

FEYNMAN SYMMETRIES OF THE MARTIN AND C_2 INVARIANTS OF REGULAR GRAPHS

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Abstract. For every regular graph, we define a sequence of integers, using the recursion of the Martin polynomial. We prove that this sequence counts spanning tree partitions and thus constitutes the diagonal coefficients of powers of the Kirchhoff polynomial. We also prove that this sequence respects all known symmetries of Feynman period integrals in quantum field theory. We show that other quantities with this property, the c_2 invariant and the extended graph permanent, are essentially determined by our new sequence. This proves the completion conjecture for the c_2 invariant at all primes, and also that it is fixed under twists. We conjecture that our invariant is perfect: Two Feynman periods are equal, if and only if, their Martin sequences are equal.

Keywords. Martin polynomial, transitions, spanning trees, point counts, Feynman integrals, integer sequences, permanent, Prüfer sequence

Mathematics Subject Classifications. 81Q30, 05C70, 05C45

1. Introduction

In this paper, graphs are undirected and allowed to have self-loops (edges that connect a vertex to itself) and multiedges (several edges connecting the same pair of vertices).

Let G be a $2k$ -regular graph, that is, every vertex has the same, even degree $2k > 0$. Following [Jae90, FGJ07], a *transition* at a vertex v is a partition $\tau(1) \sqcup \dots \sqcup \tau(k)$ of the $2k$ half-edges $e_i = vw_i$ at v into pairs $\tau(i) = \{e_{\tau'(i)}, e_{\tau''(i)}\}$ (made precise in Definition 2.1). Given a transition, we can transform G into a smaller graph, which is again $2k$ -regular:

$$G_\tau = (G \setminus v) + w_{\tau'(1)}w_{\tau''(1)} + \dots + w_{\tau'(k)}w_{\tau''(k)}$$

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is obtained by removing v together with its edges e_i , and then adding k new edges to match the neighbours w_i of v . Let $\mathcal{T}(v)$ denote the set of all $(2k - 1)!!$ transitions at v .

In this paper, we study invariants of graphs that solve the recursion

$$M(G) = \sum_{\tau \in \mathcal{T}(v)} M(G_\tau). \quad (1.1)$$

Such invariants are interesting both as pure combinatorics and because they provide a unifying perspective on previously disparate invariants that are interesting in quantum field theory on account of them having the key symmetries of Feynman period integrals.

This recursion was introduced by Martin [Mar77] to define a polynomial that counts circuit decompositions. He considered in particular the 3-term recurrence

$$M \left(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right) = M \left(\begin{array}{c} \bullet \quad \bullet \\ \hline \bullet \quad \bullet \end{array} \right) + M \left(\begin{array}{c} \bullet \quad \bullet \\ \hline \bullet \quad \bullet \end{array} \right) + M \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right) \quad (1.2)$$

for 4-regular graphs, and Las Vergnas [Las83] extended Martin's considerations to arbitrary even degrees. For matroid theorists, these polynomials can be interpreted as a kind of chromatic polynomial for the transition matroid [Tra15], instead of the usual cycle matroid. Our focus lies on a certain derivative of Martin's polynomial (see Definition 2.5) which, for graphs with 3 or more vertices, can also be defined as follows:

Definition 1.1. The *Martin invariant* $M(G)$ of a regular graph G with even degree and at least 3 vertices, is the non-negative integer defined by the rules:

1. If G has a self-loop, then $M(G) = 0$.
2. If G has three vertices (and no self-loop), then $M(G) = 1$.
3. Otherwise, pick any vertex v , and define $M(G)$ recursively by (1.1).

It follows from [Las83] that all choices of v in rule 3. produce the same result, so $M(G)$ is well-defined. The situation for graphs with fewer than 3 vertices and justification for our normalization is explained in Definition 2.5 and subsequent discussion; it is not crucial at present. To illustrate the definition, the Martin invariant of the complete graph K_5 is

$$M(K_5) = 3 M \left(\begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) = 3 M \left(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right) + 6 M \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right) = 3 \cdot 0 + 6 \cdot 1 = 6.$$

In the first expansion, all 3 transitions produce isomorphic graphs: a perfect matching added to $K_4 = K_5 \setminus v$. The second expansion produces a self-loop when the transition pairs the parallel edges with each other; the other 2 transitions create \triangle .

For 4-regular graphs, the Martin invariant was introduced in [BG96]. It was shown in particular that $M(G) > 0$ if and only if G is 4-edge connected, and that $M(G)$ simplifies into a product if G has a 4-edge cut. We generalize these properties to higher degrees:

Theorem 1.2. *Let G be a $2k$ -regular graph. Then:*

1. *If G has an edge cut of size less than $2k$, then $M(G) = 0$.*
2. *If G has no edge cut of size less than $2k$, then $M(G) > 0$.*
3. *If G has an edge cut of size equal to $2k$, then $M(G) = k! \cdot M(G_1) \cdot M(G_2)$ for the graphs G_1 and G_2 obtained by replacing one side of the cut with a single vertex:*

$$M \left(\text{Diagram 1} \right) = k! \cdot M \left(\text{Diagram 2} \right) \cdot M \left(\text{Diagram 3} \right). \tag{1.3}$$

These results are proved in Lemma 3.1, Corollary 3.3, and Lemma 3.4. In Section 3.1 we prove further results, including a lower bound on non-zero Martin invariants and a characterization of precisely when the lower bound is achieved in terms of total decomposability (Theorem 3.5) and we prove uniqueness of these decompositions (Proposition 3.8).

Note that if one side of the cut in case 3 is just a single vertex, then $G_1 = G$. This is consistent with the full definition of $M(G)$, Definition 2.5, where we find that $M(G_2) = 1/k!$ for the graph G_2 consisting of two vertices with $2k$ edges between them.

The computation of $M(G)$ therefore reduces to graphs G that are *cyclically $(2k + 2)$ -edge connected*.¹ This means that G is $2k$ -edge connected, and requires in addition that the only edge cuts of size $2k$ are the trivial ones that separate a single vertex from the rest of the graph. It is easy to see that such graphs cannot have any 1- or 2-vertex cuts. Our following two results (proven in Section 3.2) give relations from 3- and 4-vertex cuts:

- Let G be $2k$ -regular and $2k$ -edge connected, with a 3-vertex cut. Then the two sides of the cut can be turned into $2k$ -regular graphs G_1 and G_2 , by adding edges (but no self-loops) between the cut vertices. Furthermore, we have

$$M(G) = M(G_1) \cdot M(G_2). \tag{1.4}$$

- If two regular graphs of even degree are obtained from each other by a double transposition on one side of a 4-vertex cut, then their Martin invariants agree:

$$M \left(\text{Diagram 4} \right) = M \left(\text{Diagram 5} \right). \tag{1.5}$$

These operations on graphs are called *product* and *twist* in [Sch10]. For an example of the product identity, consider the 3-vertex cut highlighted in red (the 3 vertices stacked vertically) in the following graph:

$$M \left(\text{Diagram 6} \right) = M \left(\text{Diagram 7} \right) \cdot M \left(\text{Diagram 8} \right) = M(K_5) \cdot M(K_5) = 36.$$

¹In our context of $2k$ -regular graphs, the notions of cyclically $(2k + 2)$ -edge connected, essentially $(2k + 2)$ -edge connected, and internally $(2k + 2)$ -edge connected are all equivalent since the properties of each side of the cut having at least one cycle, having at least one edge, and having at least two vertices are equivalent for $2k$ -edge cuts of $2k$ regular graphs.

1.1. Spanning trees and diagonals

The Martin invariant has another combinatorial description, in terms of spanning trees. Given a graph G with n vertices, a *spanning tree* T of G is a subset of the edges of G such that T has $n - 1$ elements and forms a connected subgraph. We prove (see Section 5):

Theorem 1.3. *For every vertex v in a $2k$ -regular graph G with at least 3 vertices, the number of partitions of the edge set of $G \setminus v$ into k spanning trees is equal to $M(G)$. In particular, the number of such partitions is independent of v .*

It is worth emphasizing that this is quite surprising: the Martin polynomial, by its nature, counts circuit partitions, but it is not at all apparent at the outset that it can also be used to count spanning tree partitions.²

As an example of the theorem, consider the 4-regular complete graph $G = K_5$. Then the 6 edges of $K_5 \setminus v \cong K_4$ allow precisely $M(K_5) = 6$ partitions into pairs of spanning trees:

$$\begin{array}{cccccc} \text{[Diagram 1]} = \text{[Diagram 2]} \cup \text{[Diagram 3]} & = & \text{[Diagram 4]} \cup \text{[Diagram 5]} & = & \text{[Diagram 6]} \cup \text{[Diagram 7]} & = & \text{[Diagram 8]} \cup \text{[Diagram 9]} \\ \text{[Diagram 10]} \cup \text{[Diagram 11]} & = & \text{[Diagram 12]} \cup \text{[Diagram 13]} & = & \text{[Diagram 14]} \cup \text{[Diagram 15]} & = & \text{[Diagram 16]} \cup \text{[Diagram 17]} \end{array} \quad (1.6)$$

The theorem implies that $M(G)$ is determined by the cycle matroid of $G \setminus v$.³ We can also view the theorem as a relation between $M(G)$ and certain coefficients of a polynomial:

Definition 1.4. Given a formal power series $F = \sum_n c_n x^n$ in several variables x_i , we write $[x^n] F = c_n$ for the coefficient of any monomial $x^n = \prod_i x_i^{n_i}$.

The *Symanzik* or *dual Kirchhoff* polynomial⁴ Ψ_G of any graph G is the sum

$$\Psi_G = \sum_T \prod_{e \notin T} x_e \in \mathbb{Z}[x_1, \dots, x_m] \quad (1.7)$$

over all spanning trees T of G . The variables x_e are labelled by the m edges of G . Applied to the complement of a vertex in a $2k$ -regular graph G , we can restate Theorem 1.3 as

$$[x_1^{k-1} \cdots x_m^{k-1}] \Psi_{G \setminus v}^k = k! \cdot M(G)$$

(see Theorem 5.17 for the proof). We will use this identity to show the following symmetry under planar duality (see Section 5.3):

Theorem 1.5. *Suppose that G_1 and G_2 are 4-regular graphs that have vertices v_1, v_2 such that $G_1 \setminus v_1$ is a planar dual of $G_2 \setminus v_2$. Then $M(G_1) = M(G_2)$.*

²Thanks to a referee for drawing attention to the unexpected nature of this connection.

³However, the Martin invariant is *not* determined by the Tutte polynomial (see Section 7.1).

⁴This is sometimes called *graph polynomial*, e.g. in [BS12, BEK06].

1.2. Residues

By linearity, the residues $M(G) \pmod q$, for a fixed integer q , also solve the recursion (1.1). Two such residues coincide with invariants of graphs that have been studied previously: the *graph permanent* and the *c_2 -invariant*. These two invariants are quite different in character but both were introduced because they have key symmetries of Feynman period integrals and so should carry some quantum field theoretic information despite their purely combinatorial nature.

Let G be a $2k$ -regular graph and choose two vertices $v \neq w$ such that v has no self-loop. Call the remaining vertices $1, \dots, n$ and pick an orientation of the kn edges of $G \setminus v$. Their $n \times kn$ incidence matrix A has entries $A_{ie} = \pm 1$ if vertex $i \in \{1, \dots, n\}$ is the head or tail of edge e , and $A_{ie} = 0$ otherwise. Stacking k copies of A , we get a square $kn \times kn$ matrix, denoted $A^{[k]}$. It was shown in [CDY16] that the square of the permanent

$$\text{Perm}(G) = \text{perm } A^{[k]} \in \mathbb{Z}$$

has a well-defined residue modulo $k + 1$: this residue does not depend on the edge orientations or the choices of v and w . When $k + 1$ is composite, then $\text{Perm}(G) \equiv 0 \pmod{(k + 1)}$, so the permanent is interesting only when $k + 1$ is prime. We prove (see Theorem 4.1)

Theorem 1.6. *Let G be a $2k$ -regular graph G with n vertices. If $k + 1$ is prime, then*

$$M(G) \equiv (-1)^{n-1} \text{Perm}(G)^2 \pmod{(k + 1)}. \tag{1.8}$$

This implies that the permanent inherits all identities of the Martin invariant mentioned above, explaining all properties of the permanent obtained in [CDY16] using different methods.

The permanent of a graph was enriched in [Cru17b] to an infinite sequence of residues. Let $G^{[r]}$ denote the $2kr$ -regular graph obtained from G by replacing each edge with a bundle of r parallel edges. The *extended graph permanent* [Cru17b] consists of the permanents of $G^{[r]}$ where $kr + 1$ is prime. For example, for a 4-regular graph G , it is the sequence

$$(\text{Perm}(G), \text{Perm}(G^{[2]}), \text{Perm}(G^{[3]}), \text{Perm}(G^{[5]}), \dots) \in \mathbb{F}_3 \times \mathbb{F}_5 \times \mathbb{F}_7 \times \mathbb{F}_{11} \times \dots$$

of residues in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of the permanents of $G^{[(p-1)/2]}$ for all odd primes p .

Definition 1.7. The *Martin sequence* of a regular graph G is the list of integers $M(G^{[r]})$, indexed by all positive integers r . We denote this sequence as

$$M(G^\bullet) = \left(M(G^{[1]}), M(G^{[2]}), M(G^{[3]}), \dots \right). \tag{1.9}$$

By Theorem 4.1, the Martin sequence determines the extended graph permanent.⁵

The c_2 -invariant was introduced in [Sch11]. It is a sequence of residues $c_2^{(q)}(G) \in \mathbb{Z}/q\mathbb{Z}$ indexed by all prime powers q and defined by

$$\llbracket \Psi_G \rrbracket_q \equiv q^2 \cdot c_2^{(q)}(G) \pmod{q^3}$$

⁵Without squaring, the residues $\text{Perm}(G^{[r]}) \pmod{(rk + 1)}$ are only defined up to sign. These signs depend on the choices of v and w and the orientation of the edges.

where $[\Psi_G]_q = |\{x \in \mathbb{F}_q^N : \Psi_G(x) = 0\}|$ denotes the number of points on the hypersurface $\{\Psi_G = 0\}$ over the finite field \mathbb{F}_q with q elements. In particular, the restriction of c_2 to primes ($q = p$) determines a sequence of residues in $\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5 \times \mathbb{F}_7 \times \dots$. We will only consider c_2 restricted to primes as this is where our techniques are effective.

In many examples, the c_2 -invariant at primes ($q = p$) appears to be congruent to coefficients of modular forms [BS13, Sch21], which has been established rigorously in a few cases [BS12, Log18]. This connection hints at the deep geometric and number theoretic nature of the c_2 -invariant.

We prove (see Theorem 6.2):

Theorem 1.8. *For every 4-regular graph G with at least 6 vertices, and every prime p , and any vertex v ,*

$$c_2^{(p)}(G \setminus v) \equiv \frac{M(G^{[p-1]})}{3p} \pmod{p}.$$

This shows that the c_2 -invariant at primes is also determined by the Martin sequence. Putting these two relations side by side we have that the extended graph permanent and the c_2 -invariant capture different residues of the Martin invariants: If G is 4-regular, then

$$\begin{aligned} M(G^{[r]}) &\equiv (-1)^{n-1} \text{Perm}(G^{[r]})^2 \pmod{2r+1} \quad \text{if } 2r+1 \text{ is prime, and} \\ M(G^{[r]}) &\equiv 3p \cdot c_2^{(p)}(G \setminus v) \pmod{p^2} \quad \text{if } p = r+1 \text{ is prime.} \end{aligned}$$

For the c_2 invariant this operation of removing a vertex from a 4-regular graph is quite important. If H is obtained from a 4-regular graph G by removing a vertex we say that H is a *decompletion* of G and G is the *completion* of H . Note that the completion is unique but a graph may have many non-isomorphic decompletions. The extended graph permanent also implicitly involves the decomposition since it works with the incidence matrix of $G \setminus v$ rather than with G directly.

The c_2 invariant, like the extended graph permanent, inherits the symmetries of the Martin invariant from Theorem 6.2, some of which were not previously known (e.g. the twist). Most interestingly, although the left-hand side of the theorem involves $G \setminus v$, note that the right-hand side depends only on G but not v . Thus Theorem 1.8 implies:

Corollary 1.9 (Completion invariance at primes). *For all primes p and any two vertices v, w of a 4-regular graph G , we have $c_2^{(p)}(G \setminus v) \equiv c_2^{(p)}(G \setminus w) \pmod{p}$.*

This was conjectured by Brown and Schnetz in 2010, see [BS12, Conjecture 4]. It is indicative of a relation between the highest weight parts of the cohomology groups of the two graph hypersurfaces $\{\Psi_{G \setminus v} = 0\}$ and $\{\Psi_{G \setminus w} = 0\}$, as studied in [BEK06, BD13].

Over the last thirteen years, this conjecture was attacked by techniques from algebraic geometry, combinatorics and physics, but with very little to show for these efforts until the proof of the conjecture for $p = 2$, first only for graphs G with an odd number of vertices [Yea18] and then for all graphs [HY23] by the second author with Hu. These proofs used the combinatorial interpretation of diagonal coefficients that will figure prominently in Section 5 and Section 6 along with some intricate involutions. The connection with the Martin invariant proved in the

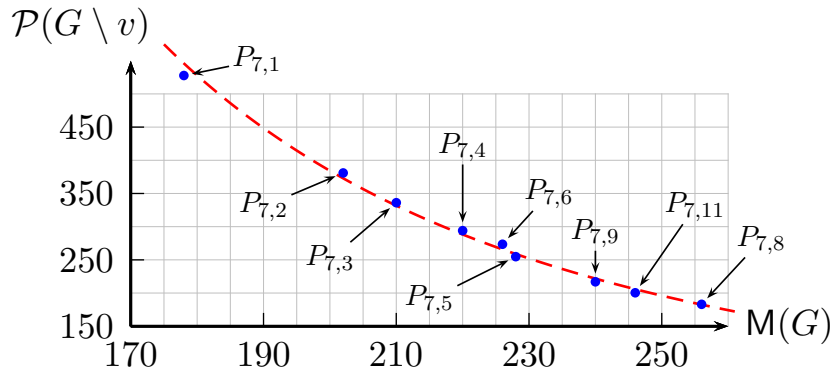


Figure 1.1: The periods and Martin invariants of primitive ϕ^4 graphs [Sch10] with 9 vertices (7 loops decompleted). The dashed curve is a fit $\mathcal{P}(G \setminus v) \approx 32150 \cdot M(G)^{-3.015}$.

present paper provides the missing piece to finally settle this conjecture from over a decade ago, for all primes. In full, [BS12, Conjecture 4] also covers c_2 at prime powers $q = p^n$ with $n > 1$, and these cases remain open.

1.3. Feynman integrals

The product and twist identities discussed above (see also Propositions 3.10 and 3.12 for precise statements) were introduced for 4-regular graphs by Schnetz [Sch10]. However, these identities were discovered not for the Martin invariant, but for a completely different function of graphs: The *period* of G is defined as the (Feynman) integral

$$\mathcal{P}(G) = \left(\prod_{e=2}^m \int_0^\infty dx_e \right) \frac{1}{\Psi_G^2} \Big|_{x_1=1} \tag{1.10}$$

whenever this converges [BEK06]. The motivation to study these numbers comes from quantum field theory, where they give contributions to the beta function that drives the running of the coupling constant. Systematic calculations of these periods go back to [BK95, Bro85].

Schnetz showed in [Sch10] that if G is 4-regular and cyclically 6-connected, then for any vertex of v , the period $\mathcal{P}(G \setminus v)$ is well-defined (convergent) and independent of the choice of v . The period therefore defines a function

$$G \mapsto \mathcal{P}(G \setminus v)$$

on the set of cyclically 6-connected 4-regular graphs. It satisfies the product, twist and duality identities [Sch10], which suggests a relation between the period and the Martin invariant. Indeed, we find two such connections.

Firstly, the Martin invariant and the period are correlated numerically. Graphs with smaller $M(G)$ tend to have larger periods. This approximate relation is illustrated in Figure 1.1 for the 4-regular graphs with 9 vertices. As the plot shows, the period can be estimated surprisingly well from $M(G)$ and a simple power law. It is remarkable that the period integral (1.10) is

so strongly correlated with the number of partitions into spanning trees (Theorem 1.3). Furthermore, the period, and hence the Martin invariant, are also correlated to the Hepp bound [Pan23], which is a tropical version of the period.

Secondly, even though the correlation just mentioned is only approximate, we find that the Martin *sequence* can detect precisely when two graphs have the same period. We calculated Martin invariants for a large number of 3- and 4-regular graphs (Section 7). Comparing with the known periods [PS17, BS22] and the Hepp bound, supports the following conjecture (we state a 3-regular version in Conjecture 7.2).

Conjecture 1.10. Cyclically 6-connected 4-regular graphs G_1 and G_2 have equal period $\mathcal{P}(G_1 \setminus v_1) = \mathcal{P}(G_2 \setminus v_2)$ if and only if they have equal Martin sequences $M(G_1^\bullet) = M(G_2^\bullet)$.

Via Theorem 1.8, our conjecture implies that the c_2 invariants of graphs with equal periods agree at all primes. This is expected, also at prime powers, according to [Sch11, Remark 2.11]. Our conjecture thus implies the case $q = p$ of [BS12, Conjecture 5].

Conjecture 1.10 also suggests that the period (and Hepp bound) of a graph should be computable from the Martin sequence. Indeed, this is proved in forthcoming work with Francis Brown. One important observation towards this is that Martin sequences are P-recursive (equivalently, their generating series are D-finite), as we prove in Corollary 5.22.

There is another, in fact classical, connection between the Martin *polynomial* and quantum field theory: the Martin polynomial appears as the symmetry factor S_N in $O(N)$ vector models. Only very recently, however, also the Martin *invariant* was noted to have a physical meaning in this context, namely it emerges at $N = -2$ for loop-erased random walks [WF19, SW20]. We summarize this connection in Section 2.1.

1.4. Origin

In fact, it was in this physics context that we discovered the Martin invariant:

The need to compile a table of symmetry factors for the calculation [KP17] triggered an, at first experimental, study of structural properties of the polynomial S_N and its correlation with the period $\mathcal{P}(G)$. Trying different values for N produced the remarkable discovery in Figure 1.1, namely that for $N = -2$, not only is the correlation strong, but the pairs $\{P_{7,4}, P_{7,7}\}$ (see Figure 5.4) and $\{P_{7,5}, P_{7,10}\}$ of graphs with the same periods also have exactly (and not just approximately) equal symmetry factors S_N . Even graphs with unknown periods supported this connection, via the Hepp bounds computed in [Pan23] which are a very good proxy for periods. The paper [BG96] suggested focusing on the recursive Definition 1.1 which we used to prove the relations between $M(G)$, $\text{Perm}(G) \pmod 3$, and $c_2^{(2)} \pmod 2$ (the argument after Example 6.15). We then applied the idea of duplicating edges from [Cru17b] to upgrade the Martin invariant to the Martin sequence, and found the proofs of Theorems 1.6 and 1.8 given in Sections 4.3 and 6.3. Only in hindsight did we note that a simpler version of our proof technique relates the Martin invariant to spanning tree partitions (Theorem 1.3), triggered by Francis Brown's suggestion to relate c_2 to the diagonal coefficients of the Symanzik polynomial—rather than the third denominator.

1.5. Conclusions

We unified the theory of several invariants of graphs that were motivated by Feynman integrals in particle physics, by relating them to a single, combinatorial invariant counting spanning tree partitions:

- All known invariants of graphs that respect the identities of the Feynman period (i.e. the c_2 invariant, the extended graph permanent, the Hepp bound, and the period integral itself) are determined by the Martin sequence.

In this paper, we prove this for the extended permanent and for the c_2 -invariant at primes. The determination of c_2 at prime powers, the Hepp bound, and the period integral, all in terms of only the Martin sequence, are work in progress with Francis Brown.

Secondly, we have learned that this integer sequence has a rich theory and satisfies a number of interesting relations. Most importantly:

- The number of spanning tree partitions fulfils Martin's recursion at a vertex.

Guided by applications in physics, we focused on regular graphs, where this observation can be interpreted as a relation between spanning tree partitions and circuit partitions (via the Martin polynomial). However, note that the sequence of spanning tree partitions, i.e. the diagonal of the Symanzik polynomial, is defined for arbitrary graphs—not just for decompletions of regular graphs. Our proof of Theorem 5.17 shows directly a recurrence relation for this diagonal, without any assumptions on regularity. Hence in hindsight, we could *define* the Martin sequence as this diagonal, and much will generalize to arbitrary graphs, or even matroids (see Remarks 4.6 and 5.25).

Apart from generalizing degrees, the Martin polynomial is also defined for *directed* Eulerian graphs, where transitions are restricted to pair incoming half-edges with outgoing half-edges. For regular digraphs with k incoming and k outgoing edges at every vertex, this polynomial is divisible by $x(x+1)\cdots(x+k-2)$, see [Las83, Théorème 3.2]. By analogy, the derivative at $x = 2 - k$ will have similar structural properties as our unoriented case (derivative at $x = 4 - 2k$). It is tempting to wonder if this invariant of digraphs is related to counting partitions into arborescences, i.e. the diagonal of an oriented version of the Symanzik polynomial, and hence oriented versions of permanents and period integrals.

