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FOUNDATIONS OF MATROIDS
PART 2: FURTHER THEORY, EXAMPLES, AND COMPUTATIONAL METHODS

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Abstract. In this sequel to "Foundations of matroids - Part 1," we establish several presentations of the foundation of a matroid in terms of small building blocks. For example, we show that the foundation of a matroid M is the colimit of the foundations of all embedded minors of M isomorphic to one of the matroids U^2_4, U^2_5, U^3_5, C_5, C^*_5, $U^2_4 \oplus U^1_2$, F_7, F^*_7, and we show that this list is minimal. We establish similar minimal lists of building blocks for the classes of 2-connected and 3-connected matroids. We also establish a presentation for the foundation of a matroid in terms of its lattice of flats. Each of these presentations provides a useful method to compute the foundation of certain matroids, as we illustrate with a number of concrete examples. Combining these techniques with other results in the literature, we are able to compute the foundations of several interesting classes of matroids, including whirls, rank-2 uniform matroids, and projective geometries. In an appendix, we catalogue various 'small' pastures which occur as foundations of matroids, most of which were found with the assistance of a computer, and we discuss some of their interesting properties.

Keywords. Matroid representation, cross ratio, inner Tutte group, foundations

Mathematics Subject Classifications. 05B35, 12K99

1. Introduction

We assume throughout this introduction that the reader is familiar with the basic theory of matroid representations over pastures, as described for example in [BL25a]. We refer the reader to the detailed introduction to [BL25a] for the definitions of, and motivation for, some of the concepts

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mentioned below (including pastures, the foundation of a matroid, and universal cross-ratios); see also Section 2 and Section 3 below for a brief summary.

In [BL25a], the authors initiated a systematic study of the *foundation* of a matroid M, an algebraic object canonically attached to M which governs the representability of M over arbitrary pastures. In particular, the foundation F_M determines the set of projective equivalence classes of representations of M over partial fields, as well as the set of reorientation classes of orientations of M. More precisely, for any pasture P, the set of (weak) P-representations of M, modulo rescaling equivalence, is canonically identified with the set of pasture homomorphisms from F_M to P. Some advantages of the foundation over the earlier concepts of "inner Tutte group", due to Dress and Wenzel, or "universal partial field", due to Pendavingh and van Zwam, include:

- Unlike the inner Tutte group, the foundation also has an additive structure rather than just a multiplicative one. This additive structure is crucial for determining the representations of M.
- If M is not representable over any field, the universal partial field does not exist, but the foundation of M is defined for every matroid M. And it carries information about representations of M over interesting pastures such as the signed or tropical hyperfields, in addition to partial fields.
- Unlike both the inner Tutte group and universal partial field, the foundation of M can be characterized intrinsically through a universal property (as the unique object representing a certain functor). Both the inner Tutte group and universal partial field, by contrast, are defined in terms of generators and relations; for the foundation, such characterizations are merely descriptions.
- The category of pastures is much more flexible and robust than the categories of fuzzy rings or partial fields; for example, it admits arbitrary limits and colimits and has both an initial and final object. In particular, one has a tensor product operation which is lacking in the earlier theories, but plays an important role in the representation theory of matroids.

In [BL25a], we made use of Tutte's homotopy theory and the companion results of Gelfand–Rybnikov–Stone to give an explicit presentation for F_M by generators ("universal cross ratios") and relations (the "GRS relations"), as well as a formula expressing F_M as the colimit of F_N over an explicit and universal set of "small" embedded minors N of M. In the case of ternary matroids, or more generally matroids without U_5^2 or U_5^3 minors (which we refer to as "large uniform minors"), we proved that F_M decomposes as a tensor product of an explicit finite set of basic "building blocks" $\{\mathbb{U},\ \mathbb{D},\ \mathbb{H},\ \mathbb{F}_3,\ \mathbb{F}_2\}$. Using this "structure theorem for matroids without large uniform minors", we were able to give new proofs and generalizations of a number of results in the matroid theory literature, for example the Lee–Scobee theorem that a matroid is both ternary and orientable if and only if it is dyadic.

In the present paper, we continue this study, complementing the results of [BL25a] with new theoretical insights which also serve as computational tools that we apply to numerous concrete examples of interest. We now summarize several of these specific enhancements to the material in [BL25a].

Foundations of direct sums

The following result (Theorem 5.1) on the foundation of the direct sum $M \oplus N$ of two matroids was stated without proof in [BL25a]:

Theorem A. $F_{M\oplus N} = F_M \otimes F_N$.

Fundamental presentations

A minor embedding $N \hookrightarrow M$ induces a morphism $F_N \to F_M$ between the respective foundations. It is shown in [BL25a] that the foundation of M is the colimit of the foundations of all embedded minors $N \hookrightarrow M$ on at most 7 elements. We explore this result in more depth in the present paper, with the goal of making it both more precise and more generalizable.

Let $\mathcal{C}$ be a class of isomorphism types of matroids and $\mathcal{E}_{M,\mathcal{C}}$ the diagram of all embedded minors $N \hookrightarrow M$ with isomorphism type in $\mathcal{C}$ together with all minor embeddings $N \hookrightarrow N'$ between such minors. Taking foundations and the induced morphisms yields a diagram $F(\mathcal{E}_{M,\mathcal{C}})$ of pastures.

It is shown in [BL25a] that F_M is the colimit of $F(\mathcal{E}_{M,\mathcal{C}_0'})$ for a certain list $\mathcal{C}_0'$ of matroids on up to 7 elements, but this list fails to be minimal. We reduce this list to $\mathcal{C}_0 = \{U_4^2,\ U_5^2,\ U_5^3,\ C_5,\ C_5^*,\ U_4^2 \oplus U_2^1,\ F_7,\ F_7^*\}$, call $\mathcal{E}_M = \mathcal{E}_{M,\mathcal{C}_0}$ the *fundamental diagram of* M, and establish the *fundamental presentation of* M (Theorem 6.3):

Theorem B. $F_M = \operatorname{colim} F(\mathcal{E}_M)$.

In fact, $\mathcal{C}_0$ is the unique minimal set such that Theorem B holds for all matroids M.

We also derive from Theorem B a description of F_M in terms of the lattice of flats of M (Theorem 6.3). For this, we establish some technical refinements of the famous "Scum Theorem" which may be of independent interest (Proposition 6.7 and Lemma 6.13). In particular, a minor embedding $N \hookrightarrow M$ defines a sublattice Λ_N of the lattice of flats of M, and the induced morphism $F_N \to F_M$ depends only on the corresponding lattice embedding.

The *fundamental lattice diagram of* M is the diagram $\mathcal{L}_M$ of all sublattices that stem from embedded minors of types U_4^2, U_5^2, U_5^3, C_5, F_7 and F_7^*, together with all inclusions of sublattices. Taking foundations yields the *fundamental lattice presentation* $F(\mathcal{L}_M)$ *of* M (Theorem 6.9):

Theorem C. $F_M = \operatorname{colim} F(\mathcal{L}_M)$.

Theorem C has some useful variants; cf. Theorem 6.11 and Theorem 6.12.

Fundamental types

More generally, we show (Theorem 8.2) that for each class $\mathcal{C}$ of matroids, there is a unique minimal set $\mathcal{C}_0$ of isomorphism classes of matroids in $\mathcal{C}$ such that the foundation of every matroid M in $\mathcal{C}$ is the colimit of the foundations of all embedded minors of M whose isomorphism type is in $\mathcal{C}_0$. We call $\mathcal{C}_0$ the *fundamental type* of $\mathcal{C}$.

We determine the fundamental type for several classes of matroids, with the most interesting examples being the classes of 2-connected matroids (Theorem 8.11) and 3-connected matroids (Theorem 8.16). Using the standard nomenclature from Oxley's book [Oxl92], we have:

Theorem D. *The fundamental type of the class of all* 2*-connected matroids is*

$$\mathcal{C}_0 \;=\; \{U_4^2,\; U_5^2,\; U_5^3,\; C_5,\; C_5^*,\; F_7,\; F_7^*\},$$

i.e. $F_M = \operatorname{colim}\ F(\mathcal{E}_{M,\mathcal{C}_0})$ *for every* 2*-connected matroid* M*.*

Theorem E. *The fundamental type of the class of all* 3*-connected matroids is*

$$\mathcal{C}_0 \;=\; \{U_4^2,\; U_5^2,\; U_5^3,\; W^3,\; Q_6,\; P_6,\; F_7,\; F_7^*\},$$

i.e. $F_M = \operatorname{colim}\ F(\mathcal{E}_{M,\mathcal{C}_0})$ *for every* 3*-connected matroid* M*.*

The proofs of these results are quite non-trivial: Theorem D requires a detailed analysis of the minimal 2-connected extensions of $U_4^2 \oplus U_2^1$, which is achieved through the Cunningham–Edmonds tree decomposition for 2-connected matroids and a rather elaborate induction; Theorem E uses a strengthening of Seymour's splitter theorem due to Bixby–Coullard and an exhaustive computer search on matroids up to 8 elements.[1]

As with Theorem C, every fundamental type has a corresponding fundamental type of sublattices; cf., for example, Theorem 8.18.

Map of examples

We illustrate how to use the various fundamental presentations of the foundation in concrete examples. The list in Table 1.1 shows which fundamental presentation of F_M enters which example.

Table 1.1: List of which fundamental presentation appears in which example.

fundamental presentation	ref. to result	matroid	ref. to example
by cross ratios	Theorem 3.5	U_5^2	Section 4.3
$F_M = \operatorname{colim}\ F(\mathcal{E}_M)$	Theorem B / Theorem 6.3	Q_6	Section 7.1
$F_M = \operatorname{colim}\ F(\mathcal{L}_M)$	Theorem C / Theorem 6.9	$AG(2,3)\backslash e$	Section 7.2
$F_M = \operatorname{colim}\ F(\mathcal{E}_M^{(2)})$	Theorem D / Theorem 8.11	whirls	Section 9.1
$F_M = \operatorname{colim}\ F(\mathcal{E}_M^{(3)})$	Theorem E / Theorem 8.16	F_7^-	Section 9.2
$F_M = \operatorname{colim}\ F(\mathcal{L}_M^{(3)})$	Theorem 8.18	P_7	Section 9.3
$F_M = \operatorname{colim}\ F(\mathcal{L}_M^{(\leqslant 3)})$	Theorem 6.11	T_8	Section 9.4

Moreover, we draw on results from the literature to determine that the foundation of the uniform matroid U_{k+3}^2 is Semple's k-regular partial field (Proposition 4.4), the foundation of the d-dimensional projective space $PG(d,q)$ is the finite field $\mathbb{F}_q$ (Proposition 4.7), and the foundation of any non-Desarguesian projective plane is $\mathbb{K}$ (Proposition 4.9).

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[1] While a number of results in this paper were verified using the Macaulay2 package Pastures, the explicit determination of the fundamental type for 3-connected matroids is the only place in the paper (outside the Appendix) which actually requires a computer-assisted proof.

A note on the examples appearing in this paper

While our techniques can be used, in principle, to algorithmically compute the foundation of any matroid, the complexity of these computations increases exponentially with the size of the matroid. So for "large" matroids, such direct calculations are best done with the aid of a computer. This has been systematically implemented by Justin Chen and the third author, who have written the Macaulay2 package PASTURES and an accompanying paper [CZ23].

Rather than relying exclusively on computer calculations in order to present interesting examples, we have decided to work out some computations of foundations "by hand" in this paper, partly in order to illustrate some of the different theoretical techniques presented here. For this, we systematically make use of tactical shortcuts, leaning on known results about the representability of the above-mentioned matroids in order to make the calculations human-readable. A typical computation of the foundation F_M of a matroid M in our list of examples might proceed as follows:

1. We leverage known results about the representability of M to gain information about which pastures can possibly appear as F_M. In many cases, M is without large uniform minors, which allows us to apply the structure theorem for this class of matroids.
2. We write down the "fundamental diagram" of M and observe, in the cases of interest, that it is connected. This implies that F_M does not decompose into the tensor product of several non-trivial factors.
3. We extract sufficient relations between the cross ratios of M from the fundamental presentation of F_M to narrow down the possibilities of F_M to a unique pasture, which concludes the proof.

Inductive approach.

We also sometimes employ an inductive approach to computing the foundation of a matroid in terms of the foundations of its minors. This approach is illustrated, for example, in the computations of the foundations of F_7^- (Proposition 9.4) and T_8 (Proposition 9.9). The inductive approach also underlies the theoretical fact that we can compute the fundamental type of a class of matroids in terms of minimal extensions of the matroids appearing in certain other fundamental types (Proposition 8.3).

At the end of Section 8.1, we formulate some open problems concerning fundamental types.

List of foundations

In Appendix A, we describe some interesting foundations, which we found partly based on theory and partly by using the Macaulay2 package PASTURES developed by Chen and the third author in [CZ23].

By Theorem 5.1 and a result whose proof appears in [BLWZ24], the foundation of a direct sum or 2-sum decomposes into a tensor product of the foundations of the 3-connected components of the matroid. We therefore concentrate on foundations of 3-connected matroids.

There are two infinite families of foundations discussed in Appendix A: the foundations of uniform matroids, which by Proposition 4.4 include Semple's k-regular partial fields, and finite fields, which by Proposition 4.7 are foundations of projective geometries.

We continue by describing various other pastures which occur as foundations, including the foundations of all 3-connected matroids on up to 8 elements with at most 7 hexagons. We see that many of the partial fields described in [PvZ10a] occur as the foundation of a matroid, while others occur as the universal partial field but not necessarily the foundation.

We conclude the appendix with some remarks on foundations of non-representable matroids.

2. Background

In this section we give a quick reminder of some basic notions from [BL25a], such as pastures, matroid representations, universal cross ratios, and the foundation of a matroid. We refer the reader to [BL25a] and [BL25b] for a more extensive discussion of these notions.

2.1. Pastures

A *pointed monoid* is a (multiplicatively written and commutative) monoid A with neutral element 1 and absorbing element 0, i.e. $1 \cdot a = a$ and $0 \cdot a = 0$ for all $a \in A$. We denote by $P^\times$ the subgroup of invertible elements. A *morphism of pointed monoids* is a multiplicative map $f : A_1 \to A_2$ that preserves 1 and 0. We denote by $\mathrm{Sym}_3(A)$ the quotient of A^3 by the permutation action of S_3 on the coefficients of $(a, b, c) \in A^3$. We denote the equivalence classes in $\mathrm{Sym}_3(A)$ by $a + b + c = [(a, b, c)]$.

A *pasture* is a pointed monoid P such that $P^\times = P - \{0\}$ together with an involution $a \mapsto -a$ and a nonempty subset N_P of $\mathrm{Sym}_3(P)$, called the *null set of* P, such that:

1. For all $a + b + c \in N_P$ and $d \in P$ we have $da + db + dc \in N_P$.
2. $a + b + 0 \in N_P$ if and only if $b = -a$.

It follows that $-0 = 0$, $(-1)^2 = 1$, and $-(-a) = a$.

A *pasture morphism* is a morphism $f : P_1 \to P_2$ of pointed monoids such that $f(a) + f(b) + f(c) \in N_{P_2}$ if $a + b + c \in N_{P_1}$. It follows that $f(-a) = -f(a)$. We denote the category of pastures by $\mathrm{Pastures}$.

We write 0 for $0 + 0 + 0$ and $a + b$ for $a + b + 0$. Note that if $a + 0 \in N_P$, then $a = 0$. We also write $a + b - c$ for $a + b + (-c)$ and $a + b = c$ if $a + b - c \in N_P$. Note also that the inversion $a \mapsto -a$ is determined by the null set N_P.

2.1.1 Examples

Some first examples of importance are:

- the *regular partial field* $\mathbb{F}_1^\pm = \{0, 1, -1\}$ with null set $N_{\mathbb{F}_1^\pm} = \{0,\ 1 + (-1)\}$, which is an initial object in $\mathrm{Pastures}$;

- the *Krasner hyperfield* $\mathbb{K} = \{0, 1\}$ with null set $N_{\mathbb{K}} = \{0,\ 1+1,\ 1+1+1\}$, which is a terminal object in $\mathrm{Pastures}$;
- the field with two elements $\mathbb{F}_2 = \{0, 1\}$ with null set $N_{\mathbb{F}_2} = \{0,\ 1+1\}$;
- the field with three elements $\mathbb{F}_3 = \{0, 1, -1\}$ with null set
$$N_{\mathbb{F}_3} \ = \ \{0,\ 1+(-1),\ 1+1+1,\ (-1)+(-1)+(-1)\};$$
- the *sign hyperfield* $\mathbb{S} = \{0, 1, -1\}$ with null set
$$N_{\mathbb{S}} \ = \ \{0, 1+(-1),\ 1+1+(-1),\ 1+(-1)+(-1)\}.$$

In fact, every partial field and every hyperfield defines a pasture. Given a partial field (G, R), where R is a ring and G is a subgroup of $R^\times$ that contains -1, we define the associated pasture as $P = G \cup \{0\}$ with null set $N_P = \{a+b-c \mid a+b = c \text{ in } R\}$. Given a hyperfield F, we define the associated pasture as $P = F$ (as a multiplicative monoid) with null set $N_P = \{a+b-c \mid c \in a \boxplus b \text{ in } F\}$. In this text, we consider partial fields and hyperfields as pastures. Note that the respective notions of morphisms for partial fields and for hyperfields coincide with the corresponding notion of pasture morphisms.

2.1.2 Free algebras and quotients

Given a pasture P and indeterminants x_i, indexed by $i \in I$, the *free algebra* $P\langle x_i \mid i \in I\rangle$ is the pasture whose unit group is the product of $P^\times$ with the free abelian group $\langle x_i \mid i \in I\rangle$ generated by the x_i, and whose null set consists of all elements of the form $da + db + dc$ where $a+b+c \in N_P$ and $d \in P\langle x_i \mid i \in I\rangle$.

Given a pasture P and a subset $\{a_j + b_j + c_j \mid j \in J\}$ of $\mathrm{Sym}^3(P)$, where we assume that $a_i, b_i \in P^\times$, we define $P /\!\!/ \langle\!\langle a_j + b_j + c_j \mid j \in J\rangle\!\rangle$ as the following pasture: its monoid is the quotient monoid $\overline{P} = P/\sim$ of P by the equivalence relation generated by the relations $da_j \sim -db_j$ for all $d \in P$ and all $j \in J$ for which $c_j = 0$. We write $[a]$ for the class of $a \in P$ in $\overline{P}$. The null set of $P /\!\!/ \langle\!\langle a_j + b_j + c_j \mid j \in J\rangle\!\rangle$ consists of all expressions of the form $[a] + [b] + [c]$ with $a + b + c \in N_P$, together with all expressions $[da_j] + [db_j] + [dc_j]$ for $d \in P$ and $j \in J$.

This allows us to write every pasture in the form
$$P \ = \ \mathbb{F}_1^\pm(x_i \mid i \in I) /\!\!/ \langle\!\langle a_j + b_j + c_j \mid j \in J\rangle\!\rangle$$
by choosing generators and relations. Indeed, to see that this is always possible, note that for any pasture P we have
$$P \ \cong \ \mathbb{F}_1^\pm(x_a \mid a \in P) /\!\!/ \langle\!\langle S\rangle\!\rangle,$$
where S consists of all binary relations $x_{a_1} \cdots x_{a_k} - 1 = 0$ corresponding to multiplicative relations of the form $a_1 \cdots a_k = 1$ in P, together with all ternary relations $x_a + x_b + x_c = 0$ corresponding to $a+b+c \in N_P$. It is easy to see that the map sending a to x_a is an isomorphism of pastures.

The following pastures, which we present via generators and relations, all play an important role in the theory of partial-field representations of matroids:

- the *near-regular partial field* $\mathbb{U} = \mathbb{F}_1^\pm(x,y) /\!\!/ \langle\!\langle x+y-1 \rangle\!\rangle$;
- the *dyadic partial field* $\mathbb{D} = \mathbb{F}_1^\pm(x) /\!\!/ \langle\!\langle x-1-1 \rangle\!\rangle$;
- the *hexagonal partial field* $\mathbb{H} = \mathbb{F}_1^\pm(\zeta_6) /\!\!/ \langle\!\langle \zeta_6^3+1,\ \zeta_6+\zeta_6^{-1}-1 \rangle\!\rangle$;
- the *golden ratio partial field* $\mathbb{G} = \mathbb{F}_1^\pm(x) /\!\!/ \langle\!\langle x^2+x-1 \rangle\!\rangle$.

2.1.3 Categorical properties

The category of pastures is complete and cocomplete, as proven in [Cre21]. For the present purposes, it suffices to understand finite colimits. To start with, we describe the *coproduct*, or *tensor product*, of two pastures P_1 and P_2. As a pointed monoid, the tensor product is

$$P_1 \otimes P_2 \;=\; \{0\} \cup (P_1^\times \oplus P_2^\times) \,/\, \{\pm(1,1)\};$$

we write $a_1 \otimes a_2 = \{(a_1,a_2),(-a_1,-a_2)\}$ for its cosets (if $a_1 \neq 0$ and $a_2 \neq 0$), and define $a_1 \otimes 0 = 0 = 0 \otimes a_2$. Its null set is

$$\begin{aligned} N_{P_1\otimes P_2} \;=\;& \Big\{ a\otimes d + b\otimes d + c\otimes d \in \mathrm{Sym}_3(P_1\otimes P_2) \,\Big|\, a+b+c \in N_{P_1},\ d \in P_2 \Big\} \\ \cup\;& \Big\{ d\otimes a + d\otimes b + d\otimes c \in \mathrm{Sym}_3(P_1\otimes P_2) \,\Big|\, d \in P_1,\ a+b+c \in N_{P_2} \Big\}. \end{aligned}$$

The tensor product comes with the canonical inclusions $\iota_1 : P_1 \to P_1 \otimes P_2$ and $\iota_2 : P_2 \to P_1 \otimes P_2$ that are defined by $\iota_1(a) = a \otimes 1$ and $\iota_2(b) = 1 \otimes b$, respectively. It satisfies the universal property of the coproduct: for every pair of morphisms $f_1 : P \to Q$ and $f_2 : P_2 \to Q$ into a pasture Q, there is a unique morphism $f : P_1 \otimes P_2 \to Q$ such that $f_1 = f \circ \iota_1$ and $f_2 = f \circ \iota_2$.

The tensor product $P_1 \otimes \cdots \otimes P_n$ of pastures $P_1,\dots,P_n$ is their categorical coproduct. It can be derived from the pairwise tensor product as

$$P_1 \otimes \cdots \otimes P_n \;=\; (((P_1 \otimes P_2) \otimes P_3) \cdots \otimes P_{n-1}) \otimes P_n.$$

When $n=0$, we define the empty tensor product to be the initial pasture $\mathbb{F}_1^\pm$.

The *colimit* of a finite diagram of $\mathcal{D}$ of pastures $\{P_i\}_{i\in I}$ and pasture morphisms $\{f_j : P_{s_j} \to P_{t_j}\}_{j\in J}$ is the quotient

$$\operatorname{colim} \mathcal{D} \;=\; \bigotimes_{i\in I} P_i \;/\!\!/\; \langle\!\langle 1 \otimes \cdots a_{s_j} \cdots \otimes 1 \;-\; 1 \otimes \cdots \underbrace{f_j(a_{s_j})}_{\in\, P_{t_j}} \cdots \otimes 1 \mid j \in J,\ a_{s_j} \in P_{s_j} \rangle\!\rangle$$

of the coproduct $\bigotimes P_i$.

Example 2.1. Consider the diagram

$$\mathcal{D} \;=\; \left(\begin{tikzcd} \mathbb{H} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "f"'] & \mathbb{H} \end{tikzcd} \right)$$

where $\mathbb{H} = \mathbb{F}_1^\pm(\zeta_6) /\!\!/ \langle\!\langle \zeta_6^3+1,\ \zeta_6+\zeta_6^{-1}-1 \rangle\!\rangle$ is the hexagonal partial field and $f : \mathbb{H} \to \mathbb{H}$ is defined by $f(\zeta_6) = \zeta_6^{-1}$. Then

$$\begin{aligned} \operatorname{colim} \mathcal{D} \;=\;& \mathbb{H} \otimes \mathbb{H} \;/\!\!/\; \langle\!\langle \zeta_6 \otimes 1 - 1 \otimes \mathrm{id}(\zeta_6),\ \zeta_6 \otimes 1 - 1 \otimes f(\zeta_6) \rangle\!\rangle \\ & \underset{(\zeta_6\otimes 1 \sim 1 \otimes \zeta_6)}{=} \; \mathbb{H} \;/\!\!/\; \langle\!\langle \zeta_6 - \zeta_6^{-1} \rangle\!\rangle \; \underset{(\zeta_6 \sim \zeta_6^3 \sim -1)}{=} \; \mathbb{F}_3. \end{aligned}$$

2.2. Matroid representations

Throughout the text, M denotes a matroid of rank r with ground set $E = \{1\dots,n\}$. We write $\mathbf{I} = (e_1,\dots,e_s)$ for elements of E^s and $I = |\mathbf{I}|$ for the subset $\{e_1,\dots,e_s\}$ of E. We write $\mathbf{I}f = (e_1,\dots,e_s,f)$ and $If = \{e_1,\dots,e_s,f\}$.

Let P be a pasture. A *weak Grassmann–Plücker function of rank r with values in P* is a function $\Delta : E^r \to P$ such that

(GP0) The support $\{|\mathbf{I}| \ : \ \Delta(\mathbf{I}) \neq 0\}$ of Δ is the set of bases of a rank r matroid on E.

(GP1) Δ is *alternating*, i.e. $\Delta(e_{\sigma(1)},\dots,e_{\sigma(r)}) = \operatorname{sign}(\sigma)\cdot\Delta(e_1,\dots,e_r)$ for all $e_1,\dots,e_r \in E$ and $\sigma \in S_r$; and

(GP2) Δ satisfies the 3-*term Plücker relations*

$$\Delta(\mathbf{J}e_1e_2)\cdot\Delta(\mathbf{J}e_3e_4) \ - \ \Delta(\mathbf{J}e_1e_3)\cdot\Delta(\mathbf{J}e_2e_4) \ + \ \Delta(\mathbf{J}e_1e_4)\cdot\Delta(\mathbf{J}e_2e_3) \ \in \ N_P$$

for all $\mathbf{J} \in E^{r-2}$ and $e_1,\dots,e_4 \in E$.

A *P-representation of M* is a weak Grassmann–Plücker function $\Delta : E^r \to P$ such that $\Delta(\mathbf{I}) \neq 0$ if and only if $|\mathbf{I}|$ is a basis of M.

A trivial but useful observation is that every matroid M has a unique $\mathbb{K}$-representation $\Delta : E^r \to \mathbb{K}$, given by setting $\Delta(\mathbf{I}) = 1$ if $|\mathbf{I}|$ is a basis of M and $\Delta(\mathbf{I}) = 0$ if not. This yields an identification of matroids with weak Grassmann–Plücker functions $\Delta : E^r \to \mathbb{K}$.

We say that M is *P-representable* if it has a P-representation $\Delta : E^r \to P$. This streamlines and extends other notions of representability: M is representable over a partial field (G,R) (in the usual sense) if and only if M is P-representable, where P is the pasture associated with (G,R). In particular, M is regular if and only if M is $\mathbb{F}_1^{\pm}$-representable, M is binary if and only if M is $\mathbb{F}_2$-representable, and M is ternary if and only if M is $\mathbb{F}_3$-representable. Moreover, M is orientable if and only if M is $\mathbb{S}$-representable.

2.3. Universal cross ratios and the inner Tutte group

Let M be a matroid. We denote by Ω_M the collection of all tuples (I,a,b,c,d), where $I \subset E$ is an $(r-2)$-subset and $a,b,c,d \in E$ such that Iac, Iad, Ibc, Ibd are bases of M. We denote by $\Omega_M^{\diamondsuit}$ the subset of *non-degenerate* tuples (I,a,b,c,d), for which also Iab and Icd are bases of M.

Let $\mathbb{T}_M$ be the Tutte group M, which is generated by symbols -1 and $T_{\mathbf{I}}/T_{\mathbf{J}}$ for $\mathbf{I},\mathbf{J} \in E^r$ such that $|\mathbf{I}|$ and $|\mathbf{J}|$ are bases of M; see [Wen91] for details. Let $\mathbb{T}_M^{(0)}$ be the inner Tutte group, which consists of the multi-degree zero elements of $\mathbb{T}_M$.

For $(I,a,b,c,d) \in \Omega_M$, we define the *universal cross ratio* as the element

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_I = \frac{T_{\mathbf{I}ac}T_{\mathbf{I}bd}}{T_{\mathbf{I}ad}T_{\mathbf{I}bc}}$$

of the inner Tutte group $\mathbb{T}_M^{(0)}$ of M, where $\mathbf{I}$ is any ordering of I. Note that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_I$ does not depend on the ordering of $\mathbf{I}$. A fundamental fact is the following:

Proposition 2.2 ([Wen91, Prop. 6.4]). *The inner Tutte group $T_M^{(0)}$ is generated by -1 and the universal cross ratios $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_I$ for $(I,a,b,c,d) \in \Omega_M^{\diamondsuit}$.*

Let Θ_M be the collection of all *modular quadruples of hyperplanes for M*, which are tuples (H_1,H_2,H_3,H_4) of hyperplanes $H_1,\dots,H_4$ of M such that $F = H_1 \cap \dots \cap H_4$ is a flat of corank 2 and such that $F = H_i \cap H_j$ for $i \in \{1,2\}$ and $j \in \{3,4\}$. Let $\Theta_M^{\diamondsuit}$ be the subset of *non-degenerate* quadruples $(H_1,\dots,H_4)$, for which also $F = H_1 \cap H_2 = H_3 \cap H_4$.

Let $\langle - \rangle$ be the closure operator of M. Then the association

$$\begin{array}{rccc} \Psi: & \Omega_M & \longrightarrow & \Theta_M \\ & (I,a,b,c,d) & \longmapsto & \big(\langle Ia\rangle, \langle Ib\rangle, \langle Ic\rangle, \langle Id\rangle\big) \end{array}$$

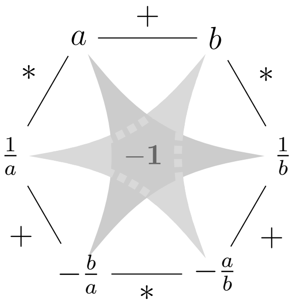
is a surjection, which restricts to a surjection $\Omega_M^{\diamondsuit} \to \Theta_M^{\diamondsuit}$. It follows from [DW89, Lemma 1.4] (see also [BL25a, Prop. 3.6]) that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_I = \left[\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right]_{I'}$ as elements of $\mathbb{T}_M^{(0)}$ if $\Psi(I,a,b,c,d) = \Psi(I',a',b',c',d')$. This allows us to define $\left[\begin{smallmatrix} H_1 & H_2 \\ H_3 & H_4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_I$ whenever $(H_1,H_2,H_3,H_4) \in \Theta_M$ with $(H_1,H_2,H_3,H_4) = \Psi(I,a,b,c,d)$.

2.4. Fundamental elements and hexagons

Let P be a pasture. A *fundamental pair of P* is an ordered pair (a,b) of elements $a,b \in P^\times$ such that $a+b-1 \in N_P$. A *fundamental element of P* is an element $a \in P^\times$ that appears in a fundamental pair (a,b). Every fundamental pair (a,b) defines the set

$$\Xi(a,b) \ = \ \Big\{ \ (a,b), \quad (b,a), \quad (\tfrac{1}{a}, -\tfrac{b}{a}), \quad (-\tfrac{b}{a}, \tfrac{1}{a}), \quad (\tfrac{1}{b}, -\tfrac{a}{b}), \quad (-\tfrac{a}{b}, \tfrac{1}{b}) \ \Big\}$$

of fundamental pairs, which are not necessarily pairwise distinct. We call such a set $\Xi(a,b)$ a *hexagon in P*, which refers to the way of illustrating the involved fundamental elements as



where an edge with label $*$ indicates that its vertices multiply to 1 and an edge with label $+$ indicates that its vertices x and y add up to 1, i.e. $x+y-1 \in N_P$. The vertices of the triangles multiply to -1.

A basic observation ([BL25b, Prop. 3.6]) is that the hexagons of P are in bijective correspondence with $N_P^\times / P^\times$, where $N_P^\times$ is the subset of all terms $a+b+c \in N_P$ with $abc \in P^\times$. Since every element in $N_P - N_P^\times$ is of the form $a-a+0$ for some $a \in P$, the null set N_P is determined by the hexagons in P.

3. The foundation of a matroid

The foundation of a matroid is defined in [BL21], and further applications to the representation theory of matroids are developed in [BL25a] and [BL25b]. In this section, we recall the definition of the foundation and its fundamental presentation in terms of generators and relations, which forms the basis for nearly all results in this paper.

3.1. The foundation

Using [BL21, Cor. 7.13], we can phrase the definition of the foundation of a matroid M as follows.

