150–171, 2022. doi:10.55016/ojs/cdm.v17i2.72886.

[Ken92] Richard Kenyon. Rigidity of planar tilings. *Invent. Math.*, 107:637–651, 1992. doi:10.1007/BF01231905.

[Ken93] Richard Kenyon. Erratum: “Rigidity of planar tilings”. *Invent. Math.*, 112(1):223, 1993. doi:10.1007/BF01232432.

[Ken96] Richard Kenyon. A group of paths in $\mathbb{R}^2$. *Trans. Amer. Math. Soc.*, 348(8):3155–3172, 1996. doi:10.1090/S0002-9947-96-01562-0.

[Mye19] Joseph Myers. Polyomino, polyhex and polyiamond tiling, 2000–2019. Accessed: February 19th, 2023. URL: https://www.polyomino.org.uk/mathematics/polyform-tiling/.

[Sch80] Doris Schattschneider. Will it tile? Try the Conway criterion! *Math. Mag.*, 53(4):224–233, 1980. doi:10.2307/2689617.

[SF88] Marjorie Senechal and George Fleck, editors. *Shaping space: A polyhedral approach*, Design Science Collection. Birkhäuser Boston, Inc., Boston, MA, 1988.

[SMKGS24] David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss. An aperiodic monotile. *Comb. Theory*, 4(1):Paper No. 6, 91, 2024. doi:10.5070/C64163843.

[ST11] Joshua E. S. Socolar and Joan M. Taylor. An aperiodic hexagonal tile. *J. Combin. Theory Ser. A*, 118(8):2207–2231, 2011. doi:10.1016/j.jcta.2011.05.001.

[ST12] Joshua E. S. Socolar and Joan M. Taylor. Forcing nonperiodicity with a single tile. *Math. Intelligencer*, 34(1):18–28, 2012. doi:10.1007/s00283-011-9255-y.