Another natural question is to identify the minimal and maximal values that $M(G)$ can take, for primitive regular graphs with a fixed degree and loop order; and to characterize the graphs that achieve those values. Proposition 7.1 for 4-regular graphs and Conjecture 7.4 for 3-regular graphs suggest that the minimizers are highly structured and could perhaps be identified for all degrees. There also appears to be a unique graph at each degree and loop order that *maximizes* the Martin invariant, see Tables 7.2 and 7.5. Those are harder to describe, but the same graphs appear to also maximize the number of connected sets [CGJ25], and they are also the unique minimizers of the Hepp bound.

The unexplained identities $M(P_{8,30}^{[r]}) = M(P_{8,36}^{[r]})$ and $M(P_{8,31}^{[r]}) = M(P_{8,35}^{[r]})$ of the graphs in Figure 7.1 remain an intriguing open problem. We verified these equalities for $r \leq 8$, but we cannot explain why these pairs of graphs share the same number of spanning tree partitions. The

currently known set of Martin sequence identities (twist, product, duality, and Fourier-split) does not connect these graphs. This strongly suggests that there exist further mechanisms that relate Martin sequences (and thus Feynman periods), and it would be interesting to discover such new symmetries of Feynman integrals.

The sections of this paper are largely independent of each other.

2. The Martin polynomial

Let G be an undirected graph where every vertex has even degree. Multiple edges between the same pair of vertices are allowed. At each vertex, we may also have any number of self-loops, that is, edges connecting the vertex to itself.

It will be helpful to think of graphs in terms of half-edges. The edges of the graph define a perfect matching of the half-edges of the graph, each half-edge is incident with exactly one vertex, and every vertex v has a *corolla* of incident half-edges. The set of the other halves of the edges incident to v will be called the *h -neighbourhood* of v . The size of this set is always equal to the degree of v ; in particular, a self-loop at v contributes both of its half-edges to the h -neighbourhood of v . The neighbourhood in the traditional graph theory sense, namely the set of vertices adjacent to v , can be recovered from the h -neighbourhood by replacing each half-edge in the h -neighbourhood with the vertex incident to that half-edge. When helpful for clarity we will call the graph theory notion of neighbourhood the *v -neighbourhood*. Whenever there are self-loops or multiedges at v , then the size of the v -neighbourhood is less than the degree of v .

Definition 2.1. A *transition* at a vertex v is a perfect matching of the half-edges of the h -neighbourhood of v . We write $\mathcal{T}_G(v)$ or simply $\mathcal{T}(v)$ for the set of all transitions at v . A *transition system* P is a choice of one transition $P(v) \in \mathcal{T}(v)$ for every vertex of G .

Note that the notion of transition could equally well be defined as a perfect matching of the corolla at v . The benefit of working with the h -neighbourhood is that for v with no self-loops, and τ a transition at v , the graph G_τ (see the beginning of Section 1) can be defined very naturally to be the graph obtained from G by removing v and its corolla and matching the half-edges of the h -neighbourhood of v into new edges according to τ .

Since the degree of every vertex v is even, the sets $\mathcal{T}(v)$ are non-empty and transition systems exist. Concretely, a v of degree d has precisely

$$|\mathcal{T}(v)| = (d-1)!! = \frac{d!}{(d/2)! \cdot 2^{d/2}} \quad (2.1)$$

transitions, and the graph G has $\prod_v (d_v - 1)!!$ transition systems. A transition $\tau \in \mathcal{T}(v)$ can be interpreted as a traffic sign that forces anyone who enters v from half-edge e , to leave by half-edge e' where e' is the partner of e in the matching τ . Traversed in this way, a transition system P determines a partition of the edges of G into circuits (closed trails). We write $|P|$ for the number of these circuits. For an example see Figure 2.1. A detailed description of circuits in terms of half-edges can be found on [Tra15, page 181].

$$P \in \left\{ \begin{array}{c} \text{Hourglass} \\ \text{Hourglass} \\ \text{Hourglass} \end{array} \right\} \quad \mathfrak{m} \left(\begin{array}{c} \text{2-rose} \\ \bullet \end{array}, x \right) = 1 + (x - 2) + 1 = x$$

Figure 2.1: The three circuit partitions of a 2-rose and its Martin polynomial.

Definition 2.2. The *Martin polynomial*⁶ is a sum over all transition systems, given by

$$\mathfrak{m}(G, x) = \sum_P (x - 2)^{|P|-1} \in \mathbb{Z}[x]. \tag{2.2}$$

See Table 2.1 for some examples. This polynomial was introduced by Martin [Mar77] and extended by Las Vergnas [Las83].⁷ In more recent literature, results for the Martin polynomial are often stated in terms of the *circuit partition polynomial*, defined as⁸

$$J(G, x) = x \cdot \mathfrak{m}(G, x + 2) = \sum_P x^{|P|} \in \mathbb{Z}[x]. \tag{2.3}$$

The coefficients of $J(G, x) = \sum_k r_k(G)x^k$ count the edge partitions into k circuits. For example, the linear coefficient $r_1(G) = J'(G, 0) = \mathfrak{m}(G, 2)$ counts the Eulerian circuits of G . The precise notion of circuit here is based on half-edges, see [Bol02, p. 262] or [EM99] for details. Further evaluations of these polynomials can be found in [Bol02, EII98]. There is also an integral representation [MR11] for the values of $J(G, x)$ at positive integers x .

It is clear from Definition 2.2 that $\mathfrak{m}(G, x)$ solves the recursion (1.1). Note however that if one applies a transition τ that pairs both ends of a self-loop, then G_τ will have a *free loop*, that is, a loop not connected to any vertex, which must still be counted. This can be avoided with the following lemma.

Lemma 2.3. *If e is a self-loop at a vertex v which has even degree $d \geq 2$, then*

$$\mathfrak{m}(G, x) = (x + d - 4) \cdot \mathfrak{m}(G \setminus e, x). \tag{2.4}$$

Proof. Given any transition $\tau' \in \mathcal{T}_{G'}(v)$ of $G' = G \setminus e$, we can pair the half-edges e', e'' of e to obtain a transition $\tau = \tau' \cup \{e', e''\} \in \mathcal{T}_G(v)$ in G . We can also pick one of the $d/2 - 1$ pairs $\{i, j\} \in \tau'$ and interlace e in either direction, to obtain two further transitions $\tau_{ij}, \tau_{ji} \in \mathcal{T}_G(v)$ of the form $\tau_{ij} = \tau' \setminus \{\{i, j\}\} \cup \{\{i, e'\}, \{j, e''\}\}$. These constructions produce every transition in $\mathcal{T}_G(v)$ exactly once.

We can thus group the sum (2.2) by transition systems of P' of G' . For each P' , we find one transition system P of G with $|P| = |P'| + 1$, and $d - 2$ with $|P| = |P'|$. Hence we conclude the claim, with the prefactor $(x - 2) + (d - 2)$. □

We can iterate Lemma 2.3 to remove all self-loops [Las83, Proposition 4.1]. Hence we can compute $\mathfrak{m}(G, x)$ without ever creating free loops. As a special case, we find

$$\mathfrak{m}(C_1^{[k]}, x) = x(x + 2) \cdots (x + 2k - 4) \tag{2.5}$$

⁶In the very degenerate case where G has no edges, $\mathfrak{m}(G, x) = 1/(x - 2)$ is not a polynomial.

⁷Our notation follows [Las83]; Martin writes $P(G, x)$.

⁸In [Bol02, ABS00] this is denoted $r_G(x)$, we use $J(x)$ following [EII98, EM04, EM99].








G							
$m(G, x)$	x	$3x$	x^2	$x^2 + 6x$	$3x^2$	$5x^2 + 12x$	$15x^2 + 36x$

Table 2.1: Martin polynomials of some 4-regular graphs.

for the graph $C_1^{[k]}$ consisting of a single vertex with $k > 1$ self-loops. This graph is called the k -rose. Induction via (1.1) with a rose as the base case shows that $m(G, x)$ is divisible by the right-hand side of (2.5) whenever G has a vertex of degree $2k$ or more.

Corollary 2.4. *If G is $2k$ -regular, has at least two vertices, and at least one self-loop, then $m(G, x)$ is divisible by $(x + 2k - 4)^2$.*

Proof. The self-loop e gives one factor of $x + 2k - 4$ directly in (2.4). Since $G \setminus e$ still has at least one vertex of degree $2k$, we get a second factor from $m(G \setminus e, x)$. \square

The Martin polynomial has numerous further interesting properties, however for our purposes we only use the above. Mainly, we conclude that the Martin invariant as defined in the introduction, Definition 1.1, makes sense: the number $M(G)$ is uniquely determined, because we can obtain it from $m(G, x)$. Using this insight we give a new direct definition of the Martin invariant and then observe its equivalence with Definition 1.1.

Definition 2.5. The *Martin invariant* of a $2k$ -regular graph ($k \geq 1$) is the derivative

$$M(G) = \frac{4 \cdot (-1)^k}{(k-2)! \cdot (2k)!} m'(G, 4-2k) \in \mathbb{Q}. \quad (2.6)$$

Since taking a derivative is a linear map, any derivative of m inherits the recursion (1.1). Note also that $m'(G, 4-2k)$ vanishes for every $2k$ -regular graph with a self-loop and more than one vertex, by Corollary 2.4. Therefore, this derivative fulfils 1. and 3. of Definition 1.1. Our choice of normalization (part 2. of Definition 1.1) determines the prefactor in (2.6). Indeed, (2.6) gives

$$M(C_1^{[k]}) = \frac{2^k}{(2k)!} \quad \text{and} \quad M(C_2^{[k]}) = \frac{1}{k!} \quad (2.7)$$

for the rose and dipole graphs. The dipole $C_2^{[k]} = K_2^{[2k]}$ is a $2k$ -fold edge, and its Martin invariant reduces to $(2k-1)!!$ times that of the rose by (1.1) and (2.1). For the rose itself, use (2.5). At three vertices, the only $2k$ -regular graph without self-loops is the k -fold triangle, and expanding it at a vertex shows that, as claimed in part 2. of Definition 1.1,

$$M(C_3^{[k]}) = k! \cdot M(C_2^{[k]}) = 1. \quad (2.8)$$

Note that only $k!$ transitions produce a dipole; all other transitions produce self-loops and can hence be discarded for the computation of the Martin invariant.

Remark 2.6. Our normalization (2.8) simplifies the 3-vertex cut product of Proposition 3.10. This maximizes the analogy between M and the Feynman period \mathcal{P} , because the latter also fulfils this product without extra factors [Sch10, Theorem 2.1]. The downside is that we move outside of integers for the Martin invariants of $C_1^{[k]}$ and $C_2^{[k]}$ (though all graphs with at least 3 vertices have integer Martin invariant) and have an explicit factor $k!$ in the $2k$ -edge cut product (1.3), which however has no direct analogue for Feynman periods, because graphs with such a cut have divergent Feynman integrals.

The Martin invariant was introduced for 4-regular graphs in [BG96]. In that paper, it is called β_4 , and a different normalization is used. If G has n vertices, then

$$M(G) = \beta_4(G) \cdot 2^{n-3}. \tag{2.9}$$

The normalization of β_4 ensures that the totally decomposable graphs (see Theorem 3.5) are precisely the graphs with $\beta_4(G) = 1$, in analogy to how the series-parallel graphs are precisely the graphs with $\beta(G) = 1$ for Crapo’s β invariant [Cra67, Proposition 8]. But in contrast to $M(G)$, for most graphs the number $\beta_4(G)$ is not an integer.

Apart from the paper [BG96], the only other result on the Martin invariant that we could find is [Eil00, Theorem 4.6]. For 4-regular graphs, it states that

$$6 M(G) = m'(G, 0) = \frac{J'(G, -2)}{-2} = \sum_C r_1(G \setminus C) \cdot (-2)^{\kappa(C)-1} \tag{2.10}$$

is a sum over all subsets C of edges such that every vertex of G has degree 0 or 2 in C . In other words, C is a vertex-disjoint union of cycles. We write $\kappa(C)$ for the number of these cycles, and $r_1(G) = m(G, 2) = J'(G, 0)$ denotes the number of Eulerian circuits.

2.1. Vector model

In theoretical physics, the Martin polynomial appears as the *symmetry factor* S_G in the $O(N)$ vector model. For details, see [KSF01, Chapter 6]. In brief, this is the theory of N real scalar fields $\phi = (\phi_1, \dots, \phi_N)$ with Euclidean Lagrangian density

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^N (\|\nabla \phi_i\|^2 + m^2 \phi_i^2) + \frac{g}{4!} (\|\phi\|^2)^2.$$

The first term describes free scalars with equal mass m . The second term with the coupling constant g encodes their interaction. This density is invariant under orthogonal transformations $\phi \mapsto T\phi$ with $T \in O(N)$. In terms of the symmetric tensor

$$\lambda_{ijkl} = \frac{1}{3} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.11}$$

the quartic interaction can be written as

$$(\|\phi\|^2)^2 = (\phi_1^2 + \dots + \phi_N^2)^2 = \sum_{i,j=1}^N \phi_i^2 \phi_j^2 = \sum_{i,j,k,l=1}^N \lambda_{ijkl} \phi_i \phi_j \phi_k \phi_l.$$

The perturbation theory of the $O(N)$ model thus differs from ϕ^4 theory ($N = 1$) only through a prefactor S_G for each Feynman graph G . This factor counts all possible assignments of the field labels $\{1, \dots, N\}$ to each edge compatibly with the expression for the interaction. Let V and E denote the sets of vertices and edges of G , and write e_v^1, \dots, e_v^4 for the four edges at a vertex. Then

$$S_G(N) = \sum_{c: E \rightarrow \{1, \dots, N\}} \prod_{v \in V} \lambda_{c(e_v^1), c(e_v^2), c(e_v^3), c(e_v^4)} \quad (2.12)$$

for every 4-regular (“vacuum”) graph G . Expanding each tensor in this product into the three terms in (2.11) corresponds to a sum over transition systems. The number of parts of the corresponding circuit partition gives the number of freely choosable field labels.

For a 4-regular graph with n vertices, we can therefore identify

$$S_G(N) = 3^{-n} \cdot J(G, N). \quad (2.13)$$

Corrections to the coupling constant are determined by graphs of the form $G \setminus v$, which have 4 free (“external”) half-edges. Their symmetry factor is [KSF01, eq. (6.77)]

$$S_{G \setminus v}(N) = \frac{3S_G(N)}{N(N+2)} = \frac{\mathfrak{m}(G, N+2)}{3^{n-1}(N+2)}.$$

Since $\mathfrak{m}(G, x)$ is a multiple of x by (2.5), it follows from (2.3) that $J(G, N)$ is a multiple of $N(N+2)$. Therefore, $S_{G \setminus v}(N)$ is a polynomial in N . One can therefore formally consider arbitrary values for N , even though the $O(N)$ vector model was defined above only for positive integers N . Upon setting $N = -2$, we obtain the Martin invariant:

$$S_{G \setminus v}(-2) = 3^{1-n} \cdot \mathfrak{m}'(G, 0) = 2 \cdot 3^{2-n} \cdot \mathfrak{M}(G). \quad (2.14)$$

This was how we discovered the connection between the Martin invariant and Feynman symmetries: analyzing the symmetry factors for graphs with up to 13 vertices, we found that $N = -2$ is the only value for which identities like twist and duality hold.

Strikingly, the formal value $N = -2$ in the $O(N)$ vector model also describes a meaningful physical system. It was recently discovered [WF19, SW20] that the formal “ $O(-2)$ ” model describes loop erased random walks (LERW). The value $N = -2$ is therefore distinguished not only by the additional mathematical properties of the Martin invariant (the full Martin polynomial does not fulfil identities like twist or duality), but it also corresponds to an exceptional field theory.

3. Connectivity

We say that a graph G is d -regular if every vertex of G has degree d . Except for Proposition 3.8, we always assume that the degree is even, and hence denote it as $2k$.

Lemma 3.1. *If G is $2k$ -regular but not $2k$ -edge connected, then $\mathfrak{M}(G) = 0$.*



Figure 3.1: Splitting off a pair (va, vb) of edges.

Proof. Consider a minimal edge cut C of G , with size $|C| = 2s < 2k$. Let G_1 and G_2 denote the two sides of the cut, so that $G \setminus C = G_1 \sqcup G_2$. If G_1 consists of only one vertex, then this vertex has $k - s > 0$ self-loops, and therefore $M(G) = 0$. Otherwise, if G_1 has several vertices, pick one and call it v . For every transition τ at v , the edges of C in G_τ define a (not necessarily minimal) edge-cut of G_τ , hence G_τ is at most $2s$ -edge connected. By induction over the total number of vertices, we may assume that $M(G_\tau) = 0$, and therefore conclude the induction step that $M(G) = 0$ as well, by (1.1). \square

Theorem 3.2. *If G is $2k$ -regular and $2k$ -edge connected, and v denotes any vertex, then there exists a transition $\tau \in \mathcal{T}(v)$ such that G_τ is $2k$ -edge connected.*

Proof. Let $\lambda_G(x, y)$ denote the maximal number of edge-disjoint walks in G , with endpoints at vertices $x \neq y$. By Menger’s theorem, the $2k$ -edge connectedness of G implies that $\lambda_G(x, y) \geq 2k$ for all pairs $x \neq y$. In fact, since G is $2k$ -regular, equality holds. Lovász’ vertex splitting-off lemma [Lov76, Theorem 1] shows that at any vertex v in any Eulerian⁹ graph G , one can find two different edges va and vb , such that the graph $G' = G - va - vb + ab$ (see Figure 3.1) has the following property: for every pair $x \neq y$ of vertices different from v , one has

$$\lambda_G(x, y) = \lambda_{G'}(x, y).$$

Iterating this lemma k times produces a transition τ with $\lambda_{G_\tau}(x, y) = \lambda_G(x, y) = 2k$, therefore G_τ is $2k$ -edge connected. \square

Corollary 3.3. *If G is $2k$ -regular and $2k$ -edge connected, then $M(G) > 0$.*

Proof. By (1.1), we have $M(G) \geq M(G_\tau)$ for any transition of G . Choosing τ as in Theorem 3.2, we can ensure that G_τ is $2k$ -edge connected. Induction over the number of vertices thus reduces the proof to the case where G is a rose; then $M(G) > 0$ by (2.7). \square

Together with Lemma 3.1, Corollary 3.3 proves that for a graph G that is $2k$ -regular, $M(G) > 0$ if and only if G is $2k$ -edge connected. This settles parts 1 and 2 of Theorem 1.2. We now show part 3:

Lemma 3.4. *Let G be $2k$ -regular and C a $2k$ -edge cut of G . Let $G'_1 \sqcup G'_2 = G \setminus C$ denote the two sides of the cut, and write G_i for the graph obtained from G'_i by adding a vertex connected to the ends of C in G'_i , see (1.3). Then $M(G) = k! \cdot M(G_1) \cdot M(G_2)$.*

⁹While not needed here, we note that vertex splitting-off generalizes to non-Eulerian graphs [Mad78, Fra92].

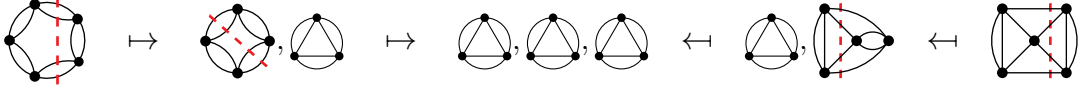


Figure 3.2: Decomposition of two totally decomposable graphs into 3 copies of $C_3^{[2]}$.

Proof. We proceed by induction over the number of vertices of G'_2 . If G'_2 has only one vertex, then G_2 is a $2k$ -fold edge and $G_1 = G$. The claim is thus trivial, $M(G) = M(G)$, because $M(G_2) = M(K_2^{[2k]}) = 1/(k!)$. If G'_2 has more than one vertex, pick any vertex v in G'_2 . Using the induction hypothesis, expanding at v yields

$$M(G) = \sum_{\tau \in \mathcal{T}(v)} M(G_\tau) = k! \sum_{\tau \in \mathcal{T}(v)} M((G_\tau)_1) \cdot M((G_\tau)_2),$$

because the edges of C define a $2k$ -edge cut in every G_τ . If τ pairs any edges in C with each other, then C induces in G_τ an edge-cut of size less than $2k$, hence such transitions can be dropped from the sum by Lemma 3.1. The remaining transitions commute with the cut: the side $(G_\tau)_1 = G_1$ is the same for all τ , and $(G_\tau)_2 = (G_2)_\tau$. Hence the right-hand side sums to the claim. Note that the dropped τ are precisely those that produce a self-loop in $(G_2)_\tau$ at the extra vertex, hence their absence causes no error to $M(G_2)$. \square

3.1. Decompositions and lower bounds

We saw above that $M(G) > 0$ if G is $2k$ -regular and $2k$ -edge connected. In this subsection, we give a combinatorial description of all such graphs which, at a given number of vertices, minimize the Martin invariant. This is closely related to factorizations with respect to the $2k$ -edge cut product from Lemma 3.4, for which we prove uniqueness (Proposition 3.8).

By repeated replacements $G \mapsto G_1, G_2$ along $2k$ -edge cuts, we can decompose every $2k$ -edge connected $2k$ -regular graph with ≥ 3 vertices into a sequence of graphs G_1, \dots, G_p until each graph in the sequence is cyclically $(2k+2)$ -connected and has at least 3 vertices. Following [BG96, FT93], we call such a sequence a *decomposition* of G . Iterating (1.3), note that

$$M(G) = (k!)^{p-1} \cdot M(G_1) \cdots M(G_p). \quad (3.1)$$

We call G *totally decomposable* [BG96] if it has a decomposition such that all G_i have only 3 vertices, see Figure 3.2. By (3.1), such a graph, with n vertices, has $M(G) = (k!)^{n-3}$. By induction one sees easily that any totally decomposable graph has at least two edges with multiplicity k . Decomposing by cutting off such an edge at each step we end with the k -fold triangle. Therefore, reversing this we get that totally decomposable graphs can be constructed from the k -fold triangle by repeatedly replacing a vertex with a k -fold edge.

For example, cutting off $i \geq 2$ consecutive vertices from a k -fold cycle,

$$C_n^{[k]} \mapsto C_{i+1}^{[k]}, C_{n-i+1}^{[k]}$$

terminates in a decomposition of $C_n^{[k]}$ into $n-2$ copies of the k -fold triangle $C_3^{[k]} = K_3^{[k]}$.

The totally decomposable graphs minimize the Martin invariant:

Theorem 3.5. *A $2k$ -regular, $2k$ -edge connected graph G with $n \geq 2$ vertices has $M(G) \geq (k!)^{n-3}$, and equality holds if and only if G is totally decomposable.*

The 4-regular case of this result was known previously [BG96, Corollary 5.6]. To prove Theorem 3.5, we first establish the lower bound, which strengthens Corollary 3.3:

Proposition 3.6. *Suppose that the graph G is $2k$ -regular, $2k$ -edge connected, and that it has $n \geq 2$ vertices. Then $M(G) \geq (k!)^{n-3}$.*

Proof. As in the proof of Theorem 3.2, let $\lambda_G(x, y)$ denote the maximum number of edge-disjoint paths between vertices x and $y \neq x$. Given G and fixing a vertex v , we call a pair (va, vb) of edges at v *admissible* if $\lambda_{G'}(x, y) \geq 2k$ for all $x \neq y$ with $x, y \neq v$, in the graph $G' = G - va - vb + ab$.

It was shown in [BJGJS99, Theorem 2.12] that if v has even degree d and $\lambda_G(x, y) \geq 2k$ for all $x \neq y$ different from v , then for any edge va , there are at least $d/2$ other edges vb such that the pair (va, vb) is admissible. Applying this argument to G' , where v has degree $2k - 2$, and iterating, we conclude that there are at least

$$\frac{2k}{2} \cdot \frac{2k-2}{2} \cdots \frac{2}{2} = k!$$

transitions $\tau \in \mathcal{T}(G)$ such that G_τ is $2k$ -edge connected. So if c_n denotes the minimum value of $M(G)$ over all $2k$ -regular, $2k$ -edge connected graphs G with n vertices, then taking only those transitions into account, the sum (1.1) shows that

$$c_n \geq k! \cdot c_{n-1}.$$

We conclude that $c_n \geq (k!)^{n-3}$ by induction over n and $c_3 = 1$. □

To finish the proof of Theorem 3.5, recall that $M(G) = (k!)^{n-3}$ if G is totally decomposable. If G is not totally decomposable, then any decomposition G_1, \dots, G_p has at least one factor with at least 4 vertices. Let G_1 be such a factor, and let n_i denote the number of vertices in G_i . Then by (3.1) and Proposition 3.6,

$$M(G) \geq (k!)^{p-1} \cdot M(G_1) \cdot (k!)^{n_2-3} \cdots (k!)^{n_p-3} = (k!)^{n-3} \cdot \frac{M(G_1)}{(k!)^{n_1-3}}$$

because $n_1 + \dots + n_p = n + 2(p-1)$. Therefore, $M(G) > (k!)^{n-3}$ follows from our next lemma.