Definition 3.1. The *foundation of* M is the pasture F_M with unit group $F_M^\times = \mathbb{T}_M^{(0)}$ and whose null set is generated by the elements $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_I + \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]_I - 1$ for all $(I, a, b, c, d) \in \Omega_M^\diamond$.

We can rescale a representation $\Delta : E^r \to P$ of M by an element $t = (d, t_1, \dots, t_n) \in (P^\times)^{n+1}$ via

$$t.\Delta(\mathbf{I}) \;=\; d \cdot \Big(\prod_{e \in |\mathbf{I}|} t_e \Big) \cdot \Delta(\mathbf{I}),$$

which defines a group action of $(P^\times)^{n+1}$ on the set of P-representations of M. We denote by $\mathcal{X}_M(P)$ the set of equivalence classes under this action, which we call the *P-rescaling classes of M*. The characterizing property of F_M (which can also be taken as the *definition* of F_M) is the following:

Theorem 3.2 ([BL21, Cor. 7.28])**.** *Let M be a matroid with foundation F_M, and let P be a pasture. Then there is a canonical bijection* $\operatorname{Hom}(F_M, P) \to \mathcal{X}_M(P)$ *which is functorial in P.*

This has the following immediate consequence, which makes the foundation a useful tool for studying matroid representations:

Corollary 3.3. *The matroid M is P-representable if and only if there is a morphism $F_M \to P$.* □

The foundation behaves well with respect to several natural matroid constructions. For instance, we recall from [BL25a, Prop. 4.8 and 4.9]:

Proposition 3.4. *Let M be a matroid with foundation F_M. Then the foundations of the dual matroid M^* and of the simplification of M are canonically isomorphic to F_M.*

3.2. A presentation of the foundation by cross ratios

We recall the description of the foundation of a matroid in terms of generators and relations from [BL25a, Thm. 4.21]. This result is derived from Gelfand, Rybnikov, and Stone's description of a complete set of relations between the universal cross ratios in [GRS95], which is itself a consequence of Tutte's homotopy theorem; cf. [Tut58].

Theorem 3.5. *Let M be a matroid with foundation F_M. Then*

$$F_M \;=\; \mathbb{F}_1^{\pm}\left\langle \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J \,\middle|\, (J; e_1, \dots, e_4) \in \Omega_M \right\rangle /\!\!/ \, S,$$

where S is defined by the multiplicative relations

$$-1 = 1 \tag{R–}$$

if the Fano matroid F_7 or its dual F_7^* is a minor of M;

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \begin{bmatrix} e_2 & e_1 \\ e_4 & e_3 \end{bmatrix}_J = \begin{bmatrix} e_3 & e_4 \\ e_1 & e_2 \end{bmatrix}_J = \begin{bmatrix} e_4 & e_3 \\ e_2 & e_1 \end{bmatrix}_J \tag{Rσ}$$

for all $(J; e_1, \dots, e_4) \in \Omega_M^{\diamondsuit}$;

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = 1 \tag{R0}$$

for all degenerate $(J; e_1, \dots, e_4) \in \Omega_M$;

$$\begin{bmatrix} e_1 & e_2 \\ e_4 & e_3 \end{bmatrix}_J = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J^{-1} \tag{R1}$$

for all $(J; e_1, \dots, e_4) \in \Omega_M^{\diamondsuit}$;

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J \cdot \begin{bmatrix} e_1 & e_3 \\ e_4 & e_2 \end{bmatrix}_J \cdot \begin{bmatrix} e_1 & e_4 \\ e_2 & e_3 \end{bmatrix}_J = -1 \tag{R2}$$

for all $(J; e_1, \dots, e_4) \in \Omega_M^{\diamondsuit}$;

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J \cdot \begin{bmatrix} e_1 & e_2 \\ e_4 & e_5 \end{bmatrix}_J \cdot \begin{bmatrix} e_1 & e_2 \\ e_5 & e_3 \end{bmatrix}_J = 1 \tag{R3}$$

for all $e_1, \dots, e_5 \in E$ and $J \subset E$ such that each of $(J; e_1, e_2, e_3, e_4)$, $(J; e_1, e_2, e_4, e_5)$ and $(J; e_1, e_2, e_5, e_3)$ is in Ω_M;

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_5} \cdot \begin{bmatrix} e_1 & e_2 \\ e_4 & e_5 \end{bmatrix}_{Je_3} \cdot \begin{bmatrix} e_1 & e_2 \\ e_5 & e_3 \end{bmatrix}_{Je_4} = 1 \tag{R4}$$

for all $e_1, \dots, e_5 \in E$ and $J \subset E$ such that $(Je_5; e_1, e_2, e_3, e_4)$, $(Je_3; e_1, e_2, e_4, e_5)$ and $(Je_4; e_1, e_2, e_5, e_3)$ are in Ω_M;

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_5} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Je_6} \tag{R5}$$

for all $e_1, \dots, e_6 \in E$ and $J \subset E$ such that $\langle Je_5 \rangle = \langle Je_6 \rangle$ and such that $(Je_5; e_1, e_2, e_3, e_4)$ and $(Je_6; e_1, e_2, e_3, e_4)$ are in $\Omega_M^{\diamondsuit}$;

as well as the additive Plücker relations

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J + \begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix}_J = 1 \tag{R+}$$

for all $(J; e_1, \dots, e_4) \in \Omega_M^{\diamondsuit}$.

Remark 3.6. It is worth noting that it is possible to have $-1 = 1$ in F_M even if M has no F_7 or F_7^*-minor; cf. Section A.3.15 and Section A.3.24 for examples.

3.3. The universal partial field

The universal partial field of a matroid is introduced in [PvZ10a] and [PvZ10b] as a tool for studying representations of matroids over partial fields, in a similar way as the foundation can be used to study representations over arbitrary pastures. In fact, the first two authors show in [BL21, Lemma 7.48] that the universal partial field can be derived in a functorial way from the foundation of the matroid.

Since this result is formulated in the language of ordered blueprints, and a direct interpretation in terms of pastures would require additional results from [BL21, sections 7.2 and 7.3], we include in this section an exposition that is independent of [BL21]. We cite certain results about pastures from [BL25b], but the proofs of those results do not require any knowledge of the theory of ordered blueprints.

Definition 3.7. Let P be a pasture. The *universal ring of* P is the ring

$$R_P \;=\; \mathbb{Z}[P^\times]\,/\,\langle N_P\rangle,$$

where we consider the elements $a+b+c$ of the null set N_P as elements of the group ring $\mathbb{Z}[P^\times]$. The association $a \mapsto [a]$ defines a multiplicative map $\rho_P : P \to R_P$ that satisfies $\rho_P(a)+\rho_P(b)+\rho_P(c)=0$ for all $a+b+c \in N_P$.

A *mock partial field* is a pasture P with $R_P \neq 0$. For a mock partial field P, we define its *associated partial field* $\Pi(P)$ as the pasture

$$\Pi(P) \;=\; \overline{P}/\!\!/\langle\!\langle a+b+c \mid a+b+c=0 \text{ in } R_P\rangle\!\rangle,$$

where $\overline{P}=\rho_P(P)$ is the image of $\rho_P : P \to R_P$. The map ρ_P restricts to a surjective pasture morphism $\pi_P : P \to \Pi(P)$.

The following summarizes Lemmas 2.12 and 2.14 of [BL25b].

Lemma 3.8. *Let P be a pasture and $f : P \to P'$ a pasture morphism into a partial field P'. Then P is a mock partial field, and there is a unique morphism $\Pi(f) : \Pi(P) \to P'$ such that $f = \pi_P \circ \Pi(f)$. In particular, Π extends to a functor from mock partial fields to the subcategory of partial fields in such a way that*

$$\operatorname{Hom}(\Pi(P),\ P') \;=\; \operatorname{Hom}(P,\ P')$$

are naturally identified. Moreover, $\pi_P : P \to \Pi(P)$ is an isomorphism if and only if P is a partial field.

The universal partial field $\mathbb{P}_M$ of a representable matroid is defined in [PvZ10a, section 4] as the partial field generated by the cross ratios inside the bracket ring $\mathbb{B}_M$, which itself is the ring generated by symbols T_B, together with their multiplicative inverses, for all bases B of M, modulo the ideal generated by the 3-term Plücker relations.

The following result gives an independent interpretation of the universal partial field $\mathbb{P}_M$ as $\Pi(F_M)$.

Proposition 3.9. *Let M be a representable matroid with foundation F_M and universal partial field $\mathbb{P}_M$. Then F_M is a mock partial field, and $\mathbb{P}_M$ is canonically isomorphic to $\Pi(F_M)$. In particular, composing a morphism $\mathbb{P}_M \to P$ with $\rho_{F_M} : F_M \to \Pi(F_M) = \mathbb{P}_M$ defines a bijection*

$$\operatorname{Hom}(\mathbb{P}_M,\ P) \ = \ \operatorname{Hom}(F_M,\ P)$$

for every partial field P.

Proof. A representation of M over a field k induces a morphism $F_M \to k$, so Lemma 3.8 implies that F_M is a mock partial field.

Comparing the definitions of the universal ring $R_M = R_{F_M}$ and the bracket ring $\mathbb{B}_M$ from [PvZ10a, section 4.1] shows that R_P is the subring of $\mathbb{B}_M$ generated by the image of $\pi_{F_M} : F_M \to R_M$, considered as a submonoid of $R_M \subset \mathbb{B}_M$.

The foundation F_M is generated over $\mathbb{F}_1^{\pm}$ by the universal cross ratios of M. The map $\pi_{F_M} : F_M \to \mathbb{B}_M$ sends universal cross ratios of F_M to cross ratios of $\mathbb{B}_M$ in the sense of [PvZ10a, section 4.2]. Since the universal partial field $\mathbb{P}_M$ is generated by such cross ratios together with -1 in $\mathbb{B}_M$, the partial field $\Pi(F_M)$ agrees with $\mathbb{P}_M$ as multiplicative submonoids of $\mathbb{B}_M$.

A comparison of $\langle N_P \rangle$ with the defining ideal of the bracket ring $\mathbb{B}_M$ shows that all defining relations stem from (possibly degenerate) 3-term Plücker relations for M. This means that the null sets of $\Pi(F_M)$ and $\mathbb{P}_M$ are equal and the identity map $\Pi(F_M) \to \mathbb{P}_M$ is an isomorphism of pastures.

The equality $\operatorname{Hom}(\mathbb{P}_M, P) = \operatorname{Hom}(F_M, P)$ follows at once from Lemma 3.8. □

4. First examples

We determine the foundation for some first classes of matroids: we recall the result for binary and regular matroids from [BL21] and determine the foundation of uniform matroids of rank 2.

4.1. The foundation of binary and regular matroids

By relation (R0), the foundation F_M of a matroid M without any U_4^2-minors is a quotient of $\mathbb{F}_1^{\pm}$. In particular, the foundation of a binary matroid is either $\mathbb{F}_1^{\pm}$ or $\mathbb{F}_2$; cf. [BL21, Thm. 7.32]. The foundations of both the Fano matroid F_7 and its dual F_7^* are isomorphic to $\mathbb{F}_2$. A matroid is regular if and only it has foundation $\mathbb{F}_1^{\pm}$; cf. [BL21, Thm. 7.35].

4.2. The foundation of U_4^2

We review the account from [BL25a, section 4.5]. The uniform matroid U_4^2, with ground set $E = \{1, 2, 3, 4\}$, has universal cross ratios

$$x \ = \ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad y \ = \ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

(up to the relations (Rσ)), which satisfy the relations

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = x^{-1}, \qquad \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = y^{-1}, \qquad \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = -x^{-1}y, \qquad \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = -xy^{-1}$$

(using the relations (R1) and (R2)) and $x + y = 1$ (relation (R+)). Thus the foundation of U_4^2 is

$$\mathbb{U} \;=\; \mathbb{F}_1^{\pm}(x,y)/\!\!/\langle\!\langle x+y-1\rangle\!\rangle.$$

4.3. The foundation of U_5^2

In this section, we establish a fact which was stated without proof in [BL25a, Prop. 5.4]. The assertion is that the foundations of U_5^2 and U_5^3 are isomorphic to the pasture

$$\mathbb{V} \;=\; \mathbb{F}_1^{\pm}(x_1,\dots,x_5)/\!\!/\langle\!\langle x_i + x_{i-1}x_{i+1} - 1 \,|\, i = 1,\dots,5\rangle\!\rangle,$$

where the subscript i has to be read 'modulo 5,' i.e., $x_0 = x_5$ and $x_6 = x_1$.

Proposition 4.1. *The foundation of U_5^2 is isomorphic to $\mathbb{V}$.*

Proof. For the proof, we choose $E = \mathbb{Z}/5\mathbb{Z} = \{1,\dots,5\}$ as the ground set for U_5^2. Since the set of bases consist of precisely all 2-subsets of E, we obtain for each tuple $(\varnothing, i, j, k, l)$ with pairwise distinct $i, j, k, l \in E$ a cross ratio

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\varnothing}.$$

Using the relations of type (Rσ), (R1), and (R2), we conclude that the foundation F of U_5^2 is generated by the cross ratios of the form

$$x_i = \begin{bmatrix} i+1 & i+2 \\ i+4 & i+3 \end{bmatrix} \qquad \text{and} \qquad y_i = \begin{bmatrix} i+1 & i+4 \\ i+2 & i+3 \end{bmatrix}$$

for $i = 1,\dots,5$, which satisfy the relations $x_i + y_i - 1 \in N_F$ by (R+). Since U_5^2 has no (dual) Fano minor, the relation (R–) does not occur. Since every 2-subset of E is a basis, there are no relations of type (R0). Since the rank of U_5^2 is smaller than 3, there are no relations of types (R4) and (R5).

We show in the following that the relations of type (R3) come down to $y_i = x_{i-1}x_{i+1}$, which concludes the proof of our claim. These relations are of the form

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix} \cdot \begin{bmatrix} i & j \\ l & m \end{bmatrix} \cdot \begin{bmatrix} i & j \\ m & k \end{bmatrix} = 1$$

for $\{i,j,k,l,m\} = E$. Note that permuting $\{i,j\}$ and $\{k,l,m\}$ yields the same relation, up to permuting the factors and possibly taking inverses of all factors. This leaves us with 10 relations of type (R3), one for each basis $\{i,j\} \subset E$.

More precisely, we choose the following representatives for each basis. For a basis of the form $\{i, i+1\}$, we consider

$$\begin{aligned} 1 \;&=\; \begin{bmatrix} i & i+1 \\ i+2 & i+3 \end{bmatrix} \cdot \begin{bmatrix} i & i+1 \\ i+3 & i+4 \end{bmatrix} \cdot \begin{bmatrix} i & i+1 \\ i+4 & i+2 \end{bmatrix} \\ &=\; x_{i+4}^{-1} \cdot x_{i+2}^{-1} \cdot y_{i+3} \end{aligned}$$

where we use the relations of types (Rσ) and (R1) to express the cross ratios in terms of the x_i and y_i. This yields at once the desired relation $y_i = x_{i-1}x_{i+1}$ and thus $x_i + x_{i-1}x_{i+1} - 1 \in N_F$.

We are left with showing that the additional 5 relations for bases of the forms $\{i, i+2\}$ do not endow additional relations between the x_i. We consider the following representatives for these relations:

$$\begin{aligned} 1 \;&=\; \begin{bmatrix} i & i+2 \\ i+3 & i+4 \end{bmatrix} \cdot \begin{bmatrix} i & i+2 \\ i+4 & i+1 \end{bmatrix} \cdot \begin{bmatrix} i & i+2 \\ i+1 & i+3 \end{bmatrix} \\ &=\; y_{i+1}^{-1} \cdot (-x_{i+3}^{-1}\, y_{i+3}) \cdot (-x_{i+4}^{-1}\, y_{i+4}), \end{aligned}$$

where we use the relations of types (Rσ), (R1) and (R2) to express the cross ratios in terms of the x_i and y_i. When we substitute y_i by $x_{i-1}x_{i+1}$ in this relation, we find that

$$y_{i+1}^{-1} \cdot (-x_{i+3}^{-1}\, y_{i+3}) \cdot (-x_{i+4}^{-1}\, y_{i+4}) \;=\; x_i^{-1}\, x_{i+2}^{-1}\, x_{i+3}^{-1}\, x_{i+2}\, x_{i+4}\, x_{i+4}^{-1}\, x_{i+3}\, x_i \;=\; 1$$

holds already in F, which shows that the latter family of relations do not imply additional relations between the cross ratios $x_1, \dots, x_5$. This concludes the proof. ☐

Corollary 4.2. *The foundation of U_5^3 is isomorphic to $\mathbb{V}$.*

Proof. This follows at once from Proposition 4.1 by taking duals; cf. Proposition 3.4 and [BL25a, Prop. 4.8]. ☐

4.4. The foundation of U_n^2

In this section, we identify the foundation of the uniform matroid U_{k+3}^2 with Semple's k-regular partial field, cf. [Sem97], [Sem98]. Interestingly, this gives a different presentation of $\mathbb{V}$ from Proposition 4.1, a fact that we comment on further at the end of this section.

Semple's k-regular partial field is defined as $\mathcal{R}_k = \big(\mathcal{A}_k, \mathbb{Q}(\alpha_1, \dots, \alpha_k)\big)$ where $\alpha_1, \dots, \alpha_k$ are algebraically independent elements over $\mathbb{Q}$ and

$$\mathcal{A}_k \;=\; \left\{ \prod_{i=1}^{s} (\beta_i - \gamma_i)^{n_i} \;\middle|\; s \geqslant 0,\ \beta_i, \gamma_i \in \{0, 1, \alpha_1, \dots, \alpha_k\},\ n_i \in \mathbb{Z} \right\}.$$

Note that $\mathcal{A}_k$ contains $0 = 0 - 0$ and $-a = (0-1)a$ for every $a \in \mathcal{A}_k$. Thus $\mathcal{R}_k$ is indeed a partial field. We define $\mathbb{U}_k$ as the pasture associated with $\mathcal{R}_k$, and (by abuse of terminology) refer to it also as the k-regular partial field.

Note that the 0-regular partial field $\mathbb{U}_0$ coincides with the regular partial field $\mathbb{F}_1^{\pm}$, and the 1-regular partial field $\mathbb{U}_1$ is isomorphic to the near-regular partial field $\mathbb{U}$.

An explicit description of the pasture $\mathbb{U}_k$ in terms of generators and relations is as follows.

Lemma 4.3. *Let* $k \geqslant 0$ *and define* $\alpha_{-1} = 0$ *and* $\alpha_0 = 1$. *Then*

$$\mathbb{U}_k \;=\; \mathbb{F}_1^{\pm}(\alpha_j - \alpha_i \mid -1 \leqslant i < j \leqslant k,\ j \neq 0) /\!\!/ \langle\!\langle S_k \rangle\!\rangle,$$

where S_k *consists of the terms*

$$\frac{\alpha_i - \alpha_l}{\alpha_j - \alpha_l} + \frac{\alpha_i - \alpha_j}{\alpha_l - \alpha_j} - 1$$

for all $-1 \leqslant i < j < l \leqslant k$ *and*

$$\frac{(\alpha_i - \alpha_l)(\alpha_j - \alpha_m)}{(\alpha_i - \alpha_m)(\alpha_j - \alpha_l)} + \frac{(\alpha_i - \alpha_j)(\alpha_l - \alpha_m)}{(\alpha_i - \alpha_m)(\alpha_l - \alpha_j)} - 1$$

for all $-1 \leqslant i < j < l < m \leqslant k$.

Proof. It is clear from the definition of $\mathcal{R}_k$ that it is generated by the terms $\alpha_j - \alpha_i$ with $-1 \leqslant i < j \leqslant k$ and -1 as a multiplicative monoid, where we can exclude $\alpha_0 - \alpha_{-1} = 1$ from the generating set. Since the multiplicative monoid of $\mathbb{U}_k$ agrees with that of $\mathcal{R}_k$ by definition, this shows that $\mathbb{U}_k$ is generated by the terms $\alpha_j - \alpha_i$ with $-1 \leqslant i < j \leqslant k$ and $j > 0$ over $\mathbb{F}_1^{\pm}$.

The null set $N_{\mathbb{U}_k}$ of $\mathbb{U}_k$ is a union of $\mathbb{U}_k^{\times}$-orbits, and by [BL25b, Prop. 3.6], $N_{\mathbb{U}_k}/\mathbb{U}_k^{\times}$ is in bijection with the set of hexagons in $\mathbb{U}_k$. We show that the hexagons in $\mathbb{U}_k$ correspond to the terms in S_k.

Semple determines in [Sem97, Thm. 3.2] the fundamental elements of $\mathcal{R}_k$ as the elements

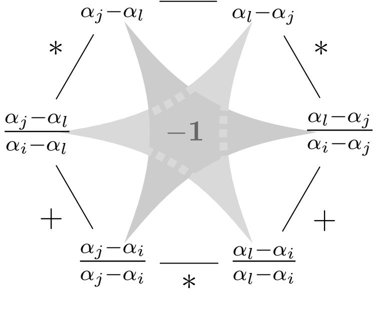
$$\frac{\alpha_i - \alpha_l}{\alpha_j - \alpha_l}$$

for pairwise distinct $i, j, l \in \{-1, \dots, k\}$ and

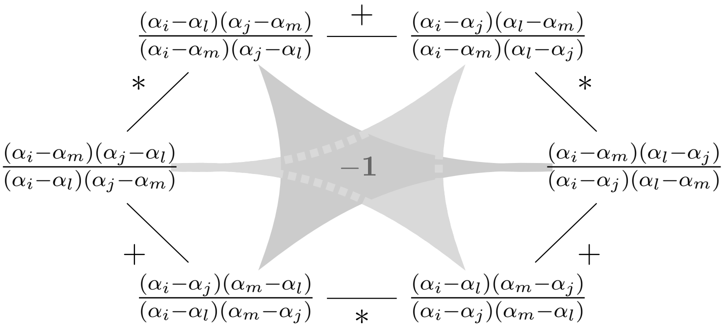
$$\frac{(\alpha_i - \alpha_l)(\alpha_j - \alpha_m)}{(\alpha_i - \alpha_m)(\alpha_j - \alpha_l)}$$

for all pairwise distinct $i, j, l, m \in \{-1, \dots, k\}$. Note that we can permute $\{i, j, l, m\}$ with elements of the Klein-four group $V = \langle (ij)(lm), (il)(jm) \rangle$ in the second expression without changing the fundamental element. Thus we can assume that i is the smallest index among i, j, l, m.

These fundamental elements are grouped into the following hexagons. The first type of fundamental element appears in the hexagon



and the second type of fundamental element appears in the hexagon



The terms in S_k correspond to the first rows in these hexagons, where we can assume that $i<j<l$ and $i<j<l<m$, respectively, after rotating or reflecting the hexagon. This shows that S_k generates the null set $N_{\mathbb{U}_k}$, and concludes the proof. $\square$

Proposition 4.4. *Let* $k \geqslant 0$. *Then the foundation of* U^2_{k+3} *is isomorphic to* $\mathbb{U}_k$.

Proof. Van Zwam shows in [vZ09, Thm. 3.3.24] that the universal partial field $\mathbb{P}_M$ of $M = U^2_{k+3}$ is $\mathcal{R}_k$ or, translated into the language of pastures, the k-regular partial field $\mathbb{U}_k$. By Proposition 3.9, we have

$$\mathbb{P}_M \;=\; \Pi(F_M) \;=\; F_M /\!\!/ \langle a-b \mid a-b \in \langle N_M\rangle_{\mathbb{Z}}\rangle,$$

where $\langle N_M\rangle_{\mathbb{Z}}$ is the ideal of the group ring $\mathbb{Z}[F_M^*]$ generated by all terms $c+d+e$ in the null set N_M of F_M. Moreover, [BL21, Cor. 7.13] identifies the unit group $F_M^\times$ of the foundation with the inner Tutte group $\mathbb{T}_M^{(0)}$ of M.

Dress and Wenzel show in [DW89, Thm. 8.1] that

$$\mathbb{T}_M^{(0)} \;\simeq\; \{\pm 1\} \times \mathbb{Z}^{\binom{k+3}{2}-(k+3)}$$

for $M = U^2_{k+3}$. And by definition, we see that $\mathcal{R}_k^\times$ is the product of $\{\pm1\}$ with the free abelian group with basis $\alpha_i - \alpha_j$, where (i,j) range over all pairs of integers with $-1 \leqslant i < j \leqslant k$ and $j \neq 0$. Thus the free rank of $\mathcal{R}_k^\times$ is

$$\#\left\{ (i,j) \in \mathbb{Z}^2 \;\middle|\; -1 \leqslant i < j \leqslant k,\ j \neq 0 \right\} \;=\; \left(\sum_{i=-1}^{k-1} (k-i) \right) - 1 \;=\; \binom{k+3}{2} - (k+3),$$

which equals the free rank of $\mathbb{T}^{(0)}$. We conclude that the surjective group homomorphism

$$\mathbb{T}^{(0)} \;\simeq\; F_M^\times \;\longrightarrow\; \mathbb{U}_k^\times \;\simeq\; \mathcal{R}_k^\times$$

is an isomorphism of groups, and thus the quotient map $F_M \to \mathbb{U}_k$ is a bijection.

In order to show that the respective null sets agree, we count the number of fundamental elements in both pastures and show that $\mathbb{U}_k$ does not have more fundamental elements than F_M. The fundamental elements of F_M are the cross ratios $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of M. Since every 2-subset of U^2_n is independent, every 4-tuple of pairwise different elements of M yields a cross ratio. Modulo the relations (Rσ), which identifies the cross ratios for 4 permutations of a, b, c and d, we count $n(n-1)(n-2)(n-3)/4 = 6\cdot\binom{n}{4}$ cross ratios.

We have to exclude further identifications of cross ratios to ensure that this is the correct count. Relation (R–) does not appear since U^2_n has no minors of types F_7 and F_7^*. Relation (R0) does not appear since all 2-subsets are independent. Relation (R4) does not appear since U^2_n has no minors of rank 3 and relation (R5) does not appear since U^2_n does not have minors with parallel elements. Relations (Rσ), (R1) and (R2) express that every U^2_4-minor leads to 6 *a priori* distinct cross ratios, which entered our count already, and (R3) are relations that are imposed by rank 2 minors on 5 elements, which are all isomorphic to U^2_5. By Proposition 4.1, none of the cross ratios of distinct U^2_4-minors are identified in the foundation of U^2_5. By [BL25a, Thm. 4.23] (also cf. Theorem 6.3 for a more concise version), the foundation of U^2_n is the colimit of the foundations of all embedded minors of types U^2_4 and U^2_5, which shows that there are no further identifications by minors on more than 5 elements.

The fundamental elements of $\mathbb{U}_k$ are described in Lemma 4.3: for each 3-subset of $\{-1,0,1,\dots,k\}$, there are 6 fundamental elements of the form

$$\frac{\alpha_i-\alpha_l}{\alpha_j-\alpha_l},$$

and for every 4-subset of $\{-1,0,1,\dots,k\}$, there are 6 fundamental elements of the form

$$\frac{(\alpha_i-\alpha_l)(\alpha_j-\alpha_m)}{(\alpha_i-\alpha_m)(\alpha_j-\alpha_l)},$$

Since $n = k+3$, the number of fundamental elements in $\mathbb{U}_k$ is at (most equal) to $6\cdot\left(\binom{n-1}{3}+\binom{n-1}{4}\right) = 6\cdot\binom{n}{4}$, which is the number of fundamental elements in F_M. This shows that the bijection $F_M \to \mathbb{U}_k$ is an isomorphism of pastures, as desired. □

Semple characterizes all automorphisms of the k-regular partial field $\mathbb{U}_k$ in [Sem97, Thm. 4.2]. The group structure of $\mathrm{Aut}(\mathbb{U}_k)$ is, however, not visible from this description. Coupling Semple's result with our methods yields the following result:

Proposition 4.5. *The automorphism group of* $\mathbb{U}$ *is isomorphic to the symmetric group* S_3*, and for* $k\geqslant 2$*, the automorphism group of* $\mathbb{U}_k$ *is isomorphic to* S_{k+3}*.*

Proof. The functoriality of the foundation with respect to minor embeddings, and in particular with respect to automorphisms, yields a group homomorphism

$$\varphi_k:\ S_{k+3}\ =\ \mathrm{Aut}(U^2_{k+3})\ \longrightarrow\ \mathrm{Aut}(\mathbb{U}_k).$$

In [BL25a, Prop. 5.6], we have shown that $\varphi_1 : S_4 = \mathrm{Aut}(U^2_4) \to \mathrm{Aut}(\mathbb{U})$ is surjective with kernel $V = \langle(12)(34),(14)(23)\rangle$, i.e. $\mathrm{Aut}(\mathbb{U})\simeq S_3$, which yields the first claim.

If $k\geqslant 2$, then every permutation $\sigma\in S_{k+3}$ that fixes j, k and l sends the cross ratio $\left[\begin{smallmatrix} i & j\\ k & l\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \sigma(i) & j\\ k & l\end{smallmatrix}\right]$, which differs from $\left[\begin{smallmatrix} i & j\\ k & l\end{smallmatrix}\right]$ if $\sigma(i)\neq i$. Since $k+3\geqslant 5$ for $k\geqslant 2$, we find such a cross ratio, which shows that φ_k is injective.

Semple's description of the automorphisms of $\mathcal{R}_k$ in [Sem97, Thm. 4.2] determines all possible images of $\alpha_1, \dots, \alpha_k$. From the outset, it is not clear that the corresponding automorphisms are pairwise distinct—and they are not for $k = 1$— but this description gives an upper bound for $\#\operatorname{Aut}(\mathcal{R}_k)$. There are 4 cases in [Sem97, Thm. 4.2]: the first 3 cases describe (up to) $(k+2)!$ automorphisms, the fourth case contains an additional element, which leads to (up to) $k \cdot (k+2)!$ automorphisms. We deduce the upper bound

$$\#\operatorname{Aut}(\mathbb{U}_k) \;=\; \#\operatorname{Aut}(\mathcal{R}_k) \;\leqslant\; 3 \cdot (k+2)! + k \cdot (k+2)! \;=\; (k+3)!.$$

Since φ_k is injective for $k \geqslant 2$ and $\#S_{k+3} = (k+3)!$, we conclude that φ_k is surjective as well and thus an isomorphism. □

Remark 4.6. Comparing Proposition 4.4 with Proposition 4.1 shows that $\mathbb{U}_2 \simeq \mathbb{V}$, which yields an arguably more symmetric description of the 2-regular partial field $\mathbb{U}_2$. An explicit isomorphism $\varphi : \mathbb{V} \to \mathbb{U}_2$ is given by sending the generators $x_1, \dots, x_5$ of $\mathbb{V}$ to

$$\varphi(x_1) = \alpha, \quad \varphi(x_2) = 1 - \beta, \quad \varphi(x_3) = \tfrac{\beta}{\alpha}, \quad \varphi(x_4) = \tfrac{\alpha-\beta}{\alpha(1-\beta)}, \quad \varphi(x_5) = \tfrac{1-\alpha}{1-\beta},$$

where we write $\alpha = \alpha_1$ and $\beta = \alpha_2$ for better readability. It is also possible to deduce $\operatorname{Aut}(\mathbb{V}) \simeq S_5$ without too much effort directly from the definition of $\mathbb{V}$, without reference to Semple's and Pendavingh–van Zwam's results.

It seems desirable to develop more "symmetric" descriptions of the higher k-regular partial fields $\mathbb{U}_k$, in the vein of the isomorphisms $\mathbb{U}_1 \simeq \mathbb{U}$ and $\mathbb{U}_2 \simeq \mathbb{V}$. One obstacle towards this goal is that van Zwam's result $\mathbb{P}_{U^2_{k+3}} \simeq \mathcal{R}_k$ relies on partial field techniques for which it is not clear how to generalize them to pastures.

4.5. Finite projective spaces

Let $PG(d, q)$ be the d-dimensional projective space over the finite field $\mathbb{F}_q$ with q elements, which is a rank $d+1$ matroid on $q^d + \dots + q + 1$ elements. Results on the inner Tutte group of $GP(d, q)$ and the universal partial of a certain minor allow us to deduce that the foundation of $PG(d, q)$ is $\mathbb{F}_q$. By a similar type of reasoning, we determine the foundation of any non-Desarguesian planes as $\mathbb{K}$.

The *extended rank* 3 *Dowling geometry* Q_q^+ *of* $\mathbb{F}_q^\times$ is the restriction of $PG(2, q)$ to $L_1 \cup L_2 \cup L_3 \cup e$, where L_1, L_2 and L_3 are three lines in $PG(2, q)$ with empty intersection and e is a point that lies on neither of these lines.

Proposition 4.7. *The foundation of the* d*-dimensional projective space* $PG(d, q)$ *is* $\mathbb{F}_q$ *for all* $d \geqslant 2$ *and all prime powers* q.