Lemma 3.7. *If G is $2k$ -regular and cyclically $2k + 2$ connected, with $n \geq 4$ vertices, then $M(G) > (k!)^{n-3}$.*

Proof. Pick any vertex v in G . Consider two edges va, vb at v with different endpoints $a \neq b$. We first show that such a pair (va, vb) is admissible:

Suppose that (va, vb) is not admissible.

Then $G' = G - va - vb + ab$ has an edge cut C' of size $|C'| < 2k$, with at least one vertex $x \neq v$ and at least one vertex $y \neq v$ on each side of C' . Let $V(G) = X \sqcup Y$ denote the vertex bipartition corresponding to C' , with $v, x \in X$ and $y \in Y$. We show that the cut C of G with this bipartition contradicts the cyclic $2k + 2$ connectivity of G :

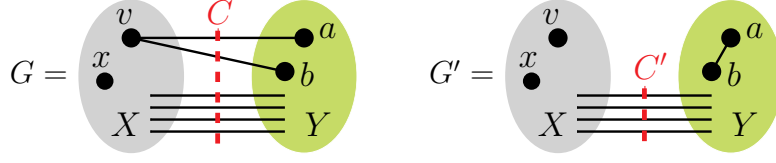


Figure 3.3: Illustration for the proof of Lemma 3.7: A non-admissible pair (va, vb) implies the existence of a forbidden cut C in G .

- If $a, b \in X$, then $C = C'$ has size $|C| = |C'| < 2k$.
- If $a \in X$ and $b \in Y$, then $ab \in C'$ and $C = C' - ab + vb$ has size $|C| = |C'| < 2k$.
- If $a, b \in Y$ (see Figure 3.3), then $C = C' + va + vb$ has size $|C| = |C'| + 2 \leq 2k + 1$. Since all vertices have even degree, $|C|$ must be even, thus $|C| \leq 2k$. Note that neither side of C is a single vertex: $|X| \geq |\{v, x\}| = 2$ and $|Y| \geq |\{a, b\}| = 2$.

The third point directly contradicts cyclic $2k + 2$ connectivity while the first two either contradict $2k$ regularity if one side of the cut is a single vertex and cyclic $2k + 2$ connectivity otherwise.

Now pick any neighbour a of v . The number s of edges between a and v is at most $k - 1$, as otherwise, the subgraph induced by a and v would constitute a cut of size $2 \cdot 2k - 2s \leq 2k$ with at least 2 vertices on each side since G has $n \geq 4$ vertices. Therefore, at least $2k - (k - 1) = k + 1$ edges at v have endpoints $b \neq a$. Since such pairs (va, vb) are admissible as we showed above, we find at least $k + 1$ admissible pairs containing va . After splitting off such a pair, we invoke [BJGJS99, Theorem 2.12] repeatedly, as in the proof of Proposition 3.6, to conclude that there are at least

$$(k + 1) \cdot (k - 1)!$$

transitions $\tau \in \mathcal{T}(v)$ such that G_τ is $2k$ -edge connected. Considering only those summands and using (1.1), we learn that $M(G) \geq (k + 1) \cdot (k - 1)! \cdot (k!)^{n-4} > (k!)^{n-3}$. \square

According to our definition earlier, a *decomposition* of a $d = 2k$ -regular graph is obtained after a maximal sequence of d -edge cuts. Such a sequence of cuts is rarely unique. For example, to decompose a k -fold cycle on 6 vertices, we can choose to cut off either 2 or 3 vertices in the first step, which leads to two different sequences of cuts:

- $C_6^{[k]} \mapsto C_5^{[k]} C_3^{[k]} \mapsto C_4^{[k]} C_3^{[k]} C_3^{[k]} \mapsto C_3^{[k]} C_3^{[k]} C_3^{[k]} C_3^{[k]}$
- $C_6^{[k]} \mapsto C_4^{[k]} C_4^{[k]} \mapsto C_4^{[k]} C_3^{[k]} C_3^{[k]} \mapsto C_3^{[k]} C_3^{[k]} C_3^{[k]} C_3^{[k]}$

However, note that the resulting decomposition is the same: we obtain four times the k -fold triangle, even though we constructed those triangles by different sequences of cuts.

We close this section by showing that decompositions $G \mapsto G_1 \cdots G_p$ are always unique. Here, *unique* is meant in the sense of multisets (lists up to reordering) of isomorphism classes of graphs.¹⁰ Our proof works for arbitrary degrees. For the cubic case $d = 3$, a proof of uniqueness was given in [FT93, Theorem 3.5].

¹⁰In the cubic case, Wormald's [Wor85, Theorem 3.1] shows uniqueness in a stronger sense, accounting also for how the G_i are arranged in G .

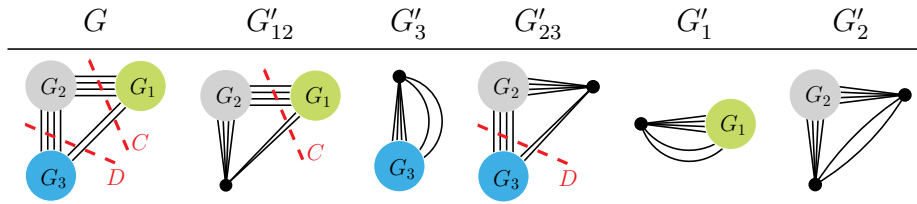


Figure 3.4: Graphs that arise in the decomposition for two cuts C and D with $G_4 = \emptyset$.

Proposition 3.8. *Decompositions of d -edge connected graphs are unique.*

Proof. Let two decompositions $A = A_1 \cdots A_p$ and $B = B_1 \cdots B_q$ of G be given. We call an edge cut *trivial* if one side of it consists of a single vertex. If G does not have any non-trivial d -edge cut, then we must have $p = q = 1$ and $A_1 = B_1 = G$ is indeed unique.

If G does have a non-trivial d -edge cut, then G is not cyclically $d + 1$ connected, hence we must have $p, q \geq 2$. Let C and D denote the d -edge cuts used in the first step of a construction of the decompositions A and B , respectively, such that

$$A = A' A'' \quad \text{and} \quad B = B' B''$$

where A' and A'' are decompositions of the two sides of C , and similarly B', B'' for D . By induction over the size of G , we may use that A', A'', B', B'' are unique.

The two cuts determine a partition of $G \setminus (C \cup D) = G_1 \sqcup G_2 \sqcup G_3 \sqcup G_4$ into four subgraphs, such that $C = [G_1 \cup G_4, G_2 \cup G_3]$ and $D = [G_1 \cup G_2, G_3 \cup G_4]$, where $[X, Y]$ denotes all edges of G with one end in X and the other end in Y . Let $m_{ij} = |[G_i, G_j]|$ denote the number of edges in G between G_i and G_j . Note that

$$m_{12} + m_{13} + m_{24} + m_{34} = |C| = d \quad \text{and} \quad m_{13} + m_{14} + m_{23} + m_{24} = |D| = d. \quad (\#)$$

If two of the subgraphs G_i are empty, we have $C = D$ and therefore $A' = B'$ and $A'' = B''$ by induction, hence $A = B$.

If only one of the subgraphs is empty, say $G_4 = \emptyset$, then A' and A'' are by definition decompositions of the graphs G'_1 and G'_{23} obtained from G_1 and $G_2 \cup G_3 \cup [G_2, G_3]$, respectively, by reconnecting the edges C to an extra vertex. Similarly, B' and B'' are decompositions of the graphs G'_3, G'_{12} obtained from G_3 and $G_1 \cup G_2 \cup [G_1, G_2]$, see Figure 3.4. Note that C defines a cut $G'_{12} \mapsto G'_1 G'_2$ and D defines a cut $G'_{23} \mapsto G'_3 G'_2$, such that

$$A'' = O B' \quad \text{and} \quad B'' = O A'$$

in terms of a decomposition O of the graph G'_2 obtained from G by replacing G_1 and G_3 by single vertices. Here we exploited the uniqueness of A'' and B'' (induction hypothesis), which allows us to compute them using any cut we like. We conclude that $A' A'' = A' O B' = B' B''$, up to order.

Finally, consider the case when all four subgraphs G_i are non-empty. Since G is d -edge connected, we have the lower bounds

$$\begin{aligned} m_{12} + m_{13} + m_{14} &\geq d, & m_{12} + m_{23} + m_{24} &\geq d, \\ m_{13} + m_{23} + m_{34} &\geq d, & m_{14} + m_{24} + m_{34} &\geq d. \end{aligned}$$

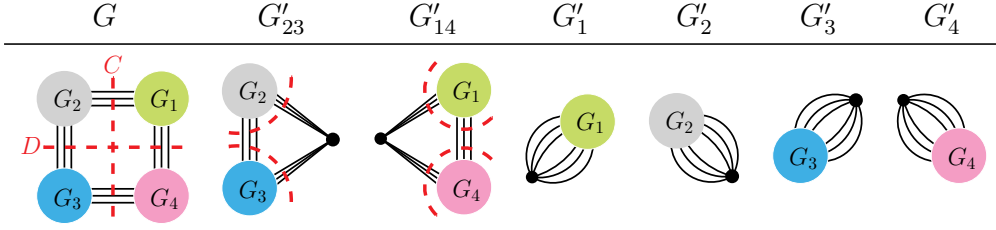


Figure 3.5: Graphs that arise in the decomposition of two cuts C, D with all $G_i \neq \emptyset$.

Combined with $(\#)$, this system has a unique solution, given by $m_{13} = m_{24} = 0$ and $m_{12} = m_{14} = m_{23} = m_{34} = d/2$. Therefore, d must be even, and the edges $[G_1, G_2 \cup G_4]$ constitute a cut of size $m_{12} + m_{14} = d$ that separates G_1 from the rest of G . Analogously, the other G_i each can be cut out along d edges. Let $G \mapsto G'_{14}G'_{23}$ denote the graphs created by the cut C , so that A' and A'' are decompositions of G'_{14} and G'_{23} , respectively. Here, G'_{ij} is obtained from $G_i \cup G_j \cup [G_i, G_j]$ by joining the cut edges to a new vertex (see Figure 3.5). Using the cuts $[G_2, G_1 \cup G_3]$ and $[G_3, G_1 \cup G_4]$ of G'_{23} , we can decompose

$$G'_{23} \mapsto G'_2G'_3C_3^{[d/2]} \quad \text{and thus find} \quad A' = O_2O_3C_3^{[d/2]}$$

where O_i denotes a decomposition of the graph G'_i obtained from G_i (see Figure 3.5). Again, we use here inductively that the decomposition of G'_{23} is unique. In the same manner, we find $A'' = O_1O_4C_3^{[d/2]}$ and therefore

$$A'A'' = O_1O_2O_3O_4C_3^{[d/2]}C_3^{[d/2]}.$$

Clearly, applying the same procedure to D , i.e. decomposing $G \mapsto G'_{12}G'_{34}$ and then $G'_{12} \mapsto G'_1G'_2C_3^{[d/2]}$ and $G'_{34} \mapsto G'_3G'_4C_3^{[d/2]}$ we obtain the same result for $B'B''$. \square

3.2. Product and twist

For the *product* identity (1.4), consider a 3-vertex cut $S = \{v_1, v_2, v_3\}$ of a $2k$ -regular graph G . More precisely, fix a bipartition $E = E_1 \sqcup E_2$ of the edges of G such that the only vertices shared by E_1 and E_2 are in S .¹¹ Let d_i denote the number of edges at v_i that belong to E_1 , then the other $2k - d_i$ edges at v_i belong to E_2 . Set

$$m_{12} = \frac{d_1 + d_2 - d_3}{2}, \quad m_{13} = \frac{d_1 + d_3 - d_2}{2}, \quad m_{23} = \frac{d_2 + d_3 - d_1}{2}. \quad (3.2)$$

These are integers, because $d_1 + d_2 + d_3$ is the size of an edge cut and hence even. Note that in any graph where every vertex has even degree, all edge cuts are even.

Lemma 3.9. *If G is $2k$ -edge connected, then $m_{12}, m_{13}, m_{23} \geq 0$.*

Proof. Suppose that $m_{12} < 0$. Then $d_1 + d_2 < d_3$, hence $d_1 + d_2 + (2k - d_3) < 2k$. So the edges at v_1, v_2 in E_1 together with the edges at v_3 in E_2 form a cut of G with size less than $2k$, contradicting the assumption on G . \square

¹¹A part E_i is not required to touch any vertices other than S .

Now we state and prove the product identity for the Martin invariant introduced in (1.4).

Proposition 3.10. *Let G be $2k$ -regular and $2k$ -edge connected, with a 3-vertex cut. Then the two sides of the cut can be turned uniquely into $2k$ -regular graphs G_1 and G_2 , by adding edges (but no self-loops) between the cut vertices. Furthermore, we have*

$$M(G) = M(G_1) \cdot M(G_2).$$

Proof. With notation as above, Lemma 3.9 implies that we can construct a $2k$ -regular graph G_2 from E_2 by adding m_{ij} edges connecting v_i and v_j , for each of the three pairs $ij = 12, 13, 23$. By a symmetric construction we get G_1 from E_1 . This proves the first claim of Proposition 3.10. In fact, (3.2) is the unique solution to the constraints $m_{ij} + m_{ik} + d_i = 2k$, so this is the only way to add non-self loop edges to E_2 that leaves a $2k$ -regular graph.

To complete the proof of Proposition 3.10, we exploit that the identity (1.4) is linear in each factor. More precisely, for any vertex $v \notin S$ that belongs to G_1 , we can apply transitions $\tau \in \mathcal{T}(v)$ at v on both sides: in G and in G_1 . Since τ only glues edges from E_1 , it leaves G_2 unchanged and the cut S remains a cut of G_τ , such that $(G_1)_\tau = (G_\tau)_1$ and $(G_\tau)_2 = G_2$. The expansion (1.1) at v therefore reduces the claim to the statement that $M(G_\tau) = M((G_\tau)_1) M((G_\tau)_2)$. An induction over the number of vertices therefore reduces the claim to trivial base cases:

- If G has a self-loop, then this self-loop is also a self-loop in G_1 or G_2 , so the identity holds trivially because both sides are zero.
- When only 3 vertices are left, namely the vertices S , and G has no self-loops, then all 3 graphs G , G_1 , and G_2 , are necessarily identical to the k -fold triangle. The identity then holds due to our normalization $M(C_3^{[k]}) = 1$. \square

With the same induction principle, we now prove the twist identity (1.5). Suppose we are given a graph G with a 4-vertex cut $S = \{v_1, v_2, v_3, v_4\}$ and a bipartition $E = E_1 \sqcup E_2$ of the edges, such that all vertices shared between E_1 and E_2 are contained in S . Let d_i denote the number of edges at v_i that belong to E_1 . Schnetz introduced the *twist* operation [Sch10, §2.6], which is illustrated in (1.5).

Definition 3.11. If $d_1 = d_2$ and $d_3 = d_4$, then the *twist* of G is the graph G' obtained from G by replacing every edge $e \in E_1$ with an endpoint in S , $e = uv_i$, by $uv_{\sigma(i)}$, where $\sigma = (1\ 2)(3\ 4)$ is the double transposition that swaps $v_1 \leftrightarrow v_2$ and $v_3 \leftrightarrow v_4$.

At a 4-vertex cut with $d_1 = d_2 = d_3 = d_4$, one can construct 3 twists, by relabeling v_i or, equivalently, using the other double transpositions $\sigma = (1\ 3)(2\ 4)$ and $\sigma = (1\ 4)(2\ 3)$. If we fix S and σ , then the twist is an involution: G is the twist of G' . Now we state and prove the twist identity for the Martin invariant introduced in (1.5).

Proposition 3.12. *If two regular graphs of even degree are obtained from each other by a double transposition on one side of a 4-vertex cut, then their Martin invariants agree:*

$$M \left(\text{Graph with cut } S \right) = M \left(\text{Twisted Graph with cut } S \right).$$

Proof. Let G, S, E, d_i be as above. As in the proof of Proposition 3.10 we can use the expansion (1.1) into transitions at any vertex $v \notin S$, to reduce the twist identity

$$M(G) = M(G')$$

to the case where G has no vertices other than S . If G contains a self-loop, so does G' , and the identity becomes trivial since both sides are zero. Hence we may assume that there are no self-loops in G . In this situation, E_1 is completely determined by the numbers m_{ij} of edges in E_1 that connect v_i with v_j . Due to the constraints

$$\begin{aligned} m_{12} + m_{13} + m_{14} &= d_1 = d_2 = m_{12} + m_{23} + m_{24} \quad \text{and} \\ m_{13} + m_{23} + m_{34} &= d_3 = d_4 = m_{14} + m_{24} + m_{34}, \end{aligned}$$

we must have $m_{13} = m_{24}$ and $m_{14} = m_{23}$. This shows that $G = G'$, because the twist changes E_1 into E'_1 with $m'_{ij} = m_{\sigma(i)\sigma(j)}$. The base case of our induction for the proof of the twist identity is therefore trivial. \square

4. Permanent

As our first case of a previously studied graph invariant with the same symmetries as the period and which can now be explained in terms of the Martin invariant, we will study a permanent-based invariant due to Iain Crump [CDY16, Cru17b]. Through Theorem 4.1 below, it inherits all properties of the Martin invariant. We therefore obtain new proofs of the permanent's invariance under completion, twist and duality [CDY16, Theorem 17–Proposition 20]; its multiplicativity for 3-vertex cuts and $2k = 4$ -edge cuts [CDY16, Corollary 23 and Theorem 24]; and the vanishing in presence of a triple edge [Cru17a, Proposition 69] (a special case of $M(G) = 0$ for a 2-edge cut). The same applies to the permanents of the graphs $G^{[r]}$ when $kr + 1$ is prime, whose invariance was first proved in [Cru17b] but now also follows from the properties of the Martin invariant. The Fourier split was not yet discovered at the time of [CDY16] and [Cru17b] but also holds for the permanent and is discussed in Remark 5.26.

To define the graph permanent, let G be a $2k$ -regular graph with $n + 2$ vertices. Then G has $kn + 2k$ edges. Pick two different vertices, called ∞ and 0 , and orient each edge. We assume that there are no self-loops at ∞ . Then the directed graph $G \setminus \infty$ has $n + 1$ vertices and kn edges. Its *reduced incidence matrix* is the $n \times kn$ matrix

$$A_{ve} = \begin{cases} +1 & \text{if } e \text{ points to } v, \\ -1 & \text{if } e \text{ comes from } v, \\ 0 & \text{if } e \text{ is not incident to } v, \end{cases} \quad (4.1)$$

whose rows are indexed by the vertices $v \neq 0, \infty$ and whose columns are indexed by the edges e of G that are not attached to ∞ . Stacking k copies of A produces a $kn \times kn$ square matrix, denoted $A^{[k]}$. Its permanent defines an integer

$$\text{Perm}(G) := \text{perm } A^{[k]} = \text{perm} \begin{pmatrix} A \\ \vdots \\ A \end{pmatrix}. \quad (4.2)$$

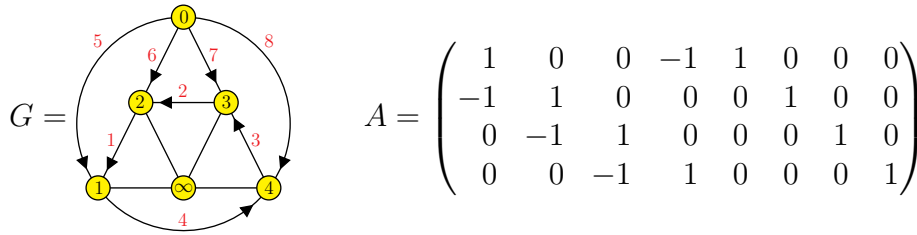


Figure 4.1: A labelled and oriented octahedron graph. Labels and orientations of edges at vertex ∞ are not shown, since they do not affect the incidence matrix A .

This number $\text{Perm}(G)$ depends on the choice of $0, \infty$, and the orientation of the edges used to write down A . However, it was shown in [CDY16] that the residue

$$\text{Perm}(G) \pmod{k + 1}$$

depends on all these choices only up to a sign. Equivalently, the residue of $\text{Perm}(G)^2$ is a well-defined invariant of the graph G . This invariant respects the known identities of Feynman periods [CDY16]. Moreover, the sequence $\text{Perm}(G^{[r]})^2 \pmod{kr + 1}$ of permanents of all duplicated graphs is a very strong invariant of G in that it distinguishes almost all Feynman periods [Cru17a, Cru17b].

In this section, we show that the permanent is determined by the Martin invariant. Note that the repetition of rows makes $\text{Perm}(G)$ a multiple of $(k!)^n$. It is therefore congruent to zero modulo $k + 1$ unless $k + 1$ is prime [CDY16, Proposition 13].

Theorem 4.1. *For every $2k$ -regular graph G with $n \geq 3$ vertices such that $p = k + 1 \geq 3$ is prime, we have*

$$M(G) \equiv (-1)^{n-1} \text{Perm}(G)^2 \pmod{p}. \tag{4.3}$$

Note that each term of the sequence $\text{Perm}(G^{[r]})^2 \pmod{kr + 1}$ for which $kr + 1$ is prime immediately also falls under this theorem and so is determined by the Martin invariant on the $G^{[r]}$.

Example 4.2. The octahedron graph has 6 vertices and it is 4-regular, so $k = 2$. With labels and orientations as in Figure 4.1, the incidence matrix gives $\text{perm} \begin{pmatrix} A \\ A \end{pmatrix} = 32$. Therefore, $(-1)^{n-1} \text{Perm}(G)^2 = -32^2 \equiv -1 \pmod{3}$. This agrees indeed with $M(G) = 14 \equiv -1 \pmod{3}$, because the Martin invariant is

$$M \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) = M(K_5) + 2 M \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) = 6 + 4 M \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) + 2 M \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) = 6 + 8 + 0.$$

We will give two proofs of Theorem 4.1. The first is elementary; the main work goes into considering how a transition, viewed as a rule for replacing a vertex with new edges joining its neighbours according to a matching, affects the permanent of $A^{[k]}$. A similar but more intricate consideration of the effect of moving between a vertex and a matching of its neighbourhood will also be key to the results of Section 5.

For the elementary proof of Theorem 4.1, we show that the right-hand side of (4.3) satisfies Definition 1.1 of $M(G)$, modulo p . Firstly, any self-loop in G corresponds to a zero column in A ,

hence $\text{perm } A^{[k]} = 0$ and (4.3) is trivial. Secondly, if G has 3 vertices and no self-loop, then G is the power of a triangle. Thus $G \setminus \infty$ is a k -fold edge and $A = (1, \dots, 1)$ is a single row of ones, if we orient the edges away from vertex 0. Hence

$$\text{Perm} \left(C_3^{[k]} \right)^2 = (k!)^2 \equiv (-1)^2 = 1 = \mathbf{M} \left(C_3^{[k]} \right) \pmod{p}$$

as required, by Wilson's theorem.¹² By induction, we can therefore conclude Theorem 4.1 by proving a third identity: If G has no self-loops and v is a vertex $v \notin \{0, \infty\}$, then

$$-\text{Perm}(G)^2 \equiv \sum_{\tau \in \mathcal{T}'(v)} \text{Perm}(G_\tau)^2 \pmod{p}, \quad (4.4)$$

where $\mathcal{T}'(v)$ denotes the subset of transitions that do not produce any self-loops at ∞ . To establish this identity in Section 4.3, and thereby finish the proof of Theorem 4.1, we first work out expansions of the permanents of G and G_τ .

For the second proof of Theorem 4.1, in Section 4.4 we show that the permanent can be read off from the diagonal coefficients of the Kirchhoff and Symanzik polynomials. Together with Theorem 5.17, this provides an independent proof of Theorem 4.1.

4.1. Expansion

Like determinants, permanents can be expanded in rows and columns. For the graph permanent, this strategy leads to explicit formulas [Cru17b, Section 4]. Here, we apply this strategy to prove the recurrence relation (4.4).

Label the vertices of G as $0, 1, \dots, n, \infty$ and let $N = \{e_1, \dots, e_{2k}\}$ denote the $2k$ edges at vertex n . Let $v_i \neq n$ denote the other end of e_i . Note that several of these edges may share the same other end $v_i = v_j$.

The non-zero entries in row n of A are -1 , located in the columns N . The expansion of $\text{perm}(A^{[k]})$ in the k copies of row n therefore eliminates k of the columns N . Let $a \subset N$ denote the leftover edges of N . Edges $e_i \in N$ incident to $v_i = \infty$ are not present as columns in A and can thus not be among the columns $N \setminus a$ eliminated during the row expansion. The expansion then reads

$$\text{Perm}(G) = k! \cdot (-1)^k \sum_{a \subset N, |a|=k} P_a \quad (4.5)$$

where we define $P_a = 0$ if there is any e_i incident to ∞ which is not in a , and otherwise define P_a to be the permanent of the $k(n-1) \times k(n-1)$ matrix that remains from $A^{[k]}$ after deleting the k copies of row n and the k columns $N \setminus a$. The factor $k!$ accounts for the different ways of assigning the columns $N \setminus a$ to the k copies of the row n .

Now expand P_a in the leftover columns, $e_i \in a$ where $v_i \neq \infty$: the non-zero entries are $+1$, located in the rows corresponding to the k copies of v_i . Let a_i denote the number of edges in a whose other vertex is i , and write P'_a for the permanent of any matrix obtained from $A^{[k]}$ by deleting all columns $e_i \in N$ with $v_i \neq \infty$, the k copies of row n , and a_i copies of row i for

¹²Wilson's theorem states that $(p-1)! \equiv -1 \pmod{p}$ if and only if p is prime.

each $1 \leq i < n$. This matrix is square with $kn - 2k + a_\infty$ rows and columns. The different ways to delete a_i copies of row i only affect the matrix up to permutation of rows which is invisible to the permanent, and so P'_a , and hence also P_a , depends only on the numbers a_0, \dots, a_∞ (the multiset of vertices a). Then the expansion of P_a in columns a reads

$$P_a = \left[\prod_{i=1}^{n-1} \frac{k!}{(k - a_i)!} \right] P'_a.$$

The prefactor takes into account the different ways to assign a_i parallel edges (identical columns) to a_i out of the k copies of vertex i . We note any column $e_i \in a$ with $v_i = 0$ becomes a zero column after deleting the copies of row n , since vertex 0 is absent in A . Therefore, $P_a = 0$ if $a_0 > 0$. Along with our convention that $P_a = 0$ if $a_\infty < N_\infty$ (where N_∞ is the number of e_i in N with $v_i = \infty$), the sum in (4.5) therefore is effectively only a sum over subsets a which contain none of the edges from n to 0, and all of the edges from n to ∞ . Squaring, we find

$$\text{Perm}(G)^2 = (k!)^2 \sum_{\substack{a, b \subset N, \\ |a|=|b|=k}} P_a P_b. \tag{4.6}$$

4.2. Expansion of a transition

Pick any transition τ at vertex n that does not produce self-loops at ∞ in G_τ . Then the incidence matrix A_τ of the graph G_τ has dimensions $(n - 1) \times k(n - 1)$. It is obtained from A by removing row n and the $2k - N_\infty$ columns N that do not connect to ∞ , and adding $k - N_\infty$ columns for the edges $e_1^\tau, \dots, e_k^\tau$ given by the perfect matching τ of N , but excluding those edges e_j^τ that connect to ∞ .

Let $\text{Or}(\tau)$ denote the set of orientations of the matching, that is, the set of size 2^k whose elements α choose for each edge e_i^τ , one of its incident vertices $\alpha(i) = v_{\alpha(i)}$, which we view as the vertex that edge e_i^τ points to under this orientation. Let α_j denote the number of edges e_i^τ which point to $v_{\alpha(i)} = v_j$.

To define the signs ± 1 in the new columns of A_τ , fix any orientation $o \in \text{Or}(\tau)$ of the new edges in G_τ . The expansion of $\text{perm}(A_\tau^{[k]})$ in one of the columns e_i^τ either picks one of the copies of row $v_{\alpha(i)}$ or one of the copies of the row for the other end of e_i^τ . The expansion in all these columns can therefore be written as

$$\text{Perm}(G_\tau) = \sum_{\alpha \in \text{Or}(\tau)} (-1)^{o\Delta\alpha} P_\alpha,$$

where $o\Delta\alpha$ denotes the number of indices i where $o(i) \neq \alpha(i)$, that is, how often an entry -1 is picked from the matrix. Whenever an orientation picks vertex $v_{\alpha(i)} = 0$ as the vertex pointed to, the corresponding contribution is zero, because this vertex is not a row in A_τ and hence does not contribute to the expansion in column e_i^τ . This is consistent with our convention $P_\alpha = 0$ whenever $\alpha_0 > 0$, as discussed in the previous section.

Furthermore, our convention that $P_\alpha = 0$ whenever $\alpha_\infty < N_\infty$, forces that all new edges e_i^τ that have one end at ∞ must be directed towards ∞ . This reduces the sum effectively to a sum

over the orientations of only those new edges e_i^τ that are not connected to ∞ . This is indeed correct, because only those edges actually appear as columns in A_τ and thus participate in the column expansion.

In conclusion, we can write the expansion of the permanent squared of G_τ as

$$\text{Perm}(G_\tau)^2 = \sum_{\alpha, \beta \in \text{Or}(\tau)} (-1)^{\alpha \Delta \beta} P_\alpha P_\beta. \quad (4.7)$$

4.3. Recurrence

The vertices chosen by an orientation $\alpha \in \text{Or}(\tau)$ define a subset $a = \{\alpha(1), \dots, \alpha(k)\} \subset N$ of size k . Conversely, by definition, an orientation $\alpha \in \text{Or}(\tau)$ is determined by the subset a . However, a defines an orientation of τ only if it picks precisely one element in each pair of τ .

Consider any pair $a, b \subset N$ with $|a| = |b| = k$, and compare the coefficient of the product $P_a P_b$ on both sides of (4.4). On the left, it appears with coefficient

$$-(k!)^2 \equiv -1 \pmod{p},$$

according to (4.6) and Wilson's theorem. On the right of (4.4), $P_a P_b$ appears as $P_\alpha P_\beta$ in (4.7) for each transition τ such that $\alpha = a$ and $\beta = b$ are orientations of τ . The collected coefficient of $P_a P_b$ on the right-hand side of (4.4) is therefore

$$(-1)^{a \Delta b} \cdot |\{\tau \in \mathcal{T}'(n) : a, b \in \text{Or}(\tau)\}|. \quad (4.8)$$

We get a $\tau \in \mathcal{T}'(n)$ with $a, b \in \text{Or}(\tau)$ by pairing the elements in $a \cap b$ with elements of $N \setminus (a \cup b)$ giving those pairs with the same orientation in a and b , and pairing the elements in $a \setminus b$ with the elements in $b \setminus a$ giving the pairs with opposite orientation in a and b . Every such choice of pairings gives a distinct $\tau \in \mathcal{T}'(n)$ with $a, b \in \text{Or}(\tau)$.

The number $s = a \Delta b$ of pairs $\tau(i)$ that get opposite orientations in a and b is $s = |a \setminus b| = |b \setminus a|$. The other $k - s$ pairs have equal orientations and go from $N \setminus (a \cup b)$ to $a \cap b$. We conclude that precisely $s! \cdot (k - s)!$ transitions contribute to (4.8), namely those that match $a \cap b$ with $N \setminus (a \cup b)$ and $a \setminus b$ with $b \setminus a$. Therefore, (4.8) equals

$$\begin{aligned} & s! \cdot (k - s)! \cdot (-1)^s \\ & \equiv s! \cdot [k + 1 - (k - s)] \cdot [k + 1 - (k - s - 1)] \cdots [k + 1 - 1] \cdot (-1)^k \pmod{(k + 1)} \\ & = s! \cdot (s + 1)(s + 2) \cdots k \cdot (-1)^k \\ & = (-1)^k \cdot k!. \end{aligned}$$

This is equal to the left-hand side $-1 \pmod{p}$ of (4.4), by Wilson's theorem and because k is even (so that $p = k + 1$ is a prime ≥ 3). Finally, note that we only need to consider a and b with $a_\infty = b_\infty = N_\infty$. Any transition τ that creates a self-loop at ∞ in G_τ , has at most $N_\infty - 1$ pairs attached to ∞ , and hence a and b cannot be orientations of such τ . Therefore, in our counting above, indeed only the transitions $\tau \in \mathcal{T}'(n)$ appear.

This completes the proof of Theorem 4.1.

4.4. From the Kirchhoff polynomial

In this section we express the permanent, which is defined via the incidence matrix A from (4.1), in terms of the graph Laplacian. Together with Theorem 5.17, this yields an alternative proof of Theorem 4.1. Recall that G is a $2k$ -regular graph with $n + 2$ vertices, labelled $0, 1, \dots, n, \infty$. In this section we assume that G has no self-loops.

Definition 4.3. Let X denote the $kn \times kn$ matrix with variables $x_i = X_{ii}$ on the diagonal and zeroes elsewhere. The *reduced graph Laplacian* of the graph $G \setminus \infty$ is the $n \times n$ matrix

$$L = AXA^T. \tag{4.9}$$

The variables x_i are associated to the edges of $G \setminus \infty$. The rows and columns of L are indexed by vertices $v \neq 0, \infty$. This matrix does not depend on orientations: diagonal entries $L_{v,v} = \sum_e x_e$ are positive sums over all edges e at v , whereas off-diagonal entries $L_{v,w} = -\sum_e x_e$ are negative sums over all edges between v and $w \neq v$.

Lemma 4.4. Denote the $kn \times kn$ matrix obtained by repeating $k \times k$ blocks of L as

$$L^{[k \times k]} = \begin{pmatrix} L & \cdots & L \\ \vdots & \ddots & \vdots \\ L & \cdots & L \end{pmatrix}. \tag{4.10}$$

Recall the notation $[x^m]$ for coefficient extraction of a monomial $x^m = \prod_i x_i^{m_i}$, see Definition 1.4. Then

$$(\text{perm } A^{[k]})^2 = [x_1 \cdots x_{kn}] \text{perm } (L^{[k \times k]}). \tag{*}$$

Proof. The block matrix on the right-hand side of (4.10) is equal to $L^{[k \times k]} = A^{[k]}X(A^{[k]})^T$. By the MacMahon master theorem, the right-hand side of (*) is therefore equal to

$$[x_1 \cdots x_{kn}] [y_1 \cdots y_{kn}] \frac{1}{\det(I_{kn} - Y A^{[k]} X (A^{[k]})^T)},$$

where Y denotes a $kn \times kn$ matrix with variables $y_i = Y_{ii}$ on the diagonal and zeroes everywhere else, and I_{kn} denotes the $kn \times kn$ identity matrix. The generalization of the MacMahon master theorem from [CDM22, Theorem 1] shows that the above expression is equal to the product of the permanents $\text{perm}(A^{[k]})$ and $\text{perm}((A^{[k]})^T) = \text{perm}(A^{[k]})$. \square

Corollary 4.5. For $p = k + 1$ prime, $(\text{perm } A^{[k]})^2 \equiv (-1)^n [x_1 \cdots x_{kn}] (\det L)^k \pmod p$.

Proof. This follows by Wilson’s theorem and the congruence $\text{perm}(L^{[k \times k]}) \equiv (k!)^n (\det L)^k \pmod{(k + 1)}$ obtained in [Gly11, Theorem 7.5].¹³ The latter holds because performing row operations (adding a multiple of one row to another) simultaneously in all k copies of any row of $L^{[k \times k]}$ does not change its permanent modulo $k + 1$ [CDY16, Corollary 6]. For R a sequence

¹³With the notation in that paper, $X_{k+1}(B) = (\text{perm } B^{[k]}) / (k!)^n$.

of such row operations such that $RL = D$ is a diagonal matrix (thus $\det L = \det D$), applying R simultaneously to the k copies of each row shows that

$$\text{perm}(L^{[k \times k]}) = \text{perm}(D^{[k \times k]}) \equiv (\det L)^k \text{perm}(I_n^{[k \times k]}) = (\det L)^k (k!)^n \pmod{p}$$

factorizes into determinants of L and the permanents of the n blocks of $I_n^{[k \times k]}$. Each of the latter blocks is a $k \times k$ matrix with each entry equal to 1, hence permanent $k!$. \square

By the matrix-tree theorem, the determinant of L is the *Kirchhoff polynomial*

$$\det L = \tilde{\Psi}_{G \setminus \infty} = \sum_T \prod_{e \in T} x_e \quad (4.11)$$

where the sum is over all spanning trees T of $G \setminus \infty$. It has degree n , and it is related to the Symanzik polynomial (1.7) with degree $(k-1)n$ by an inversion of variables,

$$\tilde{\Psi}_{G \setminus \infty}(x_1, \dots, x_{kn}) = \left(\prod_{e=1}^{kn} x_e \right) \cdot \Psi_{G \setminus \infty}(x_1^{-1}, \dots, x_{kn}^{-1}). \quad (4.12)$$

In conclusion, Corollary 4.5 expresses the square of the permanent as a diagonal coefficient of a power of the Kirchhoff or Symanzik polynomial:

$$(-1)^n \text{Perm}(G \setminus \infty)^2 \equiv [x_1 \cdots x_{kn}] \tilde{\Psi}_{G \setminus \infty}^k = [x_1^{k-1} \cdots x_{kn}^{k-1}] \Psi_{G \setminus \infty}^k \pmod{p}. \quad (4.13)$$

Remark 4.6. The results in this subsection do not require that G is $2k$ -regular. We only used that the number of edges of $G \setminus \infty$ is equal to kn , with integer k . Therefore, Corollary 4.5 proves that the graph permanent [CDY16] and its extension [Cru17b] are determined, for any graph, by the cycle matroid of that graph. This proves [Cru17a, Conjecture 3].

5. Diagonals of graph polynomials

A unifying perspective on the Martin invariant, the graph permanent, and the c_2 invariant, is that they are related to diagonal coefficients of certain polynomials defined from the graph. This point of view was crucial for recent progress on the c_2 invariant [HSSY22, HY23, EY23] and will be used again in Section 6. For the permanent, a diagonal interpretation was given in [Cru17b, § 5.1] in terms of a new polynomial, whereas in (4.13) we identified the square of the permanent with a residue of the diagonal of the Kirchhoff polynomial. In the present section, we show that not just the residue, but the diagonal coefficient itself satisfies the Martin recurrence (1.1). Consequently, the Martin invariant of a regular graph can be identified, up to a normalization, with this coefficient (Theorem 5.17).

The diagonal coefficients we are interested in have combinatorial interpretations as partitions of the edges into spanning trees (this section), or partitions into spanning trees and certain spanning forests (Section 6). We exploit this combinatorial interpretation to prove the Martin recurrence (1.1) for these diagonal coefficients. The key tool is a generalization of the Prüfer

encoding enabling us to move from spanning trees on a graph to spanning trees on the graph with a vertex resolved by a transition viewed as a matching; this is the content of Section 5.1.

After these introductory remarks, consider now concretely the Kirchhoff polynomial $\tilde{\Psi}_G$ (appeared in (4.11) for a decomposition) and the Symanzik polynomial Ψ_G from (1.7):

$$\tilde{\Psi}_G = \sum_T \prod_{e \in T} x_e \quad \text{and} \quad \Psi_G = \sum_T \prod_{e \notin T} x_e.$$

For any graph G , both $\tilde{\Psi}_G$ and Ψ_G are homogeneous, are linear in each variable, and have all monomials appearing with coefficient 1. For $\tilde{\Psi}_G$ each monomial consists of the edges of a spanning tree while for Ψ_G each monomial consists of the edges that when deleted leave a spanning tree of G . For a disconnected graph, $\Psi_G = \tilde{\Psi}_G = 0$. If G is connected, then the polynomials are non-zero (spanning trees exist), and their total degrees are

$$\deg \tilde{\Psi}_G = n - 1 \quad \text{and} \quad \deg \Psi_G = m - n + 1$$

if G has n vertices and m edges. The degree $m - n + 1$ equals the dimension of the cycle space of G (the first Betti number of G viewed as a topological space); this number is known as *nullity* or *corank* and in the quantum field theory community as the *loop number*. All these observations are immediate from the definitions (1.7) and (4.11).

Example 5.1. Let G be the triangle with one doubled edge. Let indices 1 and 2 be the doubled edge and 3 and 4 be the other edges, as in Figure 6.2. Then $n = 3$, $m = 4$, $\tilde{\Psi}_G = x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$ and $\Psi_G = x_2x_4 + x_2x_3 + x_1x_4 + x_1x_3 + x_1x_2$.

Let G be a graph with m edges. We are particularly interested in the *diagonal coefficient* of $\tilde{\Psi}_G^k$, namely $[x_1 \cdots x_m] \tilde{\Psi}_G^k$ where $[\cdot]$ denotes the coefficient extraction from Definition 1.4. Since $\tilde{\Psi}_G^k$ has degree $k(n-1)$, where n is the number of vertices of G , note that the diagonal coefficient is necessarily zero whenever $m \neq k(n-1)$. We only consider cases with $m = k(n-1)$. Then Ψ_G has degree $(k-1)(n-1)$, and so we also have a diagonal coefficient $[x_1^{k-1} \cdots x_m^{k-1}] \Psi_G^k$. Due to (4.12), they agree:

$$[x_1 \cdots x_m] \tilde{\Psi}_G^k = [x_1^{k-1} \cdots x_m^{k-1}] \Psi_G^k. \tag{5.1}$$

Consider now a $2k$ -regular graph G with n vertices and thus kn edges, and let v denote a vertex without any self-loops. Then the graph $G \setminus v$ has $n - 1$ vertices and $m = k(n - 2)$ edges, so that the above diagonal coefficients have the potential to be non-zero. In Theorem 5.17 we will show that

$$[x_1 \cdots x_m] \tilde{\Psi}_{G \setminus v}^k = k! \cdot M(G).$$

The main technical step is to prove that the diagonal coefficient on the left-hand side satisfies the Martin recurrence. We will do this using a combinatorial interpretation of the diagonal coefficient. In Section 6.3 we will apply similar considerations to the diagonal of a different but related polynomial.

Definition 5.2. A *partition* of a set S is a set $P = \{p_1, \dots, p_k\}$ of non-empty subsets $p_i \subseteq S$, called *the parts of P* , such that each element of S is in exactly one part of P .

An *ordered partition* of S is a tuple $P = (p_1, \dots, p_k)$ of distinct non-empty subsets of S that form a partition of S .

Definition 5.3. For a graph G and integer k , define $N_k(G)$ to be the number of ordered partitions of the set of edges of G into k parts, so that each part is a spanning tree of G .

We will use the following combinatorial interpretation of the diagonal coefficient.

Lemma 5.4. For a graph G with m edges, and any positive integer k ,

$$N_k(G) = [x_1 \cdots x_m] \tilde{\Psi}_G^k.$$

Proof. Every monomial which appears in $\tilde{\Psi}_G$ has coefficient 1, so $N_k(G)$ counts the ordered partitions of the variables (edges of G) into a k -tuple of parts (one part from each factor of $\tilde{\Psi}_G^k$), where each part is a monomial (spanning tree) appearing in $\tilde{\Psi}_G$. \square

Here the order of the trees T_i matters. Without the order, we get a partition $P = \{T_1, \dots, T_k\}$ of the edge set of G and require an extra combinatorial factor.

Corollary 5.5. The diagonal coefficient $N_k(G)$ of $\tilde{\Psi}_G^k$ is equal to $k!$ times the number of partitions of the edge set of G such that each part is a spanning tree.

Proof. Since the trees are edge-disjoint, the map $(T_1, \dots, T_k) \mapsto \{T_1, \dots, T_k\}$ that forgets the order is $k!$ -to-1. \square

Example 5.6. Continuing with the graph G in Example 5.1, we see that there are two partitions of the edges of G into spanning trees $\{\{1, 3\}, \{2, 4\}\}$ and $\{\{1, 4\}, \{2, 3\}\}$ and hence four ordered partitions as there are two orderings of each partition, giving $N_2(G) = 4$. This is consistent with $[x_1 x_2 x_3 x_4] \tilde{\Psi}_G^2 = [x_1 x_2 x_3 x_4] (x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4)^2 = 4$.

Remark 5.7. It is also the case that Ψ_G has every monomial with coefficient 1, so the same argument as in the proof of Lemma 5.4 interprets the diagonal coefficient of Ψ_G^k as a k -tuple of complements of spanning trees, of which there are also $N_k(G)$. This again explains (5.1), where here the act of taking the complement of a set of variables is interpreted combinatorially—instead of the equivalent algebraic formulation in (4.12).

Lemma 5.8. For a graph G with m edges and for all positive integers r and k , we have

$$N_{kr}(G^{[r]}) = [x_1 \cdots x_{rm}] \tilde{\Psi}_{G^{[r]}}^{kr} = (r!)^m [x_1^r \cdots x_m^r] \tilde{\Psi}_G^{kr}. \quad (5.2)$$

Proof. This is [Cru17b, Proposition 40]. Set $e_i = (e-1)r+i$ so that e_1, \dots, e_r label the r copies in $G^{[r]}$ of any edge $e \in \{1, \dots, m\}$ of G . For a spanning tree T' of $G^{[r]}$, replacing each $e_i \in T'$ by e defines a spanning tree T of G . In particular, T' can contain at most one copy of any edge e

of G . Thus T' is determined by T together with choices $\iota(e) \in \{1, \dots, r\}$ of one copy for each $e \in T$. Therefore,

$$\tilde{\Psi}_{G^{[r]}}(x_1, \dots, x_{rm}) = \sum_T \sum_{\iota} \prod_{e \in T} x_{e_{\iota(e)}} = \sum_T \prod_{e \in T} \sum_{i=1}^r x_{e_i} = \tilde{\Psi}_G(y_1, \dots, y_m)$$

where $y_e = x_{r(e-1)+1} + \dots + x_{er} = x_{e_1} + \dots + x_{e_r}$ is the sum of the variables associated to the copies of edge e . This identity also follows from the matrix tree theorem (4.11), since the reduced graph Laplacian L from Section 4.4 for $G^{[r]}$ is the same as that of G , except for the replacement $x_e \mapsto y_e$. Now note that

$$[x_{e_1} \cdots x_{e_r}] y_e^q = [x_{e_1} \cdots x_{e_r}] (x_{e_1} + \dots + x_{e_r})^q = \begin{cases} r! & \text{if } q = r \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

so taking the linear coefficient in the copies of edge e is the same—up to a factor $r!$ —as taking the coefficient of y_e^r . The claim follows. \square

Similarly to Lemma 5.4 and Remark 5.7, we can interpret the coefficient $[x_1^r \cdots x_m^r] \tilde{\Psi}_G^{kr} \in \mathbb{Z}$ as the number of lists (T_1, \dots, T_{kr}) of spanning trees of G such that each edge of G appears in precisely r of these trees. In contrast to Corollary 5.5, this does not imply that this coefficient is a multiple of $(kr)!$, because the T_i are not necessarily distinct. Instead, we can show

Lemma 5.9. *Let G be a graph with m edges and $n \geq 2$ vertices. Let k and r be positive integers. Then $[x_1^r \cdots x_m^r] \tilde{\Psi}_G^{kr}$ is divisible by the integer $(kr)!/(r!)^k$.*

Proof. Every spanning tree has $n - 1$ edges, so we may assume that $m = k(n - 1)$: otherwise, no edge decompositions into kr spanning trees can exist and the claim becomes trivial since the coefficient is zero.

Any tuple (T_1, \dots, T_{kr}) of spanning trees determines a multiset $t_1^{\mu_1} \dots t_p^{\mu_p}$ that records the set $\{t_1, \dots, t_p\} = \{T_1, \dots, T_{kr}\}$ of *distinct* trees and their multiplicities, where $\mu_i \geq 1$ denotes the number of indices s such that $t_i = T_s$. Note that $\mu_1 + \dots + \mu_p = kr$, so μ is necessarily a composition of kr , and therefore

$$\frac{(kr)!}{\mu_1! \cdots \mu_p!} = \binom{kr}{\mu_1, \dots, \mu_p}$$

is a multinomial coefficient. It counts the number of distinct ordered tuples that correspond to the multiset. We can therefore write the diagonal coefficient as a sum

$$[x_1^r \cdots x_m^r] \tilde{\Psi}_G^{kr} = \sum_{p, t, \mu} \frac{(kr)!}{\mu_1! \cdots \mu_p!}, \tag{†}$$

over all multisets of spanning trees with the constraint that each edge e of G appears in precisely r of the trees, counted with multiplicity. This condition can be phrased as

$$r = \mu_1 \chi_1(e) + \dots + \mu_p \chi_p(e)$$

where we set $\chi_i(e) = 1$ if $e \in t_i$ and $\chi(e) = 0$ otherwise. It follows that

$$\frac{r!}{(\mu_1!)^{\chi_1(e)} \cdots (\mu_p!)^{\chi_p(e)}} = \binom{r}{\mu_1 \chi_1(e), \dots, \mu_p \chi_p(e)} \in \mathbb{Z}$$

is an integer. Multiplying these identities for all $k(n-1)$ edges of G , we find that

$$\left(\frac{(r!)^k}{\mu_1! \cdots \mu_p!} \right)^{n-1} \in \mathbb{Z}$$

is an integer. Here we used that $\sum_e \chi_i(e) = |t_i| = n-1$ as t_i is a spanning tree. We conclude that the rational number $(r!)^k / (\mu_1! \cdots \mu_p!)$ is an integer.¹⁴ Hence, for every multiset that contributes to (\dagger) , the summand is divisible by $(kr)! / (r!)^k$. \square

Example 5.10. For the k -fold edge $G = K_2^{[k]}$ with $m = k$ edges and $n = 2$ vertices, we have $\tilde{\Psi}_G = x_1 + \cdots + x_k$ and $\Psi_G = \sum_{i=1}^k \prod_{j \neq i} x_j$. The diagonal coefficients are

$$[x_1^{r(k-1)} \cdots x_k^{r(k-1)}] \Psi_G^{kr} = [x_1^r \cdots x_k^r] \tilde{\Psi}_G^{kr} = \binom{kr}{r, \dots, r} = \frac{(kr)!}{(r!)^k}.$$

5.1. Coloured matchings and distinct representatives

The key technical step towards the proof of Theorems 5.17 and 6.2 is a bijection based on Prüfer codes between two sets called A and B , which we will now define. Throughout, we fix an integer $k > 0$ and a non-empty set S with at most $2k$ elements. We set $d = 2k - |S| \geq 0$.