Proof. By [DW90, Thm. 3.6], the inner Tutte group of $M = PG(d, q)$ is $\mathbb{F}_q^\times$, which equals the unit group of the foundation F_M of M by [BL21, Cor. 7.13]. By [BL21, Lemma 7.48], the universal partial field $\mathbb{P}_M$ of M is a quotient of F_M, and thus can have at most q elements. By [Oxl92, p. 660], $PG(2, q)$ is representable over $\mathbb{F}_q$, so there is a morphism $\mathbb{P}_M \to \mathbb{F}_q$.

The extended rank 3 Dowling geometry $N = Q_q^+$ of $\mathbb{F}_q^\times$ is a minor of $PG(2, q)$ whose universal partial field is $\mathbb{P}_N = \mathbb{F}_q$ by [vZ09, Thm. 3.3.25]. Since $PG(2, q)$ is a minor of $PG(d, q)$, we get morphisms $\mathbb{F}_q = \mathbb{P}_N \to \mathbb{P}_M \to \mathbb{F}_q$, which are isomorphisms since $\mathbb{P}_M$ has at most q elements and since every homomorphism from a field into a partial field is injective. Thus $\mathbb{P}_M = \mathbb{F}_q$, which shows that $\mathbb{P}_M$ is the trivial quotient of F_M and therefore $F_M = \mathbb{P}_M = \mathbb{F}_q$. ☐

The fact ([vZ09, Thm. 3.3.25]) that the universal partial field of extended Dowling geometries Q_q^+ is $\mathbb{F}_q$ suggests the following:

Problem 4.8. Is $\mathbb{F}_q$ the foundation of the extended rank 3 Dowling geometry Q_q^+ of $\mathbb{F}_q^\times$? More generally, what are the minimal matroids with foundation $\mathbb{F}_q$?

The answer is yes for $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$, but the extended Dowling geometries Q_3^+ and Q_4^+ are not minimal for their respective foundations; cf. Section A.2.

Proposition 4.9. *The foundation of a non-Desarguesian projective plane is* $\mathbb{K}$*.*

Proof. By [DW90, Thm. 3.7], the inner Tutte group of a non-Desarguesian projective plane is trivial. There are only two pastures P with $P^\times = \{1\}$, which are $\mathbb{F}_2$ and $\mathbb{K}$. A non-Desarguesian plane cannot be represented over any field since Desargues's theorem is violated, so we conclude that its foundation is $\mathbb{K}$. ☐

5. The foundation of a direct sum

Our goal in this section is to prove:

Theorem 5.1. *Let* M_1 *and* M_2 *be matroids. Then* $F_{M_1 \oplus M_2} \simeq F_{M_1} \otimes F_{M_2}$*.*

Before giving the proof, we note that Theorem 5.1 is a special case of a more general result on the foundation of generalized parallel connections proved in [BLWZ24]. However, the case considered here is substantially simpler, so it seems worthwhile to give a self-contained proof. A related result, also proved in [BLWZ24], is that the foundation of a 2-sum of matroids is also the tensor product of the foundations.

We begin with the following well-known facts about direct sums:

Proposition 5.2. *[Oxl92, 4.2.15] Let* M_1, M_2 *be matroids on* E_1 *and* E_2*, respectively. A subset* H *of* $E(M_1 \oplus M_2) = E_1 \sqcup E_2$ *is a hyperplane of* $M_1 \oplus M_2$ *if and only if it satisfies one of the following:*

1. $H \cap E_1$ *is a hyperplane in* M_1 *and* H *contains* E_2*.*
2. $H \cap E_2$ *is a hyperplane in* M_2 *and* H *contains* E_1*.*

Proposition 5.3. *[Oxl92, 4.2.13] With notation as in the previous proposition, if* r, r_1, r_2 *are the rank functions of* $M_1 \oplus M_2$*,* M_1*, and* M_2*, respectively, then for any subset* X *of* $E(M_1 \oplus M_2)$ *we have:*

$$r(X) = r_1(X \cap E_1) + r_2(X \cap E_2).$$

Let F be a pasture and let M be a matroid. Let $\mathcal{X}^I_M(F)$ denote the set of (weak) F-representations of M up to isomorphism, and let $\mathcal{X}^R_M(F)$ denote the set of rescaling equivalence classes of F-representations of M. In order to prove Theorem 5.1, it suffices to show that for every pasture F there is a natural bijection from $\mathcal{X}^I_{M_1\oplus M_2}(F)$ to $\mathcal{X}^I_{M_1}(F)\times\mathcal{X}^I_{M_2}(F)$, and that these bijections are functorial in F. Indeed, this immediately implies the same thing with $\mathcal{X}^I$ replaced by $\mathcal{X}^R$, and (in light of the universal property of tensor products) it also implies that both $F_{M_1\oplus M_2}$ and $F_{M_1}\otimes F_{M_2}$ represent the functor $\mathcal{X}^R_{M_1\oplus M_2}(F)$. Hence $F_{M_1\oplus M_2}$ and $F_{M_1}\otimes F_{M_2}$ are (canonically) isomorphic.[2]

Proof of Theorem 5.1. As discussed above, it suffices to define a bijection which is functorial in F from $\mathcal{X}^I_M(F)$ to $\mathcal{X}^I_{M_1}(F)\times\mathcal{X}^I_{M_2}(F)$. We use the description of $\mathcal{X}^I_M(F)$ in terms of modular systems of F-hyperplane functions for M given in [BL25a, Theorem 2.16]. (This can be thought of as a cryptomorphic description of weak F-matroids in terms of hyperplanes.)

Given a modular system $\mathcal{H}=\{f_H\}_{H\in\mathcal{H}(M)}$ of F-hyperplane functions for M, we can define corresponding sets of F-hyperplane functions $\mathcal{H}_i$ for $i=1,2$ by the formulas $f_{H_1}(e)=f_{H_1\oplus E_2}(e)$ and $f_{H_2}(e)=f_{E_1\oplus H_2}(e)$. Conversely, given a modular system $\mathcal{H}_i=\{f_H : E\to F\}_{H\in\mathcal{H}(M_i)}$ of F-hyperplane functions for M_i ($i=1,2$), we can define (using Proposition 5.2) a corresponding set of F-hyperplane functions $\mathcal{H}$ for $M:=M_1\oplus M_2$ by the formula $f_{H_1\oplus E_2}(e)=0$ if $e\in E_2$ and $f_{H_1\oplus E_2}(e)=f_{H_1}(e)$ if $e\in E_1$, and similarly $f_{E_1\oplus H_2}(e)=0$ if $e\in E_1$ and $f_{E_1\oplus H_2}(e)=f_{H_2}(e)$ if $e\in E_2$.

It is easy to see that these constructions are inverse to one another and functorial in F, so it suffices to prove that each construction yields a modular system. We verify that the second map yields a modular system, and leave the similar verification for the first map to the reader.

Suppose H_1,H_2,H_3 are a modular triple of hyperplanes for M, with mutual pairwise intersection equal to the corank 2 flat P. Up to symmetry (swapping the roles of E_1 and E_2), there are just two cases to consider:

1. Suppose E_1 is contained in H_1 and E_2 is contained in H_2. By symmetry, we can assume that H_3 contains E_1. Then, $H_3\cap H_1=P$ contains E_1, and hence H_2 contains E_1 as well, which contradicts the assumption that $E_2\subseteq H_2$. So this case does not occur.
2. Suppose E_1 is contained in both H_1 and H_2. Then E_1 is contained in H_3 as well, and so by Proposition 5.3, the restrictions H_1',H_2',H_3' of H_1,H_2,H_3 to E_2 form a modular triple of hyperplanes of M_2. Consequently, we know that the functions $f_{H_i'}$ are linearly dependent, which implies that the original functions f_{H_i} are also linearly dependent. □

5.1. Indecomposable foundations

We conclude this section with a discussion about *indecomposable foundations*, which are foundations F_M that are not isomorphic to a non-trivial tensor product $P_1\otimes P_2$ of two pastures. In this context, *non-trivial* means that neither P_1 nor P_2 is a quotient of $\mathbb{F}_1^\pm$, i.e., neither is isomorphic to a pasture in $\{\mathbb{F}_1^\pm,\mathbb{F}_2,\mathbb{F}_3,\mathbb{K}\}$. The reason for excluding quotients Q of $\mathbb{F}_1^\pm$ is that $Q\otimes Q=Q$ for such a pasture.

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[2] So, in fact, what we are really proving is that the *universal pasture* (cf. [BL25a, Definition 2.17 and Theorem 2.18]) of the direct sum is the tensor product of the universal pastures.

Theorem 5.1 tells us that the direct sum decomposition $M = M_1 \oplus M_2$ of a matroid induces a tensor decomposition $F_M \simeq F_{M_1} \otimes F_{M_2}$ of its foundation. In fact, the same is true for the decomposition into a 2-sum $M = M_1 \oplus_2 M_2$: its foundation is $F_M \simeq F_{M_1} \otimes F_{M_2}$; cf. [BLWZ24].

The foundation of many 3-connected matroids is indecomposable. For example:

Proposition 5.4. *The foundation of a* 3*-connected quarternary matroid is indecomposable.*

Proof. Let M be a quarternary matroid and let F_M be its foundation. Suppose that $F_M = P_1 \otimes P_2$ is a non-trivial decomposition into pastures P_1 and P_2, neither of which is a quotient of $\mathbb{F}_1^{\pm}$. Since F_M is generated by the universal cross ratios of M, which are fundamental elements, both P_1 and P_2 are generated by their respective fundamental elements. In particular, each factor contains a fundamental element.

The fundamental elements of $\mathbb{F}_4 = \{0, 1, a, a+1\}$ are a and $a+1$. Therefore the composition of any morphism $F_M \to \mathbb{F}_4$ with the canonical inclusion $P_i \to F_M$ (for $i = 1, 2$) is a surjection $P_i \twoheadrightarrow \mathbb{F}_4$. Composing this surjection with the nontrivial field automorphism $\mathbb{F}_4 \to \mathbb{F}_4$ yields two additional morphisms $P_i \twoheadrightarrow \mathbb{F}_4$.

The universal property of the tensor product can be expressed as a bijection

$$\mathrm{Hom}(F_M, \mathbb{F}_4) = \mathrm{Hom}(P_1, \mathbb{F}_4) \times \mathrm{Hom}(P_2, \mathbb{F}_4).$$

The previous paragraph thus guarantees that there are at least 4 distinct morphisms $F_M \to \mathbb{F}_4$. By a well-known theorem of Kahn ([Kah88]), M cannot be 3-connected, which establishes our claim. □

Problem 5.5. Instead of using Kahn's theorem to deduce Proposition 5.4, can one give a direct proof of the proposition using the theory of foundations and deduce Kahn's theorem as a corollary?

It is not true in general that the foundation of a 3-connected matroid is indecomposable (this insight was shared with us by Nathan Bowler); cf. Section A.3.4 for a 3-connected matroid with foundation $\mathbb{D} \otimes \mathbb{D}$. In so far, we wonder:

Problem 5.6. What are necessary and sufficient (or even just sufficient) conditions on a matroid M for its foundation to be indecomposable? For example, is the foundation of a 4-connected matroid indecomposable?

One can refine the notion of indecomposable foundations using the fundamental diagram of a matroid. Namely, given a diagram of pastures $\mathcal{F}$ with connected components $\mathcal{F}_1, \dots, \mathcal{F}_r$, we have $\operatorname{colim} \mathcal{F} = \otimes_{i=1}^{r} \operatorname{colim} \mathcal{F}_i$. Embedded minors of type F_7 and F_7^* appear as isolated points in the fundamental diagram $\mathcal{E}_M$, and we call them the *trivial connected components of* $\mathcal{E}_M$. We call $\mathcal{E}_M$ *essentially connected* if it has at most one nontrivial connected component. In other words, $\mathcal{E}_M$ is essentially connected if all vertices corresponding to embedded U_4^2-minors of M lie in the same connected component of $\mathcal{E}_M$. Since the foundation $\mathbb{F}_2$ of F_7 and F_7^* is a trivial tensor factor, it follows that the foundation of a matroid M with essentially connected fundamental diagram is indecomposable. We can therefore sharpen the previous problem as follows:

Problem 5.7. What are necessary and sufficient (or even just sufficient) conditions on a matroid M for its fundamental diagram to be essentially connected?

The fundamental diagram of Bowler's ternary spike mentioned above is not essentially connected, so the condition that M is ternary is not sufficient.

6. Fundamental presentations

In this section, we develop presentations of the foundation in terms of minors and in terms of sublattices of the lattice of flats of a matroid.

6.1. The fundamental presentation by embedded minors

The relations between universal cross ratios from Theorem 3.5 involve only few elements of the ground set, which means that these relations stem from minors of fairly small sizes. This has already been noted in [BL25a, Thm. 4.23], which describes the foundation of a matroid as the colimit of the foundations of its embedded minors of small size. We improve upon this result by narrowing down the list of embedded minors that we need to consider to a minimal possible set.

Let M be a matroid with ground set E. An *embedded minor of* M is a minor $M\backslash J/I$ of M with a fixed choice of subsets I and J of E, where I is coindependent and J is independent. Note that every minor of M can be expressed in this form; this is a consequence of the Scum Theorem, cf. [BL25a, section 1.3].

A *minor embedding* $N \hookrightarrow M$ is an isomorphism $N \simeq M\backslash J/I$ of N with an embedded minor $M\backslash J/I$ of M. In particular, every embedded minor comes with a tautological minor embedding $M\backslash J/I \hookrightarrow M$.

Every minor embedding $N \simeq M\backslash J/I \hookrightarrow M$ induces a morphism of foundations:

$$\begin{tikzcd}
F_N \arrow[r, "\sim"] & F_{M\backslash J/I} \arrow[r] & F_M \\
 & \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_{J'} \arrow[r, mapsto] & \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_{J'\cup J}
\end{tikzcd}$$

where $(J'; a, b, c, d)$ varies through all tuples in $\Omega^{\diamondsuit}_{M\backslash J/I}$; cf. [BL25a, Prop. 4.9]. Note that this pasture morphism is in general not injective.

Definition 6.1. Let M be a matroid. The *fundamental diagram of* M is the diagram $\mathcal{E}_M$ of all embedded minors $N = M\backslash J/I$ of M of types

$$\begin{array}{llllll}
U^2_4, & & & & & (4 \text{ elements}) \\
C_5, & C^*_5, & U^2_5, & U^3_5, & & (5 \text{ elements}) \\
U^1_2 \oplus U^2_4, & & & & & (6 \text{ elements}) \\
F_7, & F^*_7 & & & & (7 \text{ elements}),
\end{array}$$

together with all minor embeddings. We denote by $F(\mathcal{E}_M)$ the family of foundations of all embedded minors in $\mathcal{E}_M$, together with the pasture morphisms that are induced by the minor embeddings. We write $\mathcal{S} := \{U^2_4, U^2_5, U^3_5, C_5, C^*_5, U^2_4 \oplus U^1_2, F_7, F^*_7\}$, and refer to embedded minors of M isomorphic to some matroid in $\mathcal{S}$ as the *special embedded minors* of M.

Remark 6.2. By the results in Section 4.2, any minor embedding $U^2_4 \hookrightarrow M$ induces an isomorphism $\mathbb{U} \xrightarrow{\sim} F_M$ if M is any of C_5, C^*_5, or $U^1_2 \oplus U^2_4$. This follows for the series extension C_5 and the parallel extension C^*_5 at once from [BL25a, Prop. 4.9]. For $M = U^1_2 \oplus U^2_4$, this follows from Theorem 5.1: since U^1_2 is regular, its foundation is $\mathbb{F}^{\pm}_1$, and therefore the canonical inclusion $\mathbb{U} \to F_M = \mathbb{F}^{\pm}_1 \otimes \mathbb{U} = \mathbb{U}$ is an isomorphism.

Theorem 6.3. *Let M be a matroid with foundation F_M and fundamental diagram $\mathcal{E}_M$. Then the canonical morphism* $\operatorname{colim} F(\mathcal{E}_M) \to F_M$ *is an isomorphism.*

Proof. In addition to the matroids listed in the definition of the fundamental diagram, the description $F_M = \operatorname{colim} F(\mathcal{E}'_M)$ in [BL25a, Thm. 4.23] requires considering embedded minors of types $U_1^i \oplus U_4^2$ for $i = 0, 1$, as well as all embedded minors that are parallel extensions of rank 3 matroids on 5 elements with a U_4^2-minor.

The claim of this theorem follows if we can show that the colimit $\operatorname{colim} F(\mathcal{E}'_M)$ does not change when we omit these additional embedded minors from $\mathcal{E}'_M$. Our argument will make use of the following fact about colimits: if an object F_N of the diagram $F(\mathcal{E}'_M)$ is the colimit $\operatorname{colim} F(\mathcal{E}_N)$, where $\mathcal{E}_N$ is the fundamental diagram of N, then we can omit F_N from $\mathcal{E}'_M$ without changing the value of $\operatorname{colim} F(\mathcal{E}'_M)$. We verify this case by case for all embedded minors $N = M\backslash J/I$ of the mentioned types.

To start with, we consider $N = U_1^0 \oplus U_4^2$. Let 1 be the loop of the factor U_1^0. Then $\mathcal{E}_N$ contains a unique embedded minor, which is $N\backslash 1$, and thus $\operatorname{colim} F(\mathcal{E}_N) = F_{N\backslash 1}$. Since 1 is a loop, the minor embedding $N\backslash 1 \hookrightarrow N$ induces an isomorphism $\operatorname{colim} F(\mathcal{E}_N) = F_{N\backslash 1} \to F_N$ by [BL25a, Prop. 4.9]. As explained above, this shows that we can omit embedded minors of type $U_1^0 \oplus U_4^2$ from $\mathcal{E}'_M$ without changing its colimit.

The case of embedded minors of type $N = U_1^1 \oplus U_4^2$ is analogous. If 1 is the coloop of the factor U_1^1, then $\mathcal{E}_N$ consists of the unique embedded minor $N/1$ and thus $\operatorname{colim} F(\mathcal{E}_N) = F_{N/1}$. Also in this case, [BL25a, Prop. 4.9] applies and shows that we have an isomorphism $\operatorname{colim} F(\mathcal{E}_N) = F_{N/1} \simeq F_N$, as desired.

Next we turn to the parallel extensions of matroids of rank 3 on 5 elements. We begin with a classification of such parallel extensions. The isomorphism classes of rank 3 matroids on 5 elements with a U_4^2-minor are $U_1^1 \oplus U_4^2$, C_5, and U_5^3. The matroid $U_1^1 \oplus U_4^2$ has two parallel extensions: $U_2^1 \oplus U_4^2$ and $U_1^1 \oplus C_5^*$. The matroid C_5 has also two parallel extensions: one extends a series element, the other a non-series element. The uniform matroid U_5^3 has a unique parallel extension.

We continue with a case by case inspection of these parallel extensions, but for $U_2^1 \oplus U_4^2$, which we cannot omit from the fundamental diagram for the reasons explained in Remark 6.4. In each case, we identify the ground set with $E = \{1, \dots, 6\}$ and assume that 1 and 2 are parallel.

As a first case, we consider $N = U_1^1 \oplus C_5^*$. The two parallel elements 1 and 2 belong to C_5^*, and we denote by 6 be the coloop of U_1^1. We illustrate in Figure 6.1 the fundamental diagram $\mathcal{E}_N$ of N, together with the isomorphism types of the embedded minors and the resulting diagram $F(\mathcal{E}_N)$ of their foundations:

This shows that the colimit of $F(\mathcal{E}_N)$ is $\mathbb{U}$, and identifies it canonically with the foundation of $N/6$. Since 6 is a coloop, the foundation of N is isomorphic to $F_{N/6}$ (cf. [BL25a, Prop. 4.9]) and thus $\operatorname{colim} F(\mathcal{E}_N) = F_{N/6} \to F_N$ is an isomorphism, as desired.

Next we consider the parallel extension N of C_5 by a series element. We denote the parallel elements by 1 and 2 and the complementary series element of C_5 by 3. The fundamental diagram $\mathcal{E}_N$ of N and the resulting diagram $F(\mathcal{E}_N)$ of their foundations are as illustrated in Figure 6.2.

As in the previous case, the colimit of $F(\mathcal{E}_N)$ is identified canonically with the foundation of $F_{N\backslash 1}$, which is isomorphic to the foundation of the parallel extension N of $N\backslash 1$ by [BL25a,

$$\begin{tikzcd}
\begin{array}{c} U_4^2 \\ N\backslash 1/6 \end{array} \arrow[r] & \begin{array}{c} C_5^* \\ N/6 \end{array} & \begin{array}{c} U_4^2 \\ N\backslash 2/6 \end{array} \arrow[l] \\
\mathbb{U} \arrow[r, "\sim"] & \mathbb{U} & \mathbb{U} \arrow[l, "\sim"']
\end{tikzcd}$$

Figure 6.1: The fundamental diagram of $U_1^1 \oplus C_5^*$ and the associated diagram of foundations.

$$\begin{tikzcd}
\begin{array}{c} U_4^2 \\ N\backslash 1/2 \end{array} \arrow[r] & \begin{array}{c} C_5 \\ N\backslash 1 \end{array} & \begin{array}{c} U_4^2 \\ N\backslash 1/3 \end{array} \arrow[l] \arrow[r] & \begin{array}{c} C_5^* \\ N/3 \end{array} & \begin{array}{c} U_4^2 \\ N\backslash 2/3 \end{array} \arrow[l] \arrow[r] & \begin{array}{c} C_5 \\ N\backslash 2 \end{array} & \begin{array}{c} U_4^2 \\ N\backslash 2/1 \end{array} \arrow[l] \\
\mathbb{U} \arrow[r, "\sim"] & \mathbb{U} & \mathbb{U} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathbb{U} & \mathbb{U} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathbb{U} & \mathbb{U} \arrow[l, "\sim"']
\end{tikzcd}$$

Figure 6.2: The fundamental diagram of the parallel extension of C_5 by a series element and the associated diagram of foundations.

Prop. 4.9]. Thus $\operatorname{colim} F(\mathcal{E}_N) = F_{N\backslash 1} \simeq F_N$, as desired.

Next, we consider the parallel extension N of C_5 by a non-series element. We denote the parallel elements by 1 and 2 and the series elements by 5 and 6. The fundamental diagram $\mathcal{E}_N$ of N, and the resulting diagram $F(\mathcal{E}_N)$ of their foundations, are as illustrated in Figure 6.3:

$$\begin{tikzcd}
 & N\backslash 1/5 \arrow[dl] \arrow[r] & N/5 & N\backslash 2/5 \arrow[l] \arrow[dr] & \\
\begin{array}{c} C_5 \\ N\backslash 1 \end{array} & U_4^2 & C_5^* & U_4^2 & \begin{array}{c} C_5 \\ N\backslash 2 \end{array} \\
 & N\backslash 1/6 \arrow[ul] \arrow[r] & N/6 & N\backslash 2/6 \arrow[l] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{U} \arrow[dl, "\sim"'] \arrow[r, "\sim"] & \mathbb{U} & \mathbb{U} \arrow[l, "\sim"'] \arrow[dr, "\sim"] & \\
\mathbb{U} & & & & \mathbb{U} \\
 & \mathbb{U} \arrow[ul, "\sim"] \arrow[r, "\sim"] & \mathbb{U} & \mathbb{U} \arrow[l, "\sim"'] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Figure 6.3: The fundamental diagram of the parallel extension of C_5 at a non-series element and the associated diagram of foundations.

In conclusion, the colimit of $F(\mathcal{E}_N)$ is a quotient of $\mathbb{U}$ by an automorphism group that stems from the monodromy of the diagram. As a parallel extension of $N\backslash 1 \simeq C_5$, the foundation of N is isomorphic to $F_{N\backslash 1} \simeq \mathbb{U}$. Since the diagram $F(\mathcal{E}'_N)$, which includes the foundation $F_N \simeq \mathbb{U}$ of N, is commutative, we conclude that the monodromy of $F(\mathcal{E}_N)$ is trivial and thus $\operatorname{colim} F(\mathcal{E}_N) \to F_N$ is an isomorphism, as claimed.

The last case under investigation is the parallel extension N of U_5^3. The fundamental diagram $\mathcal{E}_N$ of N and the resulting diagram $F(\mathcal{E}_N)$ of their foundations are as in Figure 6.4. By [BL25a, Prop. 4.9], the embedding of $N\backslash 1$ into its parallel extension N induces an isomorphism $\mathbb{V} = F_{N\backslash 1} \simeq F_N$. We aim to show that the canonical inclusion $F_{N\backslash 1} \to \operatorname{colim} F(\mathcal{E}_N)$ is an isomorphism. This follows if we can show that the morphisms

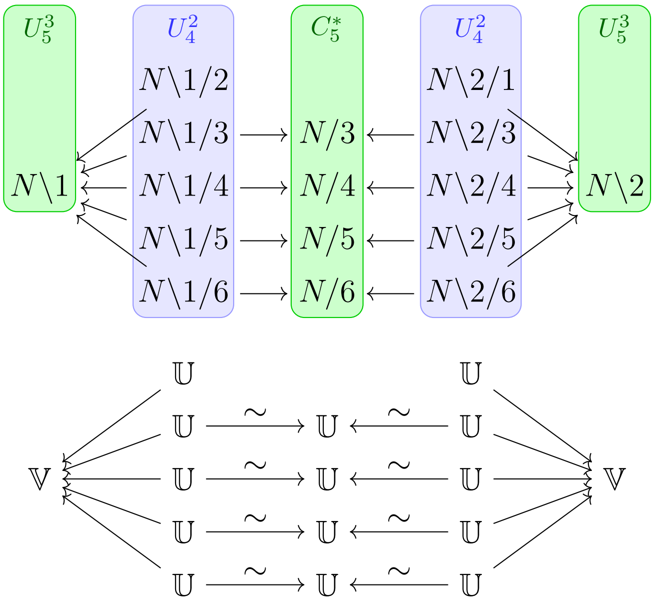

Figure 6.4: The fundamental diagram $\mathcal{E}_N$ of the parallel extension of U_5^3 and the induced diagram of foundations.

$$\Psi_i : \mathbb{U}^{\otimes 4} \simeq \bigotimes_{j=3}^{6} F_{N\backslash i/k} \longrightarrow F_{N\backslash i} \simeq \mathbb{V}$$

are epimorphisms for $i = 1, 2$, since in this case, $F_{N\backslash 1}$ and $F_{N\backslash 2}$ will be identified in the colimit of $F(\mathcal{E}_N)$. By symmetry, it suffices to show that Ψ_1 is an epimorphism. By Proposition 4.1, we have

$$F_{N\backslash 1} = \mathbb{F}_1^{\pm}(x_2, \dots, x_6) /\!\!/ \langle\!\langle x_i + x_{i-1}x_{i+1} - 1 \mid i = 2, \dots, 6 \rangle\!\rangle$$

for $x_i = \left[\begin{smallmatrix} i+1 & i+2 \\ i+3 & i+4 \end{smallmatrix}\right]$ and $y_i = x_{i-1}x_{i+1} = \left[\begin{smallmatrix} i+1 & i+3 \\ i+2 & i+4 \end{smallmatrix}\right]$, where we consider $i + k$ modulo 5 as an element of $\{2, \dots, 6\}$. For $i = 3, \dots, 6$, the cross ratio x_i of $F_{N\backslash 1}$ is the image of the corresponding cross ratio of $F_{N\backslash i/k} \simeq \mathbb{U}$. Since

$$x_2 = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} = x_5^{-1} \cdot y_6,$$

where x_5^{-1} stems from $F_{N\backslash 1/5}$ and y_6 stems from $F_{N\backslash 1/6}$, x_2 also lies in the image of the map Ψ_1. This shows that Ψ_1 is surjective, and therefore an epimorphism of pastures. This concludes our argument that $\operatorname{colim} F(\mathcal{E}_N) = F_{N\backslash 1} \simeq F_N$, as desired. □

Remark 6.4. To see that we cannot remove embedded minors of type $U_2^1 \oplus U_4^2$ from the fundamental diagram, let us consider $M = U_2^1 \oplus U_4^2$, whose fundamental diagram $\mathcal{E}_M$, together with the induced diagram $F(\mathcal{E}_M)$ of the associated foundations, is as illustrated in Figure 6.5:

$$\begin{tikzcd}
\overset{U_4^2}{M\backslash 1/2} \arrow[r] & \overset{U_2^1 \oplus U_4^2}{M} & \overset{U_4^2}{M\backslash 2/1} \arrow[l] \\
\mathbb{U} \arrow[r, "\sim"] & \mathbb{U} & \mathbb{U} \arrow[l, "\sim"']
\end{tikzcd}$$

Figure 6.5: The fundamental diagram of $U_2^1 \oplus U_4^2$ and the associated diagram of foundations.

Since $\mathbb{U}$ is a terminal object of the diagram $F(\mathcal{E}_M)$, we conclude that $F_M \simeq \mathbb{U}$. If we were to omit the embedded minors of type $U_2^1 \oplus U_4^2$ from $\mathcal{E}_M$, then the resulting diagram of foundations would consist of two isolated copies of $\mathbb{U}$ whose colimit is $\mathbb{U} \otimes \mathbb{U}$. This shows that we cannot omit minors of this type in general.

Also none of the other minors of types U_4^2, U_5^2, U_5^3, C_5, C_5^*, F_7 and F_7^* can be omitted from the fundamental diagram without violating the result of Theorem 6.3. This can be seen as follows: all proper minors of U_4^2 are regular, so if we omit $M = U_4^2$ from the list, then $\operatorname{colim} \mathcal{E}_M = \mathbb{F}_1^\pm \neq \mathbb{U} = F_M$. Similarly, all proper minors of $M = F_7$ are regular, and omitting F_7 from the fundamental type yields $\operatorname{colim} \mathcal{E}_M = \mathbb{F}_1^\pm \neq \mathbb{F}_2 = F_M$. Since the fundamental diagram and the fundamental presentation are both symmetric under duality, the same holds for F_7^*. The matroid $M = C_5$ has two non-regular proper minors, which are both of type U_4^2. Therefore the fundamental presentation is $\mathbb{U} \to \mathbb{U} \leftarrow \mathbb{U}$, equal to that of $\mathbb{U}_4^2 \oplus \mathbb{U}_2^1$. Omitting C_5 from $\mathcal{E}_M$ yields $\operatorname{colim} \mathcal{E}_M = \mathbb{U} \otimes \mathbb{U} \neq \mathbb{U} = F_M$. By duality, the same holds for C_5^*. The matroid U_5^2 has 5 non-regular proper minors, which are of type U_4^2. Thus omitting $M = U_5^2$ from the fundamental diagram yields $\operatorname{colim} \mathcal{E}_M = \mathbb{U}^{\otimes 5} \neq \mathbb{V} = F_M$. By duality, the same holds for U_5^3.

6.2. The fundamental lattice presentation

In this section, we describe a presentation of the foundation in terms of the lattice of flats of a matroid. This approach has computational advantages, because several types of embedded minors correspond to the same sublattice, which leads to a more compact presentation of the foundation as a colimit.

By [BL25a, Prop. 4.9], the foundation F_M of a matroid M only depends on its simplification and therefore is determined by the lattice of flats Λ of M, which justifies the notation $F_\Lambda = F_M$. As explained in Section 2.3, the cross ratios $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_I = \left[\begin{smallmatrix} \langle Ia\rangle & \langle Ib\rangle \\ \langle Ic\rangle & \langle Id\rangle \end{smallmatrix}\right]$ in $F_M = F_\Lambda$ only depend on the hyperplanes $\langle Ia\rangle$, $\langle Ib\rangle$, $\langle Ic\rangle$ and $\langle Id\rangle$, which are corank 1 elements of Λ. This yields intrinsic generators of F_Λ in terms of Λ.

Definition 6.5. An *upper sublattice* is a matroid sublattice Λ' (i.e. atomistic and semimodular) whose rank r' equals the corank of its bottom element F as an element in Λ (or as a flat in M).

Definition 6.6. If Λ is a lattice and S is a subset of atoms of Λ, we define the *sublattice* Λ_S *induced by* S to be the set of all $x \in \Lambda$ such that x is a join of elements in S. If Λ is a lattice and $x, y \in \Lambda$, we denote by $[x, y]$ the *interval* $\{z \in \Lambda \mid x \leqslant z \leqslant y\}$.

According to the "Scum Theorem"[3] of D. A. Higgs, Λ_N is isomorphic to an upper sublattice of Λ_M for every minor N of M, and every upper sublattice of Λ_M is isomorphic to Λ_N for some minor N of M. However, this is not a one-to-one correspondence. The following result, phrased in terms of embedded minors rather than minors, makes the correspondence more precise. To state the result, let Emb_M denote the set of embedded minors of a matroid M, and let USL_Λ denote the set of upper sublattices of a lattice Λ.