Given a partition $P = \{p_1, \dots, p_t\}$ of S , we write $|P| = t$ for the number of parts. A subset $X \subseteq S$ is called a *system of distinct representatives* (SDR) of P if it contains exactly $|X \cap p| = 1$ element in each part $p \in P$.

Given any sequence P_1, \dots, P_k of partitions of S , we want to count families (X_1, \dots, X_k) of pairwise disjoint systems of distinct representatives X_i of P_i that cover $S = X_1 \cup \dots \cup X_k$. We can encode such as a map $f: S \rightarrow \{1, \dots, k\}$ with $X_i = f^{-1}(i)$, and hence define

$$A(P_1, \dots, P_k) = \{f: S \rightarrow \{1, \dots, k\} : f^{-1}(i) \text{ is a SDR of } P_i \text{ for each } i\}. \quad (5.3)$$

In our application, the set S will consist of the h -neighbourhood of a vertex u after deleting a vertex v . Since u began with degree $2k$, d is the number of edges from u to the deleted vertex v . In our application, the set $A(P_1, \dots, P_k)$ encodes all ways to extend forests of the graph without u inducing the partitions P_1, \dots, P_k into trees using each edge around u exactly once.

Example 5.11. We will run an example through this section. Let $S = \{1, 2, \dots, 12\}$ and $k = 8$. In the context where we will ultimately use the bijection that we'll define in Proposition 5.15, S will be the h -neighbourhood of a vertex u . For instance, the particular values of S and k could occur as in Figure 5.1.

¹⁴An elementary argument shows that $q^n \in \mathbb{Z}$, for $n \in \mathbb{Z}_{>0}$ and rational q , implies that $q \in \mathbb{Z}$.

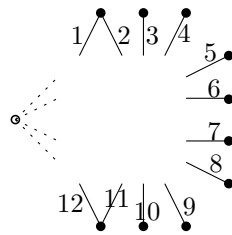


Figure 5.1: An illustration of S as an h -neighbourhood of a central vertex u which has been removed. The dotted half edges and empty vertex on the left represent the connections to the deleted v , and are not included in S .

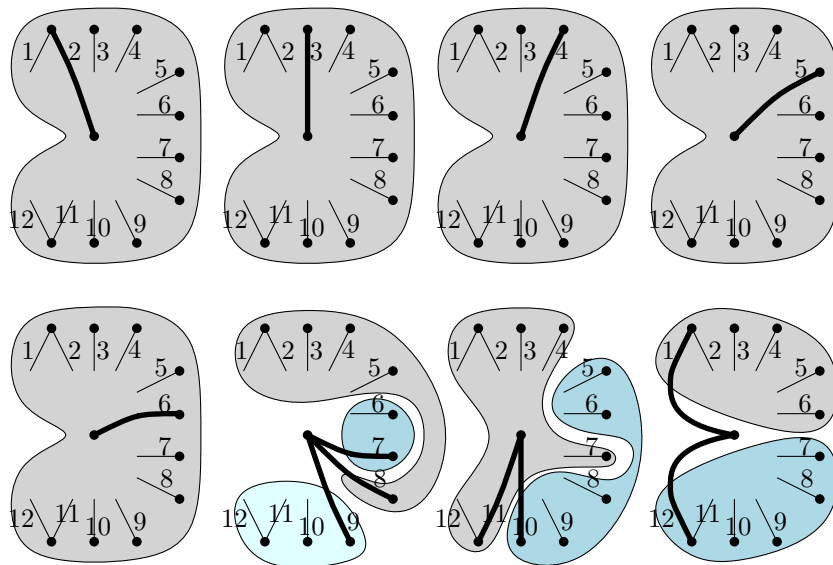


Figure 5.2: The systems of distinct representatives given by f illustrated in the context of S as the h -neighbourhood of the central vertex u .

Take the partitions $P_1 = P_2 = P_3 = P_4 = P_5 = \{S\}$ to be trivial (one part) and let

$$\begin{aligned}
 P_6 &= \{\{1, 2, 3, 4, 5, 8\}, \{6, 7\}, \{9, 10, 11, 12\}\}, \\
 P_7 &= \{\{1, 2, 3, 4, 7, 11, 12\}, \{5, 6, 8, 9, 10\}\}, \\
 P_8 &= \{\{1, 2, 3, 4, 5, 6\}, \{7, 8, 9, 10, 11, 12\}\}.
 \end{aligned}$$

For our context involving the h -neighbourhood of u , each partition of S is induced from a partition of the vertices adjacent to u , and so with the vertices as illustrated in Figure 5.1, the partitions must all have 1 and 2 in the same part and likewise for 11 and 12.

Then one example of an element of $A(P_1, \dots, P_8)$ in this case is given by $f(1) = f(12) = 8$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$, $f(5) = 4$, $f(6) = 5$, $f(7) = f(8) = f(9) = 6$, $f(10) = f(11) = 7$. Figure 5.2 illustrates the partitions as coloured blobs and f with bold edges. We shall return to this example after the next definitions.

A *matching* of S is a set of disjoint pairs of elements of S . Its *support* is the set $\text{supp } m \subseteq S$

of all elements of S that appear in some pair $\{a, b\} \in m$.

Given a partition $P = \{p_1, \dots, p_t\}$ of S , we can view a set m of pairs of elements of S as a graph whose vertices are the parts of P , by interpreting each element $\{a, b\} \in m$ as an edge between the vertices p_i and p_j corresponding to the parts that contain $a \in p_i$ and $b \in p_j$. If this graph happens to be a tree, then we say that m defines a tree on P .

Definition 5.12. For any sequence P_1, \dots, P_k of partitions of S , let $B(P_1, \dots, P_k)$ denote the set of tuples (m, z) that consist of k matchings $m = (m_1, \dots, m_k)$ of S and d elements $z = (z_1, \dots, z_d) \in S^d$ such that:

1. the matchings have $|m_1| + \dots + |m_k| = k - d$ pairs in total,
2. m_i defines a tree on P_i for each $1 \leq i \leq k$, and
3. $S = \text{supp } m_1 \cup \dots \cup \text{supp } m_k \cup \{z_1, \dots, z_d\}$.

Since $|\text{supp } m_i| = 2|m_i|$ and $|S| = 2k - d$, the first condition implies that the union in 3. is disjoint. Therefore, all z_i are necessarily distinct, and $m_1 \sqcup \dots \sqcup m_k$ is a k -coloured perfect matching of $S \setminus \{z_1, \dots, z_d\}$.

In our application the set $B(P_1, \dots, P_k)$ will encode the ways to extend forests of the graph without vertex u (and without v), inducing the partitions P_1, \dots, P_k , into trees in such a way that the new edges are formed from a matching of the half edges at u , i.e. a transition. Each z_i labels a half-edge that is paired with the deleted vertex v .

Example 5.13. Continuing Example 5.11, if we set $z = (3, 4, 5, 6)$ and define the matchings $m_1 = m_2 = m_3 = m_4 = m_5 = \emptyset$,

$$m_6 = \{\{2, 7\}, \{8, 9\}\}, \quad m_7 = \{\{10, 11\}\}, \quad m_8 = \{\{1, 12\}\}$$

then this gives an element of $B(P_1, \dots, P_8)$ and is illustrated in Figure 5.3.

Lemma 5.14. $A(P_1, \dots, P_k)$ and $B(P_1, \dots, P_k)$ are empty whenever $\sum_{i=1}^k |P_i| \neq |S|$.

Proof. For A , we must choose exactly one element from each part and use each element of S exactly once, so the sum of the number of parts in each P_i must be equal to $|S|$. For B , since a tree has one fewer edges than vertices, we have $\sum_{i=1}^k (|P_i| - 1) = k - d$. Since $|S| = 2k - d$ by definition, this is the same as the condition for A . \square

Let us recall the classic Prüfer code [Prü18]. Fix a set P of size $n \geq 2$ and equip it with a total order. Given a tree t with vertex set P , start with $i = 1$ and repeat:

1. pick the smallest leaf $a_i \in P$ of t and let $b_i \in P$ denote its neighbour,
2. remove a_i from t and increment i .

After $n - 2$ steps, this produces the Prüfer sequence $b = (b_1, \dots, b_{n-2})$. The b_i need not be distinct, in fact every tuple $b \in P^{n-2}$ arises exactly once this way. This is one way to prove Cayley's formula that there are n^{n-2} labelled trees on n vertices.

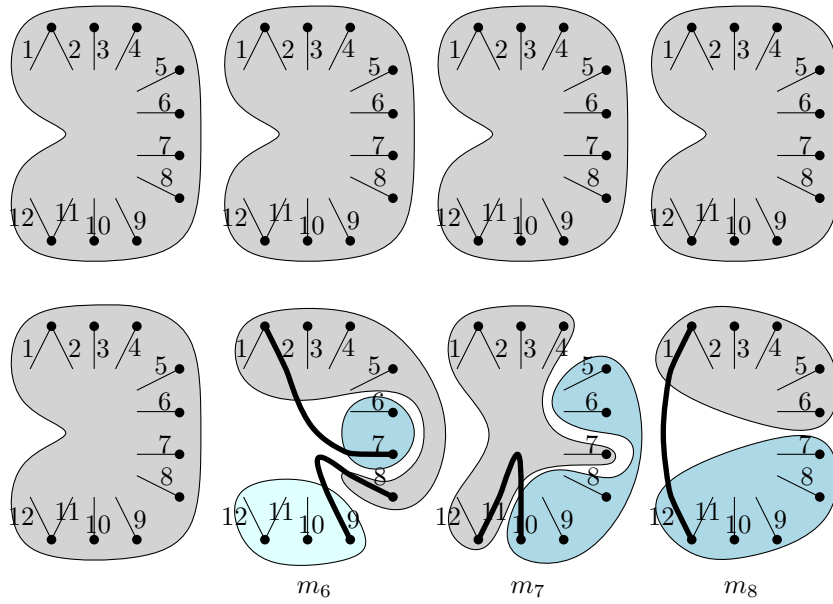


Figure 5.3: The matchings giving a B set. The unmatched half-edges 3, 4, 5, 6 then form the vector z and in this graph context would connect to the deleted vertex v .

Proposition 5.15. *For any choice of $k > 0$, any non-empty set S of size $|S| \leq 2k$, and any sequence P_1, \dots, P_k of partitions of S , we have*

$$|A(P_1, \dots, P_k)| = |B(P_1, \dots, P_k)|.$$

Proof. Let r denote the number of partitions P_i with only one part. We may place these trivial partitions first, so that $P_1 = \dots = P_r = \{S\}$ and then $|P_i| \geq 2$ for $i > r$. By Lemma 5.14, we may also assume that $\sum_{i=1}^k |P_i| = |S| = 2k - d$, and therefore

$$r = d + \sum_{i=r+1}^k (|P_i| - 2).$$

For each partition P_i , fix some order of its parts. We will now define a bijection $\alpha: B \rightarrow A$ between the sets $B = B(P_1, \dots, P_k)$ and $A = A(P_1, \dots, P_k)$. Let $(m, z) \in B$.

For each $i > r$, consider the tree t_i defined by m_i on P_i . We run the Prüfer algorithm for t_i to build a sequence $y_i \in S^{|P_i|-2}$ and a set $X_i \subseteq S$ of size $|P_i|$ as follows:

- Loop for $|P_i| - 2$ steps:
 - at the j th step take the remaining leaf which is first in the order on P_i and let $\{a, b\}$ be the element of m_i corresponding to the unique edge adjacent to this leaf where a is in the leaf part and b is in the other part, then
 - put b in the j th slot of y_i ,
 - add a to X_i , and
 - remove this leaf from the tree.

- Let $\{a_1, a_2\}$ be the element of m_i corresponding to the last remaining edge of the tree, then add a_1 and a_2 to X_i .

Since elements are added to X_i when the part they are contained in is a leaf of the tree and then the leaf is removed or the algorithm terminates, there will be exactly one element of each part in X_i . Therefore, X_i is a SDR for P_i . We also note that the tree t_i can be reconstructed from y_i , because the classical Prüfer code of t_i is obtained by replacing each entry of y_i with the part of P_i that contains it.

Let $y \in S^r$ denote the sequence of length r obtained by concatenation of the sequences y_{r+1}, \dots, y_k, z . We define a function $f: S \rightarrow \{1, \dots, k\}$ as follows:

- for each $i > r$ set $f(a) = i$ for each $a \in X_i$,
- for each $i \leq r$ set $f(b) = i$ for b the i th entry of y .

By the observations above, this construction defines an element $f \in A$. We have thus defined a map $\alpha(m, z) = f$ from A to B . We will now define a map $\beta: A \rightarrow B$ and then show that this is a two-sided inverse for α .

Given $f \in A$, set $X_i = f^{-1}(i)$. Since X_i is a SDR for P_i , we have $|X_i| = |P_i|$. Thus the first r sets $X_i = \{x_i\}$ are singletons and define a sequence $y = (x_1, \dots, x_r) \in S^r$. Cut y into pieces of lengths $|P_{r+1}| - 2, \dots, |P_k| - 2, d$ to define sequences y_{r+1}, \dots, y_k, z . Set $m_1 = \dots = m_r = \emptyset$ and define matchings m_i of S for $i > r$ as follows:

- Let t_i denote the tree on P_i corresponding to the Prüfer code obtained by replacing the elements of y_i by the parts of P_i that contain them.
- Run the Prüfer algorithm for t_i . In the j th step, for $p \in P_i$ the smallest leftover leaf, add $\{a, b\}$ to m_i , where $\{a\} = X_i \cap p$ and b is the j th entry of y_i .
- For the last edge $\{p_1, p_2\}$ of t_i , add $\{a_1, a_2\}$ to m_i , where $\{a_j\} = p_j \cap X_i$.

We define $\beta(f) = (m_1, \dots, m_k, z)$. From this construction it is clear that applying first α and then β is the identity on A . Applying first β then α observe that in specifying the last edge for each m_i , the two remaining elements will be one in each part not corresponding to a part already removed as a leaf, so this does give a well defined element of B and all other steps are direct inverses. \square

Example 5.16. Let us work through an example of α and β described above. We will see in hindsight that Example 5.11 and Example 5.13 were chosen with this in mind.

As noted above, with S interpreted as the specific h -neighbourhood of a vertex u as illustrated in Figure 5.1, the partitions must all have 1 and 2 in the same part and likewise for 11 and 12. The fact that half-edges 1 and 2 have the same incident vertex in the figure, and likewise for 11 and 12 has no effect on the maps α and β but can occur in the context where we will use Proposition 5.15 and so is useful to illustrate in the example.

The defect $d = 2k - |S| = 4$ indicates that in the 16-regular completion of the graph, u has 4 edges connecting it to the deleted vertex v as illustrated in Figure 5.1.

Take the partitions to be as in Example 5.11, and as illustrated by the coloured blobs in Figure 5.2 and Figure 5.3. Set $z = (3, 4, 5, 6)$ along with the matchings $m_1 = m_2 = m_3 = m_4 = m_5 = \emptyset$,

$$m_6 = \{\{2, 7\}, \{8, 9\}\}, \quad m_7 = \{\{10, 11\}\}, \quad m_8 = \{\{1, 12\}\}$$

as in Example 5.13.

Applying α to this (m, z) we obtain f with $f(1) = f(12) = 8, f(2) = 1, f(3) = 2, f(4) = 3, f(5) = 4, f(6) = 5, f(7) = f(8) = f(9) = 6, f(10) = f(11) = 7$ which is the f we began with in Example 5.11.

5.2. Diagonals and the Martin invariant

We apply the results of Section 5.1 to show that the diagonal coefficient satisfies the Martin recurrence and hence with suitable normalization agrees with the Martin invariant.

Theorem 5.17. *Let G be a $2k$ -regular graph with n vertices. Then for any vertex v without self-loops,*

$$[x_1 \cdots x_{k(n-2)}] \tilde{\Psi}_{G \setminus v}^k = k! \cdot M(G). \tag{5.4}$$

Proof. By Lemma 5.4 the left-hand side, $N_k(G \setminus v)$, is the number of ordered partitions $E = T_1 \sqcup \dots \sqcup T_k$ of the edge set E of $G \setminus v$ into k edge-disjoint spanning trees T_i .

If G has a self-loop, then $M(G) = 0$ and by hypothesis on v , $G \setminus v$ also has a self-loop and so $N_k(G \setminus v) = 0$ since no spanning tree can include any self-loop. Therefore, (5.4) holds whenever G has one or more self-loops.

From now on, assume that G has no self-loops. Then G must have at least two vertices. If G has exactly 2 vertices, then $G \cong K_2^{[k]}$ is a k -fold edge with $k! \cdot M(G) = 1$, see (2.7), and $G \setminus v$ is a single vertex without edges and thus $N_k(G \setminus v) = 1$ from the partition $T_1 = \dots = T_k = \emptyset$. So (5.4) holds for $n = 2$.¹⁵

To complete the proof, we will show that the diagonal coefficient $N_k(G \setminus v)$ satisfies the Martin recurrence (1.1): Let $w \neq v$ be another vertex of G , then we will prove that

$$N_k(G \setminus v) = \sum_{\tau \in \mathcal{T}'(w)} N_k(G_\tau \setminus v) \tag{5.5}$$

where $\mathcal{T}'(w)$ is the subset of transitions at w that does not create a self-loop at v . This induction step reduces (5.4) from n to $n - 1$ vertices: if $N_k(G_\tau \setminus v) = k! \cdot M(G_\tau)$ is already known, then the right-hand side of (5.5) is $k! \cdot M(G)$. The transitions $\tau \in \mathcal{T}(w) \setminus \mathcal{T}'(w)$ missing from (1.1) only produce graphs with $M(G_\tau) = 0$ due to self-loops at v .

We now prove (5.5). Let S be the h -neighbourhood (see Section 2) of w in $G \setminus v$.

Let T_1, \dots, T_k be a partition of E into spanning trees of $G \setminus v$. For each T_i , let $X_i = T_i \cap S$ and X'_i be the set of edges incident to w which contain a half-edge of X_i . The remaining

¹⁵A direct proof for $n = 3$ is also straightforward (though not necessary), since then $G \cong K_3^{[k]}$ with $M(G) = 1$ and the claim follows from the $r = 1$ case of Example 5.10.

edges $F_i = T_i \setminus X'_i$ determine a spanning forest of $G \setminus \{v, w\}$. Such a forest F_i induces a partition P_i of S as follows: two half-edges from the h -neighbourhood of w are in the same part of P_i if the corresponding neighbours of w are in the same component of F_i . Since $T_i = X'_i \sqcup F_i$ and X_i is in bijection with X'_i , X_i includes precisely one half-edge in each part of P_i , so it is an SDR of P_i . Since the T_i partition E , the SDRs X_1, \dots, X_k partition S . Conversely, given any list of forests F_i that induce P_i , and any partition of S into SDRs X_i of P_i , then each T_i formed by taking $F_i = X'_i \sqcup F_i$ (where X'_i is again the set of edges which contain a half-edge of X_i) is a spanning tree of $G \setminus v$ and these trees partition E . Therefore, we can write

$$N_k(G \setminus v) = \sum_P |A(P_1, \dots, P_k)| \cdot |C(P_1, \dots, P_k)| \quad (\#)$$

as a sum over all ordered sequences $P = (P_1, \dots, P_k)$ of partitions of S . Here A is defined in (5.3) and counts the choices of X_i . To count the choices of F_i , we define $C(P)$ as the set of lists (F_1, \dots, F_k) of spanning forests $F_i \subseteq E'$ of $G \setminus \{v, w\}$ such that each F_i induces P_i , and such that all F_i together form a partition of the edge set E' of $G \setminus \{v, w\}$.

Now consider a transition τ of G at w . This is a matching of the h -neighbourhood of w in G . This neighbourhood differs from S by the $d = 2k - |S|$ half-edges e_1, \dots, e_d incident to v . Suppose additionally that τ does not create any self-loops at v . Then let $z = (z_1, \dots, z_d)$ be the elements of S which are paired in τ with e_1, \dots, e_d . Let m be the perfect matching of $S \setminus \{z_1, \dots, z_d\}$ given by the rest of the transition. We can thus identify the set $\mathcal{T}'(w)$ with pairs (m, z) of a d -tuple z and a matching m of the above form. Note that the elements of m are canonically identified with edges in $G_\tau \setminus v$, such that the edge set of $G_\tau \setminus v$ is the disjoint union of E' and m .

Let T'_1, \dots, T'_k be a partition of the edges of $G_\tau \setminus v$ into spanning trees. Similarly to before, we can decompose each $T'_i = m_i \sqcup F_i$ into the edges $m_i = T'_i \cap m$ that belong to m and a spanning forest $F_i = T'_i \setminus m_i$ of $G \setminus \{v, w\}$. Let P_i be the partition of S induced by F_i . Then m_i defines a tree on P_i in the sense explained before Definition 5.12. Conversely, given any forest F_i that induces P_i , together with a matching m_i defining a tree on P_i , we obtain a spanning tree $T'_i = F_i \sqcup m_i$ of $G_\tau \setminus v$. Therefore, we can write

$$\sum_{\tau \in \mathcal{T}'(w)} N_k(G_\tau \setminus v) = \sum_P |B(P_1, \dots, P_k)| \cdot |C(P_1, \dots, P_k)|, \quad (b)$$

as a sum over sequences $P = (P_1, \dots, P_k)$ of partitions of S . Here, $B(P)$ from Definition 5.12 counts the choices of τ (that is, m and z) and also the choice of partition $m = m_1 \sqcup \dots \sqcup m_k$; whereas $C(P)$ counts the spanning forest decompositions $E' = F_1 \sqcup \dots \sqcup F_k$ as before.

By Proposition 5.15, (#) and (b) agree and hence we proved (5.5). \square

Corollary 5.18. *For a $2k$ -regular graph G and a vertex v without self-loops, $M(G)$ is equal to the number of partitions of the edge set of $G \setminus v$ into k spanning trees. Hence, the number of these partitions is independent of v .*

This combinatorial interpretation, stated as Theorem 1.3 in the introduction, follows from Corollary 5.5. It is illustrated for the example $M(K_4) = 6$ in (1.6). The claim remains true even

if there are self-loops at v , for then $M(G) = 0$ but also $N_k(G \setminus v) = 0$, because $G \setminus v$ must then have more than kn edges (where G has $n + 2$ vertices) and hence does not admit partitions into k spanning trees (each of which contains n edges).

Remark 5.19. From this point of view, the condition $M(G) > 0$ is equivalent to the existence of an edge partition of $G \setminus v$ into spanning trees. Necessary and sufficient conditions for the existence of such a partition were given by Nash-Williams [NW61] and Tutte [Tut61]. In fact, for our case of $2k$ -regular graphs G , it follows easily from [NW61, Lemma 5] that $G \setminus v$ has k edge-disjoint spanning trees if and only if G is $2k$ -edge connected. This gives an alternative proof of Corollary 3.3, which says that $M(G) > 0$ in this case. Also for more general graphs, the maximal number of edge-disjoint spanning trees is related to the edge-connectivity via upper and lower bounds [Gus83, Kun74].

Corollary 5.20. *Let G be a $2k$ -regular graph with n vertices, and let v denote a vertex without self-loops. Then for every positive integer r ,*

$$M(G^{[r]}) = \frac{(r!)^{k(n-2)}}{(kr)!} [x_1^r \cdots x_{k(n-2)}^r] \tilde{\Psi}_{G \setminus v}^{kr}. \tag{5.6}$$

Proof. Combine Theorem 5.17 with Lemma 5.8. □

Corollary 5.21. *For a $2k$ -regular graph G with $n \geq 3$ vertices and any positive integer r , the Martin invariant $M(G^{[r]})$ is an integer that is divisible by $(r!)^{k(n-3)}$.*

Proof. Combine Lemma 5.9 with Corollary 5.20. □

The identity (5.6) shows that the sequence $M(G^\bullet)$ of Martin invariants of all duplications $G^{[r]}$ of a fixed graph encodes the diagonal coefficients of all powers of the Kirchhoff polynomial. These coefficients are related to each other by a difference equation:

Corollary 5.22. *For any $2k$ -regular graph G , there exists an integer $d > 0$ and univariate polynomials $P_0, \dots, P_d \in \mathbb{Z}[t]$ with $P_d \neq 0$ such that, for all $r \geq 1$,*

$$P_0(r) \cdot M(G^{[r]}) + P_1(r) \cdot M(G^{[r+1]}) + \dots + P_d(r) \cdot M(G^{[r+d]}) = 0. \tag{5.7}$$

Proof. If a rational function $f(x_1, \dots, x_m)$ of several variables is non-singular at $x = 0$, then it has a Taylor expansion at $x = 0$. The diagonal coefficients of this expansion define a univariate formal power series $\text{Diag}(f) = \sum_{r \geq 0} z^r [x_1^r \cdots x_m^r] f$ called *the diagonal* of f . Such diagonals are D -finite functions [Lip88], hence the sequence of diagonal coefficients is P -recursive [Sta80] (namely it fulfils a polynomial recurrence).