Proposition 6.7. *Define* $\Psi : \mathrm{Emb}_M \to \mathrm{USL}_{\Lambda_M}$ *by sending* $M\backslash J/I$ *to* $[\langle I\rangle, E]_S$ *where* $S = \{\langle I, e\rangle \mid e \notin I \cup J\}$ *are the atoms of* $[\langle I\rangle, E]$ *that are generated by the elements of* $M\backslash J/I$. *Then* Ψ *is surjective and* $\Psi(N) \cong \Lambda_N$ *for every embedded minor* N *of* M.

Proof. According to Proposition 3.3.7 in [Oxl92], the flats of $M\backslash J$ are the subsets of E of the form $F - J$ such that F is a flat of M, and the flats of M/I are subsets F' of E such that $F' \cup I$ is a flat of M. Thus a subset F' of $E - (I \cup J)$ is a flat of $N = M\backslash J/I$ if and only if $F' \cup I = F - J$ for some flat F of M. (The flat F is determined by F' since the closure of $F - J$ is equal to F as $E - J$ is spanning.)

It follows that for every embedded minor $N = M\backslash J/I$ and $S = \{\langle I, e\rangle \mid e \notin I \cup J\}$, the lattice $[\langle I\rangle, E]_S$ is an upper sublattice of Λ_M that is isomorphic to Λ_N. The surjectivity of Ψ follows since every upper sublattice of Λ_M is of the form $[F, E]_S$ for some $F \in \Lambda$ and some subset S of atoms of $[F, E]$, and since $[F, E]_S = \Psi(M\backslash J/I)$ for I and J such that $F = \langle I\rangle$ and $S = \{\langle I, e\rangle \mid e \notin I \cup J\}$. □

Given an upper sublattice Λ' of Λ_M, we find an embedded minor $N = M\backslash J/I$ of M with sublattice $\Lambda_N = \Lambda'$ by Proposition 6.7. Consequently, the lattice inclusion $\Lambda' \hookrightarrow \Lambda$ induces a morphism of foundations $F_{\Lambda'} = F_N \to F_M = F_\Lambda$, which can be described intrinsically in terms of the tautological association

$$\begin{tikzcd}
F_{\Lambda'} \arrow[r] & F_\Lambda \\
\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \arrow[r, mapsto] & \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}
\end{tikzcd}$$

for modular quadruples (H_1, H_2, H_3, H_4) of hyperplanes of Λ'. We say that an upper sublattice Λ' of Λ is an N-*sublattice*, or *of type* N, if Λ' is isomorphic to the lattice of flats of the matroid N.

Definition 6.8. Let M be a matroid with lattice Λ_M. The *fundamental lattice diagram* $\mathcal{L}_M$ *of* M is the diagram of all upper sublattices Λ' of M of types U^2_4, U^2_5, U^3_5, C_5, F_7 and F^*_7 (as illustrated in Figure 6.6), together with all lattice inclusions $\Lambda' \hookrightarrow \Lambda''$. The $\mathcal{L}_M$-presentation of M is the induced diagram $F(\mathcal{L}_M)$ of foundations of the lattices in $\mathcal{L}_M$.

Theorem 6.9. *Let* M *be a matroid with fundamental lattice diagram* $\mathcal{L}_M$. *Then* $F_M \simeq \operatorname{colim} F(\mathcal{L}_M)$.

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[3] The name comes from the fact that, like scum which rises to the top of a pond, the lattice of flats of any minor of M can be found "hanging from the top" of the lattice of flats of M.

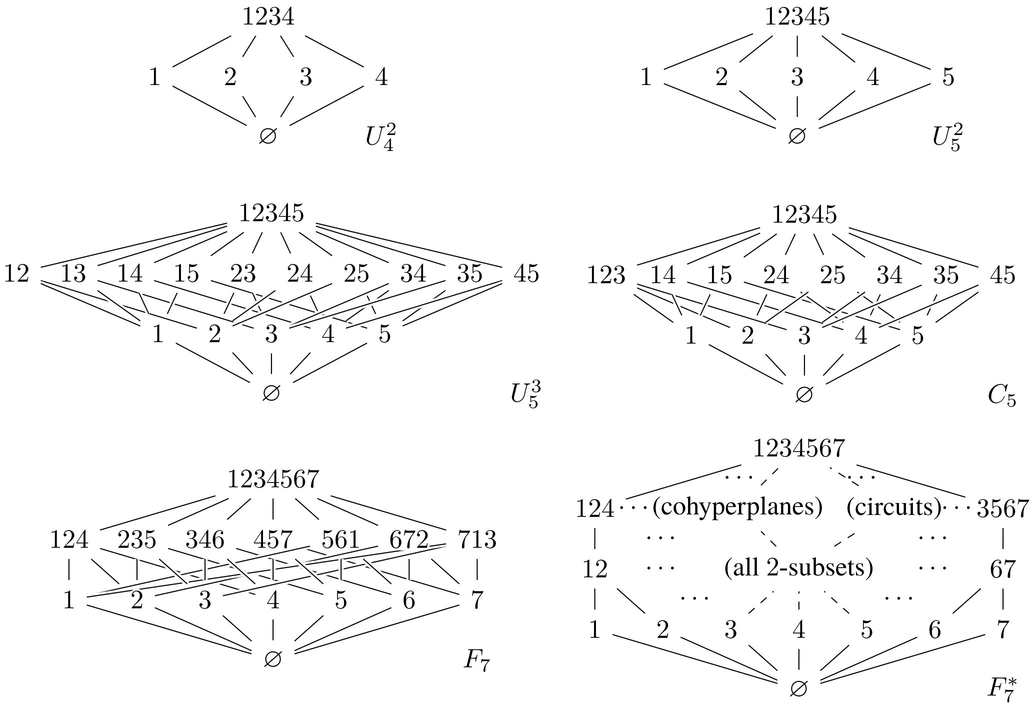


Figure 6.6: Lattices of types U_4^2, U_5^2, U_5^3, C_5, F_7 and F_7^*.

Proof. By Theorem 6.3, the foundation F_M is the colimit of $F(\mathcal{E}_M)$, where $\mathcal{E}_M$ is the fundamental diagram of M. By our previous considerations, the association $\mathcal{E}_M \mapsto F(\mathcal{E}_M)$ factors through the diagram $\mathcal{L}'_M$ of upper sublattices Λ_N of M that correspond to the embedded minors $N = M\backslash J/I$ in $\mathcal{E}_M$. Thus Λ'_M consists of all upper sublattices of M of types U_4^2, U_5^2, U_5^3, C_5, C_5^*, $U_2^1 \oplus U_4^2$, F_7 and F_7^* and $F_M = \operatorname{colim} F(\mathcal{E}_M) = \operatorname{colim} F(\mathcal{L}'_M)$. Since the lattice of $N = C_5^*$ is equal to the lattice of type U_4^2, this case is subsumed by type U_4^2.

For upper sublattices of type $N = U_2^1 \oplus U_4^2$, we proceed as in the proof of Theorem 6.3: we show that the foundation F_{Λ_N} is the colimit of $F(\mathcal{L}_N)$, where $\mathcal{L}_N$ is the fundamental lattice diagram of N. Let $E = \{1, \dots, 6\}$ be the ground set of N, where we assume that 1 and 2 are the parallel elements corresponding to the direct summand U_2^1. The lattice of N is as illustrated in Figure 6.7:

The fundamental lattice diagram $\mathcal{L}_N$ of N consists of a unique upper sublattice Λ' of type U_4^2, which is characterized by its hyperplanes 123, 124, 125, and 126. Since the foundation $F_N \simeq \mathbb{U}$ of N is isomorphic to the foundation of U_4^2, we obtain the desired isomorphism $\operatorname{colim} F(\mathcal{L}_N) = \mathbb{U} \simeq F_N$. □

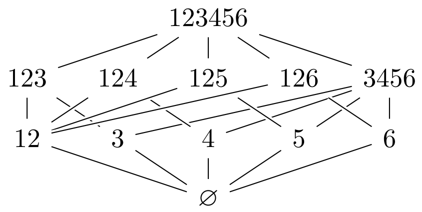


Figure 6.7: The lattice of $U_2^1 \oplus U_4^2$.

The nice thing about the lattice presentation $F(\mathcal{L}_M)$ of the foundation F_M, in contrast to the embedded minor presentation $F(\mathcal{E}_M)$, is that it is more economical and therefore better for explicit computations; cf. Section 7.2.

Remark 6.10. *A posteriori,* we can associate with the different types of sublattices in $\mathcal{L}_M$ relations between non-degenerate cross ratios $\left[\begin{smallmatrix} H_1 & H_2 \\ H_3 & H_4 \end{smallmatrix}\right]$, as follows. (We denote by H_F the hyperplane that appear as 'F' in the illustration in Figure 6.6.)[4]

- Type U_4^2 corresponds to the relations

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} + \begin{bmatrix} H_1 & H_3 \\ H_2 & H_4 \end{bmatrix} = 1; \tag{H+}$$

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} = \begin{bmatrix} H_2 & H_1 \\ H_4 & H_3 \end{bmatrix} = \begin{bmatrix} H_3 & H_4 \\ H_1 & H_2 \end{bmatrix} = \begin{bmatrix} H_4 & H_3 \\ H_2 & H_1 \end{bmatrix}; \tag{Hσ}$$

$$\begin{bmatrix} H_1 & H_2 \\ H_4 & H_3 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}^{-1}; \tag{H1}$$

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \cdot \begin{bmatrix} H_1 & H_3 \\ H_4 & H_2 \end{bmatrix} \cdot \begin{bmatrix} H_1 & H_4 \\ H_2 & H_3 \end{bmatrix} = -1. \tag{H2}$$

- Type U_5^2 corresponds to the relation

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \cdot \begin{bmatrix} H_1 & H_2 \\ H_4 & H_5 \end{bmatrix} \cdot \begin{bmatrix} H_1 & H_2 \\ H_5 & H_3 \end{bmatrix} = 1. \tag{H3}$$

- Type U_5^3 corresponds to the relation

$$\begin{bmatrix} H_{15} & H_{25} \\ H_{35} & H_{45} \end{bmatrix} \cdot \begin{bmatrix} H_{13} & H_{23} \\ H_{43} & H_{53} \end{bmatrix} \cdot \begin{bmatrix} H_{14} & H_{24} \\ H_{54} & H_{34} \end{bmatrix} = 1. \tag{H4}$$

---

[4] To be precise, we will reorder the elements of a flat F when this yields a more systematic description of a relation. The relations have to read in the sense that all indices 1, 2, 3, 4 (and 5 in (H3) and (H4)) can be permuted. In (H4'), the permutation of elements must preserve the circuit 123, so we cannot exchange 4 and 5 by any of 1, 2 and 3 in this relation.

- Type C_5 (with circuit 123) corresponds to the relation

$$\begin{bmatrix} H_{14} & H_{24} \\ H_{34} & H_{54} \end{bmatrix} = \begin{bmatrix} H_{15} & H_{25} \\ H_{35} & H_{45} \end{bmatrix}. \tag{H4'}$$

- Types F_7 and F_7^* correspond to the relation

$$-1 = 1. \tag{H–}$$

As a consequence of Theorem 6.9, we deduce the following presentation for the foundation, which is helpful for inductive computations of foundations of matroids of large rank in terms of matroids of smaller rank (but with possibly more elements). For a concrete application, see Proposition 9.9.

Let M be a matroid with lattice Λ_M. A *full upper sublattice of* Λ_M is an upper sublattice Λ' of Λ_M for which an inclusion $F' \subset F$ of flats $F' \in \Lambda'$ and $F \in \Lambda_M$ implies that $F \in \Lambda'$. In other words, if F_0 is the bottom of Λ', then Λ' consists of precisely all flats $F \in \Lambda_M$ that contain F_0. In particular, the full upper sublattices of Λ_M are in bijection with the elements of Λ_M.

We write $\mathcal{L}_M^{\leqslant r}$ for the diagram of all full upper sublattices of Λ_M of rank less or equal to r that contain a sublattice of type U_4^2, F_7 or F_7^*, together with all lattice inclusions. Note that every upper sublattice in $\mathcal{L}_M$ contains an upper sublattice of type U_4^2, F_7 or F_7^*.

Theorem 6.11. *Let M be a matroid with lattice Λ_M. Then the canonical morphism* $\operatorname{colim} F(\mathcal{L}_M^{\leqslant 4}) \to F_M$ *is an isomorphism. If M either contains a minor of type F_7 or* does not *contain a minor of type F_7^*, then the canonical morphism* $\operatorname{colim} F(\mathcal{L}_M^{\leqslant 3}) \to F_M$ *is an isomorphism.*

Proof. These claims follow from general considerations about colimits. First of all, we observe that every every upper sublattice Λ' of $\mathcal{L}_M$ embeds into a (unique) full upper sublattice $\overline{\Lambda'}$ with the same bottom element, which is in $\Lambda_M^{\leqslant 4}$. Moreover, every lattice inclusion $\Lambda' \to \Lambda''$ in $\mathcal{L}_M$ extends to a lattice inclusion $\overline{\Lambda'} \to \overline{\Lambda''}$ in $\Lambda_M^{\leqslant 4}$.

Consequently, these lattice inclusions induce a morphism $F_M \simeq \operatorname{colim} F(\mathcal{L}_M) \to \operatorname{colim} F(\mathcal{L}_M^{\leqslant 4})$ that is a section to the canonical morphism $\operatorname{colim} F(\mathcal{L}_M^{\leqslant 4}) \to F_M$. For a given upper sublattice Λ', we denote by $\mathcal{L}_{\Lambda'}^{\mathrm{rk}}$ the family of all sublattices Λ'' of Λ' of the same rank that are in $\mathcal{L}_M$. The inclusion $\Lambda'' \to \Lambda'$ induces a morphism $F_{\Lambda''} \to F_{\Lambda'}$ and thus a morphism

$$\bigotimes_{\Lambda'' \in \mathcal{L}_{\Lambda'}^{\mathrm{rk}}} F_{\Lambda''} \longrightarrow F_{\Lambda'}.$$

This morphism is surjective, and thus an epimorphism, since $F_{\Lambda'} = \operatorname{colim} F(\mathcal{L}_{M(\Lambda')})$ (by Theorem 6.9), and since $\mathcal{L}_{\Lambda'}^{\mathrm{rk}}$ contains all maximal elements of $\mathcal{L}_{M(\Lambda')}$, where $M(\Lambda')$ is the simple matroid with lattice Λ'. We conclude that the section $F_M \to \operatorname{colim} F(\mathcal{L}_M^{\leqslant 4})$ is an epimorphism, and thus an isomorphism that is inverse to $\operatorname{colim} F(\mathcal{L}_M^{\leqslant 4}) \to F_M$. The first claim follows.

If M has no minor of type F_7^*, then every sublattice in $\mathcal{L}_M$ has rank at most 3 and the above arguments hold with $\mathcal{L}_M^{\leqslant 4}$ replaced by $\mathcal{L}_M^{\leqslant 3}$. If M has a minor of type F_7, then the corresponding upper sublattice is of rank 3 and induces the relation $1 = -1$ on F_M. Therefore we can remove all upper sublattices of type F_7^* from $\mathcal{L}_M$ without changing the colimit and, once again, the above arguments yield the desired result with $\mathcal{L}_M^{\leqslant 4}$ replaced by $\mathcal{L}_M^{\leqslant 3}$. The second claim follows. □

Similar arguments lead to the following variant of Theorem 6.11 (we omit a formal proof):

Theorem 6.12. *Let M be a matroid of rank $\geqslant 3$ and Λ_M its lattice of flats. Let $\mathcal{L}_M^{3+}$ be the diagram of all upper sublattices of Λ_M of the types U_4^2 and F_7^* and all full upper sublattices of rank 3, together with all lattice inclusions. Then $F_M \simeq \operatorname{colim} F(\mathcal{L}_M^{3+})$.*

Finally we mention the following result for future reference.

Lemma 6.13. *Let M be a matroid with lattice Λ_M, let $N = M\backslash J/I \hookrightarrow M$ be an embedded minor with lattice Λ_N, and let $\Lambda_N \to \Lambda'$ be an inclusion of upper sublattices of Λ_M. If Λ' has the same rank as Λ_N, then there exists an embedded minor $N' = M\backslash J'/I$ with $N \hookrightarrow N'$, i.e. $J' \subset J$ and $\Lambda' = \Lambda_{N'}$ as upper sublattices of Λ_M.*

Proof. Let $G = \langle I \rangle$ be the bottom element of Λ_N, which is the smallest flat of M that is contained in Λ_N. Since Λ' is an upper sublattice of the same rank as Λ_N and $\Lambda_N \subset \Lambda'$, the flat G is also the bottom element of Λ'. Consider an atom F of Λ', which is a flat of M that covers G and is contained in Λ'. If F is not in Λ_N, then $F - G \subset J$. We choose, for every atom F of $\Lambda' - \Lambda_N$, an element e_F in $F - G$ and define

$$J' \;=\; J \;-\; \{\, e_F \mid \text{atoms } F \text{ of } \Lambda' - \Lambda_N \,\}$$

and $N' = M\backslash J'/I$. Then $F = \langle I e_F \rangle$ is an atom of $\Lambda_{N'}$ and all other atoms of $\Lambda_{N'}$ are atoms of Λ_N. This shows that the atoms of $\Lambda_{N'}$ and Λ' agree. Since upper sublattices are atomistic, this shows that $\Lambda' = \Lambda_{N'}$. ☐

Remark 6.14. The condition that Λ' and Λ_N have the same rank cannot be dropped in Lemma 6.13 in general. For instance, for $M = U_3^2$ with ground set $E = \{1, 2, 3\}$, the upper sublattice Λ_N of the minor $N = M/12$ consists of the single element 123 and thus is a sublattice of $\Lambda' = \Lambda_{M/3}$. But the lattice inclusion $\Lambda_N \to \Lambda'$ is not induced by a minor embedding $N \hookrightarrow N'$ for any embedded minor N' of M.

We conclude with the observation that equalities between universal cross ratios are invariant under automorphisms of the matroid and simultaneous permutation of the 4 entries in a cross ratio.

Proposition 6.15. *Let M be a matroid with ground set E and foundation F_M. Let $\varphi : E \to E$ be an automorphism of M and $\sigma \in S_4$ a permutation. Let $H_1, \dots, H_4$ and $H'_1, \dots, H'_4$ be two modular quadruples of hyperplanes such that $\left[\begin{smallmatrix} H_1 & H_2 \\ H_3 & H_4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} H'_1 & H'_2 \\ H'_3 & H'_4 \end{smallmatrix}\right]$ as elements of F_M. Then*

$$\begin{bmatrix} \varphi(H_{\sigma(1)}) & \varphi(H_{\sigma(2)}) \\ \varphi(H_{\sigma(3)}) & \varphi(H_{\sigma(4)}) \end{bmatrix} \;=\; \begin{bmatrix} \varphi(H'_{\sigma(1)}) & \varphi(H'_{\sigma(2)}) \\ \varphi(H'_{\sigma(3)}) & \varphi(H'_{\sigma(4)}) \end{bmatrix}.$$

Proof. By Theorem 6.9 and Remark 6.10, all relations between cross ratios are a concatenation of the relations of types (H+)–(H–). Since an automorphism of M induces an automorphism of the lattice Λ_M of flats of M, which permutes the upper sublattices of M in an inclusion and type preserving way, it also preserves equalities between cross ratios.

To establish our assertion for a permutation σ of the 4 entries of cross ratios, we note that any equality between cross ratios stems from a chain of relations of types (Hσ) and (H4'). Type (Hσ) is evidently invariant under σ. To deduce the invariance for (H4'), we note that the inclusions $\Lambda'/i \to \Lambda' \leftarrow \Lambda'/j$ of the two upper sublattices Λ'/i and Λ'/j of types U_4^2 into an upper sublattice Λ' of type C_5 yields isomorphisms $\mathbb{U} \to \mathbb{U} \leftarrow \mathbb{U}$. We conclude that all cross ratios of Λ'/i are identified with those of Λ/j, in a way that is stable under permutations of the four entries of cross ratios. $\square$

6.3. Foundations of matroids without large uniform minors

A matroid is *without large uniform minors* if it has no minors of type U_5^2 or U_5^3. These are the only types of minors in the fundamental presentation $\mathcal{E}_M$ of M whose foundation is $\mathbb{V}$.

Thus, for a matroid M without large uniform minors, the fundamental diagram $F(\mathcal{E}_M)$ consists of copies of $\mathbb{F}_2$ and $\mathbb{U}$. More precisely, copies of $\mathbb{F}_2$ appear as isolated vertices in $F(\mathcal{E}_M)$, and any morphism between two copies of $\mathbb{U}$ is an isomorphism. Thus, the colimit of a connected component $\mathcal{C}$ of $F(\mathcal{E}_M)$ is either $\mathbb{F}_2$ or a *symmetry quotient of* $\mathbb{U}$, by which we mean a quotient of $\mathbb{U}$ by a group of automorphisms.

The automorphism group of $\mathbb{U}$ is $\mathrm{Aut}(\mathbb{U}) = S_3$ (cf. [BL25a, Prop. 5.6]), and the non-trivial symmetry quotients of $\mathbb{U}$ are

$$\begin{aligned}
\mathbb{D} \;&=\; \mathbb{F}_1^\pm(x)/\!\!/\langle\!\langle x-1-1\rangle\!\rangle,\\
\mathbb{H} \;&=\; \mathbb{F}_1^\pm(\zeta_6)/\!\!/\langle\!\langle \zeta_6^3+1,\ \zeta_6+\zeta_6^{-1}-1\rangle\!\rangle,\\
\mathbb{F}_3 \;&=\; \mathbb{F}_1^\pm/\!\!/\langle\!\langle 1+1+1\rangle\!\rangle
\end{aligned}$$

(cf. [BL25a, Prop. 5.8]). Since the colimit of a diagram is the coproduct of the colimits of its connected components, this recovers the *structure theorem for foundations of matroids without large uniform minors* (cf. [BL25a, Thm. 5.9]):

Theorem 6.16. *Let M be a matroid without large uniform minors, F_M its foundation, and r the number of connected components of the fundamental diagram of M. Then*

$$F_M \;\simeq\; F_1 \otimes \cdots \otimes F_r \tag{6.1}$$

with $F_1, \dots, F_r \in \{\mathbb{U},\ \mathbb{D},\ \mathbb{H},\ \mathbb{F}_3,\ \mathbb{F}_2\}$. $\square$

Note that (the underlying graph of) the fundamental lattice diagram $\mathcal{L}_M$ is a contraction of (the underlying graph of) the fundamental diagram $\mathcal{E}_M$ and has, in particular, the same number of connected components. A result that we apply in the computations of several examples in Section 7 and Section 9 is the following.

Corollary 6.17. *Let M be a matroid without minors of types U_5^2, U_5^3, F_7 or F_7^*, and assume that its fundamental (lattice) diagram is connected and non-empty. Then the foundation of M is a symmetry quotient of $\mathbb{U}$.* $\square$

7. More examples

We apply the methods from the previous sections to compute the foundation of Q_6 (following Oxley's notation in [Oxl92, p. 641]) and the one element restriction $AG(2,3)\backslash e$ of the ternary affine plane (cf. [Oxl92, p. 653]).

7.1. The foundation of Q_6

The matroid Q_6 is the rank 3 matroid on 6 elements whose 3-circuits are illustrated in Figure 7.1.

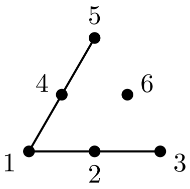


Figure 7.1: The circuits of Q_6.

Proposition 7.1. *The foundation of* Q_6 *is* $\mathbb{V}$.

Proof. The matroid $M = Q_6$ has one embedded minor of type U^2_5, which is $M/6$, and one embedded minor of type U^3_5, which is $M\backslash 1$. It has several embedded minors of types U^2_4, C_5, and C_5^*, and none of types F_7 and F_7^*. The fundamental diagram is illustrated in Figure 7.2:

$$\begin{tikzcd}
 & U^2_4 & C_5 & U^2_4 & C_5^* & U^2_4 & \\
 & M\backslash 2/6 \arrow[r] \arrow[ddl] & M\backslash 2 & M\backslash 2/3 \arrow[l] \arrow[r] & M/3 & M\backslash 1/3 \arrow[l] \arrow[ddr] & \\
U^2_5 & M\backslash 3/6 \arrow[r] \arrow[dl] & M\backslash 3 & M\backslash 3/2 \arrow[l] \arrow[r] & M/2 & M\backslash 1/2 \arrow[l] \arrow[dr] & U^3_5 \\
M/6 & M\backslash 4/6 \arrow[r] \arrow[l] & M\backslash 4 & M\backslash 4/5 \arrow[l] \arrow[r] & M/5 & M\backslash 1/5 \arrow[l] \arrow[r] & M\backslash 1 \\
 & M\backslash 5/6 \arrow[r] \arrow[ul] & M\backslash 5 & M\backslash 5/4 \arrow[l] \arrow[r] & M/4 & M\backslash 1/4 \arrow[l] \arrow[ur] & \\
 & & & M\backslash 1/6 \arrow[uulll] \arrow[uurrr] & & &
\end{tikzcd}$$

Figure 7.2: The fundamental diagram $\mathcal{E}^{\min}_{Q_6}$ of embedded minors of Q_6.

In order to compute the colimit of $F(\mathcal{E}_M)$, we investigate how the foundations of the embedded minors become identified. Recall from Proposition 4.1 that the foundation of $M/6 \simeq U^2_5$ is isomorphic to

$$\mathbb{V} \;=\; \mathbb{F}_1^{\pm}(x_i \mid i = 1, \dots, 5) /\!\!/ \langle\!\langle x_i + x_{i-1}x_{i+1} - 1 \mid i = 1, \dots, 5 \rangle\!\rangle$$

via the isomorphism $\alpha : \mathbb{V} \to F_{M/6}$ with

$$x_1 \mapsto \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}_6, \quad x_2 \mapsto \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}_6, \quad x_3 \mapsto \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}_6, \quad x_4 \mapsto \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}_6, \quad x_5 \mapsto \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}_6.$$

Similarly, there is an isomorphism $\beta : \mathbb{V} \to F_{M\backslash 1}$ with

$$x_1 \mapsto \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}_6, \quad x_2 \mapsto \begin{bmatrix} 2 & 5 \\ 6 & 4 \end{bmatrix}_3, \quad x_3 \mapsto \begin{bmatrix} 5 & 4 \\ 3 & 6 \end{bmatrix}_2, \quad x_4 \mapsto \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}_5, \quad x_5 \mapsto \begin{bmatrix} 6 & 3 \\ 5 & 2 \end{bmatrix}_4.$$

The isomorphisms α and β are compatible with the minor embeddings $M/6 \hookrightarrow M$ and $M\backslash 1 \hookrightarrow M$ in the sense that the respective images of the x_i become identified in F_M, as follows from the computation below. Recall from [BL25a, Prop. 5.3] that the relations (R0), (R1), and (R4) imply that the cross ratios of the two embedded U^2_4-minors of a C_5-minor are pairwise identified, e.g.,

$$\begin{bmatrix} 3 & i \\ j & k \end{bmatrix}_6 = \begin{bmatrix} 6 & i \\ j & k \end{bmatrix}_3$$

in $M\backslash 2$, which also applies if the four entries of the cross ratios are simultaneously permuted. Similarly, we find identifications

$$\begin{bmatrix} i & j \\ k & 1 \end{bmatrix}_2 = \begin{bmatrix} i & j \\ k & 3 \end{bmatrix}_2$$

in $M/2$; cf. also [BL25a, Prop. 5.3]. We label these equalities by the respective embedded minors that induce them in the following equations.

Note that every cross ratio stems from a (unique) embedded minor of type U^2_4. We verify that

$$\begin{aligned}
\alpha(x_1) &= \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}_6 = \beta(x_1), \\
\alpha(x_2) &= \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}_6 \underset{M\backslash 2}{=} \begin{bmatrix} 6 & 4 \\ 1 & 5 \end{bmatrix}_3 \underset{M/3}{=} \begin{bmatrix} 6 & 4 \\ 2 & 5 \end{bmatrix}_3 \underset{(\mathrm{H}\sigma)}{=} \begin{bmatrix} 2 & 5 \\ 6 & 4 \end{bmatrix}_3 = \beta(x_2), \\
\alpha(x_3) &= \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}_6 \underset{M\backslash 3}{=} \begin{bmatrix} 4 & 5 \\ 6 & 1 \end{bmatrix}_2 \underset{M/2}{=} \begin{bmatrix} 4 & 5 \\ 6 & 3 \end{bmatrix}_2 \underset{(\mathrm{H}\sigma)}{=} \begin{bmatrix} 5 & 4 \\ 3 & 6 \end{bmatrix}_2 = \beta(x_3), \\
\alpha(x_4) &= \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}_6 \underset{M\backslash 4}{=} \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}_5 \underset{M/5}{=} \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}_5 \underset{(\mathrm{H}\sigma)}{=} \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}_5 = \beta(x_4), \quad \text{and} \\
\alpha(x_5) &= \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}_6 \underset{M\backslash 5}{=} \begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix}_4 \underset{M/4}{=} \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}_4 \underset{(\mathrm{H}\sigma)}{=} \begin{bmatrix} 6 & 3 \\ 5 & 2 \end{bmatrix}_4 = \beta(x_5)
\end{aligned}$$

as elements of F_M.

Since every upper sublattice of type U^2_4 is contained in either $\Lambda/6$ or $\Lambda\backslash 1$, the foundation F_M is generated by the images of $F_{\Lambda/6}$ and $F_{\Lambda\backslash 1}$. Since we have taken all relations into consideration (note that the $y_i \in \mathbb{V}$ are the unique elements with $x_i + y_i = 1$, so $\alpha(y_i)$ equals $\beta(y_i)$ in F_M as well), it follows that the canonical morphisms $F_{\Lambda/6} \to F_M$ and $F_{\Lambda\backslash 1} \to F_M$ are isomorphisms, and thus $F_M \simeq \mathbb{V}$. □

7.2. The foundation of the Möbius-Kantor configuration $AG(2,3)\backslash e$

The Möbius-Kantor configuration is the single-element deletion of the ternary affine plane $AG(2,3)$. Since the automorphism group of $AG(2,3)$ acts transitively on its points, all of the single-element deletions of $AG(2,3)$ are isomorphic to each other. We write $AG(2,3)\backslash e$ for one of these minors. Figure 7.3 contains a depiction of its 3-circuits, and Figure 7.4 illustrates its lattice of flats.

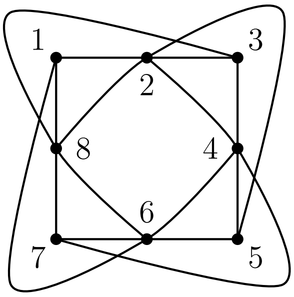


Figure 7.3: The 3-circuits of $AG(2,3)\backslash e$.

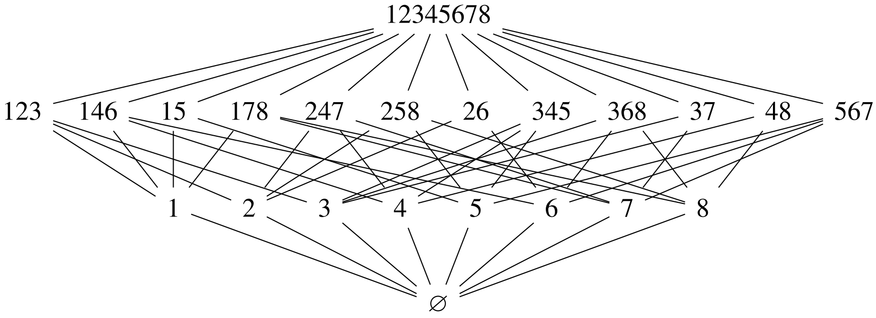


Figure 7.4: The lattice of flats of $AG(2,3)\backslash e$.

Proposition 7.2. *The foundation of* $M = AG(2,3)\backslash e$ *is isomorphic to* $\mathbb{H}$.