Therefore, the diagonal coefficients $M(G^{[r]}) \cdot (kr)! / (r!)^{k(n-2)}$ of the geometric series $f = 1 / (1 - \tilde{\Psi}_{G \setminus v}) = \sum_{i \geq 0} \tilde{\Psi}_{G \setminus v}^i$ fulfil a recurrence of the form (5.7). Multiplying this recurrence with $((r + d)!)^{k(n-2)} / (kr)!$ produces the claimed recurrence for $M(G^{[r]})$ alone, by the transformation $P_i(r) \mapsto P_i(r) \cdot [(r + i + 1) \cdots (r + d)]^{k(n-2)} \cdot (kr + 1) \cdots (kr + ki)$. □

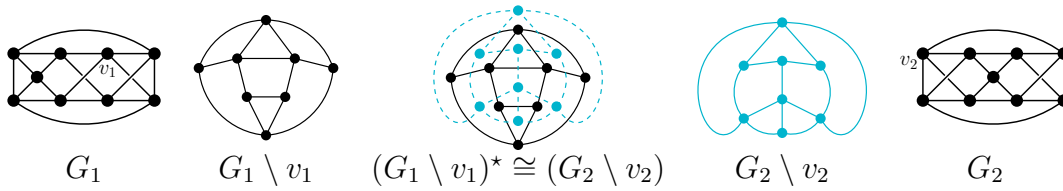


Figure 5.4: Planar embeddings of decompletions $G_i \setminus v_i$ that are planar duals of each other. The completed graphs are labelled $G_1 = P_{7,4}$ and $G_2 = P_{7,7}$ in [Sch10].

5.3. Duality

As an application of the connection between the Martin invariant and diagonal coefficients, we can relate the Martin sequences of graphs that are planar duals of each other.

Suppose that G is a graph with a plane embedding and G^* denotes the corresponding dual graph. Then the edges of G are in canonical bijection with the edges of G^* . Under this correspondence, a set of edges defines a spanning tree in G precisely when its complement defines a spanning tree in G^* . Therefore, we have

$$\Psi_{G^*} = \tilde{\Psi}_G. \quad (5.8)$$

Recall that the Martin invariant of a regular graph G can be computed by the diagonal coefficient of the Kirchhoff polynomial of a decomposition $G \setminus v$. For two regular graphs G_1 and G_2 with vertex degrees d_1 and d_2 , respectively, a duality $(G_1 \setminus v)^* \cong G_2 \setminus w$ between decompletions implies that $(d_1, d_2) \in \{(3, 6), (4, 4), (6, 3)\}$. This follows from the constraints on the edge- and vertex numbers imposed by regularity and duality. In such cases, (5.8) translates into a relation between the Martin sequences of G_1 and G_2 . An example is shown in Figure 5.4.

Corollary 5.23. *If G_1 and G_2 are 4-regular graphs such that there exist vertices v_1 and v_2 without self-loops such that $G_1 \setminus v_1$ is a planar dual of $G_2 \setminus v_2$, then the Martin sequences $M(G_1^\bullet) = M(G_2^\bullet)$ agree.*

Proof. Let n denote the number of vertices of G_1 . Then $G_1 \setminus v_1$ and $G_2 \setminus v_2$ each have $n - 1$ vertices, $m = 2n - 4$ edges, and $n - 2$ loops. From (4.12) and (5.8) we see that

$$[x_1^r \cdots x_m^r] \tilde{\Psi}_{G_1 \setminus v_1}^{2r} = [x_1^r \cdots x_m^r] \Psi_{G_2 \setminus v_2}^{2r} = [x_1^r \cdots x_m^r] \tilde{\Psi}_{G_2 \setminus v_2}^{2r}$$

and thus the claim follows from Corollary 5.20. \square

Corollary 5.24. *Let G_3 be a 3-regular graph and G_6 a 6-regular graph, such that there exist vertices v and w , without self-loops, such that $G_3 \setminus v$ is a planar dual of $G_6 \setminus w$. Let n denote the number of vertices of G_6 . Then for every positive integer r , we have*

$$M\left(G_3^{[2r]}\right) = \left(\frac{(2r)!}{r!}\right)^{3(n-2)} M\left(G_6^{[r]}\right). \quad (5.9)$$

Proof. The graphs $G_3 \setminus v$ and $G_6 \setminus w$ each have $m = 3(n - 2)$ edges. Setting (r, k) to $(2r, 3/2)$ in (5.2), we find that $M(G_3^{[2r]}) \cdot (3r)! = [(2r)!]^m [x_1^{2r} \cdots x_m^{2r}] \tilde{\Psi}_{G_3 \setminus v}^{3r}$ from Theorem 5.17.¹⁶ From (4.12) and (5.8) we see that

$$[x_1^{2r} \cdots x_m^{2r}] \tilde{\Psi}_{G_3 \setminus v}^{3r} = [x_1^{2r} \cdots x_m^{2r}] \Psi_{G_6 \setminus w}^{3r} = [x_1^r \cdots x_m^r] \tilde{\Psi}_{G_6 \setminus w}^{3r},$$

and thus the claim follows from Corollary 5.20. □

Through Theorem 4.1, the above dualities imply relations between the graph permanents. Those have been noted before, see [Cru17b, Proposition 28].

Remark 5.25. The identity (5.8) extends beyond the realm of planar graphs G to all matroids M and their duals M^* , if we replace “spanning tree of G ” in the definition of $\tilde{\Psi}_G$ and Ψ_G by “basis of M ”. The *matroid polynomial* $\tilde{\Psi}_M$ thus defined is still linear in each variable, and its degree is the rank of M . The degree of Ψ_M is the corank (formerly loop number) of M . The combinatorial interpretation from Lemma 5.4 identifies the diagonal coefficient of $\tilde{\Psi}_M^k$ with decompositions into bases. The existence of such decompositions has been characterized in [Edm65, Theorem 1]. For regular matroids, $\tilde{\Psi}_M$ is the determinant of a matrix, similar to the graph Laplacian (4.11).

Consequently, any function of the diagonal coefficient of $\tilde{\Psi}_M$ is invariant under matroid duality, for all matroids in the domain of the function—not just cycle matroids of planar graphs. For example, the permanent of regular matroids is invariant under duality. This follows from Section 4.4 or the matrix manipulations in [CDY16], since those apply without change to regular matroids.¹⁷ For the c_2 invariant, the situation is less clear: its connection to the diagonal coefficient is known only in the presence of a 3-valent vertex, see Remark 6.18. Moreover, invariance of c_2 under duality is not even known for all graphic matroids [Dor18].

Remark 5.26. An identity of Feynman periods called *Fourier split* was introduced in [HSSY22, §2]. It operates by taking a planar dual on one side of a 3-vertex cut in a decomposition $G \setminus v$. This identity also holds for the Martin invariant. The same proof strategy as in Section 3.2 applies: the Martin recursion (1.1) at a vertex on the other side of the 3-vertex cut reduces the statement by induction to smaller graphs. The remaining base case is the ordinary duality relation (Fourier identity).

We hence conclude via Theorems 4.1 and 6.2 that also the extended graph permanent and the c_2 invariant at primes respect the Fourier split identity.

6. c_2 invariant

For any power $q = p^s$ of a prime p , let \mathbb{F}_q denote the finite field with q elements. Given any polynomial $P \in \mathbb{Z}[x_1, \dots, x_m]$ in m variables x_i , we denote by

$$[[P]]_q = |\{x \in \mathbb{F}_q^m : P(x) = 0\}| \in \mathbb{Z}_{\geq 0}$$

¹⁶We may set k to rational values in Lemma 5.8, as long as the exponent rk stays positive integer.

¹⁷An example of a permanent of a neither graphic nor co-graphic matroid R_{10} is computed in [Cru17a, § 7.5].

the number of points on the hypersurface $\{P = 0\}$ over \mathbb{F}_q . In the special case $P = \Psi_G$ of Symanzik polynomials (1.7) of graphs, the study of point-counts goes back to [Sta98, Ste98]. For all graphs with less than 14 edges [Sch11], the point-counting function is a polynomial,

$$q \mapsto \llbracket \Psi_G \rrbracket_q = \sum_{i=0}^m c_i(G) q^i. \quad (*)$$

Starting at 14 edges, there exist graphs whose point-count is not polynomial [Dor11, Sch11]. In fact, point-counting functions of graph hypersurfaces $\{\Psi_G = 0\}$ can be as complicated as the point-counting functions of any variety, as made precise and proved in [BB03].

Lemma 6.1. *Let G be a graph with at least 3 vertices and at least 2 edges. Then for every prime power q , the point count $\llbracket \Psi_G \rrbracket_q$ is divisible by q^2 .*

Proof. This is proved in [Sch11, Corollary 2.8] and [BS12, Proposition-Definition 18] when G is connected. If G is disconnected, then $\Psi_G = 0$ and hence $\llbracket \Psi_G \rrbracket_q = q^m$ where m is the number of edges of G . \square

We will only consider point-counts of graphs with $n \geq 3$ vertices and $m = 2(n - 1)$ edges, hence Lemma 6.1 applies and we can write

$$\llbracket \Psi_G \rrbracket_q \equiv q^2 \cdot c_2^{(q)}(G) \pmod{q^3} \quad (6.1)$$

for well-defined residues $c_2^{(q)}(G) \in \mathbb{Z}/q\mathbb{Z}$. The c_2 -invariant [Sch11] is the sequence of these residues, one for each prime power, given by¹⁸

$$c_2^{(q)}(G) \equiv \frac{\llbracket \Psi_G \rrbracket_q}{q^2} \pmod{q}. \quad (6.2)$$

Our main result in this section extracts c_2 at primes from the Martin invariant:

Theorem 6.2. *Let G be a 4-regular graph with at least 6 vertices. Then for every vertex v of G , and every prime p , we have*

$$c_2^{(p)}(G \setminus v) \equiv \frac{M(G^{[p-1]})}{3p} \pmod{p}. \quad (6.3)$$

In Section 6.1 we give a proof for primes $p \neq 3$, by relating c_2 to the diagonal coefficients of powers of Ψ_G as in Section 5. In Sections 6.2 to 6.4 we present another proof, which is more technical, but which works for all primes—including $p = 3$.

Remark 6.3. The relation Equation (6.3) does not extend to prime powers. Only a few relations for c_2 are known at prime powers, see [EY23, SY21, Dor17a].

¹⁸When the point-count happens to be a polynomial, then $c_0(G) = c_1(G) = 0$ and $c_2^{(q)}(G) \equiv c_2(G) \pmod{q}$ is determined by the coefficient of q^2 in (*).

Corollary 6.4. *Let G be a 4-regular graph G with at least 6 vertices, and let v, w denote two vertices. Then $c_2^{(p)}(G \setminus v) \equiv c_2^{(p)}(G \setminus w) \pmod p$ for all primes p .*

This proves the *completion conjecture*, [BS12, Conjecture 4], for all prime fields. This conjecture is a special case of the stronger [BS12, Conjecture 5] from [Sch11, Remark 2.11]:

Conjecture 6.5. Whenever $\mathcal{P}(G_1) = \mathcal{P}(G_2)$ for two primitive ϕ^4 graphs, then $c_2^{(q)}(G_1) \equiv c_2^{(q)}(G_2) \pmod q$ for all prime powers q .

Here a graph G is called *primitive ϕ^4* if it can be obtained from a cyclically 6-connected 4-regular graph G' by deleting a vertex: $G \cong G' \setminus v$. We then call G' *the completion* of G , and we also say that G is a *decompletion* of G' , as originally defined in Section 1.2. Every ϕ^4 graph has a unique completion, but a 4-regular graph can have several non-isomorphic decompletions.

The *completion identity* $\mathcal{P}(G \setminus v) = \mathcal{P}(G \setminus w)$ for cyclically 6-connected 4-regular graphs G is only one of several known period identities proved in [Sch10]. The twist and duality identities lead to further special cases of Conjecture 6.5. Combining our results on the Martin invariant (Proposition 3.12 and Corollary 5.23) with Theorem 6.2, we also obtain proofs of the corresponding identities for c_2 . Specifically, for all primes p :

- If G_1 is a twist of G_2 , then $c_2^{(p)}(G_1 \setminus v) \equiv c_2^{(p)}(G_2 \setminus w) \pmod p$.
- If G is planar, and G^* a planar dual, then $c_2^{(p)}(G) \equiv c_2^{(p)}(G^*) \pmod p$.

The duality relation was known previously, even for prime powers [Dor17a]. Our twist identity for c_2 is a new result, however our proof of the completion identity is arguably more significant. The completion conjecture has been repeatedly stated explicitly [BS12, BS13] and has had many unsuccessful or only partially successful attempts at a proof.

To apply the main theorem, we use some divisibility properties, which we record here and which are proved in Section 6.4. In Corollary 6.31 and Lemma 6.35 we show:

Lemma 6.6. *Let G denote a 4-regular graph with at least 4 vertices. Then for every prime p , $M(G^{[p-1]})$ is divisible by p . For $p = 3$ and at least 5 vertices, we also have that $M(G^{[2]})$ is divisible by 3^2 . Hence, the right-hand side of Equation (6.3) is well-defined.*

In practice, for the purpose of computing $c_2^{(p)}$, one can simplify the Martin recursion and ignore all graphs that have a $\geq p$ fold edge:

Lemma 6.7. *If G is $4r$ -regular with $p = r + 1$ prime, and G has at least 6 vertices and at least one edge with multiplicity p or more, then $M(G)$ is divisible by p^2 (or p^3 in the case $p = 3$) and hence such a graph does not contribute to c_2 .*

Proof. Lemma 6.30 shows that applying the Martin recurrence around either end of an edge with multiplicity $\geq p$ gives a factor divisible by p multiplying the Martin invariant of each G_τ . Applying the previous lemma to $M(G_\tau)$ gives the result. \square

Remark 6.8. The properties of the Martin invariant and Theorem 6.2 also explain other c_2 identities which are known from [BSY14] and [BS12, Corollary 34]:

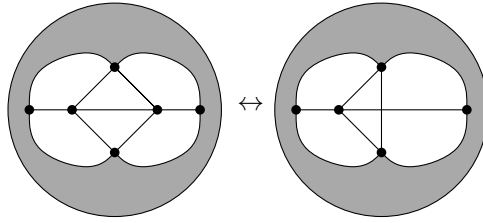


Figure 6.1: The double triangle operation.

- If G has a 2-edge cut, a 4-edge cut, or a 3-vertex cut, then $c_2^{(p)}(G \setminus v) \equiv 0 \pmod{p}$.
- If G and G' are related as in Figure 6.1, then $c_2^{(p)}(G) \equiv c_2^{(p)}(G') \pmod{p}$.

The first follows from Theorem 1.2 and the factors of p in $M(G^{[p-1]})$ (see Lemma 6.6) producing p^2 in the products (1.3) and Proposition 3.10. The double-triangle relation follows from Lemma 6.7, since for either vertex incident to the edge shared between two triangles in G , after $(p-1)$ -duplicating the only transitions that do not lead to self-loops or edges of multiplicity p or larger, are precisely those that produce G' .

We can exploit Lemma 6.7 to simplify the calculation of $c_2^{(p)}$ in recursive families of graphs. A recursive family [CY19, §5] is in the simplest case a sequence G_1, G_2, \dots of graphs where G_{n+1} is obtained from G_n through a fixed local modification that affects only a finite number of vertices (call these the *active vertices*).

Lemma 6.9. *For a recursive family of ϕ^4 graphs G_1, G_2, \dots and any fixed integer r , the sequence of Martin invariants $M(G_1^{[r]}), M(G_2^{[r]}), \dots$ fulfils a linear recurrence with constant coefficients.*

Proof. Applying the Martin recurrence to the finite number of vertices in G_n that are involved in moving from G_{n-1} to G_n produces a linear combination of finitely many graphs which have the same recursive structure as G_{n-1} and the same active vertices but with potentially different connections between the active vertices. The Martin recurrence can likewise be applied to these graphs and so on. There are only finitely many possibilities for connecting the active vertices so there are only finitely many such graphs to consider. Collecting the Martin invariants of these graphs into a vector a_n , we obtain a system of recurrences $a_n = T a_{n-1}$ for some matrix T . \square

For example, the prisms $K_2 \times C_n$ can be constructed using 4 active vertices, by adding one rung at a time. We derive the corresponding recurrence for $M(G_n^{[2]})$ at the end of Section 7.2. For the family of circulants $C_{1,2}^n$ we compute $M(G_n)$ in (7.4).

Through Theorem 6.2, the sequence $c_2^{(p)}(G_n \setminus v)$ at a fixed prime fulfils the same linear recurrence as $M(G^{[p-1]})$. Lemma 6.7 shows that we can drop all graphs with $\geq p$ -fold edges from the vector a_n , to obtain a shorter recurrence for c_2 . It could be interesting to compare this approach with the method of [Yea16], since the latter produces recurrences that are considerably larger than necessary [Yea20]. Experiments for circulants $C_{1,3}^n$ and $C_{2,3}^n$ at primes $p = 2, 3$ suggest that indeed the approach above produces shorter recurrences.

In order to prove Theorem 6.2, we relate c_2 to diagonal coefficients of some polynomials. One such relation was known previously, and its combinatorial interpretation (Lemma 6.14) was

key to the results of [Yea18, HY23]. We will use this interpretation to prove Theorem 6.2 for all primes, including $p = 3$. For $p \neq 3$, we will also give a simpler proof using the diagonal coefficient of Ψ . We will need the following polynomials from [BY11]:

Definition 6.10. Let G be a graph and let P be a partition of a subset of the vertices of G . The associated *spanning forest polynomial* is defined to be

$$\Phi_G^P = \sum_F \prod_{e \notin F} x_e$$

where the sum is over spanning forests F of G which are *compatible* with P in the sense that there is a bijection between the trees of F and the parts of P such that each tree of F contains all vertices that belong to the corresponding part of P .

Lemma 6.11. Let G be a 4-regular graph with at least 5 vertices and let v and w be two adjacent vertices, both without self-loops. Let a, b, c be the three neighbours of w which are not v and let x_4, \dots, x_m be the variables associated to the edges of $G \setminus \{v, w\}$. Then

$$c_2^{(p)}(G \setminus v) \equiv - [x_4^{p-1} \cdots x_m^{p-1}] \left(\Phi_{G \setminus \{v, w\}}^{\{a, b\}, \{c\}} \Psi_{G \setminus \{v, w\}} \right)^{p-1} \pmod p$$

for all primes p .¹⁹ Accordingly, the right-hand residue is invariant when permuting a, b, c .

Proof. This is a consequence of a chain of results from [BS12] and [BSY14]. To see them all laid out in detail to obtain the statement of the lemma, see [EY23, §2].²⁰ □

Remark 6.12. With G as in the lemma, and with 1, 2, 3 the edges from w to a, b, c respectively, then the two spanning forest polynomials in the lemma can also be written in terms of Dodgson polynomials, which were introduced in [Bro09]. Specifically

$$\begin{aligned} \Phi_{G \setminus \{v, w\}}^{\{a, b\}, \{c\}} &= \Psi_{G \setminus v, 3}^{1, 2} \\ \Psi_{G \setminus \{v, w\}} &= \Psi_{G \setminus v}^{13, 23}. \end{aligned}$$

Proving these two facts is one of the steps in the chain of results proving the lemma, coming from results of [BSY14] (see also [EY23, Example 2] or [HY23, §2.2] for the calculation exhibited pedagogically). We will not need to use Dodgson polynomials in the present paper, so we will not give their definition, but may use the two displayed above simply as an alternate notation for the corresponding spanning forest polynomials.

Definition 6.13. For a graph H with 3 vertices marked a, b, c and an integer r define $N_{r,r}(H)$ to be the number of ordered partitions of the set of edges of H into $2r$ parts, the first r of which are spanning trees of H , and the last r of which are spanning forests compatible with the partition $\{\{a, b\}, \{c\}\}$.

¹⁹Such an identity also holds at prime powers $q = p^s$, but only modulo p not q (as detailed in [EY23]).

²⁰The lower bound of 5 vertices is required to apply [BS12, Lemma 24] to $G \setminus v$. With only 4 vertices, the lemma fails. The graph from Remark 6.17 provides a counter-example.

The following combinatorial interpretation of c_2 was proven and put to productive use in [Yea18, HY23, EY23]. It will be used in our second proof of Theorem 6.2.

Lemma 6.14. *Let G be as in Lemma 6.11. Then for every integer r ,*

$$(r!)^{m-3} \cdot [x_4^r \cdots x_m^r] \left(\Phi_{G \setminus \{v,w\}}^{\{a,b\},\{c\}} \Psi_{G \setminus \{v,w\}} \right)^r = N_{r,r}(G^{[r]} \setminus \{v,w\}).$$

Consequently, $c_2^{(p)}(G \setminus v) \equiv N_{r,r}(G^{[r]} \setminus \{v,w\}) \pmod{p}$ for every prime $p = r + 1$.

Proof. For the first claim, apply the argument from the proof of Lemma 5.8 to relate the left-hand side to the diagonal coefficient of the polynomials for the duplicated graph $G^{[r]} \setminus \{v,w\}$; the interpretation as $N_{r,r}$ then follows as in Lemma 5.4, using the observation from Remark 5.7 that taking the complement of each part does not change the count. The second claim is then a corollary of Lemma 6.11 and Wilson's theorem $r! \equiv -1 \pmod{p}$, noting that the number $m = 2n - 4$ of edges in $G \setminus v$ is even (here n is the number of vertices in G). \square

Example 6.15. To compute $c_2^{(2)}(G)$, the Martin recursion has only 3 terms, which are again 4-regular graphs, hence we get a simple recursion as in (1.2):

$$c_2^{(2)} \left(\text{graph 1} \right) \equiv c_2^{(2)} \left(\text{graph 2} \right) + c_2^{(2)} \left(\text{graph 3} \right) + c_2^{(2)} \left(\text{graph 4} \right) \pmod{2}. \quad (6.4)$$

This recursion of c_2 specifically for $p = 2$ admits a quick proof using *denominator reduction*, see [Bro09, §10]. The *denominator* $D_G^r(e_1, \dots, e_r)$ of a graph arises as the denominator of the Feynman integral after integrating out the variables corresponding to the edges e_1, \dots, e_r . It is a polynomial in the remaining edge variables and can be defined by purely algebraic means. For example, the second denominator is always a square, $D_G^2(1, 2) = P^2$, of a certain (Dodgson) polynomial ($P = \Psi_G^{1,2}$). The third denominator is the product $D_G^3(1, 2, 3) = ([x_3^1] P)([x_3^0] P)$, i.e. the expression $\Psi_G^{13,23} \Psi_G^{1,2}$ used in Lemma 6.11 via Remark 6.12. Higher denominators can be computed as follows: whenever $D_G^r(e_1, \dots, e_r) = PQ$ with $r \geq 3$ is a product of two polynomials that are both linear in x_e , then $D_G^{r+1}(e_1, \dots, e_r, e) = [P, Q]_e$ is the resultant $[P, Q]_e = P^e Q_e - P_e Q^e$. We use the shorthand notations $P^e = [x_e^1] P$ and $P_e = [x_e^0] P$ for the coefficients of a polynomial P that is linear in x_e .

Denominators starting from the third determine c_2 by taking the point count:

$$c_2^{(q)}(G) \equiv (-1)^r \llbracket D_G^r(e_1, \dots, e_r) \rrbracket_q \pmod{q}$$

is shown in [BS12, Theorem 29]. The first case $r = 3$ is used in the proof of Lemma 6.11.

Let G (left-hand side), G_s , G_t , G_u be the four graphs in the order illustrated in (6.4). As described at the beginning of [BSY14, §6.1], the square roots of the second denominators after reducing the two explicitly drawn edges of G_s , G_t , and G_u are given by $(A - B)$, $(B - C)$, and $(C - A)$ respectively where A , B , and C are certain explicit spanning forest polynomials. Furthermore letting e be any edge in the common part of G , G_s , G_t , and G_u , by [BSY14, Lemma 42], the denominator after reducing the 4 illustrated edges of G along with edge e is equal to

$$D_G^5 = [A, B]_e + [B, C]_e + [C, A]_e.$$

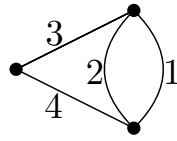


Figure 6.2: This graph shows the necessity of requiring at least 4 vertices in Lemma 6.16.