Proof. Let $\Lambda = \Lambda_M$ be the lattice of flats of $M = AG(2,3)\backslash e$ as illustrated in Figure 7.4. It contains 8 upper sublattices of type U_4^2, whose respective bottom elements are precisely the 8 atoms of Λ. We denote the U_4^2-sublattice with bottom element i by Λ/i for $i = 1,\dots,8$. The lattice Λ contains 32 upper sublattices of type C_5, which can be obtained from $AG(2,3)\backslash e$ by deleting 3 elements i, j and k that either form a 3-circuit (e.g. $\{i,j,k\} = \{1,2,3\}$) or such that $|i-j| = 4$ (e.g. $\{i,j,k\} = \{1,5,6\}$). We denote these sublattices by $\Lambda\backslash jkl$. These are all

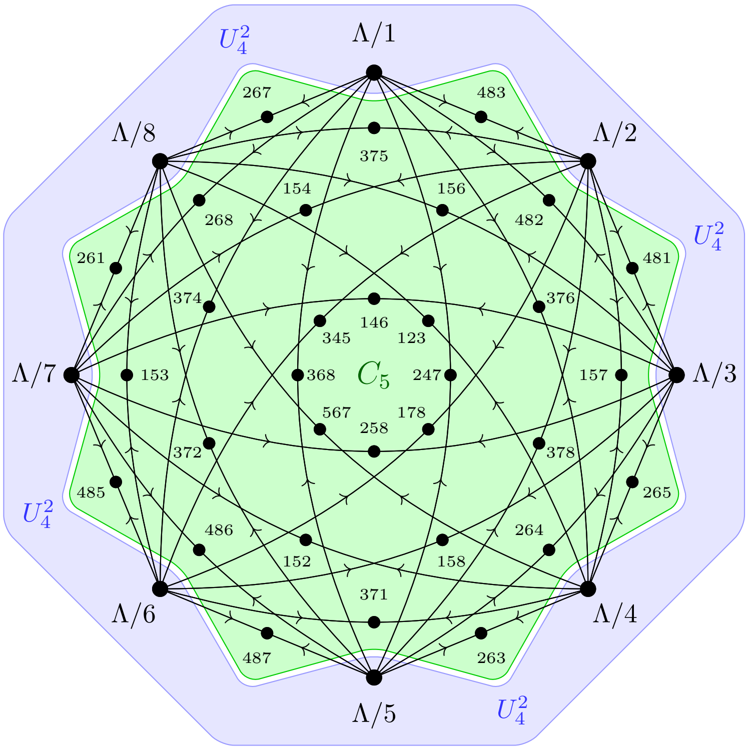


Figure 7.5: The fundamental lattice diagram of $AG(2,3)\backslash e$.

upper sublattices that appear in the fundamental lattice diagram $\mathcal{L}_M$ of M, which is illustrated in Figure 7.5, where we label the sublattices $\Lambda\backslash ijk$ of type C_5 by "ijk" due to space limitations.

Since the fundamental lattice diagram $\mathcal{L}_M$ is connected and does not contain upper sublattices of types U_5^2, U_5^3, F_7 or F_7^*, Corollary 6.17 implies that the foundation F_M of M is a symmetry quotient of $\mathbb{U}$.

By [Oxl92, p. 653], $AG(2,3)$ is quarternary, and so is $AG(2,3)\backslash e$. Therefore F_M cannot be $\mathbb{D}$ nor $\mathbb{F}_3$, which leaves only the possibilities $\mathbb{U}$ and $\mathbb{H}$. We exhibit the defining relation $x \sim y^{-1}$ of $\mathbb{H} = \mathbb{U}/\!\!/\{x - y^{-1}\}$ in the following.

Consider the map $\alpha : \mathbb{U} \simeq F_{\Lambda/1} \to F_M$ with $\alpha(x) = \left[\begin{smallmatrix}123 & 146\\ 15 & 178\end{smallmatrix}\right]$, and thus $\alpha(y^{-1}) = \left[\begin{smallmatrix}123 & 15\\ 178 & 146\end{smallmatrix}\right]$. We use the following part of $\mathcal{L}_M$ to identify $\alpha(x)$ with $\alpha(y^{-1})$ in F_M:

$$\begin{tikzcd}
 & & \Lambda/1 \arrow[dl] \arrow[dr] & & \\
 & \Lambda\backslash 267 & & \Lambda\backslash 483 & \\
\Lambda/8 \arrow[ur] \arrow[rr] & & \Lambda\backslash 375 & & \Lambda/2 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

In the following computation, we label an equality of cross ratios in F_M that is induced by an upper sublattice of type C_5 by the corresponding sublattice. We have

$$\alpha(x) = \begin{bmatrix} 123 & 146 \\ 15 & 178 \end{bmatrix} \underset{\Lambda\backslash 267}{=} \begin{bmatrix} 368 & 48 \\ 258 & 178 \end{bmatrix} \underset{\Lambda\backslash 375}{=} \begin{bmatrix} 26 & 247 \\ 258 & 123 \end{bmatrix} \underset{\Lambda\backslash 483}{=} \begin{bmatrix} 146 & 178 \\ 15 & 123 \end{bmatrix}$$

$$\underset{(\mathrm{H}\sigma)}{=} \begin{bmatrix} 123 & 15 \\ 178 & 146 \end{bmatrix} = \alpha(y^{-1}),$$

which proves our assertion that $F_M \simeq \mathbb{U}/\!\!/\{x - y^{-1}\} = \mathbb{H}$. ☐

8. Fundamental types

8.1. The fundamental type of a class of matroids

In Section 6.1, we have seen that the foundation of a matroid M can be presented as the colimit of the foundations of all special embedded minors of M, which consists of the isomorphism types of U^2_4, U^2_5, U^3_5, C_5, C^*_5, $U^2_4 \oplus U^1_2$, F_7 and F^*_7. In fact, this presentation is minimal: the foundation of none of the matroids in this list can be written as the colimit of the foundations of its proper minors; cf. Remark 6.4.

If we are given a class $\mathcal{C}$ of matroids, it is natural to wonder if there is a minimal set $\mathcal{C}_0$ of isomorphism classes of matroids in $\mathcal{C}$ such that the foundation of every matroid M in $\mathcal{C}$ is the colimit of the foundations of all embedded minors of M whose isomorphism type is in $\mathcal{C}_0$.

Given a matroid M and a family $\mathcal{C}_0$ of isomorphism types of matroids, we denote by $\mathcal{E}_{M,\mathcal{C}_0}$ the diagram of all embedded minors N of M whose isomorphism class belongs to $\mathcal{C}_0$, together with all minor embeddings. We denote by $\Omega_{M,\mathcal{C}_0} : \mathrm{colim}\ F(\mathcal{E}_{M,\mathcal{C}_0}) \to F_M$ the canonical morphism from the colimit of the foundations of embedded minors of M with isomorphism type $\mathcal{C}_0$ into the foundation of M.

Definition 8.1. Let $\mathcal{C}$ be a class of matroids. A *fundamental type of* $\mathcal{C}$ is a minimal family $\mathcal{C}_0$ of isomorphism classes of matroids such that $\Omega_{M,\mathcal{C}_0} : \mathrm{colim}\ F(\mathcal{E}_{M,\mathcal{C}_0}) \to F_M$ is an isomorphism for all $M \in \mathcal{C}$.

It turns out that every class of matroids possesses a unique fundamental type. Before we prove this, let us consider some examples:

- The fundamental type of all matroids consists of U^2_4, U^2_5, U^3_5, C_5, C^*_5, $U^2_4 \oplus U^1_2$, F_7 and F^*_7 (by Theorem 6.3).
- The fundamental type of all regular matroids is empty since the foundation of a regular matroid is $\mathbb{F}^\pm_1$ (by [BL21, Thm. 7.35]).
- The fundamental type of all binary matroids consists of F_7 and F^*_7, since the foundation of a binary matroid is $\mathbb{F}^\pm_1$ or $\mathbb{F}_2$ (by [BL21, Thm. 7.32]).
- The fundamental type of matroids without U^2_5 and U^3_5-minors consists of U^2_4, C_5, C^*_5, $U^2_4 \oplus U^1_2$, F_7 and F^*_7 (by Theorem 6.3; cf. also [BL25a, Thm. 5.9]).

In Section 8.2 (resp. Section 8.3), we describe the fundamental types of the classes of all 2-connected (resp. 3-connected) matroids.

The next result shows that a fundamental type always exists, and is uniquely determined through an explicit description. For M in $\mathcal{C}$, we denote by $\mathcal{E}^{<}_{M,\mathcal{C}}$ the diagram of all *proper* embedded minors $N = M\backslash J/I$ (i.e. $N \neq M$) that are in $\mathcal{C}$, together with all minor embeddings. We denote by $\Omega^{<}_{M,\mathcal{C}} : \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}}) \to F_M$ the canonical morphism.

Theorem 8.2. *Every class $\mathcal{C}$ of matroids has a unique fundamental type $\mathcal{C}_{\text{fund}}$, which consists of all isomorphism classes of matroids M in $\mathcal{C}$ for which $\Omega^{<}_{M,\mathcal{C}} : \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}}) \to F_M$ is* not *an isomorphism.*

Proof. Let $\mathcal{C}_{\text{fund}}$ be the collection of isomorphism classes of matroids M in $\mathcal{C}$ for which the canonical map $\Omega^{<}_{M,\mathcal{C}} : \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}}) \to F_M$ is not an isomorphism. As a first step we show that $\mathcal{C}_{\text{fund}}$ is contained in every fundamental type $\mathcal{C}_0$ of $\mathcal{C}$.

Let M be a matroid that does not belong to $\mathcal{C}_0$. Then the diagram $\mathcal{E}_{M,\mathcal{C}_0}$ of embedded minors of M with isomorphism class in $\mathcal{C}_0$ is a subdiagram of $\mathcal{E}^{<}_{M,\mathcal{C}}$, which induces a morphism $\Phi_M : \operatorname{colim} F(\mathcal{E}_{M,\mathcal{C}_0}) \to \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}})$ that commutes with the canonical morphisms to F_M:

$$\begin{tikzcd}
\operatorname{colim} F(\mathcal{E}_{M,\mathcal{C}_0}) \arrow[rr, "\Phi_M"] \arrow[dr, "\sim", "\Omega_{M,\mathcal{C}_0}"'] & & \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}}) \arrow[dl, "\Omega^{<}_{M,\mathcal{C}}"] \\
& F_M &
\end{tikzcd}$$

If we define $\widetilde{\Phi}_M = \Phi_M \circ (\Omega_{M,\mathcal{C}_0})^{-1}$, then this means that $\widetilde{\Phi}_M \circ \Omega^{<}_{M,\mathcal{C}} = \operatorname{id}_{F_M}$.

In order to show that $\Omega^{<}_{M,\mathcal{C}} \circ \widetilde{\Phi}_M$ is the identity on $\operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}})$, we note that the foundation F_N of every embedded minor N of M is generated by the universal cross ratios of N, and that every universal cross ratio stems from a U^2_4-minor of N. Since $\Omega_{N,\mathcal{C}_0} : \operatorname{colim} F(\mathcal{E}_{N,\mathcal{C}_0}) \to F_N$ is an isomorphism, every embedded U^2_4-minor of N is contained in a (minimal) embedded minor N' in $\mathcal{E}_{N,\mathcal{C}_0}$. Thus the canonical morphism

$$\pi : \bigotimes_{N \in \mathcal{E}^{\min}_{M,\mathcal{C}_0}} F_N \longrightarrow \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}})$$

is an epimorphism, where $\mathcal{E}^{\min}_{M,\mathcal{C}_0}$ is the collection of minimal embedded minors of $\mathcal{E}_{M,\mathcal{C}_0}$ that contain a U^2_4-minor. Since $\mathcal{E}^{\min}_{M,\mathcal{C}_0}$ is contained in $\mathcal{E}_{M,\mathcal{C}_0}$, we have naturally that

$$\Omega^{<}_{M,\mathcal{C}} \circ \widetilde{\Phi}_M \circ \pi \;=\; \pi \;=\; \operatorname{id}_{\operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}})} \circ \pi.$$

Since π is an epimorphism, this implies that $\Omega^{<}_{M,\mathcal{C}} \circ \widetilde{\Phi}_M$ is the identity on $\operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}})$.

We conclude that $\Omega^{<}_{M,\mathcal{C}} : \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}}) \to F_M$ is an isomorphism and that the isomorphism class of M does not belong to $\mathcal{C}_{\text{fund}}$. This shows that $\mathcal{C}_{\text{fund}}$ is contained in $\mathcal{C}_0$ as claimed. Once we have shown that $\mathcal{C}_{\text{fund}}$ is a fundamental type, the uniqueness of $\mathcal{C}_{\text{fund}}$ follows by the minimality of fundamental types.

Given a matroid M, we write $\mathcal{E}^{\text{fund}}_{M,\mathcal{C}} = \mathcal{E}_{M,\mathcal{C}_{\text{fund}}}$ and $\Omega^{\text{fund}}_{M,\mathcal{C}} = \Omega_{M,\mathcal{C}_{\text{fund}}}$. We aim to show that $\Omega^{\text{fund}}_{M,\mathcal{C}} : \operatorname{colim} F(\mathcal{E}^{\text{fund}}_{M,\mathcal{C}}) \to F_M$ is an isomorphism for all $M \in \mathcal{C}$. If $\Omega^{<}_{M,\mathcal{C}}$ is not an isomorphism, then M belongs to $\mathcal{E}^{\text{fund}}_{M,\mathcal{C}}$ and forms its terminal object. Therefore $\Omega^{\text{fund}}_{M,\mathcal{C}}$ is tautologically an isomorphism.

Assume that $\Omega^{<}_{M,\mathcal{C}}$ is an isomorphism. Let $\mathcal{E}$ be a full subdiagram of $\mathcal{E}^{<}_{M,\mathcal{C}}$, i.e., whenever it contains two embedded minors N and N' of M such that N is an embedded minor of N', then the minor embedding $N \hookrightarrow N'$ is in $\mathcal{E}$. The inclusion of diagrams $\mathcal{E} \hookrightarrow \mathcal{E}^{<}_{M,\mathcal{C}}$ induces a pasture morphism

$$\Phi_{\mathcal{E}} : \operatorname{colim} F(\mathcal{E}) \longrightarrow \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}}) \simeq F_M.$$

Since the isomorphism type of M does not belong to $\mathcal{C}_{\text{fund}}$, the diagram $\mathcal{E}^{\text{fund}}_{M,\mathcal{C}}$ is a subdiagram of $\mathcal{E}^{<}_{M,\mathcal{C}}$. We show by induction on the number k of embedded minors in $\mathcal{E}^{<}_{M,\mathcal{C}}$ that are not in $\mathcal{E}$ that $\Phi_{\mathcal{E}}$ is an isomorphism provided that $\mathcal{E}$ contains $\mathcal{E}^{\text{fund}}_{M,\mathcal{C}}$. If $k = 0$, then $\mathcal{E} = \mathcal{E}^{<}_{M,\mathcal{C}}$, so $\Phi_{\mathcal{E}}$ is the identity, which establishes the base case of the induction.

Let $k > 0$. Let $N \in \mathcal{E}^{<}_{M,\mathcal{C}}$ be an embedded minor of minimal size in the complement of $\mathcal{E}$. Then all proper embedded minors of N are in $\mathcal{E}$, i.e., $\mathcal{E}^{<}_{N,\mathcal{C}}$ is a subdiagram of $\mathcal{E}$. Since N is not in $\mathcal{E}^{\text{fund}}_{M,\mathcal{C}} \subset \mathcal{E}$, the canonical morphism $\operatorname{colim} F(\mathcal{E}^{<}_{N,\mathcal{C}}) \to F_N$ is an isomorphism. Let $\mathcal{E}'$ be the full subdiagram of $\mathcal{E}^{<}_{M,\mathcal{C}}$ that contains $\mathcal{E}$ and N. Thus $\Phi_{\mathcal{E}}$ factors into

$$\Phi_{\mathcal{E}} : \operatorname{colim} F(\mathcal{E}) \xrightarrow{\sim} \operatorname{colim} F(\mathcal{E}') \xrightarrow{\Phi_{\mathcal{E}'}} F_M.$$

By the inductive hypothesis, $\Phi_{\mathcal{E}'}$ is an isomorphism, and so is $\Phi_{\mathcal{E}}$, which completes the inductive step.

Since the number of embedded minors of M is finite, this shows that the morphism $\Phi_{\mathcal{E}^{\text{fund}}_{M,\mathcal{C}}} : \operatorname{colim} F(\mathcal{E}^{\text{fund}}_{M,\mathcal{C}}) \to F_M$ is an isomorphism, which concludes the proof. □

Given a matroid N and a class of matroids $\mathcal{C}$, we call a matroid M in $\mathcal{C}$ together with a minor embedding $N \simeq M\backslash J/I \hookrightarrow M$ a *minimal $\mathcal{C}$-extension of N* if all intermediate embedded minors $N \hookrightarrow N' \hookrightarrow M$ with $N' \neq M$ are not in $\mathcal{C}$. The following result allows us to determine new fundamental types from known ones in terms of their minimal extensions.

Proposition 8.3. *Let $\mathcal{D} \subset \mathcal{C}$ be classes of matroids with respective fundamental types $\mathcal{D}_{\text{fund}}$ and $\mathcal{C}_{\text{fund}}$. Then $\mathcal{D}_{\text{fund}}$ consists of the isomorphism types of all minimal $\mathcal{D}$-extensions M of matroids N with isomorphism class in $\mathcal{C}_{\text{fund}}$ for which* $\operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{D}}) \to F_M$ *is not an isomorphism.*

Proof. By Theorem 8.2, the isomorphism class of M belongs to the fundamental type of $\mathcal{D}$ if and only if the canonical morphism $\Phi_M : \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{D}}) \to F_M$ is not an isomorphism. Thus the claim follows at once for the minimal $\mathcal{D}$-extensions M of matroids with isomorphism type in $\mathcal{C}_{\text{fund}}$.

Due to the characterization of $\mathcal{D}_{\text{fund}}$ in Theorem 8.2, we are left with verifying that $\Omega^{<}_{M,\mathcal{D}} : \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{D}}) \to F_M$ is an isomorphism if M is in $\mathcal{D}$ and not a minimal $\mathcal{D}$-extension of any of its embedded minors N with isomorphism class in $\mathcal{C}_{\text{fund}}$. Note that this means in particular that the isomorphism class of M is not in $\mathcal{C}_{\text{fund}}$, and thus $\Omega^{<}_{M,\mathcal{C}} : \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}}) \to F_M$ is an isomorphism.

Let $\mathcal{E}^{\mathrm{fund}}_{N,\mathcal{C}} = \mathcal{E}_{N,\mathcal{C}_{\mathrm{fund}}}$ be the diagram of all embedded minors of a matroid N with isomorphism class in $\mathcal{C}_{\mathrm{fund}}$, and let $F^{\mathrm{fund}}_{N,\mathcal{C}} = \operatorname{colim} F(\mathcal{E}^{\mathrm{fund}}_{N,\mathcal{C}})$. Then by the defining property of the fundamental type, the canonical morphism $\Omega^{\mathrm{fund}}_{N,\mathcal{C}} : F^{\mathrm{fund}}_{N,\mathcal{C}} \to F_N$ is an isomorphism.

A minor embedding $N \hookrightarrow N'$ induces an inclusion of diagrams $\mathcal{E}^{\mathrm{fund}}_{N,\mathcal{C}} \hookrightarrow \mathcal{E}^{\mathrm{fund}}_{N',\mathcal{C}}$ and a morphism $F^{\mathrm{fund}}_{N,\mathcal{C}} \to F^{\mathrm{fund}}_{N',\mathcal{C}}$ between the colimits of the respective diagrams of foundations, which corresponds to the canonical morphism $F_N \to F_{N'}$ under the isomorphisms $\Omega^{\mathrm{fund}}_{N,\mathcal{C}}$ and $\Omega^{\mathrm{fund}}_{N',\mathcal{C}}$. If $\mathcal{F}^{<}_{M,\mathcal{D}}$ is the diagram of all colimits $F^{\mathrm{fund}}_{N,\mathcal{C}}$, together with the morphisms $F^{\mathrm{fund}}_{N,\mathcal{C}} \to F^{\mathrm{fund}}_{N',\mathcal{C}}$ induced by the inclusions $N \hookrightarrow N'$ of embedded minors N and N' of M that are in $\mathcal{D}$, then the previous observations imply that the colimit over the isomorphisms $\Omega^{\mathrm{fund}}_{N,\mathcal{C}}$ yields an isomorphism $\Theta_{M,\mathcal{D}} : \operatorname{colim} \mathcal{F}^{<}_{M,\mathcal{D}} \to \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{D}})$.

Since M is not a minimal $\mathcal{D}$-extension of any matroid N for which $[N] \in \mathcal{C}_{\mathrm{fund}}$, we find for every embedded minor N of M with $[N] \in \mathcal{C}_{\mathrm{fund}}$ a proper embedded minor $N' \in \mathcal{D}$ of M that contains N. Thus the union of the diagrams $\mathcal{E}^{\mathrm{fund}}_{N,\mathcal{C}}$, where N ranges over all proper embedded minors of M that are in $\mathcal{D}$, is all of $\mathcal{E}^{\mathrm{fund}}_{M,\mathcal{C}}$. Therefore we have a canonical identification $\operatorname{colim} F(\mathcal{E}^{\mathrm{fund}}_{M,\mathcal{C}}) \cong \operatorname{colim} \mathcal{F}^{<}_{M,\mathcal{D}}$. This yields a commutative diagram

$$\begin{tikzcd}
\operatorname{colim} F(\mathcal{E}^{\mathrm{fund}}_{M,\mathcal{C}}) \arrow[r, "\sim"] \arrow[d, "\Omega^{\mathrm{fund}}_{M,\mathcal{C}}"', "\sim"] & \operatorname{colim} \mathcal{F}^{<}_{M,\mathcal{D}} \arrow[r, "\sim", "\Theta_{M,\mathcal{D}}"'] & \operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{D}}) \arrow[d, "\Omega^{<}_{M,\mathcal{D}}"] \\
F_M \arrow[rr, "\mathrm{id}"] & & F_M
\end{tikzcd}$$

which establishes our claim that $\Omega^{<}_{M,\mathcal{D}}$ is an isomorphism and completes the proof. □

If $\mathcal{C}$ is the class of all matroids, Proposition 8.3 reads as follows:

Corollary 8.4. *Let $\mathcal{D}$ be a class of matroids. Then its fundamental type consists of the isomorphism types of all minimal $\mathcal{D}$-extensions M of*

$$U^2_4, \qquad U^2_5, \qquad U^3_5, \qquad C_5, \qquad C^*_5, \qquad U^2_4 \oplus U^1_2, \qquad F_7 \qquad \textit{and} \qquad F^*_7$$

for which $\operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{D}}) \to F_M$ *is not an isomorphism.*

Proof. This follows at once from Proposition 8.3 applied to the class $\mathcal{C}$ of all matroids and Theorem 6.3, which characterizes the fundamental type in this case as the isomorphism classes of U^2_4, U^2_5, U^3_5, C_5, C^*_5, $U^2_4 \oplus U^1_2$, F_7 and F^*_7 and satisfies the hypotheses of Proposition 8.3. □

Remark 8.5. In general, two different classes $\mathcal{C}$ and $\mathcal{C}'$ can have the same fundamental type. Since their union has the same type, there is a maximal class for each given fundamental type. Examples of such maximal classes are regular matroids, binary matroids, and matroids without U^2_5 or U^3_5-minors. The classes $\mathcal{C}^{(k)}$ of k-connected matroids (for $k \geqslant 2$) are not maximal, since we can add a loop to a matroid in $\mathcal{C}^{(k)}$ without changing the foundation. This leads us to ask:

Problem 8.6. Can we characterize which families of matroids are the maximal classes for some fundamental type?

Not all fundamental types are finite, as there are infinite families of matroids that are not minors of each other, such as the class of excluded minors for orientable matroids. This suggests:

Problem 8.7. Is there a simple set of sufficient conditions, or, more ambitiously, both necessary and sufficient conditions, for a class of matroids to have a finite fundamental type?

Remark 8.8. Fundamental types are in some sense complementary to excluded minors when applied to the class $\mathcal{C}$ of matroids representable over some pasture F: while excluded minors are isomorphism types of minimal matroids that do not appear in $\mathcal{C}$, the fundamental type of $\mathcal{C}$ consists of isomorphism types of certain matroids that *do* appear in $\mathcal{C}$.

The difficulty of these complementary approaches is, however, of a very different nature. While it is in general hard to find a complete list of excluded minors, the fundamental type of $\mathcal{C}$ is easily determined—the difficulty lies rather in the structure of the fundamental diagram.

More explicitly, the fundamental type of $\mathcal{C}$ consists of the isomorphism types of those matroids among U_4^2, U_5^2, U_5^3, C_5, C_5^*, $U_4^2 \oplus U_2^1$, F_7 and F_7^* that are representable over F. This follows from Corollary 8.4 and the fact that if a matroid N from this list is *not* representable over F, then no extension of N is F-representable. In particular, this means that the fundamental type of the class of F-representable matroids is finite for every pasture F.

For connectivity considerations, however, the question of determining fundamental types is more interesting:

Problem 8.9. Can one explicitly determine the fundamental type for the class of 4-connected, or vertically 4-connected, matroids?

8.2. The 2-connected fundamental presentation

If M is 2-connected, then it turns out that we can omit the embedded minors of type $U_4^2 \oplus U_2^1$ from the fundamental diagram without changing the fundamental presentation $\operatorname{colim}\ F(\mathcal{E}_M)$ of the foundation F_M of M. The key idea for the proof of this claim was communicated to us by Nathan Bowler.

Definition 8.10. The 2*-connected fundamental diagram of* M is the diagram $\mathcal{E}_M^{(2)}$ of embedded minors of isomorphism types

$$U_4^2, \qquad U_5^2, \qquad U_5^3, \qquad C_5, \qquad C_5^*, \qquad F_7, \qquad F_7^*,$$

together with all minor embeddings.

Note that $\mathcal{E}_M^{(2)}$ is a subdiagram of the fundamental diagram $\mathcal{E}_M$. More precisely, it consists of all embedded minors of $\mathcal{E}_M$ that are 2-connected. If M is 2-connected, then we call the associated diagram $F(\mathcal{E}_M^{(2)})$ of foundations the 2*-connected fundamental presentation of* M, which is motivated by the following result. Let $\mathcal{C}^{(2)}$ be the class of all 2-connected matroids.

Theorem 8.11. *The fundamental type of* $\mathcal{C}^{(2)}$ *consists of the isomorphism types of the matroids* U_4^2, U_5^2, U_5^3, C_5, C_5^*, F_7 *and* F_7^**. This is, if* M *is a* 2*-connected matroid with foundation* F_M *and* 2*-connected fundamental diagram* $\mathcal{E}_M^{(2)}$*, then* $F_M \simeq \operatorname{colim}\ F(\mathcal{E}_M^{(2)})$.

The rest of this section is dedicated to the proof of this theorem. By Corollary 8.4, we only need to determine which of the minimal 2-connected extensions of U_4^2, U_5^2, U_5^3, C_5, C_5^*, $U_4^2 \oplus U_2^1$, F_7 and F_7^* belong to the fundamental type $\mathcal{C}_{\text{fund}}^{(2)}$ of $\mathcal{C}^{(2)}$. Since each of U_4^2, U_5^2, U_5^3, C_5, C_5^*, F_7 and F_7^* is 2-connected, they are each their own unique minimal 2-connected extension. And since the foundation of none of these matroids is a colimit of foundations of proper embedded minors, each belongs to $\mathcal{C}_{\text{fund}}^{(2)}$.

We are left with an investigation of the minimal 2-connected extensions of $U_4^2 \oplus U_2^1$. We assume that the reader is familiar with Cunningham and Edmond's canonical tree decomposition of 2-connected matroids; for details see [Cun73] and [Oxl92, Thm. 8.3.10].

Lemma 8.12. *Let M be a minimal 2-connected extension of $N = U_4^2 \oplus U_2^1$ with $N = M\backslash J/I$. Then $n = \#E_M \geqslant 7$, and up to duality, the tree decomposition of M is*

$$M_1 \xrightarrow{\ e_1\ } M_2 \xrightarrow{\ e_2\ } \cdots \xrightarrow{\ e_{n-5}\ } M_{n-4},$$

where $M_i \simeq U_3^2$ for $i \geqslant 2$ even, $M_i \simeq U_3^1$ for $i \geqslant 2$ odd, and $M_1 \simeq U_5^2$. Moreover,

$$J \ = \ E_M \ \cap \bigcup_{2 \leqslant i \leqslant n-5 \text{ odd}} E_{M_i} \qquad \textit{and} \qquad I \ = \ E_M \ \cap \bigcup_{2 \leqslant i \leqslant n-5 \text{ even}} E_{M_i}.$$

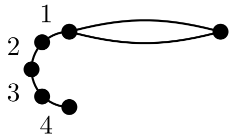
Figure 8.1: Minimal 2-connected extensions of $U_4^2 \oplus U_2^1$.

Remark 8.13. The minimal 2-connected extensions of $N = U_4^2 \oplus U_2^1$ with $M_1 = U_5^2$ are illustrated in Figure 8.1 for $n = \#E_M$ equal to 12 and to 13, which exemplify the cases $M_{n-4} = U_3^2$ and $M_{n-4} = U_3^1$, respectively.

The black half-circle with edges 1, 2, 3 and 4 stands symbolically for the 2-summand $M_1 = U_5^2$, and might be seen as a stylized version of the common illustration of the point line configuration of U_5^2. We use the convention $E_M \cap E_{M_1} = \{1, \dots, 4\}$, as well as $E_M \cap E_{M_i} = \{i+3\}$ for $i = 2, \dots, n-5$ and $E_M \cap E_{M_{n-4}} = \{n-1, n\}$. The elements of J are illustrated as green edges and the elements of I are illustrated as red edges.

Since M is a parallel-series extension of U_5^2, we can illustrate the deletion and contraction of elements $i \geqslant 5$ of M in terms of deleting or contracting the corresponding edge. In particular, a subset J of $E_M - \{1, \dots, 4\}$ is coindependent if it is not an edge cut and a subset I of $E_M - \{1, \dots, 4\}$ is independent if $I \cup \{1, \dots, 4\}$ is a forest.

Excluding the green edges and contracting the red edges results in both cases of Figure 8.1 in



which illustrates the matroid $N_0 = U_4^2 \oplus U_2^1$.

Proof of Lemma 8.12. Let us identify E_M with $\{1, \dots, n\}$ such that $E_{N_0} = \{1, \dots, 4, a, b\}$ with $1, \dots, 4$ being the elements of the summand U_4^2 of $N_0 = U_4^2 \oplus U_2^1$, and $a = n-1$ and $b = n$ being elements of U_2^1. Since N_0 is not 2-connected and has 6 elements, it is clear that $n \geqslant 7$.

Let $M_1, \dots, M_r$ be the 2-summands of the canonical tree decomposition of M. This means that each M_i is either 3-connected, a circuit (i.e. isomorphic to U_k^1) or a cocircuit (i.e. isomorphic to U_k^{k-1}). Since U_4^2 is 3-connected and appears as a minor of N_0, and thus of M, we conclude that $1, \dots, 4$ belong to the same 2-summand of M, say M_1. Since U_4^2 is not contained in a circuit or cocircuit, we conclude that M_1 is 3-connected.

The assumption that M is a minimal 2-connected extension of N_0 has a series of implications. Since the removal of a single element of a 3-connected matroid results in a 2-connected minor, none of $M_2, \dots, M_r$ are 3-connected, and thus each element $e \in E_M - E_{N_0}$ is contained in a circuit $M_i \simeq U_{n_i}^1$ or in a cocircuit $M_i \simeq U_{n_i}^{n_i-1}$. Note that deleting e from a circuit $M_i \simeq U_{n_i}^1$, as well as contracting e from a cocircuit $M_i \simeq U_{n_i}^{n_i-1}$, leaves M 2-connected. By the minimality of M, we conclude that if $e \in M_i \simeq U_{n_i}^1$, then $e \in I$, and if $e \in M_i \simeq U_{n_i}^{n_i-1}$, then $e \in J$. Moreover, since J is coindependent and I is independent by the definition of N_0 as an embedded minor, we conclude that M_i contains at most one element that is also in M.

Since $U_4^2 \oplus U_2^1$ is not 2-connected, a and b are not in M_1. Since $N_0 \backslash 12/34 \simeq U_2^1$ is 2-connected, but $M \backslash e$ for $e \in J$ and M/e for $e \in I$ are not, we conclude that a and b are contained in the same component M_i, say M_r, and that M_r does not contain any other element of M. Once again, by the minimality of M, we conclude that all other components $M_2, \dots, M_{r-1}$ must lie on a path between M_1 and M_r, i.e., that the tree decomposition of M is of the form

$$\begin{tikzcd} M_1 \arrow[r, dash, "e_1"] & M_2 \arrow[r, dash, "e_2"] & \cdots \arrow[r, dash, "e_{r-1}"] & M_r \end{tikzcd}$$

after reordering the indices appropriately. Thus M_1 has elements $1, 2, 3, 4, e_1$ and is isomorphic to U_5^2 or U_5^3 as a 3-connected matroid on 5 elements. After taking duals (as allowed in the claim of the lemma), we can assume that $M_1 \simeq U_5^2$. Since U_2^1 does not appear in a canonical tree decomposition, we conclude that that every component $M_2, \dots, M_{r-1}$ contains precisely 1 element of M and M_r has elements a, b, e_{r-1}. Thus M_i is isomorphic to U_3^1 or to U_3^2 for every $i = 2, \dots, r$. Counting elements, we conclude that $r = n - 4$.

Let 5 be the unique element in $E_{M_2} \cap E_M$. Since $M \backslash (J-5)/(I-5)$ has a U_5^2-minor by [Oxl92, Prop. 7.1.21], but $M \backslash J/I$ is not 2-connected, we conclude that $5 \in I$. In conclusion, $M_2 \simeq U_3^2$. Since adjacent 2-summands cannot both be circuits, nor can they both be cocircuits, in the canonical tree decomposition of a matroid, this determines the isomorphism types of $M_3, \dots, M_{n-4}$, which proves all claims of the lemma. □

In the following arguments, we use a chain of series extensions of U^2_5 that appear as embedded minors of M. We denote the series extension of U^2_5 by D_6. We fix the ground set $E_{D_6} = \{1, \dots, 6\}$, where 5 and 6 are the series elements.

Lemma 8.14. *For every permutation $\sigma \in S_4$, we have*

$$\begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_5 = \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_6$$

in D_6.

Proof. The dual D_6^* of D_6 is a parallel extension of U^3_5 with parallel elements 5 and 6. Thus $\left[\begin{smallmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{smallmatrix}\right]_5 = \left[\begin{smallmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{smallmatrix}\right]_6$ in D_6^* by relation (R5) of Theorem 3.5. Applying the canonical isomorphism $\varphi : F_{D_6} \to F_{D_6^*}$ to the cross ratios in question yields

$$\varphi\left(\begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_5\right) = \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_6 = \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_5 = \varphi\left(\begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_6\right),$$

and thus the relation claimed in the lemma. □

We use the previous insights to study the foundation of a fixed minimal 2-connected extension M of N_0. Since foundations are insensitive to dualization, we can assume that M is a parallel-series extension of $M_1 = U^2_5$. We label the elements of M according to the conventions of Figure 8.1: if

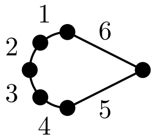
$$M_1 \overset{e_1}{\text{———}} M_2 \overset{e_2}{\text{———}} \cdots \overset{e_{n-5}}{\text{———}} M_{n-4}$$

is the tree decomposition of M, then

$$E_{M_1} \cap M = \{1,2,3,4\}, \qquad E_{M_i} \cap E_M = \{i+3\}, \qquad E_{M_{n-4}} \cap E_M = \{a,b\}$$

for $i = 2, \dots, n-5$, and where $a = n-1$ and $b = n$.

Since $D_6 \simeq U^2_5 \oplus_2 U^2_3$, we can illustrate D_6 as



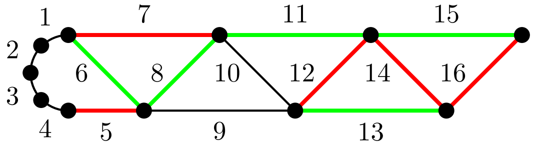
following the conventions of Remark 8.13 and Figure 8.1. Thus an embedded minor $N = M\backslash J/I$ that contains $\{1, \dots, 4\}$ is of type D_6 if excluding the edges in J and contracting the edges labeled by I results in the above picture.

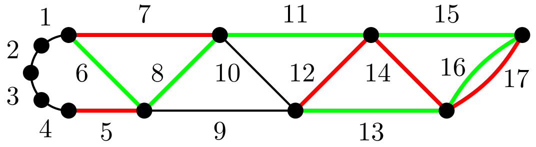
For odd k between 5 and $n-2$, we define

$$\begin{aligned} J_k &= \big\{ i \in \{5, \dots, n-2\} \,\big|\, i \text{ odd if } i < k \text{ and } i \text{ even if } i \geqslant k \big\}, \\ I_k &= \big\{ i \in \{5, \dots, n-2\} \,\big|\, i \text{ even if } i < k \text{ and } i \text{ odd if } i \geqslant k \big\}, \end{aligned}$$

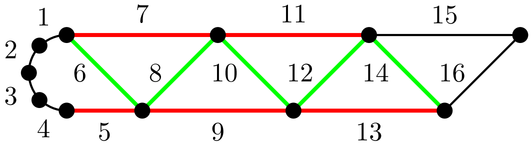
as well as $J'_k = J_k - k$ and $I'_k = I_k - (k+1)$. Then all of the following embedded minors are of type D_6, as visible from the corresponding illustrations where the green edges are deleted and the red edges are contracted.

1. $M\setminus J'_k a/I'_k b$ and $M\setminus J'_k b/I'_k a$ for $k = 5, 7, \ldots, n-3$ (note that a and b are symmetric, so we only illustrate the first case for $k = 9$, $n = 16$ and $k = 9$, $n = 17$):

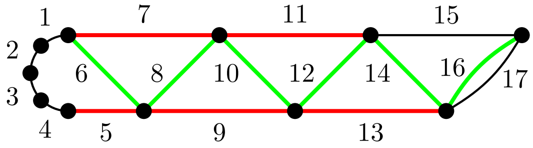


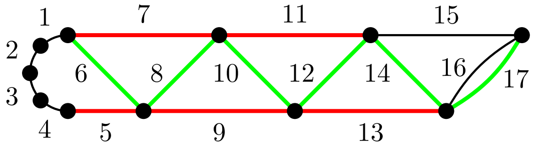


2. $M\setminus J'_{n-1}/I'_{n-1}$ for even n:



3. $M\setminus J'_{n-2}a/I'_{n-2}$ and $M\setminus J'_{n-2}b/I'_{n-2}$ for odd n:





With this, we are prepared to prove Theorem 8.11. We are left with showing that for all minimal 2-connected extensions M of $N_0 = U_4^2 \oplus U_2^1$, the natural map $\operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}^{(2)}}) \to F_M$ is an isomorphism. We show this by induction on $n = \#E_M$. Since there is no 2-connected extension of $N_0 = U_4^2 \oplus U_2^1$ with 6 elements, the base case for $n = 6$ is trivially true.

Let $n \geqslant 7$. By Theorem 6.3, the natural map $\operatorname{colim} F(\mathcal{E}^{<,+}_{M,\mathcal{C}^{(2)}}) \to F_M$ is an isomorphism, where $\mathcal{E}^{<,+}_{M,\mathcal{C}^{(2)}}$ is the diagram $\mathcal{E}^{<}_{M,\mathcal{C}^{(2)}}$ enriched with all embedded minors of type $U_4^2 \oplus U_2^1$.

By Lemma 8.12, $N_0 = M\setminus J_0/I_0$ is the unique embedded minor of M of type $U_4^2 \oplus U_2^1$ such that M is a minimal 2-connected extension. Thus every other embedded $U_4^2 \oplus U_2^1$-minor $N \hookrightarrow M$ is contained in a proper 2-connected embedded minor M' of M. By our inductive hypothesis, the natural map $\operatorname{colim} F(\mathcal{E}^{<}_{M',\mathcal{C}^{(2)}}) \to F_{M'}$ is an isomorphism. Since the map $F_N \to F_M$ factors through $F_{M'}$, we conclude that we can exclude N from $\mathcal{E}^{<,+}_{M,\mathcal{C}^{(2)}}$ without changing the colimit of $F(\mathcal{E}^{<,+}_{M',\mathcal{C}^{(2)}})$.

We are left with N_0, whose effect on the cross ratios of M consists of the relations $\begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_0 a} = \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_0 b}$ for all $\sigma \in S_4$. These relations are also implied by a sequence of embedded D_6-minors, as considered above in (1)–(3), which identify cross ratios according to Lemma 8.14. Namely, if n is even, then

$$\begin{aligned} \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_0 a} &= \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_5 a} = \cdots = \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_{n-4} a} = \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_{n-2}} \\ &= \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_{n-4} b} = \cdots = \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_5 b} = \begin{bmatrix} \sigma(1) & \sigma(2) \\ \sigma(3) & \sigma(4) \end{bmatrix}_{I_0 b}. \end{aligned}$$

If n is odd, then

$$\begin{aligned}\begin{bmatrix}\sigma(1) & \sigma(2)\\ \sigma(3) & \sigma(4)\end{bmatrix}_{I_0a} &= \begin{bmatrix}\sigma(1) & \sigma(2)\\ \sigma(3) & \sigma(4)\end{bmatrix}_{I_5a} = \cdots = \begin{bmatrix}\sigma(1) & \sigma(2)\\ \sigma(3) & \sigma(4)\end{bmatrix}_{I_{n-1}a}\\ &= \begin{bmatrix}\sigma(1) & \sigma(2)\\ \sigma(3) & \sigma(4)\end{bmatrix}_{I_{n-1}b} = \cdots = \begin{bmatrix}\sigma(1) & \sigma(2)\\ \sigma(3) & \sigma(4)\end{bmatrix}_{I_5b} = \begin{bmatrix}\sigma(1) & \sigma(2)\\ \sigma(3) & \sigma(4)\end{bmatrix}_{I_0b}.\end{aligned}$$

We see that in either case, the relations between the cross ratios of N_0 are already induced by other embedded minors of F_M of type D_6. This allows us to exclude N_0 from $\mathcal{E}^{<,+}_{M,\mathcal{C}^{(2)}}$ without changing the colimit of $F(\mathcal{E}^{<,+}_{M',\mathcal{C}^{(2)}})$. We deduce that the natural map $\operatorname{colim} F(\mathcal{E}^{<}_{M,\mathcal{C}^{(2)}}) \to F_M$ is an isomorphism, which concludes the proof of Theorem 8.11. ☐

8.3. The 3-connected fundamental presentation

If M is 3-connected, then we can present its foundation F_M as the colimit of 3-connected embedded minors of certain isomorphism types. In comparison with the 2-connected fundamental diagram $\mathcal{E}^{(2)}_M$, we replace the embedded minors of types C_5 and C_5^*, which are not 3-connected, by their minimal 3-connected extensions, which are the rank 3 whirl W^3 and the matroids Q_6 and P_6 (following Oxley's notation in [Oxl92, p. 641]). This presentation of F_M results from a version of the splitter theorem "for a marked element" by Bixby and Coullard ([BC87]), together with an exhaustive computer search.

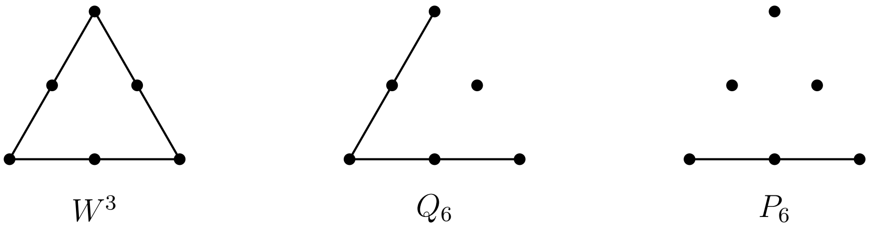


Figure 8.2: The 3-circuits of W^3, Q_6 and P_6.

The matroids W^3, Q_6 and P_6 are all rank 3 matroids on 6 elements. Their respective 3-circuits are illustrated in Figure 8.2. The foundations of these three matroids are isomorphic to the foundations of the first rank 2 uniform matroids:

$$F_{W^3} \simeq F_{U^2_4} = \mathbb{U}, \qquad F_{Q_6} \simeq F_{U^2_5} = \mathbb{V}, \qquad F_{P_6} \simeq F_{U^2_6}.$$

See Section 7.1 for the computation of the foundation of Q_6 and Section 9.1 for W^3. We have verified that the foundations of P_6 and U^2_6 are isomorphic with the assistance of the Macaulay2 package Pastures; however, it also follows from the fact (suggested to us by Nathan Bowler) that the foundation of a matroid M is equal to the foundation of any delta-wye exchange M' of M; this is proved in [BLWZ24].

Definition 8.15. The 3-*connected fundamental diagram of* M is the diagram $\mathcal{E}_M^{(3)}$ of embedded minors of isomorphism types

$$U_4^2, \qquad U_5^2, \qquad U_5^3, \qquad W^3, \qquad Q_6, \qquad P_6, \qquad F_7, \quad \text{and} \quad F_7^*,$$

together with all minor embeddings.

If M is 3-connected, we call the associated diagram $F(\mathcal{E}_M^{(3)})$ of foundations the 3-*connected fundamental presentation of* M, which is motivated by the following result. Let $\mathcal{C}^{(3)}$ be the class of all 3-connected matroids.

Theorem 8.16. *The fundamental type of* $\mathcal{C}^{(3)}$ *consists of the isomorphism types of the matroids* U_4^2, U_5^2, U_5^3, W^3, Q_6, P_6, F_7 *and* F_7^*. *In other words, if* M *is a* 3-*connected matroid with foundation* F_M *and* 3-*connected fundamental diagram* $\mathcal{E}_M^{(3)}$, *then* $F_M \simeq \operatorname{colim} F(\mathcal{E}_M^{(3)})$.

Proof. By Proposition 8.3 applied to $\mathcal{C} = \mathcal{C}^{(2)}$ and Theorem 8.11, it suffices to show that the minimal 3-connected extensions of C_5 and C_5^* are W^3, P_6, and Q_6. Since each of the latter three matroids is self-dual, it suffices to establish the result for C_5.

Let e be a series element of C_5, so that $C_5/e = U_4^2$. Let M be a minimal 3-connected extension of C_5, which is the same as a minimal 3-connected extension of $U_4^2 = C_5/e$ that uses e. By [OW98, Theorem 12.3.6], M has at most 4 elements that are not in U_4^2. An exhaustive search among all matroids with up to 8 elements using the Macaulay2 package PASTURES then shows that W^3, Q_6, and P_6 are the only minimal 3-connected extensions of C_5. □

Definition 8.17. Let M be a matroid with lattice of flats Λ. The 3-*connected fundamental lattice diagram* is the diagram $\mathcal{L}_M^{(3)}$ of all upper sublattices of Λ of types U_4^2, U_5^2, U_5^3, W^3, Q_6, P_6, F_7, and F_7^*, together with all inclusions as sublattices.

Theorem 8.18. *Let* M *be a matroid with foundation* F_M *and* 3-*connected fundamental lattice diagram* $\mathcal{L}_M^{(3)}$. *Then* $F_M \simeq \operatorname{colim} F(\mathcal{L}_M^{(3)})$.

Proof. This follows at once from Theorem 8.16, using the observation that the foundations of two embedded minors with the same upper sublattice are identified in $\operatorname{colim} F(\mathcal{E}^{(3)}) \simeq F_M$, and thus this isomorphism factors through $\operatorname{colim} F(\mathcal{L}_M^{(3)})$. □

If we exclude certain minors from the fundamental type of $\mathcal{C}^{(3)}$, then these minors get replaced by their minimal proper 3-connected extensions. In this way, we can derive new results for subclasses of $\mathcal{C}^{(3)}$. For example, the minimal proper 3-connected extensions of U_5^2 are U_6^2, U_6^3, W^3, Q_6, and P_6. The same holds for U_5^3 if we replace U_6^2 by U_6^4 in this list. By excluding some of these matroids, we find fundamental types for various subclasses of $\mathcal{C}^{(3)}$. One such result that will prove useful in a follow-up paper is the following.

Corollary 8.19. *Let* $\mathcal{C}$ *be the class of all* 3-*connected matroids of type* U_4^2, *or* 3-*connected matroids with at least* 6 *elements that are without minors of type* U_6^2, U_6^3, U_6^4, *or* P_6. *Then the fundamental class of* $\mathcal{C}$ *consists of the isomorphism types of* U_4^2, W^3, Q_6, F_7, *and* F_7^*.

Proof. Let M be in $\mathcal{C}$ and let $N = M\backslash J/I$ an embedded minor of type U_5^2 or U_5^3. By the Splitter Theorem ([Oxl92, Thm. 12.1.2]), N is embedded in a 3-connected N'-minor of M on 6 elements, which must be of type Q_6 since we excluded all other possible types in the hypothesis of the corollary. By Proposition 7.1, the induced morphism $F_N \to F_{N'}$ is an isomorphism, and the map $F_N \to F_M$ factors through $F_N \to F_{N'}$. Since P_6 is excluded, we deduce from Theorem 8.16 that the fundamental type of $\mathcal{C}$ consists of U_4^2, W^3, Q_6, F_7, and F_7^*, as claimed. ☐

9. Final examples

In this section, we compute the foundation of whirls ([Oxl92, p. 659]), the non-Fano matroid F_7^- ([Oxl92, p. 643]), and the matroids P_7 and T_8 ([Oxl92, p. 644, 649]).

9.1. The foundation of whirls

Let us recall the definition of whirls. Let W_r be the r-spoked wheel, considered as a matroid. Let $a_1, \ldots, a_r$ correspond to the rim edges and $b_1, \ldots, b_r$ to the spoke edges, as illustrated in Figure 9.1. The rim $\{a_1, \ldots, a_r\}$ is the unique circuit-hyperplane of W_r, and the rank r whirl W^r is the corresponding relaxation of W_r. Note that wheels are regular, so the foundation of W_r is $\mathbb{F}_1^\pm$, cf. Section 4.1.

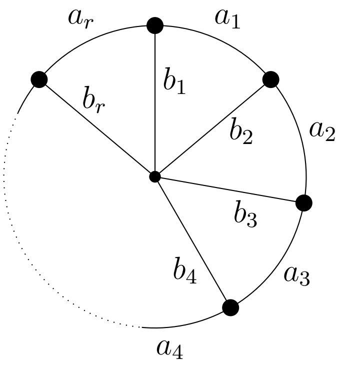


Figure 9.1: The graphic representation of the r-spoked wheel.

Proposition 9.1. *The foundation of the whirl W^r is isomorphic to $\mathbb{U}$ for all $r \geqslant 2$.*

Proof. By [Oxl92, p. 660], whirls are 2-connected and near-regular. In particular, they do not contain any minors of types U_5^2 and U_5^3. Thus by the structure theorem for foundations of matroids without large uniform minors (cf. Theorem 6.16), the foundation of W^r is a tensor product of copies of $\mathbb{U}$, $\mathbb{D}$, $\mathbb{H}$, $\mathbb{F}_3$, and $\mathbb{F}_2$. Since W^r is near-regular, but none of $\mathbb{D}$, $\mathbb{H}$, $\mathbb{F}_3$, and $\mathbb{F}_2$ allow for a morphism to $\mathbb{U}$, we conclude that the foundation of W^r is a tensor power of $\mathbb{U}$. Moreover, if $\mathcal{E}^{(2)}(W^r)$ is connected then there is only one tensor factor.

We prove that $\mathcal{E}^{(2)}(W^r)$ is connected by induction on r. The 2-whirl W^2 is isomorphic to U^2_4 and thus $\mathcal{E}^{(2)}(W^2)$ consists of a single vertex (of type U^2_4), which establishes the base case.

Assume that $r > 2$. We need to show that every pair of embedded U^2_4-minors of W^r lie in the same connected component of $\mathcal{E}^{(2)}(W^r)$. Let $E = \{a_1, \dots, a_r, b_1, \dots, b_r\}$ be the common ground set of W_r and W^r and $i \in \{1, \dots, r\}$. As a first step, we observe that the bases W^r/b_i are those subsets B of $E - b_i$ for which $B \cup \{b_i\}$ is a basis of W^r. These subsets agree with the bases of the regular matroid W_r/b_i, which shows that W^r/b_i is regular and does not contain any U^2_4-minor.

Since the rank of W^r is $r > 2$, we conclude that every embedded U^2_4-minor of W^r is contained in W^r/a_i for some i. Since b_i and b_{i+1} are parallel in W^r/a_i (where we use $b_{r+1} = b_1$), every U^2_4-minor of W^r/a_i is contained in either $W^r\backslash b_i/a_i$ or $W^r\backslash b_{i+1}/a_i$, which are both isomorphic to W^{r-1}. By the inductive hypothesis, the 2-connected fundamental diagram of W^{r-1} is connected, so we are left with showing that the 2-connected fundamental diagrams of $W^r\backslash b_i/a_i$ or $W^r\backslash b_{i+1}/a_i$ (for varying i) are connected as subdiagrams of $\mathcal{E}^{(2)}(W^r)$.

The subdiagrams $W^r\backslash b_i/a_i$ and $W^r\backslash b_{i+1}/a_i$ are connected by the embedded C^*_5-minor

$$W^r \setminus b_1 \dots b_{i-2}\; b_{i+2} \dots b_r \;/\; a_1 \dots a_{i-2}\; a_i\; a_{i+2} \dots a_r$$

(where we read indices modulo r, as appropriate) and the subdiagrams $W^r\backslash b_i/a_i$ and $W^r\backslash b_i/a_{i-1}$ are connected by the embedded C_5-minor

$$W^r \setminus b_1 \dots b_{i-2}\; b_i\; b_{i+2} \dots b_r \;/\; a_1 \dots a_{i-2}\; a_{i+2} \dots a_r,$$

which shows that $\mathcal{E}^{(2)}(W^r)$ is connected as claimed. □

Example 9.2. We determine $\mathcal{E}^{(3)}_M$ and $\mathcal{L}^{(3)}_M$ for the rank 3 whirl $M = W^3$ as a first example. Its 3-circuits and its lattice of flats Λ are illustrated in Figure 9.2.

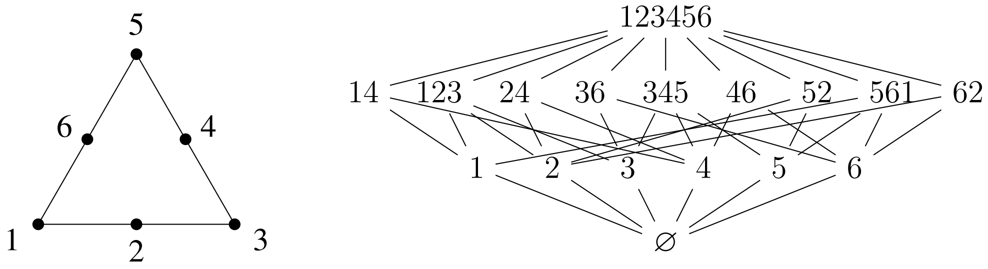


Figure 9.2: The 3-circuits and the lattice of flats of the 3-whirl W^3.

The embedded minors that appear in $\mathcal{E}^{(3)}_M$ are M itself and the U^2_4-minors $M\backslash j/i$ with j even, i odd, both contained in a common 3-circuit. The upper sublattice Λ/i defined by $M\backslash j/i$ is insensitive to j, and therefore Λ/i corresponds to two distinct embedded U^2_4-minors. The diagrams $\mathcal{E}^{(3)}_M$ and $\mathcal{L}^{(3)}_M$ are illustrated in Figure 9.3.

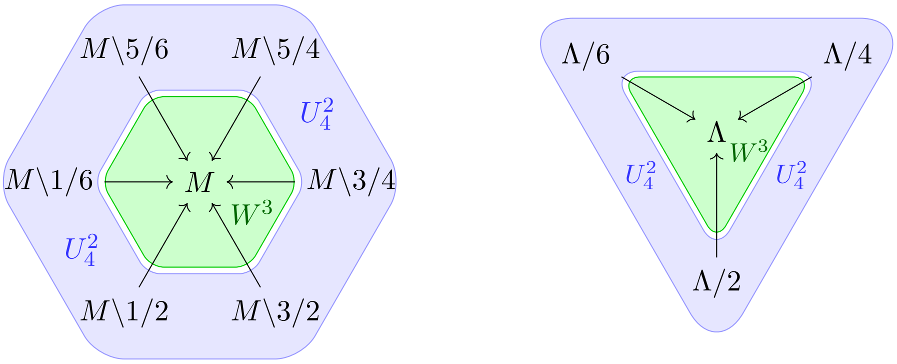


Figure 9.3: The 3-connected fundamental diagram and lattice diagram of W^3.

Since the foundation of the 3-whirl W^3 is $\mathbb{U}$ by Proposition 9.1, the embedding of each minor $W^3\backslash j/i$ of type U_4^2 into W^3 induces an isomorphism $F_{W^3\backslash j/i} \to F_{W^3}$. This means that the cross ratios stemming from the six U_4^2-minors of W^3 are identified in F_{W^3}. Explicitly, these identifications are as follows.

Lemma 9.3. *The cross ratios in the foundation of W^3 satisfy the following relations (assuming the enumeration of elements as in Figure 9.2):*

$$\begin{bmatrix} 123 & 24 \\ 25 & 26 \end{bmatrix} = \begin{bmatrix} 345 & 46 \\ 14 & 24 \end{bmatrix} = \begin{bmatrix} 156 & 26 \\ 36 & 46 \end{bmatrix}.$$

Proof. For each pair of cross ratios in this equation, all of their hyperplanes are contained in an upper sublattice of W^3 of type C_5, thus the claim follows directly from (H4'). □

9.2. The foundation of F_7^-

In this section, we compute the foundation of the non-Fano matroid F_7^-, whose 3-circuits are as illustrated in Figure 9.4.

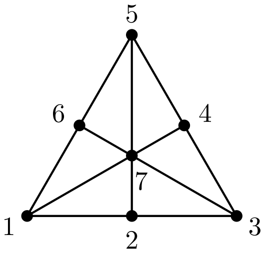


Figure 9.4: The 3-circuits of the non-Fano matroid F_7^-.

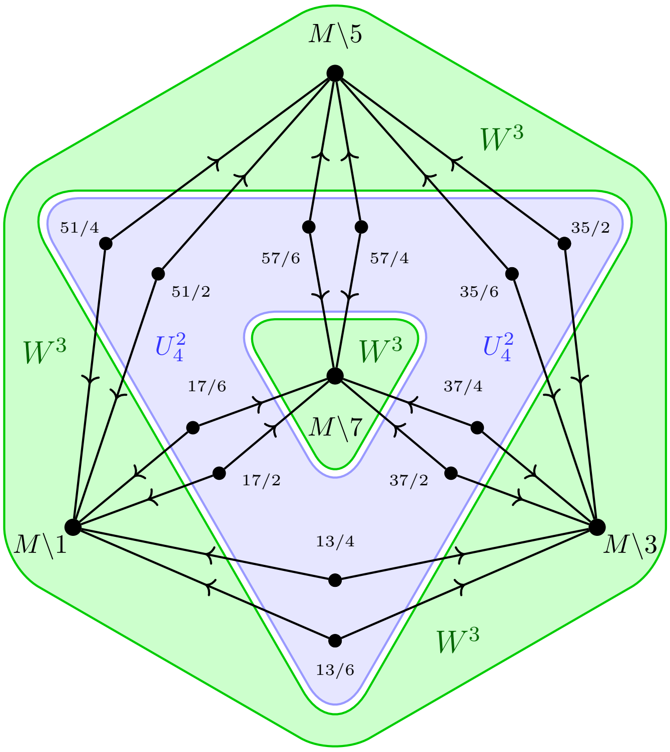


Figure 9.5: The 3-connected fundamental diagram of F_7^-.

Proposition 9.4. *The foundation of F_7^- is isomorphic to $\mathbb{D}$.*

Proof. Since F_7^- is 3-connected, we can use the 3-connected fundamental diagram $\mathcal{E}^{(3)}(F_7^-)$ of F_7^-, as illustrated in Figure 9.5, to compute F_M by Theorem 8.16

Since $\mathcal{E}^{(3)}(F_7^-)$ is connected and F_7^- does not have any minors of types U_5^2, U_5^3, F_7 and F_7^*, Corollary 6.17 implies that the foundation $F_{F_7^-}$ of F_7^- is a symmetry quotient of $\mathbb{U}$.

By [Oxl92, p. 644], the non-Fano matroid F_7^- is dyadic, i.e., $\mathbb{D}$-representable. Since neither $\mathbb{H}$ nor $\mathbb{F}_3$ map to $\mathbb{D}$, the foundation F_M is either $\mathbb{U}$ or $\mathbb{D}$. In order to exhibit the defining relation of $\mathbb{D} = \mathbb{U}/\!\!/\{x-y\}$, we consider the subdiagram

$$\begin{tikzcd}
 & & M\backslash 7 & & \\
 & M\backslash 17/2 \arrow[ur] \arrow[dl] & & M\backslash 37/4 \arrow[ul] \arrow[dr] & \\
M\backslash 1 & & M\backslash 13/6 \arrow[ll] \arrow[rr] & & M\backslash 3
\end{tikzcd}$$

of $\mathcal{E}^{(3)}(F_7^-)$ and the morphism $\alpha : \mathbb{U} \simeq F_{W^3\backslash 13/6} \to F_{W^3}$ with $\alpha(x) = \begin{bmatrix} 367 & 46 \\ 26 & 56 \end{bmatrix}$ and $\alpha(y) = \begin{bmatrix} 367 & 26 \\ 46 & 56 \end{bmatrix}$. By Lemma 9.3, we find the following chain of equalities (where we label the equalities by the

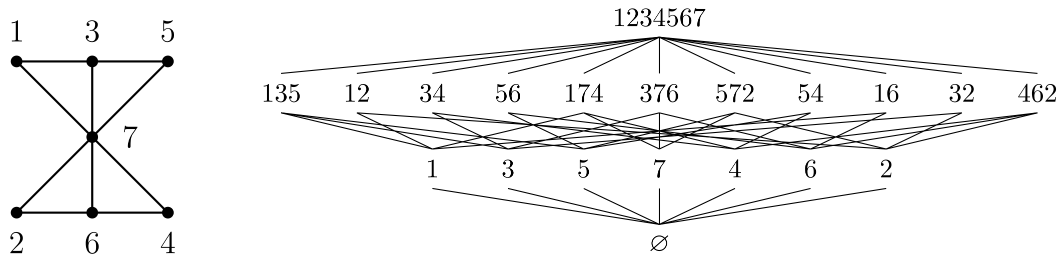


Figure 9.6: The 3-circuits and the lattice of flats of P_7.

corresponding W^3-minor to which we apply the lemma):

$$\alpha(x) \;=\; \begin{bmatrix} 367 & 46 \\ 26 & 56 \end{bmatrix} \underset{M\backslash 1}{=} \begin{bmatrix} 123 & 24 \\ 26 & 25 \end{bmatrix} \underset{M\backslash 7}{=} \begin{bmatrix} 147 & 24 \\ 46 & 45 \end{bmatrix} \underset{M\backslash 3}{=} \begin{bmatrix} 367 & 26 \\ 46 & 56 \end{bmatrix} \;=\; \alpha(y),$$

which shows that $\alpha(x) = \alpha(y)$ in $F_{F_7^-}$ and thus $F_{F_7^-} \simeq \mathbb{U}/\!\!/\{x-y\} = \mathbb{D}$, as claimed. □

Remark 9.5. Note that it is known that F_7^- is not near-regular: by [Oxl92, p. 644]), F_7^- is not representable over any field of characteristic 2. This could have been used as a shortcut in last step of the proof of Proposition 9.4.

Lemma 9.6. *There is a unique isomorphism* $\alpha : \mathbb{D} = \mathbb{F}_1^{\pm}(x)/\!\!/\langle\!\langle x+x-1\rangle\!\rangle \to F_{F_7^-}$*, and the following equalities between cross ratios hold in* F_7^- *(assuming the enumeration of elements as in Figure 9.4):*

$$\begin{aligned}
\alpha(x) &= \begin{bmatrix} 367 & 46 \\ 26 & 156 \end{bmatrix} = \begin{bmatrix} 367 & 26 \\ 46 & 156 \end{bmatrix}; \\
\alpha(x^{-1}) &= \begin{bmatrix} 367 & 26 \\ 156 & 46 \end{bmatrix} = \begin{bmatrix} 367 & 46 \\ 156 & 26 \end{bmatrix}; \\
\alpha(-1) &= \begin{bmatrix} 367 & 156 \\ 46 & 26 \end{bmatrix} = \begin{bmatrix} 367 & 156 \\ 26 & 46 \end{bmatrix}.
\end{aligned}$$

Proof. The existence of $\alpha : \mathbb{D} \to F_{F_7^-}$ is the content of Proposition 9.4, and its uniqueness follows from the fact that $\mathbb{D}$ does not have any nontrivial automorphisms, since x is the unique element of $\mathbb{D}$ that satisfies $x + x - 1 \in N_{\mathbb{D}}$. The first row of equalities of cross ratios reflects the computation in the proof of Proposition 9.4, and the second and third row are deduced from this by Proposition 6.15. □

9.3. The foundation of P_7

The matroid P_7, following Oxley's notation in [Oxl92, p. 644], is the rank 3 matroid on 7 elements whose 3-circuits and lattice of flats are as illustrated in Figure 9.6.

Proposition 9.7. *The foundation of* P_7 *is isomorphic to* $\mathbb{U}$*.*

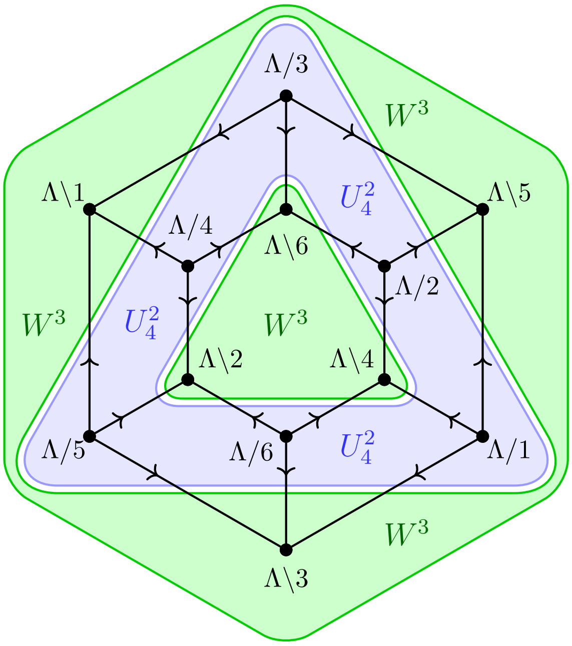


Figure 9.7: The 3-connected fundamental lattice diagram $\mathcal{L}_{P_7}^{(3)}$ of P_7.