For the third denominator for G_s we get

$$(A - B)^e(A - B)_e = A^e A_e + B^e B_e - A^e B_e - A_e B^e = A^e A_e + B^e B_e - [A, B]_e - 2A_e B^e.$$

Adding the third denominators for $G_s, G_t,$ and G_u we get

$$\begin{aligned} & 2A^e A_e + 2B^e B_e + 2C^e C_e - [A, B]_e - [B, C]_e - [C, A]_e - 2A_e B^e - 2B_e C^e - 2C_e A^e \\ & \equiv [A, B]_e + [B, C]_e + [C, A]_e \pmod{2}. \end{aligned}$$

So the fifth denominator of G equals modulo 2 the sum of the third denominators of $G_s, G_t,$ and $G_u,$ but since denominators from the third on up compute $c_2,$ by taking the diagonal coefficient (point count \pmod{p}) we obtain

$$c_2^{(2)}(G) \equiv c_2^{(2)}(G_s) + c_2^{(2)}(G_t) + c_2^{(2)}(G_u) \pmod{2}$$

as desired. This provides a very efficient recurrence to compute $c_2^{(2)}$ and gives a short alternative proof of [HY23]. Discovering this relation between the Martin invariant and c_2 at $p = 2$ was the starting point that ultimately led to the result below for arbitrary primes.

6.1. Primes $p \neq 3$ from the diagonal

Lemma 6.16. *Consider a graph G with $n \geq 4$ vertices and $m = 2(n - 1)$ edges. Let $p = r + 1$ be prime. Then the diagonal coefficient of Ψ_G^{2r} fulfils*

$$[x_1^r \cdots x_m^r] \Psi_G^{2r} \equiv -3p^2 \cdot c_2^{(p)}(G) \pmod{p^3}. \tag{6.5}$$

Remark 6.17. The lemma fails for graphs with less than 4 vertices: The graph G in Figure 6.2 has $[\Psi_G]_q = q^3$ and thus $c_2^{(p)}(G) \equiv 0,$ whence the right-hand side is $0 \pmod{p^3}.$ This point-count of $\Psi_G = x_1 x_2 + (x_1 + x_2)(x_3 + x_4)$ is obtained using [Ste98, Sch11]. The diagonal coefficient of Ψ_G^{2r} on the left-hand side, however, is equal to $\binom{2r}{r}^2 \equiv p^2 \pmod{p^3}.$

Proof. Pick a 3-valent vertex $w,$ let 1, 2, 3 be its incident edges, and let a, b, c denote the vertices on the other ends of these edges. Then the spanning forest polynomials denoted

$$f_0 = \Psi_{G \setminus w}, \quad f_1 = \Phi_{G \setminus w}^{\{a\}, \{b, c\}}, \quad f_2 = \Phi_{G \setminus w}^{\{a, c\}, \{b\}}, \quad f_3 = \Phi_{G \setminus w}^{\{a, b\}, \{c\}}, \quad f_{123} = \Phi_{G \setminus w}^{\{a\}, \{b\}, \{c\}}$$

are subject to $f_0 f_{123} = f_1 f_2 + f_1 f_3 + f_2 f_3$ [BS12, Lemma 22]. Furthermore, we have

$$\Psi_G = f_0(x_1 x_2 + x_1 x_3 + x_2 x_3) + x_1(f_2 + f_3) + x_2(f_1 + f_3) + x_3(f_1 + f_2) + f_{123}.$$

This implies $f_0\Psi_G = y_1y_2 + y_1y_3 + y_2y_3$ with $y_i = f_0x_i + f_i$. Expanding the trinomial,

$$f_0^{2r}\Psi_G^{2r} = \sum_{i+j+k=2r} \binom{2r}{i, j, k} (f_0x_1 + f_1)^{j+k} (f_0x_2 + f_2)^{i+k} (f_0x_3 + f_3)^{i+j}$$

and then extracting the coefficient of $x_1^r x_2^r x_3^r$, we see that if $f_0 \neq 0$ then we can cancel $2r$ powers of f_0 from each side and get

$$[x_1^r x_2^r x_3^r] \Psi_G^{2r} = f_0^r \sum_{i+j+k=2r} \binom{2r}{i, j, k} \binom{i+j}{r} \binom{i+k}{r} \binom{j+k}{r} f_1^{r-i} f_2^{r-j} f_3^{r-k}.$$

This identity remains true when $f_0 = 0$, because in this case, the left-hand side vanishes because Ψ_G^{2r} has degree $2r$ in the variables x_1, x_2, x_3 .

Note that the first binomial coefficient is always divisible by p . The second binomial is divisible by p when $k < r$ (so that $r < i + j = 2r - k$), and similarly the remaining binomials are divisible by p when $j < r$ and $i < r$, respectively.

In conclusion, the summand is divisible by p^3 unless two summation indices are equal to r . Therefore, only the indices $(i, j, k) \in \{(0, r, r), (r, 0, r), (r, r, 0)\}$ contribute $\pmod{p^3}$:

$$\begin{aligned} [x_1^r x_2^r x_3^r] \Psi_G^{2r} &\equiv \binom{2r}{r}^2 ((f_0 f_1)^r + (f_0 f_2)^r + (f_0 f_3)^r) \pmod{p^3} \\ &\equiv p^2 ((f_0 f_1)^r + (f_0 f_2)^r + (f_0 f_3)^r) \pmod{p^3}. \end{aligned}$$

Now take the diagonal coefficient $[x_4^r \cdots x_m^r]$ in the remaining variables, on both sides. Since each of the three products $f_0 f_i$ is the polynomial on the right-hand side of Lemma 6.11 for some permutation of $\{a, b, c\}$, the claim follows. \square

This result converts the diagonal coefficient of the more complicated polynomials of Lemma 6.11 into the diagonal coefficient of Ψ^{2r} —at least modulo p^3 . For $p \neq 3$, by using (4.12) (or the combinatorial argument of Remark 5.7) to move between Ψ and $\tilde{\Psi}$ and using Lemma 5.8 to move between G and $G^{[r]}$, we see that Lemma 6.16 and Theorem 5.17 together give the desired relationship between the Martin invariant and the c_2 invariant. This proves Theorem 6.2 for all primes $p \neq 3$.

Unfortunately, the explicit factor 3 in Lemma 6.16 does not allow us to draw the same conclusion when $p = 3$. In this case we would need the congruence in Lemma 6.16 to hold modulo p^4 . Lacking a direct method to establish this congruence, we prove Theorem 6.2 for $p = 3$ using a different approach, detailed in the following subsections. The essence of this second proof is to avoid the troublesome explicit 3 appearing in Lemma 6.16 by instead recreating the argument used in Theorem 5.17 directly on the polynomials of Lemma 6.11 using Lemma 6.14. This second proof applies for all primes, although even there the case $p = 3$ requires some special treatment in the base case of the induction.

As a consequence of this second proof, having established Theorem 6.2 for $p = 3$ we can infer indirectly that indeed, for $p = 3$ the statement of Lemma 6.16 holds modulo p^4 , provided that G has at least 8 edges and 5 vertices.²¹

²¹For 6 edges, a counter-example is discussed after Lemma 6.35.

Remark 6.18. The c_2 invariant and the diagonal coefficient of $\tilde{\Psi}$ are defined for arbitrary graphs (with at least 5 edges and 4 vertices, say). It thus makes sense to ask if the statement of Lemma 6.16 holds more generally. Note that our proof requires the existence of a 3-valent vertex, which may not exist in more general graphs. Almost nothing is known, however, for c_2 invariants for graphs without 3-valent vertices. The only paper making a dent in this direction is [Dor17b].

6.2. Structure of the proof for all primes

Our second proof of Theorem 6.2 proceeds by showing that the diagonal coefficient $N_{r,r}$ from Lemma 6.14 is congruent to the Martin invariant. Concretely, we will show that

$$N_{r,r}(G \setminus \{v, w\}) \equiv \frac{M(G)}{3p} \pmod{p} \tag{6.6}$$

for every prime $p = r + 1$ and every $4r$ -regular graph G with at least 6 vertices, provided w is a vertex with four neighbours $\{a, b, c, v\}$, each connected to w by an r -fold multiedge. Applied to the special case where G is the r -fold duplication of a 4-regular graph, we obtain Theorem 6.2 as the consequence of Lemma 6.14. This approach works for all primes p and thus fills the gap at $p = 3$, which is not covered by the method of Section 6.1.

The key step to prove (6.6) is to establish the Martin recurrence (1.1) for the left-hand side. Like the diagonal coefficient N_k of Ψ^k from Section 5—see (5.5)—the diagonal coefficient $N_{r,r}$ also satisfies the Martin recurrence *exactly*: For a vertex $u \notin \{v, w, a, b, c\}$,

$$N_{r,r}(G \setminus \{v, w\}) = \sum_{\tau \in \mathcal{T}'(u)} N_{r,r}(G_\tau \setminus \{v, w\}) \tag{6.7}$$

is the sum over all transitions at u which do not produce a self-loop at v . The residue modulo p in (6.6) is needed only to connect $N_{r,r}$ with $M(G)$ in the base cases (Section 6.4), and with the c_2 invariant when G is the r -fold duplication of a 4-regular graph.

At its core the idea to prove (6.7) is the same as for Theorem 5.17, namely we interpret a transition as going from sets counted by A (5.3) to those counted by B (Definition 5.12) and then use the Prüfer based Proposition 5.15. However, this is more intricate for $N_{r,r}$ than for N_k , since we are now counting partitions of the edges into r trees and r forests consisting of two trees each, one containing $\{a, b\}$ and the other containing $\{c\}$. We thus need to keep track of where the marked vertices $\{a, b, c\}$ are. This requires augmenting the constructions of the proof of Theorem 5.17 by 3 coloured vertices.

As in the proof of Theorem 5.17, applying the recurrence (1.1) will lead us outside the world of r -duplicated graphs and we need to consider arbitrary graphs with the same vertex degrees. To this end, by H we denote a graph with

- two vertices (a and b) marked with one colour, say red, and one (c) with a second colour, say blue, all three of which have degree at most $3r$,
- all other vertices have degree at most $4r$, and

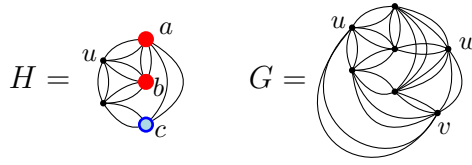


Figure 6.3: The graphs for a running example. The large filled vertices, a and b , are the two red vertices and the large empty vertex, c , is the blue vertex.

- the number of edges is $r(2n - 3)$, where $n \geq 4$ is the total number of vertices of H .

Then let G be the graph obtained from H by adding a new vertex w with r edges to each of the three coloured vertices of H , then adding a new vertex v with r edges to w and edges to each remaining vertex of degree less than $4r$ so as to raise the degree to $4r$, and finally forgetting the colours. The above conditions on H ensure that G is $4r$ -regular, v and w have no self-loops, and $H = G \setminus \{v, w\}$. Setting $H_\tau = G_\tau \setminus \{v, w\}$, we can state (6.7) as $N_{r,r}(H) = \sum_\tau N_{r,r}(H_\tau)$.

Consequently, the structure of our inductive proof is as follows: we first show the Martin recurrence (6.7) for the number of partitions of edges of H into r spanning trees and r spanning forests compatible with the colouring in Section 6.3. Then in Section 6.4 we compute these numbers when H has only the three coloured vertices, and compare them with the Martin invariant of the corresponding graph G . This completes our second proof of Theorem 6.2 and in particular proves the theorem in the case $p = 3$.

6.3. Proof of the recurrence for all primes

Let H , G , v , w , and r be as in Section 6.2. In this section, we prove the recurrence (6.7).

Recall the definitions of h -neighbourhood and v -neighbourhood from Section 2. Throughout this section, let u be an uncoloured vertex of H , and S its h -neighbourhood in H .

Example 6.19. We will use as a running example in this section the graph $H = G \setminus \{v, w\}$ illustrated on the left-hand side of Figure 6.3. In this case G is obtained from a doubled 4-regular graph; this is not necessary for the proof, but the application we care about.

Given a partition of the edges of H into r spanning trees and r spanning forests compatible with the vertex colours, consider what happens when we remove u . As in the proof of Theorem 5.17, for each spanning tree, if u has degree k in the tree then upon removing u the tree is split into k trees and this induces a partition of S into k parts. For each spanning forest, the vertex u is in one of the trees of the forest and if u has degree k in that tree, then upon removing u this tree is split into k trees and the other tree of the forest is unchanged, leaving a spanning forest of $k + 1$ trees of $H \setminus u$. There are then two possibilities for the resulting induced partition of the h -neighbourhood S of u : if all the vertices of the v -neighbourhood of u were originally in the same tree as u , then locally around u we only see one tree and so we get a partition of the h -neighbourhood S of u into k parts as before. However, if some of the v -neighbourhood of u was originally in the other tree, then there will additionally be a $(k + 1)$ th part in the induced partition of the h -neighbourhood corresponding to the other tree. In order to keep track of this we will need to introduce coloured partitions in this case.

We will first consider the case where for all the r spanning forests the whole v -neighbourhood of u is in the same tree. Here the local situation is exactly as for Theorem 5.17 and so the same argument applies. We now briefly go through this argument again in order to set notation that will be useful when both trees interact around u .

Given a partition P of the h -neighbourhood of u , say a spanning forest is compatible with both P and the vertex colouring if it has $|P| + 1$ trees, one of which involves all vertices of one colour and no vertices of P or of the other colour and where the remaining $|P|$ trees are compatible with P . As observed in the proof of Theorem 5.17, given a partition P be a partition of the h -neighbourhood of u , any spanning forest of $H \setminus u$ compatible with P can be extended to a spanning tree T by choosing a system of distinct representative (SDR, as defined in Section 5.1) for P and then adding the edges containing each of these half edges in H to T ; likewise for any spanning forest of $H \setminus u$ compatible with P and the vertex colouring. Every spanning tree or spanning forest which induces the partition P on the h -neighbourhood of u is obtained in this way.

We call such an SDR a u -extension of P .

For the other side of the Martin recurrence, given a transition τ of G which does not create self-loops at v , write H_τ for the graph obtained by applying the transition restricted to H and removing the unmatched half-edges (those that were matched to v in G). On each H_τ we are then counting partitions of the edges into r spanning trees and r spanning forests compatible with the coloured vertices. By breaking the edges matched by τ , each such tree or forest induces a partition on the v -neighbourhood of u . For such a spanning forest, if all of the v -neighbourhood of u is in the same tree of the forest, then locally we are only working with trees and so as in Theorem 5.17 for any partition P of the h -neighbourhood of u with at most $\deg(u) - 2r + 1$ parts, any matching that defines a tree on P in the sense of Definition 5.12 serves to extend any spanning forest of $H \setminus u$ compatible with P or compatible with P and the vertex colouring. Furthermore, every such matching comes from at least one transition τ with no self loops at v since the constraint on the number of parts guarantees enough half edges left to connect to v without loops, and every spanning tree or spanning forest with all of the v -neighbourhood of u is obtained in this way.

We call such a choice of matching an m -extension of P .

Let S be the h -neighbourhood of u in H and let P_1, \dots, P_{2r} be partitions of S . For $H' = H$ or $H' = H_\tau$ for some transition at u , write

$$N_{H'}(P_1, \dots, P_{2r})$$

for the number of partitions of the edges of H' into r spanning trees and r spanning forests compatible with the coloured vertices so that the i th spanning tree induces the same partition on the v -neighbourhood of u that P_i does and the j th spanning forest induces the same partition on the v -neighbourhood of u that P_{r+j} does and only one tree of the spanning forest is involved in the induced partition. Then what we need from Theorem 5.17 is the following:

Proposition 6.20. *Let P_1, \dots, P_{2r} be partitions of the h -neighbourhood of u in H . Then*

$$N_H(P_1, \dots, P_{2r}) = \sum_{\tau} N_{H_\tau}(P_1, \dots, P_{2r})$$

where the sum is over transitions τ of u in G which do not create self loops at v .

Proof. Consider spanning forests F_i compatible with P_i on $H \setminus u$ for each $1 \leq i \leq r$ and spanning forests F_j for each $p \leq i \leq 2r$ so F_j has $|P_j| + 1$ trees of which $|P_j|$ of them are compatible with P_j and also contain the vertices of one of the two colours and the last tree contains the vertex of the other colour. Let $C(P_1, \dots, P_{2r})$ be the set of possible choices for the F_i .

If $\sum_{i=1}^{2r} |P_i| = \deg(u)$ and each partition has at most $\deg(u) - 2r + 1$ parts, then as in Theorem 5.17

$$N_H(P_1, \dots, P_{2r}) = |C(P_1, \dots, P_{2r})| \cdot |A(P_1, \dots, P_{2r})|$$

and collecting the sets of m -extensions by their z s,

$$\sum_{\tau} N_{H_{\tau}}(P_1, \dots, P_{2r}) = |C(P_1, \dots, P_{2r})| \cdot |B(P_1, \dots, P_{2r})|$$

where the sum is over transitions τ of u in G which do not create self loops at v . Here we have used that for τ to have no self loops at v we must have

$$|z| = \deg(u) - \sum_{i=1}^r 2(|P_i| - 1) = 4r - \deg(u)$$

which is the same condition as $\sum_{i=1}^{2r} |P_i| = \deg(u)$. Applying Proposition 5.15

$$N_H(P_1, \dots, P_{2r}) = \sum_{\tau} N_{H_{\tau}}(P_1, \dots, P_{2r})$$

as desired.

Finally if $\sum_{i=1}^{2r} |P_i| \neq \deg(u)$ or some partition has more than $\deg(u) - 2r + 1$ parts then both sides are 0: the left side because the u -extensions are SDRs and each P_i has at least one part and the right side because all compatible transitions would have self loops at v . \square

Example 6.21. Let us continue Example 6.19 with a case where Proposition 6.20 applies. Write $S = \{1, 2, 3, 4, 5, 6\}$ for the h -neighbourhood of u where half edges 1 and 2 are incident to vertex a , half edges 3 and 4 are incident to vertex b , and half edges 5 and 6 are incident to the unnamed fifth vertex of H . One possible partition of the edges into two spanning forest compatible with the colouring and two spanning trees is given in Figure 6.4. The corresponding partitions of S are $P_1 = P_2 = \{\{1, 2, 3, 4, 5, 6\}\}$ and $P_3 = P_4 = \{\{1, 2, 5, 6\}, \{3, 4\}\}$, which we can also visualize locally around u in the same manner as in the examples of Section 5.1, as shown in Figure 6.5.

These u -extensions are given by the function $f: 1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 3, 4 \mapsto 4, 5 \mapsto 3, 6 \mapsto 2$. Then under Proposition 6.20 these u -extensions correspond to the matching of Figure 6.6, where using the notation of the Prüfer bijection we have $y = (1, 6)$, $X_3 = \{3, 5\}$, $X_4 = \{2, 4\}$. The transition τ in this case is the one pairing $\{2, 4\}$ and $\{3, 5\}$, while 1 and 6 each get paired to v in G and so are no longer present in H_{τ} . The partition of the edges of H_{τ} into the corresponding two spanning forests compatible with the colouring and two spanning trees is illustrated in Figure 6.7.

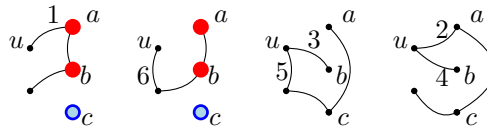


Figure 6.4: A partition of the edges of H into two spanning forests compatible with the colouring and two spanning trees.

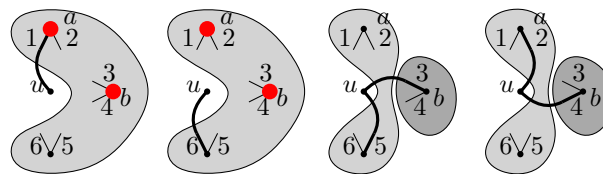


Figure 6.5: A local view of the partitions around u obtained from Figure 6.4.

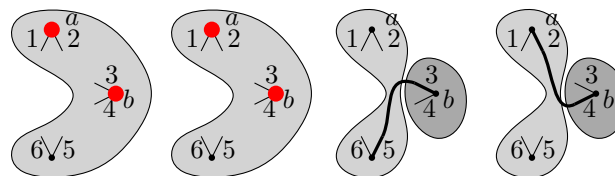


Figure 6.6: The corresponding matching of the h -neighbourhood of u .

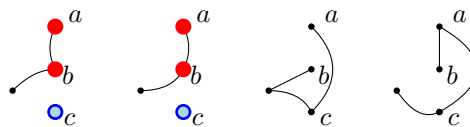


Figure 6.7: The corresponding partition of the edges of H_τ into two spanning forests compatible with the colouring and two spanning trees.

Now we extend Proposition 6.20 and the associated set-up to the case where the v -neighbourhood of u may involve both trees in one or more of the spanning forests. For such a spanning forest, the vertices of the v -neighbourhood of u which were originally in the same tree as u are now partitioned into k parts, but there may additionally be a part containing vertices in the v -neighbourhood of u which are in the other tree of the spanning forest. Furthermore, if the tree corresponding to the red vertices is the tree which contains u , then the two red vertices may be in the same tree after removing u or they may be in two different trees after removing u . Thus, in the case that not all vertices of the v -neighbourhood of u are in the same tree of the spanning forest, we need to mark the parts of the partition of the v -neighbourhood obtained by removing u according to whether or not the tree associated to that part (after removing u) contained a red vertex or a blue vertex. This will result in one or two red parts, one blue part, and any other parts uncoloured. The partition of the h -neighbourhood is induced from the partition of the v -neighbourhood and also inherits the colours. Call this a *coloured partition*.

The extensions are now more intricate to describe:

Lemma 6.22. *Let P be a coloured partition of the h -neighbourhood of a vertex u of H (that is P has one or two parts marked as red, one part marked as blue, and the rest uncoloured). Any spanning forest of $H \setminus u$ compatible with P can be extended to a spanning forest F of H where the red part(s) are in one tree and the blue part in the other tree by*

- *choosing one half edge from each red part of P and each unmarked part of P and then adding the edges containing each of these half edges in H to F , or*
- *if there is only one red part in P , choosing one half edge from the blue part of P and from each unmarked part of P and then adding the edges containing each of these half edges in H to F .*

Every spanning forest of H which is compatible with the coloured vertices of H and which induces the coloured partition P on the h -neighbourhood of u is obtained in this way.

We again call such a choice of edges a u -extension of the coloured partition P .

Proof. In any case, all uncoloured components must join through u as otherwise we would have 3 or more trees in the final forest. If there are two red parts then they must join through u in order to be connected and so only the first option is possible. With one red and one blue part we are free to join either one through u .

For coloured and uncoloured parts, the ways to join through u are as in the tree case. □

Now consider the other side of a Martin-type recurrence. Given a transition τ of u in G which does not create self-loops around v and with H_τ as before. A spanning forest of H_τ again results in a partition of the vertices of H and hence the h -neighbourhood of u and if both trees of the forest appear in the h -neighbourhood of u then we need to mark the parts of the partition of the h -neighbourhood of u red or blue if their tree contained a red or blue vertex after splitting all edges of the matching, resulting in a coloured partition of the h -neighbourhood of u .

Again we only need this local information to extend forests.

Lemma 6.23. *Let u be an uncoloured vertex of H and let P be a coloured partition of the h -neighbourhood of u with at most $\deg(u) - 2r + 2$ parts. Any spanning forest of $H \setminus u$ compatible with P can be extended to a spanning forest F of some H_τ where the red part(s) are in one tree and the blue part in the other tree by choosing a forest structure X where the vertex set of X is the set of parts of P and the forest is compatible with the colours coming from the parts of P , and then for each end of an edge of X choosing a distinct half edge from the corresponding part of P .*

Every spanning forest of every H_τ which is compatible with the coloured vertices of H and which induces the coloured partition P on the h -neighbourhood of u is obtained in this way.

We again call such a choice of edges an m -extension of the coloured partition P .

Proof. As for the tree case but so as to obtain a compatible spanning forest. The condition on the number of parts likewise makes it always possible for τ to avoid self-loops at v where the spanning forest structure on X uses one fewer edge than the tree structure did. \square

Extend the meaning of $N_H(P_1, \dots, P_{2r})$ where P_p, \dots, P_{2r} may be coloured to be the number of partitions of the edges of H into r spanning trees and r spanning forests compatible with the coloured vertices so that the i th spanning tree induces the same partition on the v -neighbourhood of u that P_i does and the j th spanning forest induces the same partition on the v -neighbourhood of u that P_{r+j} does, with the inherited colouring if multiple trees of the forest interact with the v -neighbourhood of u .

When some of the P_i may be coloured partitions, we bootstrap off Proposition 6.20.

Proposition 6.24. *Let P_1, \dots, P_{2r} be partitions of the h -neighbourhood of u where for $p \leq i \leq 2r$ the partition P_i may be a coloured partition. Then*

$$N_H(P_1, \dots, P_{2r}) = \sum_{\tau} N_{H_\tau}(P_1, \dots, P_{2r})$$

where the sum is over transitions τ of u in G which do not create self loops at v .