Proof. Since $M = P_7$ is near-regular (cf. [Oxl92, p. 644]), its foundation F_M is isomorphic to $\mathbb{U} \otimes \ldots \otimes \mathbb{U}$ by [BL25a, Thm. 5.9]. Our result $F_M \simeq \mathbb{U}$ follows from the fact that $\mathcal{L}_M^{(3)}$ is connected, as visible in Figure 9.7. □

Lemma 9.8. *The following equalities between cross ratios hold in the foundation of P_7 (assuming the enumeration of elements as in Figure 9.6):*

$$\begin{bmatrix} 135 & 147 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} 135 & 45 \\ 257 & 56 \end{bmatrix} = \begin{bmatrix} 135 & 24 \\ 23 & 367 \end{bmatrix} = \begin{bmatrix} 246 & 147 \\ 45 & 34 \end{bmatrix} = \begin{bmatrix} 246 & 12 \\ 257 & 23 \end{bmatrix} = \begin{bmatrix} 246 & 16 \\ 56 & 367 \end{bmatrix}.$$

Proof. As visible in Figure 9.7, every U_4^2-minor of P_7 is contained in a W^3-minor. This allows us to deduct the lemma by a repeated application of Lemma 9.3. □

9.4. The foundation of T_8

In this section, we determine the foundation of the ternary spike T_8 (using Oxley's notation in [Oxl92, p. 649]) as $\mathbb{F}_3$ using the fundamental presentation by upper sublattices of rank $\leqslant 3$ as in Theorem 6.11.

We realize T_8 as the matroid on $E = \{1, \ldots, 8\}$ whose 4-circuits are

$$1238,\ 1247,\ 1346,\ 2345,\ 1256,\ 1357,\ 1458,\ 2367,\ 2468,\ 3478,\ 5678$$

and whose other circuits have all 5 elements.

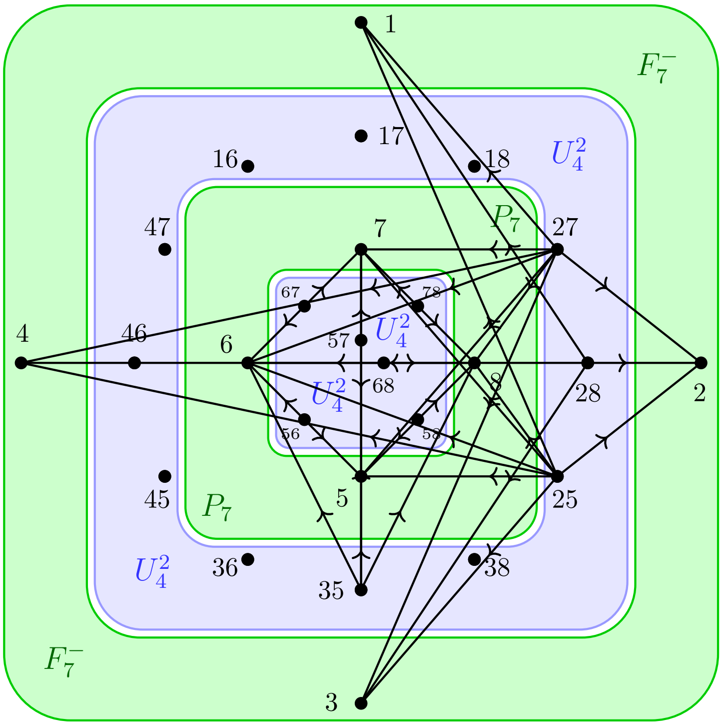


Figure 9.8: The fundamental lattice diagram $\mathcal{L}^{(\leqslant 3)}$ of T_8.

Proposition 9.9. *The foundation of* T_8 *is isomorphic to* $\mathbb{F}_3$.

Proof. As a ternary matroid, $M = T_8$ does not have minors of type F_7^*, and thus Theorem 6.11 implies that $F_M = \operatorname{colim} F(\mathcal{L}_M^{\leqslant 3})$ where $\mathcal{L}_M^{\leqslant 3}$ consists of all full upper sublattices of $\Lambda = \Lambda_M$ of rank less or equal to 3 that contain an upper sublattice of type U_4^2.

The lattice Λ of flats of T_8 contains the following elements: its atoms are $1, \dots, 8$, its 2-flats are all 2-subsets, and its hyperplanes are all 4-circuits and all 3-subsets ijk with $1 \leqslant i \leqslant 4 < j < k \leqslant 8$ with $j - i \neq 4 \neq k - i$. The full upper sublattice Λ/ij is of type U_4^2 if $j \geqslant 5$ and $j - i \neq 4$; otherwise it is regular. For $i \leqslant 4$, the lattice Λ/i is of type F_7^-, while for $i \geqslant 5$, it is of type P_7. The fundamental diagram $\mathcal{L}^{(\leqslant 3)}$ is illustrated in Figure 9.8. Because of space limitations, we label the upper sublattices Λ/i and Λ/ij by their respective bottom elements i and ij.

Since $\mathcal{L}_M^{\leqslant 3}$ is connected and since, as a ternary matroid, M is without minors of types U_5^2, U_5^3, F_7 and F_7^*, Corollary 6.17 implies that F_M is a symmetry quotient of $\mathbb{U}$.

By Proposition 9.4, the foundation of the upper sublattices Λ/i of type F_7^- (for $i = 1, \dots, 4$) have foundation $\mathbb{D}$. In conclusion, F_M is isomorphic to a symmetry quotient of $\mathbb{D}$, which is either $\mathbb{D}$ or $\mathbb{F}_3$.

In order to exhibit the defining relation of $\mathbb{F}_3 \simeq \mathbb{D}/\!\!/\{x+1\}$, we consider the chain of lattice inclusions

$$\Lambda/1 \longleftarrow \Lambda/18 \longrightarrow \Lambda/8 \longleftarrow \Lambda/28 \longrightarrow \Lambda/2,$$

which induces a chain of morphisms

$$\begin{tikzcd}\mathbb{D} & \mathbb{U} \arrow[l] \arrow[r, "\sim"] & \mathbb{U} & \mathbb{U} \arrow[l, "\sim"'] \arrow[r] & \mathbb{D}\end{tikzcd}$$

between the respective foundations. The 3-circuits of $\Lambda/1$, $\Lambda/8$ and $\Lambda/2$ are depicted in Figure 9.9.

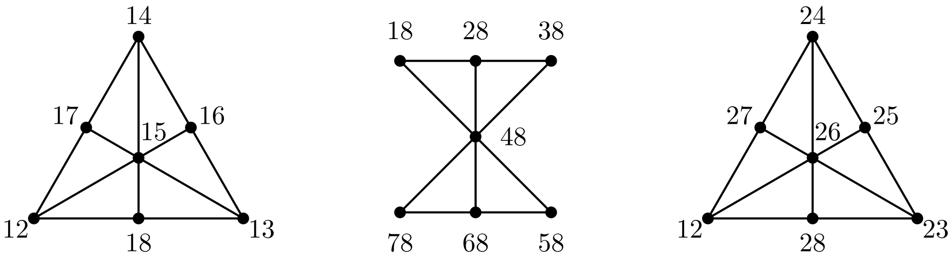


Figure 9.9: The 3-circuits of $T_8/1$, $T_8/8$ and $T_8/2$.

Let $\alpha_i : \mathbb{D} \to F_{\Lambda/i} \to F_M$ (for $i = 1, 2$) be the composition of the unique isomorphisms $\mathbb{D} \to F_{\Lambda/i}$ (cf. Lemma 9.6) with the morphism induced by the lattice inclusion $\Lambda/i \hookrightarrow \Lambda$. Let $x \in \mathbb{D}$ be the unique element with $x + x - 1 \in N_{\mathbb{D}}$. Then

$$\alpha_1(x) \underset{\Lambda/1}{=} \begin{bmatrix} 1458 & 178 \\ 168 & 1238 \end{bmatrix} \underset{\Lambda/8}{=} \begin{bmatrix} 258 & 278 \\ 2468 & 1238 \end{bmatrix} \underset{\Lambda/2}{=} \alpha_2(-1),$$

where we label the equalities with the upper sublattice Λ/i to which we apply one of Lemma 9.6 and Lemma 9.8, depending on the type of Λ/i. This equation shows that x and -1 are identified in F_M, and thus $F_M \simeq \mathbb{D}/\!\!/\langle\!\langle x+1 \rangle\!\rangle \simeq \mathbb{F}_3$, as claimed. ☐

Remark 9.10. Note that we can replace the last step in the proof of Proposition 9.9, which exhibits the relation $\alpha_1(x) = \alpha_2(-1)$, by the fact that T_8 is not representable over any field of characteristic different from 3 by [Oxl92, p. 649], which rules out the possibility that its foundation is $\mathbb{D}$.

10. The structure theorem for matroids without large uniform minors revisited

A matroid is *without large uniform minors* if it does not have any minors of type U^2_5 or U^3_5. A central result of the first two author's paper [BL25a] is that the foundation of a matroid without large uniform minors decomposes into a tensor product of pastures that are isomorphic to $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{U}$, $\mathbb{D}$ and $\mathbb{H}$. The previous computations show that every such tensor product occurs as the foundation of a matroid without large uniform minors.

Theorem 10.1. *The isomorphism classes of the foundations of matroids without large uniform minors are represented by all pastures of the form* $F_1 \otimes \ldots \otimes F_r$ *for some* $r \geqslant 0$ *and* $F_i \in \{\mathbb{F}_2, \mathbb{F}_3, \mathbb{U}, \mathbb{D}, \mathbb{H}\}$.

Proof. By [BL25a, Thm. 5.9], every foundation of a matroid without large uniform minor is of the described form. Thus we are left with showing that every pasture of the form $F_1 \otimes \ldots \otimes F_r$ with $F_i \in \{\mathbb{F}_2, \mathbb{F}_3, \mathbb{U}, \mathbb{D}, \mathbb{H}\}$ is isomorphic to the foundation of a matroid without large uniform minors. By Theorem 5.1, it suffices to show that each of $F_i \in \{\mathbb{F}_2, \mathbb{F}_3, \mathbb{U}, \mathbb{D}, \mathbb{H}\}$ appears as such a foundation. This is indeed the case: the foundation of F_7 is $\mathbb{F}_2$ (Section 4.1), the foundation of T_8 is $\mathbb{F}_3$ (Proposition 9.9), the foundation of U^2_4 is $\mathbb{U}$ (Section 4.2), the foundation of F_7^- is $\mathbb{D}$ (Proposition 9.4), and the foundation of $AG(2,3)\backslash e$ is $\mathbb{H}$ (Proposition 7.2). □

A. Some interesting foundations

Not every pasture appears as the foundation of a matroid. A concrete example is the pasture $P = \mathbb{F}_1^\pm(x)/\!\!/\langle\!\langle x^4 + x - 1 \rangle\!\rangle$. On the one hand, there is no morphism from $\mathbb{V}$ to P, and hence a matroid with a large uniform minor (i.e., a minor isomorphic to U^2_5 or U^3_5) cannot have foundation P. But, on the other hand, it is easy to check that P is not a tensor product of copies of $\mathbb{U}$, $\mathbb{D}$, $\mathbb{H}$, $\mathbb{F}_3$, and $\mathbb{F}_2$, so by Theorem 6.16 P cannot be the foundation of a matroid without large uniform minors. In general, it is a wide open problem to characterize which pastures are foundations.

In this appendix, we list some foundations, many of which we found with the help of the Macaulay2 package Pastures developed by Chen and the third author; cf. [CZ23]. Since the foundation of a direct sum and of a 2-sum of two matroids is the tensor product of the foundations of the summands (cf. Theorem 5.1 and [BLWZ24]), we concentrate on the description of foundations of 3-connected matroids, which can be thought of as the building blocks for all foundations.

The following list contains descriptions of all small foundations of 3-connected matroids on up to 8 elements, where "small" means that the foundation has at most 7 hexagons, along with various other examples of interest. We discuss notable properties of these foundations, in particular whether they admit morphisms into a field, into the sign hyperfield $\mathbb{S}$, or into the tropical hyperfield $\mathbb{T}$. We call a pasture *rigid* if every morphism to $\mathbb{T}$ factors through $\mathbb{K}$, i.e. its image is contained in $\{0,1\}$. By [BL24, Prop. B.1], a matroid is rigid if and only if its foundation is rigid. We gather the information about representability in Table A.1.

We freely use Oxley's notation from [Oxl92] throughout. Where we lack a better description of a matroid M, we list its *short circuits*, which are all circuits of size less than or equal to the rank r of M. Note that the whole circuit set $\mathcal{C}_M$ of M can be recovered from the subset $\mathcal{C}_M^{\mathrm{rk}}$ of short circuits by adding all $(r+1)$-subsets that do not contain a short circuit.

Some of the examples of foundations that we mention below are too large for a complete description to be meaningful. In these cases, we restrict ourselves to a *numerical description* of the foundation F_M which mentions:

- the *rank of* F_M, which is the free rank of the unit group $F_M^\times$;
- the *torsion of* F_M, which is the torsion subgroup of $F_M^\times$;
- whether $-1 = 1$ or not; note that in many cases the torsion of F_M is generated by -1, in which case it equals $\{1\}$ or $\{1, -1\}$;
- the number of *hexagons* of F_M.

Some open problems

Before discussing the list of foundations, we would like to mention a few interesting problems:

Problem A.1. For a pasture F, let $\mathcal{C}_F^{\min}$ be the class of minimal isomorphism types of matroids with foundation F. For which F is $\mathcal{C}_F^{\min}$ finite?

Note that this problem is intimately related to Rota's conjecture, which can be separated into two parts:

1. Is there a finite list $\mathcal{L}(q)$ of *excluded foundations* for every finite field $\mathbb{F}_q$? (More precisely, we're asking here for a list with the following property: if M is a matroid with foundation F_M such that no foundation $F \in \mathcal{L}(q)$ maps to F_M, then there is a morphism $F_M \to \mathbb{F}_q$.)
2. Is $\mathcal{C}_F^{\min}$ finite for every finite field $\mathbb{F}_q$ and $F \in \mathcal{L}(q)$?

We do not know the answer to either part.

Another highly interesting task is to develop a better understanding of which pastures appear as the foundation of a matroid. Results in this direction can be used as tools to study matroid representations. Thus we formulate the task:

Problem A.2. Find (easily verifiable) criteria that imply that a given pasture is a foundation or that it is *not* a foundation.

A.1. Uniform foundations

We call the foundation of a uniform matroid U_n^r a *uniform foundation* and denote it by $\mathbb{U}_n^r$.

The first example of a uniform foundation is the regular partial field $\mathbb{F}_1^\pm$, which is the foundation of every uniform matroid U_n^r with $r \in \{0, 1, n-1, n\}$. The unique minimal matroid with foundation $\mathbb{F}_1^\pm$ is the trivial (empty) matroid.

The second (class of) examples of uniform matroids are the k-regular partial fields $\mathbb{U}_k = \mathbb{U}_{k+3}^2 = \mathbb{U}_{k+3}^{k+1}$ for $k \geqslant 1$, which appear as the foundation of U_{k+3}^2 and U_{k+3}^{k+1}, and of no other uniform matroid. These two matroids are minimal matroids with this foundation. In particular, we have $\mathbb{U}_1 = \mathbb{U}$ and $\mathbb{U}_2 = \mathbb{V}$. The unique minimal matroid with foundation $\mathbb{U}$ is U_4^2, and the unique minimal matroids with foundation $\mathbb{V}$ are U_5^2 and U_5^3. For $k \geqslant 3$, the regular partial field $\mathbb{U}_k$ is the foundation of minimal matroids that are not uniform; e.g. P_6 is a minimal matroid with foundation $\mathbb{U}_3$.

The smallest example of a uniform foundation that is not a k-regular partial field is $\mathbb{U}_6^3$, which has rank 14, whose torsion group is generated by $-1 \neq 1$, and which has 30 hexagons.

In general, the unit group of the uniform matroid $\mathbb{U}_n^r$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{\binom{r}{n}-n}$ by [DW89, Thm. 8.1] and [BL21, Cor. 7.13]. This shows that the rank of uniform foundations grows very quickly with the size of the ground set. Note that the rank of every non-uniform foundation of a rank r matroid on n elements is strictly smaller than $\binom{r}{n} - n$.

The foundations of all 3-connected matroids on up to 6 elements are uniform. The first examples of non-uniform foundations appear for 3-connected matroids on 7 elements, cf. Section A.5.

A.2. Finite fields

By Proposition 4.7, every finite field $\mathbb{F}_q$ is a foundation, namely of the projective geometry $PG(d,q)$ of any dimension $d \geqslant 2$.

In the case of $\mathbb{F}_2$, the Fano plane $F_7 = PG(2,2)$ and its dual F_7^* are the unique minimal matroids with foundation $\mathbb{F}_2$. This follows from the fact that a matroid is binary if and only if it does not contain a U_4^2-minor, which is equivalent with its foundation being equal to $\mathbb{F}_1^\pm$ or $\mathbb{F}_2$; coupling this with Theorem 6.3 yields our claim.

Since $PG(2,q)$ is a proper minor of $PG(d,q)$ for $d \geqslant 3$, none of the higher dimensional projective spaces is minimal with foundation $\mathbb{F}_q$. Since $\mathbb{F}_q$ is the universal partial field of the extended Dowling geometry for $\mathbb{F}_q^\times$, which is a proper minor of $PG(2,q)$ for $q > 2$, we conjecture that $F_7 = PG(2,2)$ is the only (Desarguesian) projective plane that is minimal for its foundation.

Minimal matroids with foundation $\mathbb{F}_3$ are T_8 and R_9. Minimal size matroids for the foundations $\mathbb{F}_4$, $\mathbb{F}_5$, $\mathbb{F}_7$ and $\mathbb{F}_8$ all have 9 elements. Two minimal matroids with foundation $\mathbb{F}_4$ are those represented by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & b \\ 0 & 1 & 0 & 1 & 1 & a & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & a & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & a & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & a & 0 & a & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

over $\mathbb{F}_4 = \{0,1,a,b\}$. Two minimal matroids with foundation $\mathbb{F}_5$ are those represented by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

over $\mathbb{F}_5$. A minimal size matroid with foundation $\mathbb{F}_7$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 3 \end{pmatrix}$$

over $\mathbb{F}_7$. A minimal size matroid with foundation $\mathbb{F}_8$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & a+1 & 1 \\ 0 & 0 & 1 & 0 & 1 & a & a^2+1 & a^2+1 & a^2+a+1 \\ 0 & 0 & 0 & 1 & 0 & 1 & a^2+a+1 & a & 1 \end{pmatrix}$$

over $\mathbb{F}_8 = \mathbb{F}_2(a)$ where $a \in \mathbb{F}_8$ is a primitive element over $\mathbb{F}_2$ with minimal polynomial T^3+T+1.

Let p be a prime number. Then [Sil24, Thm. 4.1.1] determines a rank 3 matroid whose universal partial field is $\mathbb{F}_p$ in terms of an explicit matrix representation. Namely, let $l = \lfloor \log_2(p+1) \rfloor$ and $b_i = \lfloor (p+1)/2^{l-i+1} \rfloor$ for $i = 1,\dots,l$. Then the matroid represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & \dots & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & \dots & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & b_1 & b_1 & \dots & b_l & b_l \end{pmatrix}$$

over $\mathbb{F}_p$ has universal partial field $\mathbb{F}_p$. We do not know at the time of writing if the foundation of this matroid is equal to $\mathbb{F}_p$ for all primes p.

A.3. Some small foundations

In the following, we describe a series of "small" foundations of 3-connected matroids. We searched exhaustively among all matroids on up to 8 elements and list all foundations with up to 7 hexagons (for 3-connected matroids on up to 8 elements). There are several examples of foundations of larger matroids. We order these examples by (h, r) (as indicated in the section headers) where h is the number of hexagons and r is the rank of the foundation.

A.3.1 (1,0)

The *Krasner hyperfield* $\mathbb{K} = \mathbb{F}_2 \otimes \mathbb{F}_3$ appears as the foundation of several matroids on 9 elements (we found at least 44 such matroids). There are no matroids with foundation $\mathbb{K}$ that have less than 9 elements. A concrete example of a matroid on 10 elements with foundation $\mathbb{K}$ is the modular sum of F_7 with R_9 along a common 3-circuit as indicated in Figure A.1 (where the circled vertices of the generalized parallel connection are deleted).

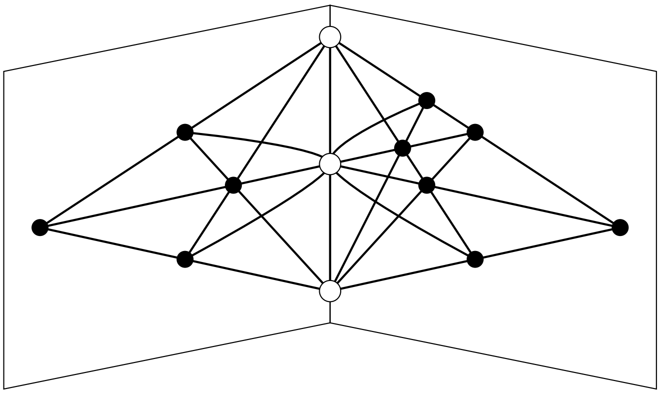


Figure A.1: A modular sum of F_7 with R_9.

A.3.2 (1,0)

The sign hyperfield $\mathbb{S} = \mathbb{F}_1^{\pm} /\!\!/ \langle\!\langle 1 + 1 - 1 \rangle\!\rangle$ appears as the foundation of the rank 4 matroid on 9 elements with short circuits

$$\begin{array}{ccccccc} 1234, & 1256, & 1357, & 1369, & 1468, & 1589, & 2358, \\ 2379, & 2457, & 2469, & 2678, & 3456, & 4789, & 5679. \end{array}$$

This matroid is 3-connected and of minimal size for foundation $\mathbb{S}$. The sign hyperfield is orientable and rigid, but not representable over any field.

Note that this implies that also the weak sign hyperfield

$$\mathbb{W} \;=\; \mathbb{F}_1^\pm /\!\!/ \langle\!\langle 1+1+1,\; 1+1-1 \rangle\!\rangle \;=\; \mathbb{F}_3 \otimes \mathbb{S}$$

(cf. [BB19, BJ87, Wag89]) is a foundation, e.g. for the direct sum of the above matroid with $PG(2,3)$. We do not know if the weak sign hyperfield appears as the foundation of a 3-connected matroid.

A.3.3 (1,0)

The *hexagonal partial field*

$$\mathbb{H} \;=\; \mathbb{F}_1^\pm(\zeta_6) /\!\!/ \langle\!\langle \zeta_6^3+1,\; \zeta_6+\zeta_6^{-1}-1 \rangle\!\rangle$$

is of importance for its role in the structure theorem for foundations of matroids without large uniform minors (Theorem 10.1). It also appears under the name of the *sixth-root-of-unity partial field* (or $\sqrt[6]{1}$ *partial field*), see, e.g., [PvZ10a]. The smallest matroids with foundation $\mathbb{H}$ have 8 elements, and there are 3 of them: $AG(2,3)\backslash e$, its dual, and the self-dual rank 4 matroid with short circuits

$$\begin{array}{ccccccc} 1234, & 1235, & 1236, & 1237, & 1238, & 1245, & 1345, \\ 1467, & 1578, & 2345, & 2468, & 2567, & 3478, & 3568. \end{array}$$

The hexagonal partial field is representable (in all fields with a solution to $x^2-x+1=0$) and rigid, but not orientable.

A.3.4 (1,1)

The *dyadic partial field*

$$\mathbb{D} \;=\; \mathbb{F}_1^\pm(x) /\!\!/ \langle\!\langle x-1-1 \rangle\!\rangle$$

is another building block for foundations of matroids without large uniform minors (Theorem 10.1). The smallest matroids with foundation $\mathbb{D}$ are the non-Fano matroid F_7^- and its dual. There is one minimal matroid for $\mathbb{D}$ with 8 elements, which is the self-dual matroid P_8. All other minimal matroids with foundation $\mathbb{D}$ have more than 8 elements.

Interestingly, certain tensor products of $\mathbb{D}$ with other pastures appear as the foundation of 3-connected matroids. For instance, $\mathbb{D}\otimes\mathbb{F}_2 = \mathbb{F}_2(x)/\!\!/\langle\!\langle x+x+1\rangle\!\rangle$ is the foundation of $AG(3,2)'$, and $\mathbb{D}\otimes\mathbb{D}$ is the foundation the rank 4 spike on 8 elements with short circuits

$$\begin{array}{cccccc} 1256, & 1278, & 1357, & 1368, & 1458, & 1467, \\ 2367, & 2468, & 2358, & 2457, & 3456, & 3478. \end{array}$$

The dyadic partial field is the only pasture that we know which is not a quotient of $\mathbb{F}_1^\pm$ but appears as a non-trivial tensor factor of the foundation of a 3-connected matroid.

In so far, we wonder (cf. the discussion about indecomposable foundations in Section 5.1):

Problem A.3. Let M be a 3-connected matroid whose foundation $F_M \simeq P_1 \otimes P_2$ is the tensor product of two pastures P_1 and P_2. Is then one of P_1 and P_2 necessarily a tensor product of copies of $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{S}$, and/or $\mathbb{D}$?

A.3.5 (1,1)

The *golden ratio partial field*

$$\mathbb{G} \;=\; \mathbb{F}_1^{\pm}(x)/\!\!/\langle\!\langle x^2+x-1\rangle\!\rangle$$

plays a prominent role in the representation theory of matroids; cf. [PvZ10a] and [PvZ10b]. It is the quotient of $\mathbb{V} = \mathbb{F}_1^{\pm}(x_1,\dots,x_5)/\!\!/\langle\!\langle x_i + x_{i-1}x_{i+1} - 1\rangle\!\rangle$ obtained by identifying all generators $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_5$. The smallest matroid with foundation $\mathbb{G}$ is the self-dual rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1368, \quad 1478, \quad 2358, \quad 2457, \quad 2678, \quad 3456,$$

and this is the only matroid with foundation $\mathbb{G}$ that has less than 9 elements. A minimal matroid for foundation $\mathbb{G}$ with 9 elements is the minor $B_{11}\backslash 23$ of the Betsy Ross matroid B_{11} (cf. [vZ09, Figure 3.3]), where 123 is a 3-circuit and 1 is the "center" of M. Also the Betsy Ross matroid itself has foundation $\mathbb{G}$.

The golden ratio partial field is representable (over all fields with a solution to $x^2+x=1$), orientable, and rigid.

A.3.6 (2,0)

The pasture

$$F_{A.3.6} \;=\; \mathbb{S}(x)/\!\!/\langle\!\langle x^2+1,\; x+1-1\rangle\!\rangle$$

appears as the foundation of the 3-connected rank 4 matroid on 9 elements with short circuits

$$\begin{array}{ccccccc} 1234, & 1256, & 1289, & 1357, & 1368, & 1469, & 1478, \\ 2358, & 2379, & 2457, & 2678, & 3456, & 3489, & 5689, \end{array}$$

which is a matroid of minimal size for $F_{A.3.6}$. The foundation $F_{A.3.6}$ is rigid, but not orientable nor representable (over any field).

A.3.7 (2,1)

The pasture

$$F_{A.3.7} \;=\; \mathbb{F}_1^{\pm}(x)/\!\!/\langle\!\langle x^3-x-1,\; x^5-x^4-1\rangle\!\rangle$$

appears as the foundation of the 3-connected rank 4 matroid on 9 elements with short circuits

$$\begin{array}{cccccccc} 1234, & 1235, & 1245, & 1267, & 1345, & 1368, & 1478, & 1579, \\ 2345, & 2369, & 2489, & 2568, & 3467, & 3789, & 4569. & \end{array}$$

which is a matroid of minimal size for $F_{A.3.7}$. Since

$$(x^2-x+1)\cdot(x^3-x-1) \;=\; x^5-x^4-1,$$

the foundation $F_{A.3.7}$ injects into its the universal ring $R = \mathbb{Z}[x]/\langle x^3-x-1\rangle$ (note that $x\cdot(x^2-1) = 1$, so x is a unit of R). Solving the S-unit equation $a+b = 1$ in R

(with SageMath) exhibits (x^3, x) and $(x^5, -x^4)$ as its only solutions, up to S_3-conjugates. Therefore $F_{A.3.7}$ is a partial field, and is thus equal to its universal partial field. The foundation $F_{A.3.7}$ is representable (over all fields with a root of $T^3 - T - 1$), orientable, and rigid.

The pastures A.3.7 and $\mathbb{G}$ are the only two examples we currently know of infinite rigid pastures which occur as foundations of 3-connected matroids.

Problem A.4. Are there infinitely many different infinite rigid pastures?

A.3.8 (2,1)

The *Gaussian partial field*

$$\mathbb{H}_2 \;=\; \mathbb{F}_1^{\pm}(i,x)/\!\!/\langle\!\langle i^2+1, \quad x-i-1, \quad x^2-i-i\rangle\!\rangle$$

(cf. [PvZ10a, p. 543]) is the foundation of the 3-connected rank 4 matroid on 9 elements with short circuits

$$\begin{matrix} 1234, & 1267, & 1235, & 1245, & 1345, & 1368, & 1379, & 1489, \\ 2345, & 2469, & 2568, & 2789, & 3467, & 3589, & 4578, & 5679, \end{matrix}$$

which is minimal for this foundation. It embeds into its universal ring $\mathbb{Z}[i,1/2]$ via $i \mapsto i$ and $x \mapsto i+1$. The partial field $\mathbb{H}_2$ is representable (over all fields of characteristic $\neq 2$ with a square root of -1), but it is neither orientable nor rigid.

A.3.9 (2,1)

The Dowling lift of $\mathbb{F}_4$

$$F_{A.3.9} \;=\; \mathbb{H}(x)/\!\!/\langle\!\langle x-\zeta_6-1\rangle\!\rangle$$

(cf. [PvZ10a, p. 543]) appears as the foundation of the 3-connected rank 4 matroid on 9 elements with short circuits

$$\begin{matrix} 1234, & 1235, & 1245, & 1267, & 1268, & 1278, & 1345, & 1389, & 1469, & 1579, \\ 1678, & 2345, & 2379, & 2489, & 2569, & 2678, & 3467, & 3568, & 4578, & \end{matrix}$$

which is a matroid of minimal size for $F_{A.3.9}$. It embeds into its universal ring $\mathbb{Z}[\zeta_6,\ (1+\zeta_6)^{-1}]$ via $\zeta_6 \mapsto \zeta_6$ and $x \mapsto 1+\zeta_6$ and is equal to its universal partial field. The Dowling lift of $\mathbb{F}_4$ is representable (over all fields with a primitive 3rd root of unity). It is neither orientable nor rigid.

A.3.10 (2,1)

The pasture

$$F_{A.3.10} \;=\; \mathbb{F}_4(x)/\!\!/\langle\!\langle x+a+1\rangle\!\rangle,$$

where $\mathbb{F}_4 \;=\; \{0,1,a,b\}$, appears as the foundation of the 3-connected rank 4 matroid on 9 elements with short circuits

$$\begin{matrix} 123, & 1245, & 1267, & 1345, & 1367, & 1468, & 1789, & 2345, \\ 2367, & 2469, & 2568, & 2579, & 3478, & 3689, & 4567, & 4589, \end{matrix}$$

which is a matroid of minimal size for $F_{A.3.10}$. It surjects onto its universal ring $\mathbb{F}_4 = \mathbb{F}_4[x^{\pm 1}]/\langle x + a + 1\rangle$ (with $x \mapsto a + 1$), which is equal to the universal partial field of $F_{A.3.10}$. The foundation $F_{A.3.10}$ is representable (over field extensions of $\mathbb{F}_4$), but it is neither orientable nor rigid.

A.3.11 (2,1)

The pasture

$$F_{A.3.11} \;=\; \mathbb{F}_3(x) /\!\!/ \langle\!\langle x + 1 - 1 \rangle\!\rangle \;=\; \mathbb{F}_3 \otimes \Big(\mathbb{F}_1^{\pm}(x) /\!\!/ \langle\!\langle x + 1 - 1 \rangle\!\rangle \Big)$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1368, \quad 1458, \quad 2358, \quad 2367, \quad 2457, \quad 3456, \quad 4678,$$

which is minimal for this foundation. It is neither representable, nor orientable, nor rigid.

A.3.12 (2,1)

The pasture

$$F_{A.3.12} \;=\; \mathbb{S}(x) /\!\!/ \langle\!\langle x + 1 - 1 \rangle\!\rangle \;=\; \mathbb{S} \otimes \Big(\mathbb{F}_1^{\pm}(x) /\!\!/ \langle\!\langle x + 1 - 1 \rangle\!\rangle \Big)$$

appears as the foundation of the 3-connected rank 4 matroid on 9 elements with short circuits

$$\begin{array}{ccccccc} 1234, & 1256, & 1279, & 1357, & 1389, & 1459, & 1468, \\ 2358, & 2367, & 2457, & 2489, & 3456, & 3479, & 6789. \end{array}$$

which is a matroid of minimal size for $F_{A.3.12}$. The foundation $F_{A.3.12}$ is orientable, but is neither representable nor rigid.

A.3.13 (2,1)

The pasture

$$F_{A.3.13} \;=\; \mathbb{K}(x) /\!\!/ \langle\!\langle x + 1 + 1 \rangle\!\rangle \;=\; \mathbb{K} \otimes \Big(\mathbb{F}_1^{\pm}(x) /\!\!/ \langle\!\langle x + 1 - 1 \rangle\!\rangle \Big)$$

appears as the foundation of the 3-connected rank 4 matroid on 9 elements with short circuits

$$\begin{array}{cccccccc} 1234, & 1256, & 1289, & 1357, & 1468, & 1479, & 2358, & 2379, \\ 2457, & 2469, & 2678, & 3456, & 3689, & 4589, & 5679. & \end{array}$$

which is a matroid of minimal size for $F_{A.3.13}$. The foundation $F_{A.3.13}$ is neither representable, nor orientable, nor rigid.

A.3.14 (2,1)

The pasture

$$F_{A.3.14} \;=\; \mathbb{F}_2(x) /\!\!/ \langle\!\langle x+x+1,\; x^2+x^2+1 \rangle\!\rangle$$

appears as the foundation of the 3-connected rank 4 matroid on 9 elements with short circuits

$$\begin{array}{ccccccccc} 1234, & 1235, & 1245, & 1267, & 1345, & 1368, & 1479, & 1589, & 2345, \\ 2378, & 2469, & 2568, & 2579, & 3467, & 3489, & 3569, & 4578, & 6789. \end{array}$$

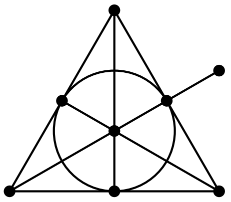
which is a matroid of minimal size for $F_{A.3.14}$. The foundation $F_{A.3.14}$ is neither representable, nor orientable, nor rigid.

A.3.15 (2,2)

The pasture

$$F_{A.3.15} \;=\; \mathbb{F}_2(x,y) /\!\!/ \langle\!\langle x+y+1, \quad x^2+y^2+1 \rangle\!\rangle$$

is the foundation of the matroid F_7^+ (cf. [vZ09, p. 91]), which can be depicted as



Another minimal 3-connected matroid for this foundation is represented over $\mathbb{F}_4 = \{0,1,a,b\}$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & b & a & 1 & 1 \\ 0 & 1 & 0 & 0 & a & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & a & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Note that $-1 = 1$ in $F_{A.3.15}$ even though this matroid has no minors of type F_7 and F_7^*.

The foundation $F_{A.3.15}$ embeds into its universal ring $\mathbb{F}_2[x^{\pm 1}, y^{\pm 1}]/(x+y-1)$ via the tautological map $x \mapsto x$ and $y \mapsto y$ (note that $x^2+y^2-1 \in (x+y-1)$), and its universal partial field is $\mathbb{F}_2(x,y) /\!\!/ \langle\!\langle x^{2^k} + y^{2^k} - 1 \mid k \geqslant 0 \rangle\!\rangle$ (which recovers [vZ09, Thm. 3.3.27.(iv)]). The foundation $F_{A.3.15}$ is representable (in all fields of characteristic 2 with at least 4 elements), but is neither orientable nor rigid.

A.3.16 (3,2)

The pasture

$$F_{A.3.16} \;=\; \mathbb{F}_2(x,y) /\!\!/ \langle\!\langle x+1+1, \quad y+1+1, \quad xy+1+1 \rangle\!\rangle$$

is the foundation of (the 3-connected rank 4 matroid) F_8, which is of minimal size for this foundation. It is neither representable, nor orientable, nor rigid.

A.3.17 (3,2)

Let ζ_6 be a fundamental element of $\mathbb{H}$ (cf. Section A.3.3). The pasture

$$F_{A.3.17} \;=\; \mathbb{H}(x,y)/\!\!/\langle\!\langle x-\zeta_6-1, \quad y-\zeta_6^{-1}-1\rangle\!\rangle$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1278, \quad 1357, \quad 1368, \quad 2358, \quad 2457, \quad 3456, \quad 4678,$$

which is minimal for this foundation. It maps to its universal ring $\mathbb{Z}[\zeta_6, 1/3]$ via $x \mapsto \zeta_6+1$ and $y \mapsto \zeta_6^{-1}+1$ (since $(\zeta_6+1)(\zeta_6^{-1}+1) = 3$, the integer 3 is invertible). Note that $\zeta_6 \cdot (\zeta_6^{-1}+1) = \zeta_6+1$, which shows that this map is not injective and that its universal partial field is $F_{A.3.9} = \mathbb{H}(x)/\!\!/\langle\!\langle x-\zeta_6-1\rangle\!\rangle$, the *Dowling lift of* $\mathbb{F}_4$ (cf. Section A.3.9). The foundation $F_{A.3.17}$ is representable (over all fields that contain a primitive third root of unity), but is neither orientable nor rigid.

A.3.18 (3,2)

The *non-Gersonides foundation*[5]

$$F_{A.3.18} \;=\; \mathbb{F}_1^{\pm}(x,y)/\!\!/\langle\!\langle x-1-1, \quad y-x-1, \quad x^2-y-1\rangle\!\rangle$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1278, \quad 1357, \quad 1368, \quad 1458, \quad 2367, \quad 2457, \quad 3456, \quad 4678,$$

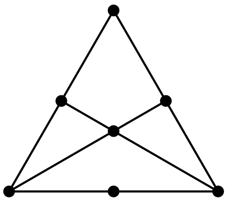
which is minimal for this foundation. It maps injectively into its universal ring $\mathbb{Z}[1/6]$ via $x \mapsto 2$ and $y \mapsto 3$, and its universal partial field is the *Gersonides partial field* $\mathbb{GE} = \mathbb{F}_1^{\pm}(x,y)/\!\!/\langle\!\langle x-1-1,\ y-x-1,\ x^2-y-1,\ y^2-x^3-1\rangle\!\rangle$ (cf. [PvZ10a, p. 543]), which acquires the additional relation $y^2-x^3-1=0$. This means that the quotient map $F_{A.3.18} \to \mathbb{GE}$ is a bijection, but not an isomorphism. The non-Gersonides foundation is representable (over all fields of characteristic $\neq 2,3$) and orientable, but not rigid.

A.3.19 (3,3)

The 2*-cyclotomic partial field*

$$\mathbb{K}_2 \;=\; \mathbb{F}_1^{\pm}(x,y,z)/\!\!/\langle\!\langle y-x-1, \quad z-y-1, \quad y^2+xz-1\rangle\!\rangle$$

(cf. [PvZ10a, p. 542]), which embeds into $\mathbb{Z}[x^{\pm 1}, \frac{1}{x+1}, \frac{1}{x+2}]$ via $x \mapsto x$, $y \mapsto x+1$ and $z \mapsto x+2$, appears as the foundation the rank 3 matroid on 7 elements that is depicted as



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[5] The Gersonides partial field $\mathbb{GE}$ is named after Gersonides, who determined all solutions to $x+y=1$ in powers of -1, 2 and 3 as $2-1=1$, $3-2=1$, $4-3=1$ and $9-8=1$. Since the last relation is missing in the pasture $F_{A.3.18}$, we call it the non-Gersonides foundation.

This matroid is of minimal size for $\mathbb{K}_2$ and 3-connected, and its dual is the only other matroid on 7 elements with foundation $\mathbb{K}_2$. The 2-cyclotomic partial field is representable (in all fields with at least 4 elements) and orientable, but not rigid. It plays a role in [PvZ10a, Thm. 4.17]: a matroid is quarternary and representable over the Gaussian partial field $\mathbb{H}_2$ (cf. Section A.3.8) if and only if it is $\mathbb{K}_2$-representable.

A.3.20 (4,2)

The pasture

$$F_{A.3.20} = \mathbb{F}_1^{\pm}(x,y) /\!\!/ \langle\!\langle x+1+1, \quad y+1-1, \quad xy+1-1, \quad x^2y+1-1 \rangle\!\rangle$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1278, \quad 1357, \quad 1468, \quad 2358, \quad 2467, \quad 3456, \quad 4578,$$

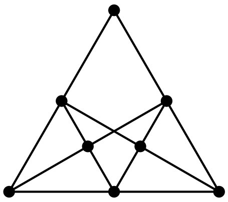
which is minimal for this foundation. This foundation is neither representable nor rigid, but is orientable.

A.3.21 (4,3)

The *Hydra* 3 *partial field*

$$\mathbb{H}_3 = \mathbb{F}_1^{\pm}(x,y,z) /\!\!/ \langle\!\langle x+y-1, \quad xy+z-1, \quad x+y^2-z, \quad x^2+y-z \rangle\!\rangle$$

(cf. [PvZ10a, p. 543]) is the foundation of the 3-connected rank 3 matroid on 8 elements, depicted as



which is minimal for this foundation. The foundation $\mathbb{H}_3$ is representable (in fields with at least 5 elements) and orientable, but not rigid.

A.3.22 (5,3)

Let ζ_6 be a fundamental element of $\mathbb{H}$ (cf. Section A.3.3). The pasture

$$F_{A.3.22} = \mathbb{H}(x,y,z) /\!\!/ \langle\!\langle z-x+1, \quad x-y+\zeta_6^2, \quad y-z+\zeta_6^4, \quad x+\zeta_6^2 y+\zeta_6^4 z \rangle\!\rangle$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1678, \quad 2378, \quad 2467, \quad 3456, \quad 4578,$$

which is minimal for this foundation. The foundation $F_{A.3.22}$ embeds into its universal ring $R_M = \mathbb{Z}[\zeta_6,\ x^{\pm 1},\ (x-1)^{-1},\ (x+\zeta_6^2)^{-1}]$ via $\zeta_6 \mapsto \zeta_6$, $x \mapsto x$, $y \mapsto x+\zeta_6^2$ and $z \mapsto x-1$. Since all solutions to the S-unit equation $a+b-1=0$ for R_M and $S = \{x,\ x-1,\ x+\zeta_6^2\}$ are in the null set of $F_{A.3.22}$, the foundation $F_{A.3.22}$ is a partial field and is equal to the universal partial field of M. The foundation $F_{A.3.22}$ is representable (in every field with a root of T^2-T+1 and at least 4 elements), but is neither orientable nor rigid.

A.3.23 (5,3)

The pasture

$$F_{A.3.23} \;=\; \mathbb{F}_1^{\pm}(x,y,z)/\!\!/\langle\!\langle x-1-1,\;\; y-1-1,\;\; z+1-1,\;\; xz+1-1,\;\; yz+1-1\rangle\!\rangle$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$\begin{array}{cccccc} 1234, & 1256, & 1278, & 1357, & 1368, & 1458, \\ 2367, & 2457, & 2468, & 3456, & 3478, & \end{array}$$

which is minimal for this foundation. This foundation is neither representable nor rigid, but is orientable.

A.3.24 (5,3)

The pasture

$$F_{A.3.24} \;=\; \mathbb{F}_2(x,y,z)/\!\!/\langle\!\langle x+y+1,\;\; x^2+y^2+1,\;\; z+1+1,\;\; xz+1+1,\;\; yz+1+1\rangle\!\rangle$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1468, \quad 2358, \quad 2467, \quad 3456, \quad 3478, \quad 5678,$$

which is minimal for this foundation. Another minimal (3-connected) matroid for this foundation is the rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1468, \quad 2368, \quad 2457, \quad 3456, \quad 5678.$$

Interestingly, this matroid has no minors of type F_7 and F_7^*, even though $-1=1$ holds in $F_{A.3.24}$. This foundation is neither representable, nor orientable, nor rigid.

A.3.25 (5,4)

The partial field

$$\mathbb{P}_4 \;=\; \mathbb{F}_1^{\pm}(x,y,z,w)/\!\!/\langle\!\langle x+y-1,\;\; z-x-1,\;\; w-y-1,\;\; x^2+yz-1,\;\; y^2+xw-1\rangle\!\rangle$$

(cf. [PvZ10a, p. 543]) is the foundation of the 3-connected matroid $M_{8591}^{Y\Delta}$ (cf. [vZ09, p. 91]). It is representable (over every field with at least 4 elements) and orientable, but not rigid.

A.3.26 (6,3)

The pasture $F_{A.3.26} = \mathbb{F}_1^{\pm}(x,y,z)/\!\!/\langle\!\langle S\rangle\!\rangle$, where S consists of

$$x-1-1, \quad y+1-1, \quad xy+1-1, \quad yz+1-1, \quad z-x-1, \quad x^2-z-1,$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1368, \quad 1478, \quad 2358, \quad 2457, \quad 3456, \quad 5678,$$

which is minimal for this foundation. It is neither representable, nor rigid, nor orientable.

A.3.27 (6,3)

The pasture $F_{A.3.27} = \mathbb{F}_1^{\pm}(x,y,z) /\!\!/ \langle\!\langle S \rangle\!\rangle$, where S consists of

$$x - 1 - 1, \quad y - x - 1, \quad z - x - 1, \quad x^2 - y - 1, \quad x^2 - z - 1, \quad yz - x^3 - 1,$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1458, \quad 1678, \quad 2358, \quad 2367, \quad 2457, \quad 3456,$$

which is minimal for this foundation. The foundation $F_{A.3.27}$ maps (non-injectively) to its universal ring $\mathbb{Z}[\frac{1}{6}]$ via $x \mapsto 2$, $y \mapsto 3$ and $z \mapsto 3$. Its universal partial field is the Gersonides partial field $\mathbb{GE} = \mathbb{F}_1^{\pm}(x,y) /\!\!/ \langle\!\langle x-1-1,\ y-x-1,\ x^2-y-1,\ y^2-x^3-1 \rangle\!\rangle$ (cf. Section A.3.18). The foundation $F_{A.3.27}$ is representable (in characteristic $\neq 2,3$) and orientable, but not rigid.

A.3.28 (6,3)

The pasture $F_{A.3.28} = \mathbb{F}_2(x,y,z) /\!\!/ \langle\!\langle S \rangle\!\rangle$, where S consists of

$$x + 1 + 1, \quad y + 1 + 1, \quad z + 1 + 1, \quad xy + 1 + 1, \quad xz + 1 + 1, \quad yz + 1 + 1,$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$\begin{array}{cccccc} 1234, & 1256, & 1368, & 1357, & 1458, & 2358, \\ 2367, & 2457, & 3456, & 3478, & 5678, & \end{array}$$

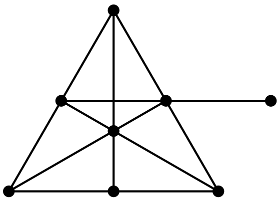
which is minimal for this foundation. It is neither representable, nor orientable, nor rigid.

A.3.29 (6,4)

The pasture $F_{A.3.29} = \mathbb{F}_1^{\pm}(x,y,z,w) /\!\!/ \langle\!\langle S \rangle\!\rangle$, where S consists of

$$w - 1 - 1, \quad y - x - 1, \quad z - y - 1, \quad y^2 - xz - 1, \quad \tfrac{wy}{x} - \tfrac{z}{x} - 1, \quad \tfrac{x}{z} + \tfrac{w}{z} - 1,$$

is the foundation of the 3-connected rank 3 matroid on 8 elements depicted as



which is minimal for this foundation. It embeds into its universal ring $\mathbb{Z}[\frac{1}{2}, x^{\pm 1}, \frac{1}{x+1}, \frac{1}{x+2}]$ via $w \mapsto 2$, $x \mapsto x$, $y \mapsto x+1$, and $z \mapsto x+2$, but is not equal to its universal partial field, which is isomorphic to the universal partial field of $F_{A.3.30}$ (cf. Section A.3.30). The foundation $F_{A.3.29}$ is representable (in all fields of characteristic $\neq 2$ with at least 5 elements) and orientable, but not rigid.

A.3.30 (7,4)

The pasture $F_{A.3.30} = \mathbb{F}_1^{\pm}(x,y,z,w) /\!\!/ \langle\!\langle S \rangle\!\rangle$, where S consists of

$$\begin{array}{llll} w-1-1, & z-y-1, & \frac{wy}{x}-\frac{z}{x}-1, & \frac{x^2}{z^2}+\frac{w^2y}{z^2}-1, \\ y-x-1, & y^2-xz-1, & \frac{x}{z}+\frac{w}{z}-1, & \end{array}$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1478, \quad 2368, \quad 2457, \quad 3456, \quad 5678,$$

which is minimal for this foundation. It embeds into its universal ring $\mathbb{Z}[\frac{1}{2},\ x^{\pm 1},\ \frac{1}{x+1},\ \frac{1}{x+2}]$ via $w \mapsto 2$, $x \mapsto x$, $y \mapsto x+1$, and $z \mapsto x+2$, and is equal to its universal partial field (verified with the method of [VdlH24]). The foundation $F_{A.3.30}$ is representable (over every field with at least 5 elements) and orientable, but not rigid.

A.3.31 (7,4)

The pasture $F_{A.3.31} = \mathbb{F}_1^{\pm}(x,y,z,w) /\!\!/ \langle\!\langle S \rangle\!\rangle$, where S consists of

$$\begin{array}{llll} x-y-1, & \frac{x^2}{w}+\frac{y}{w}-1, & \frac{z}{w}+\frac{xy}{w}-1, & \frac{x^3}{yw}-\frac{z}{yw}-1, \\ \frac{z}{x}-\frac{y}{x}-1, & \frac{z}{x^2}+\frac{y^2}{x^2}-1, & \frac{xz}{w}-\frac{y^2}{w}-1, & \end{array}$$

is the foundation of the 3-connected rank 4 matroid on 8 elements with short circuits

$$1234, \quad 1256, \quad 1357, \quad 1468, \quad 2358, \quad 2457, \quad 2678, \quad 3456,$$

which is minimal for this foundation. The foundation $F_{A.3.31}$ embeds into its universal ring $\mathbb{Z}[x^{\pm 1},\ (x-1)^{-1},\ (2x-1)^{-1},\ (x^2+x-1)^{-1}]$ via $x \mapsto x$, $y \mapsto x-1$, $z \mapsto 2x-1$ and $w \mapsto x^2+x-1$. It is not equal to its universal partial field

$$\Pi(F_{A.3.31}) \;=\; F_{A.3.31} /\!\!/ \langle\!\langle \tfrac{xw}{y^3} - \tfrac{z^2}{y^3} - 1 \rangle\!\rangle,$$

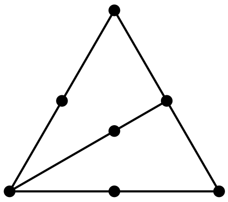
whose null set features an additional relation (verified with the method from [VdlH24]). The foundation $F_{A.3.31}$ is representable (over every field with at least 5 elements) and orientable, but not rigid.

A.3.32 (9,7)

The *Hydra* 4 *partial field* $\mathbb{H}_4 = \mathbb{F}_1^{\pm}(x,y,z,s,t,w) /\!\!/ \langle\!\langle S \rangle\!\rangle$, where S consists of

$$\begin{array}{lll} x+s-1, & \frac{s}{z}+\frac{xt}{z}-1, & \frac{xt}{w}+\frac{ys}{w}-1, \\ y+t-1, & \frac{t}{z}+\frac{ys}{z}-1, & \frac{w}{yz}-\frac{xt^2}{yz}-1, \\ xy+z-1, & \frac{w}{z}+\frac{st}{z}-1, & \frac{w}{xz}-\frac{ys^2}{xz}-1, \end{array}$$

(cf. [PvZ10a, p. 543]) is the foundation of the 3-connected rank 3 matroid on 7 elements depicted as



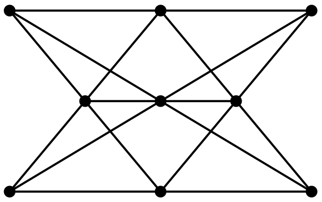
which is minimal for this foundation. The Hydra 4 partial field is representable (over every field with at least 5 elements) and orientable, but not rigid.

A.3.33 (11,7)

The *Pappus foundation* $F_{A.3.33} = \mathbb{F}_1^{\pm}(x,\ y,\ z,\ s,\ t,\ v,\ w)/\!\!/\langle\!\langle S\rangle\!\rangle$, where S consists of

$$\begin{array}{lll} s+x-1, & \frac{w}{x}-\frac{st}{x}-1, & \frac{x}{z}-\frac{y}{z}-1, \\ t+y-1, & \frac{v}{y}-\frac{st}{y}-1, & \frac{t}{z}-\frac{s}{z}-1, \\ v+xt-1, & \frac{v}{xy}-\frac{s}{xy}-1, & \frac{xt}{z}-\frac{ys}{z}-1, \\ w+ys-1, & \frac{w}{xy}-\frac{t}{xy}-1, & \end{array}$$

is the foundation of the Pappus matroid, depicted as



which is a minimal matroid for this foundation. The Pappus foundation embeds into its universal ring

$$\mathbb{Z}\big[x^{\pm1},\ y^{\pm1},\ (1-x)^{-1},\ (1-y)^{-1},\ (x-y)^{-1},\ (1-x+xy)^{-1},\ (1-y+xy)^{-1}\big]$$

via

$$\begin{array}{llll} x\mapsto x, & s\mapsto 1-x, & v\mapsto 1-x+xy, & z\mapsto x-y, \\ y\mapsto y, & t\mapsto 1-y, & w\mapsto 1-y+xy. & \end{array}$$

The Pappus foundation differs from its universal partial field

$$\Pi(F_{A.3.33}) \ = \ F_{A.3.33}/\!\!/\langle\!\langle \tfrac{w}{z}-\tfrac{v}{z}-1, \quad \tfrac{xtw}{z}-\tfrac{ysv}{z}-1, \quad \tfrac{tv}{xyz}-\tfrac{sw}{xyz}-1, \quad \tfrac{xv}{stz}-\tfrac{yw}{stz}-1\rangle\!\rangle$$

(verified with the method from [VdlH24]). It is representable (over $\mathbb{F}_4$ and all fields with at least 7 elements) and orientable, but not rigid.

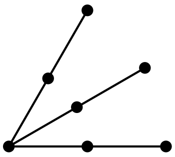
A.4. Morphisms of foundations into other pastures

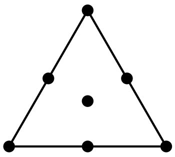
A matroid M is representable over a pasture P if and only if there is a morphism from the foundation F_M of M into P. This allows us to study matroid representations in terms of algebraic properties, a technique that has been applied successfully in the previous papers [BL25a] and [BL25b].

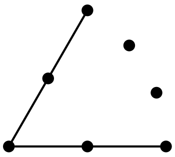
In Table A.1, we display for a range of foundations F (in the leftmost column) and pastures P (in the top row) whether there exists a morphism from F to P. The second column ("repr.") indicates if the foundation is representable over any field, the third column ("PF") indicates if the foundation is a partial field, and the last column ("rigid") indicates if the foundation is rigid.

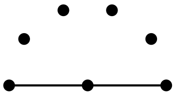
A.5. Numerical description of foundations of 3-connected matroids on 7 and 8 elements

The pastures we've encountered already which appear as foundations of 3-connected matroids on 7 elements are $\mathbb{F}_2$, $\mathbb{D}$, $\mathbb{K}_2$, $\mathbb{H}_4$, $\mathbb{U}_7^2$, and $\mathbb{U}_7^3$. There are 4 additional pastures that appear as foundation of 3-connected matroids on 7 elements; they occur for the following rank 3 matroids:


$(18, 10)$


$(22, 10)$


$(41, 15)$


$(69, 21)$

These matroids are labeled by the numerical type (h, r) of their foundations (where h is the number of hexagons and r is the rank), which are too large for a meaningful explicit description. In each case, the torsion is generated by -1 and $-1 \neq 1$.

An exhaustive search on 3-connected matroids on 8 elements reveals (at least) 196 different isomorphism classes of foundations. For up to 20 hexagons, we verified that there are precisely 66 isomorphism types, but beyond 20 hexagons, the available computational power limited us to a search for numerical types: there are 130 different ones, and these might very well split into different isomorphism classes. This means that there might be more than 196 isomorphism classes.

A.6. Large non-representable foundations

According to Nelson's theorem ([Nel18]), most matroids are not representable. Accordingly, we expect that most foundations are not representable. We have seen already a significant number of small non-representable foundations in Section A.3: $\mathbb{K}$, $\mathbb{S}$, $F_{A.3.6}$, $F_{A.3.11}$, $F_{A.3.12}$, $F_{A.3.13}$, $F_{A.3.14}$, $F_{A.3.16}$, $F_{A.3.20}$, $F_{A.3.23}$, $F_{A.3.24}$, $F_{A.3.26}$, and $F_{A.3.28}$.

The foundations of some prominent non-representable matroids are much larger than the examples in this appendix: the non-Pappus matroid has numerical type $(29, 9)$; the Vámos matroid has numerical type $(76, 21)$; the Desarguesian matroid has numerical type $(95, 17)$.

In so far, we wonder if the analog of Nelson's theorem for foundations holds:

Table A.1: Morphisms from foundations into other pastures.

found.	repr.	PF	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{F}_4$	$\mathbb{F}_5$	$\mathbb{F}_7$	$\mathbb{F}_8$	$\mathbb{F}_9$	$\mathbb{G}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{S}$	$\mathbb{P}$	rigid
$\mathbb{F}_1^{\pm}$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\mathbb{U}$	✓	✓	—	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	—
$\mathbb{V}$	✓	✓	—	—	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	—
$\mathbb{F}_2$	✓	✓	✓	—	✓	—	—	✓	—	—	—	—	—	—	✓
$\mathbb{F}_3$	✓	✓	—	✓	—	—	—	—	✓	—	—	—	—	—	✓
$\mathbb{K}$	×	—	—	—	—	—	—	—	—	—	—	—	—	—	✓
$\mathbb{S}$	×	—	—	—	—	—	—	—	—	—	—	—	✓	✓	✓
$\mathbb{H}$	✓	✓	—	✓	✓	—	✓	—	✓	—	—	✓	—	✓	✓
$\mathbb{D}$	✓	✓	—	✓	—	✓	✓	—	✓	—	✓	✓	✓	✓	—
$\mathbb{G}$	✓	✓	—	—	✓	✓	—	—	✓	✓	✓	✓	✓	✓	✓
$F_{A.3.6}$	×	—	—	—	—	—	—	—	—	—	—	—	—	—	✓
$F_{A.3.7}$	✓	✓	—	—	—	✓	✓	✓	—	—	✓	✓	✓	✓	✓
$\mathbb{H}_2$	✓	✓	—	—	—	✓	—	—	✓	—	—	✓	—	✓	—
$F_{A.3.9}$	✓	✓	—	—	✓	—	✓	—	—	—	—	✓	—	✓	—
$F_{A.3.10}$	✓	—	—	—	✓	—	—	—	—	—	—	—	—	—	—
$F_{A.3.11}$	×	—	—	—	—	—	—	—	—	—	—	—	—	—	—
$F_{A.3.12}$	×	—	—	—	—	—	—	—	—	—	—	—	✓	—	—
$F_{A.3.13}$	×	—	—	—	—	—	—	—	—	—	—	—	—	—	—
$F_{A.3.14}$	×	—	—	—	—	—	—	—	—	—	—	—	—	—	—
$F_{A.3.15}$	✓	—	—	✓	—	—	—	—	—	—	—	—	✓	✓	—
$F_{A.3.16}$	×	—	—	—	—	—	—	—	—	—	—	—	—	—	—
$F_{A.3.17}$	✓	—	—	—	—	✓	—	—	—	—	—	✓	—	✓	—
$F_{A.3.18}$	✓	—	—	—	✓	✓	—	—	—	—	✓	✓	✓	✓	—
$\mathbb{K}_2$	✓	✓	—	✓	✓	✓	✓	✓	—	—	✓	✓	✓	✓	—
$F_{A.3.20}$	×	—	—	—	—	—	—	—	—	—	—	—	✓	✓	—
$\mathbb{H}_3$	✓	✓	—	—	—	✓	✓	✓	✓	✓	✓	✓	✓	✓	—
$F_{A.3.22}$	✓	✓	—	—	✓	✓	✓	✓	✓	✓	✓	✓	—	✓	—
$F_{A.3.23}$	×	—	—	—	—	—	—	—	—	—	—	—	✓	✓	—
$F_{A.3.24}$	×	—	—	—	—	—	—	—	—	—	—	—	—	—	—
$\mathbb{P}_4$	✓	✓	—	—	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	—
$F_{A.3.26}$	×	—	—	—	—	—	—	—	—	—	—	—	✓	✓	—
$F_{A.3.27}$	✓	—	—	—	—	✓	✓	—	—	✓	✓	✓	✓	✓	—
$F_{A.3.28}$	×	—	—	—	—	—	—	—	—	—	—	—	—	—	—
$F_{A.3.29}$	✓	—	—	—	—	✓	✓	—	✓	✓	✓	✓	✓	✓	—
$F_{A.3.30}$	✓	✓	—	—	—	✓	✓	—	✓	✓	✓	✓	✓	✓	—
$F_{A.3.31}$	✓	—	—	—	—	✓	✓	✓	✓	✓	✓	✓	✓	✓	—
$\mathbb{H}_4$	✓	✓	—	—	—	✓	✓	✓	✓	✓	✓	✓	✓	✓	—
$F_{A.3.33}$	✓	—	—	—	✓	—	✓	✓	✓	✓	✓	✓	✓	✓	—

Problem A.5. Are almost all foundations non-representable? This is, let $\mathcal{F}(n)$ be set of isomorphism classes of foundations of matroids on n elements and $\mathcal{N}(n) \subset \mathcal{F}(n)$ the subset of non-representable classes. Is

$$\lim_{n\to\infty} \frac{\#\mathcal{N}(n)}{\#\mathcal{F}(n)} \;=\; 1\,?$$

Many examples of non-representable foundations contain 1 as a fundamental element, which is an obstruction for representability. A minimal size non-representable matroid for which 1 is not a fundamental element is the matroid R_9^A (see [PvZa13] for the notation), as observed in [CZ23, Section 5]: it has foundation is $\mathbb{H}(\epsilon, x) /\!\!/ \langle\!\langle \epsilon^2 - 1,\ S \rangle\!\rangle$, where S consists of

$$\begin{array}{lllll} \epsilon + \zeta_6\epsilon x - 1, & \zeta_6^2 - 1 - 1, & \zeta_6^2\epsilon x - 1 - 1, & \epsilon/x + 1/x - 1, & -\epsilon - \epsilon x - 1. \\ \epsilon - \zeta_6\epsilon x - 1, & -\zeta_6^2\epsilon - \zeta_6 - 1, & -\zeta_6\epsilon x - 1 - 1, & \zeta_6/x + \zeta_6/x - 1, & \end{array}$$

and where ζ_6 is a fundamental element of $\mathbb{H}$ (cf. Section A.3.3). The element 1 does not appear as a fundamental element; the reason for non-representability are the relations

$$\epsilon + \zeta_6\epsilon x - 1, \qquad \epsilon - \zeta_6\epsilon x - 1, \qquad \zeta_6^2 - 1 - 1.$$

Indeed, suppose we had a homomorphism $\varphi : F_{R_9^A} \to K$ from the foundation of R_9^A to a field K. The first two displayed relations imply that $2\varphi(\zeta_6\epsilon x) = 0$, and thus $2 = 0$ in K, while the third relation implies that $2 = \varphi(\zeta_6^2) \neq 0$ in K, a contradiction.

Chen and the third author of this paper find in [CZ23] a criterion for non-representability that holds in all known cases. Namely, consider

$$P \;=\; \mathbb{F}_1^{\pm}(x, y, z, w) /\!\!/ \langle\!\langle x + y - 1,\ \ x + z - 1,\ \ w + y/z - 1 \rangle\!\rangle,$$

which is a pasture that does not map into any field, but does map into every non-representable foundation known to us. The first part of the following problem is stated in [CZ23, Section 5].

Problem A.6. Is there a morphism $P \to F$ for every non-representable foundation F? If not, can we describe an explicit (and possibly even finite?) list of pastures $P_1, P_2, \ldots$ such that a foundation F is non-representable if and only if there is a morphism $P_i \to F$ for some i?

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