Proof. Proposition 6.20 gives the result when all of the partitions are uncoloured and so we now reduce to that case.

With P_i as in the statement, let $A(P_1, \dots, P_{2r})$, generalizing (5.3), denote the set of u -extensions for each P_i (in the coloured sense where appropriate) which partition the edges of H around u . Likewise, let $B(P_1, \dots, P_{2r})$, generalizing Definition 5.12, denote the set of m -extensions for each P_i (in the coloured sense where appropriate) for which the union of the matchings gives the matching of τ on S .

We will show that

$$|A(P_1, \dots, P_{2r})| = |B(P_1, \dots, P_{2r})|. \tag{6.8}$$

Multiplying by the number of possible choices for the forests in $H \setminus u$ which give P_i will then give the result.

Suppose P_i is a coloured partition with one red part and one blue part. For any u -extension of this coloured partition we take either an edge from the red part or an edge from the blue part

and that choice is independent of the choices for the other parts. Therefore the u -extensions of the coloured partition P_i are the same as the u -extensions of the uncoloured partition resulting from merging the red and blue parts of P_i and then forgetting the colours. Similarly for any m -extension of P_i , we obtain a spanning forest on P_i with two trees, one containing the vertex associated to the red part and one containing the vertex associated to the blue part. Merging the red and blue parts then the same edge and end information gives an m -extension of the uncoloured partition obtained from merging the red and blue parts and forgetting the colour and every such tree becomes a forest and a valid m -extension upon splitting this part to reobtain P_i . If P_i had no more than $\deg(u) - 2r + 2$ parts before merging then it has no more than $\deg(u) - 2r + 1$ parts after, and vice versa.

Therefore, for the purposes of (6.8), whenever there is a coloured partition with one red and one blue part, merge those two parts and proceed with an uncoloured partition instead.

It remains to consider partitions with two red parts and one blue part. Suppose P_i is such a partition and let p be the blue part and q and s the two red parts. Similarly to the above, every u -extension of P_i is also a u -extension after merging p and q and forgetting the colours, and is also a u -extension after merging p and s and forgetting the colours. However, merging p and q also allows u -extensions which use an edge from p and an edge from s , and merging p and s also allows u -extensions which use an edge from p and an edge from q . These are exactly the u -extensions obtained from merging q and s and forgetting the colours. Therefore, indicating the merged parts as superscripts, for example $P_i^{p \cup q} = (P_i \setminus \{p, q\}) \cup \{p \cup q\}$, we see that

$$\begin{aligned} & 2|A(P_1, \dots, P_i, \dots, P_{2r})| \\ &= |A(P_1, \dots, P_i^{p \cup q}, \dots, P_{2r})| + |A(P_1, \dots, P_i^{p \cup s}, \dots, P_{2r})| - |A(P_1, \dots, P_i^{q \cup s}, \dots, P_{2r})| \end{aligned}$$

where the counts on the right-hand side involve one coloured partition fewer than on the left-hand side.

The situation is similar for the m -extensions. Every m -extension of P_i is also an m -extension after merging p and q and forgetting the colours, and is also an m -extension after merging p and s and forgetting the colours. However, merging p and q also allows m -extensions where s is in the tree with p after re-splitting p and q , and merging p and s also allows m -extensions where s is in the tree with q after re-splitting p and s . These are exactly the m -extensions obtained from merging q and s and forgetting the colours. The bounds on the numbers of parts correspond as before. Therefore we have

$$\begin{aligned} & 2|B(P_1, \dots, P_i, \dots, P_{2r})| \\ &= |B(P_1, \dots, P_i^{p \cup q}, \dots, P_{2r})| + |B(P_1, \dots, P_i^{p \cup s}, \dots, P_{2r})| - |B(P_1, \dots, P_i^{q \cup s}, \dots, P_{2r})| \end{aligned}$$

which lines up term by term with what was obtained for the u -extensions.

Similarly expanding for each remaining coloured partition, we can write $A(P_1, \dots, P_{2r})$ and $B(P_1, \dots, P_{2r})$ as identical signed sums involving only uncoloured partitions of S . Therefore the expansions agree term by term by Proposition 5.15, proving (6.8) and hence proving the proposition. \square

Summing both sides of Proposition 6.24 over all choices of the partitions P_i , we obtain the Martin recursion $N_{r,r}(H) = \sum_{\tau} N_{r,r}(H_{\tau})$ as claimed in Equation (6.7).

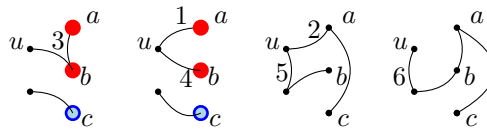


Figure 6.8: Another partition of the edges of H into two spanning forests compatible with the colouring and two spanning trees.

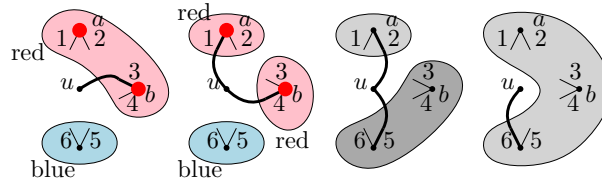


Figure 6.9: A local view of the partitions around u obtained from Figure 6.8.

Example 6.25. Let us continue Example 6.21 with a case where Proposition 6.24 applies. One possible partition of the edges into two spanning forest compatible with the colouring and two spanning trees for which Proposition 6.24 is needed is given in Figure 6.8. The corresponding partitions of S are $P_1 = \{\{1, 2, 3, 4\}_{\text{red}}, \{5, 6\}_{\text{blue}}\}$, $P_2 = \{\{1, 2\}_{\text{red}}, \{3, 4\}_{\text{red}}, \{5, 6\}_{\text{blue}}\}$, $P_3 = \{\{1, 2\}, \{3, 4, 5, 6\}\}$, and $P_4 = \{\{1, 2, 3, 4, 5, 6\}\}$, with the colours indicated by the subscripts. We can visualize this situation locally around u as shown in Figure 6.9.

Now we need to convert to uncoloured partitions in order to reduce to cases where Proposition 6.20 applies. Consider first P_1 . As described in the proof of Proposition 6.24, in this case we can simply merge the red and blue parts obtaining in this case the partition with only one part. P_2 is more complicated. Label the parts as follows $P_2 = \{p, q, s\}$ with $s = \{1, 2\}_{\text{red}}$, $q = \{3, 4\}_{\text{red}}$, $p = \{5, 6\}_{\text{blue}}$. Considering both the uncoloured partitions with $p \cup q$ and $p \cup s$ we see that the spanning trees and forests we started with are consistent with both, as illustrated in Figure 6.10. Note that every u -extension of the coloured partition of Figure 6.9 also appears as a u -extension for each of the partitions of Figure 6.10. However, some u -extensions of each of the partitions of Figure 6.10 do not give u -extensions of Figure 6.9. For example taking edges 1 and 6 satisfies the requirements for P_2 with $p \cup q$ as in the upper set of partitions of Figure 6.10 but not in Figure 6.9. However, 1 and 6 does work upon taking $q \cup s$. This is an example of the general fact that $q \cup s$ gives exactly the terms that need subtracting, as described in the proof of Proposition 6.24.

Under Proposition 6.20, the two SDRs of Figure 6.10 correspond to the two matchings of Figure 6.11. The transition τ in both cases is the one pairing $\{1, 4\}$ and $\{2, 5\}$ while 3 and 6 each get paired to v in G and so are no longer present in H_τ . The argument to reduce coloured partitions to partitions by merging parts and subtracting for overcounting is the same as it was in H . In this case we obtain the matching of the coloured partitions illustrated in Figure 6.12 and hence the partition of the edges of H_τ into two spanning forests compatible with the colouring and two spanning trees as illustrated in Figure 6.13.

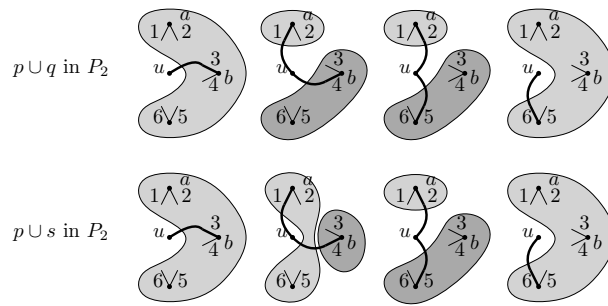


Figure 6.10: The partitions around u after merging as described in Proposition 6.24.

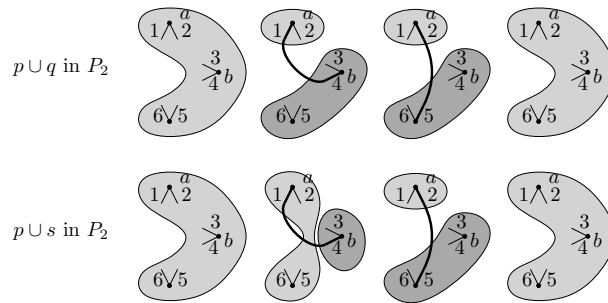


Figure 6.11: The corresponding matchings.

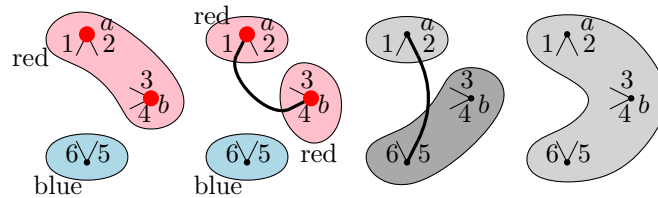


Figure 6.12: The corresponding matching of the h -neighbourhood of u .

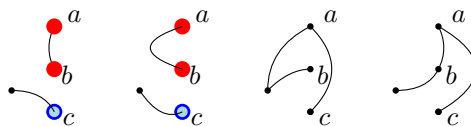


Figure 6.13: The corresponding partition of the edges of H_τ into two spanning forests compatible with the colouring and two spanning trees.

6.4. Base cases

Recall that $p = r + 1$ is prime. Given a graph H with three marked vertices called $\{a, b, c\}$, with $(2|V(H)| - 3)r$ edges, and with no vertex of degree more than $4r$, recall from Section 6.2 that H determines a $4r$ -regular graph G by adding two vertices, labelled $\{v, w\}$, adding r edges between w and each of $\{a, b, c, v\}$, and finally adding $3r$ edges between v and the original vertices of H in the unique way such that the resulting graph G is $4r$ -regular. With this construction, $H = G \setminus \{v, w\}$.

Recall that $N_{r,r}(H)$ from Definition 6.13 counts the ordered partitions of the edges of H into r spanning trees and r spanning forests compatible with the partition $\{a, b\}, \{c\}$.

For the base case of our inductive proof, we need to show that

$$N_{r,r}(H) \equiv \frac{M(G)}{3p} \pmod{p} \tag{6.9}$$

for all marked graphs H as above on a fixed small number of vertices. The argument is easier for G and H smaller, and so for $p \neq 3$ we will prove (6.9) for H with 3 vertices (only the marked vertices) and hence G with 5 vertices. For $p = 3$ we will need to move to H with 4 vertices and hence G with 6 which remains sufficient for Theorem 6.2.

Suppose, then, that H is a marked graph with only 3 vertices—namely, the marked vertices—and $3r$ edges, and G as above.

Consider first the left-hand side.

Lemma 6.26. *For any graph $H = G \setminus \{v, w\}$ with 3 vertices and $3r$ edges and $p = r + 1$ prime and G as explained above, we have*

$$N_{r,r}(H) \equiv \begin{cases} -1 \pmod{p}, & \text{if } G \cong K_5^{[r]}, \text{ and} \\ 0 \pmod{p}, & \text{otherwise.} \end{cases} \tag{6.10}$$

Proof. Any self-loop in H cannot belong to any tree or forest, so the presence of self-loops forces $N_{r,r}(H) = 0$. Thus suppose that H has no self-loops. Let m_{ij} denote the number of edges between vertices i and j . The 2-forests of H that separate $\{a, b\}$ from $\{c\}$ consist of a single edge between a and b . Hence there are precisely $\binom{m_{ab}}{r}$ ways to assign r of these edges to 2-forests. The remaining edges are to be grouped into α spanning trees of the form $\{ab, ac\}$, β spanning trees $\{ab, bc\}$, and γ spanning trees $\{ac, bc\}$. The constraints $m_{ab} - r = \alpha + \beta$, $m_{ac} = \alpha + \gamma$, and $m_{bc} = \beta + \gamma$ have the unique solution $\alpha = r - m_{bc}$, $\beta = r - m_{ac}$, $\gamma = 2r - m_{ab}$ where we note $\alpha + \beta + \gamma = r$. After choosing α out of the $m_{ab} - r$ edges ab and α out of the m_{ac} edges ac , there are $\alpha!$ ways to pair them up into spanning trees of the first type, etc. Finally, accounting for the $r!$ ways to order the spanning trees and 2-forests, we find

$$N_{r,r}(H) = r! \binom{m_{ab}}{r} r! \alpha! \beta! \gamma! \binom{m_{ab} - r}{\alpha} \binom{m_{ac}}{\alpha} \binom{m_{bc}}{\beta} = \frac{r! \cdot m_{ab}! \cdot m_{ac}! \cdot m_{bc}!}{\alpha! \beta! \gamma!}. \tag{6.11}$$

We need $m_{ab} \geq r$ for $N_{r,r}(H)$ to be non-zero. Then $\gamma \leq r$ and we always have $\alpha, \beta \leq r$, hence the denominator is not divisible by p . So the numerator, and hence $N_r(H)$, become divisible by p if any $m_{ij} > r$. For this not to happen, we need each $m_{ij} \leq r$. But then we must

have $m_{ab} = m_{ac} = m_{bc} = r$, due to the constraint $m_{ab} + m_{ac} + m_{bc} = 3r$. Then all edges in G are r -fold, and we get $N_{r,r}(H) = (r!)^3 \equiv -1 \pmod{p}$ by Wilson's theorem. \square

We now turn to the right-hand side of (6.9).

Consider the effect of multiedges on the expansion (1.1) of the Martin invariant at a vertex u without self-loops. Let w_1, \dots, w_s denote the neighbours of u , and say that there are d_i edges uw_i between u and w_i . Then G_τ depends only on the numbers D_{ij} of pairs $\{uw_i, uw_j\}$ in τ . Hence different transitions with the same numbers D_{ij} will repeat the same contributions $M(G_\tau)$.

Definition 6.27. Let D be a symmetric matrix D of non-negative integers with diagonal $D_{ii} = 0$ and row sums $\sum_j D_{ij} = d_i$ equal to the multiplicities of the edges between u and its neighbours w_i . We write $G_D = G_\tau$ for any transition $\tau \in \mathcal{T}(u)$ with precisely D_{ij} matches $\{uw_i, uw_j\}$ between any pair of neighbours. We also introduce the notation

$$d! = d_1! \cdots d_s! \quad \text{and} \quad D! = \prod_{1 \leq i < j \leq s} (D_{ij}!).$$

We only consider zero diagonals, because $D_{ii} > 0$ would produce self-loops in G_D , which thus do not contribute to $M(G)$. The row sum constraint arises because a transition has to pair up all half-edges at u .

Lemma 6.28. Let G be a regular graph without self-loops and u any vertex. Let $d = (d_1, \dots, d_s)$ denote the edge multiplicities at u . Then

$$M(G) = \sum_D \frac{d!}{D!} M(G_D) \tag{6.12}$$

where the sum is over all matrices D as in Definition 6.27. The coefficients are integers,

$$\frac{d!}{D!} = D! \prod_{i=1}^s \binom{d_i}{D_{i1}, \dots, D_{is}}. \tag{6.13}$$

Proof. With the notation introduced above, this is just a rewriting of the vertex expansion (1.1), including only transitions that do not produce self-loops in G_τ . The coefficient of $M(G_D)$ is the number of transitions with the number of pairings prescribed by D . For every neighbour w_i , a transition partitions the d_i edges uw_i into a part of size D_{i1} (which is paired with an edge uw_1), a part of size D_{i2} (to be paired with uw_2), and so on. These partitions are counted by the multinomial coefficients in (6.13). The remaining factors $D_{ij}!$ in $D!$ count the ways to pair the thus chosen D_{ij} copies of uw_i with the $D_{ji} = D_{ij}$ chosen copies of uw_j into the new edges $\{uw_i, uw_j\}$ in G_D . \square

Example 6.29. Let $K_4^{[\alpha, \beta, \gamma]}$ denote the graph on 4 vertices $\{1, 2, 3, 4\}$ with $m_{12} = m_{34} = \alpha$, $m_{13} = m_{24} = \beta$, and $m_{14} = m_{23} = \gamma$, where m_{ij} denotes the number of edges between vertices i and j . This is a regular graph of degree $\alpha + \beta + \gamma$. Suppose that $\alpha + \beta + \gamma = 2k$ is even. Then there is at most a single summand in (6.12), because a matrix D for the transitions at

vertex 4 is uniquely determined by the row sums $(d_1, d_2, d_3) = (\gamma, \beta, \alpha)$; it must be $D_{12} = k - \alpha$, $D_{13} = k - \beta$, $D_{23} = k - \gamma$. Therefore

$$M\left(K_4^{[\alpha, \beta, \gamma]}\right) = \begin{cases} \frac{\alpha! \beta! \gamma!}{(k-\alpha)!(k-\beta)!(k-\gamma)!} & \text{if } \alpha, \beta, \gamma \leq k \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

Lemma 6.30. *If p is a prime and u a vertex with a neighbour w_i with edge multiplicity $d_i \geq p$, then for every summand in (6.12), the coefficient $d!/D!$ is divisible by p .*

Proof. If D has an entry $D_{ij} \geq p$, the divisibility follows from the $D!$ factor on the right-hand side of (6.13). So assume that all entries of D are less than p . Then $D!$ is not divisible by p , but $d_i!$, and thus $d!/D!$, is. \square

Corollary 6.31. *For every $4r$ -regular graph G on at least 4 vertices, with $p = r + 1$ prime, $M(G)$ is divisible by p .*

Proof. If G has self-loops, then the claim is trivial by $M(G) = 0$. For graphs with more than 4 vertices, the claim follows inductively by vertex expansion (1.1) to smaller graphs. Hence we only need to consider G with 4 vertices and no self-loops, that is, $G \cong K_4^{[\alpha, \beta, \gamma]}$ for some integers α, β, γ with $\alpha + \beta + \gamma = 4r$. Since at least one of $\{\alpha, \beta, \gamma\}$ must exceed $r = p - 1$, Lemma 6.30 implies the claim. \square

Corollary 6.32. *Let $p = r + 1$ be prime. Then for every $4r$ -regular graph G with 5 vertices, which is not isomorphic to $K_5^{[r]}$, we have that $M(G)$ is divisible by p^2 .*

Proof. If G has any edge of multiplicity greater than r , expansion at a vertex with such an edge writes $M(G)$ as a linear combination of $(d!/D!) M(G_D)$. The factor $d!/D!$ is divisible by p due to Lemma 6.30, and $M(G_D)$ is divisible by p due to Corollary 6.31. So we conclude that $M(G)$ is divisible by p^2 whenever G has an edge with multiplicity $\geq p$. So, assume that all edges have multiplicity $\leq r$. To reach degree $4r$ at every vertex, we must then have $G \cong K_5^{[r]}$. \square

Lemma 6.33. *The Martin invariant of the r -fold power of the complete graph K_5 is*

$$M\left(K_5^{[r]}\right) = (r!)^4 \sum_{\alpha+\beta+\gamma=r} \frac{(r+\alpha)!(r+\beta)!(r+\gamma)!}{(\alpha!\beta!\gamma!)^2(r-\alpha)!(r-\beta)!(r-\gamma)!} \tag{6.15}$$

and congruent to $M(K_5^{[r]}) \equiv -3p \pmod{p^2}$ whenever $p = r + 1$ is prime.

Proof. The expansion (6.12) at any vertex produces

$$M\left(K_5^{[r]}\right) = \sum_{\alpha+\beta+\gamma=r} \frac{r!r!r!r!}{(\alpha!\beta!\gamma!)^2} M\left(K_4^{[r+\alpha, r+\beta, r+\gamma]}\right).$$

Inserting (6.14) gives (6.15). Now consider $p = r + 1$ prime. The denominator of the summand in (6.15) is not divisible by p . The numerator is divisible by p^2 unless at

most one of α, β, γ is non-zero. We conclude that modulo p^2 , only the three summands with $(\alpha, \beta, \gamma) \in \{(0, 0, r), (0, r, 0), (r, 0, 0)\}$ contribute. Therefore,

$$M\left(K_5^{[r]}\right) \equiv 3(r!)^2(2r)! = 3(r!)^3 p(p+1) \cdots (2p-2) \equiv -3p \cdot (r!)^4 \pmod{p^2}$$

and the claim follows from Wilson's theorem $r! \equiv -1 \pmod{p}$. \square

Note how the three explicit summands contributing in the last step of the proof are what leads to the explicit factor of 3, causing extra trouble for $p = 3$.

Corollary 6.34. *For every $4r$ -regular graph with 5 vertices and $r + 1$ prime, we have $M(G) \equiv 3pN_{r,r}(H) \pmod{p^2}$.*

Proof. For every graph other than $K_5^{[r]}$ the claim follows from Corollary 6.32 and Lemma 6.26. For $G = K_5^{[r]}$, see Lemma 6.33 and Lemma 6.26. \square

This establishes the base case (6.9) for all primes $p \neq 3$. Now consider the special case $p = 3$. Then (6.9) is equivalent to

$$M(G) \equiv 9N_{2,2}(H) \pmod{27}, \quad (6.16)$$

that is, we have to compute the Martin invariants $\pmod{p^3}$. We first note that the expression $M(G)/9 \pmod{3}$ on the right-hand side of (6.9) makes sense.

Lemma 6.35. *The Martin invariant of every 8-regular graph with at least 5 vertices is divisible by 9.*

Proof. By vertex expansion (1.1), the claim reduces to the case of graphs G with 5 vertices. If $G = K_5^{[2]}$, then $M(G) \equiv -3p \equiv 0 \pmod{p^2}$ for $p = 3$. For all other graphs, see Corollary 6.32. \square

We thus have $M(G) \equiv 9j \pmod{27}$ for some integer j , but for graphs with 5 vertices, j is not necessarily congruent to $N_{2,2}(H)$. For example, the totally decomposable graph

$$M\left(\left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right)^{[2]}\right) = (4!)^2 \equiv 9 \pmod{27}$$

has $j \equiv 1 \pmod{3}$; but we saw in Lemma 6.26 that $N_{2,2}(H) \equiv 0 \pmod{3}$. Therefore, for prime $p = 3$ and 5 vertices, the c_2 invariant cannot be read off the Martin invariant modulo p^3 . We need to base the induction one step up, at 6 vertices.

So consider an 8-regular graph G with 6 vertices. Recall that G has 5 special vertices labelled $\{v, w, a, b, c\}$ with $H = G \setminus \{v, w\}$, such that w is connected only to $\{a, b, c, v\}$, via a double edge to each. It follows that the sixth vertex, call it u , is not connected to w and has at most 4 neighbours $\{a, b, c, v\}$. If any of these connections to u has multiplicity 3 or higher, then Lemma 6.30 shows that $N_{2,2}(H) \equiv 0 \pmod{3}$, since $N_{2,2}(H)$ fulfils the Martin recursion at u . By Lemma 6.30 and Lemma 6.35, we also find $M(G) \equiv 0 \pmod{27}$, hence we have established (6.16) in this case.

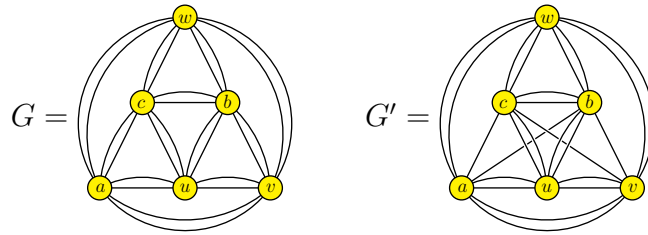


Figure 6.14: The doubled octahedron $G = (C_{1,2}^6)^{[2]}$ and the graph G' .

(D_{ab}, D_{ac}, D_{bc})	$(2, 0, 0)$	$(0, 2, 0)$	$(0, 0, 2)$	$(1, 1, 0)$	$(1, 0, 1)$	$(0, 1, 1)$
$d!/D!$	4	4	4	16	16	16
G_D	$K_5^{[2]}$					
G'_D				$K_5^{[2]}$		

Table 6.1: Contributions to the vertex expansion at u of the graphs from Figure 6.14.

It remains to consider graphs G where u is connected to each of $\{a, b, c, v\}$ via a double edge. So we are only left to arrange 8 edges between $\{a, b, c, v\}$ to obtain an 8-regular graph G . Let m_{ij} denote these multiplicities. The 8-regularity implies that $m_{ab} = m_{cv}$, $m_{ac} = m_{bv}$, and $m_{av} = m_{bc}$. As before, any edge of multiplicity $m_{ij} \geq 3$ will force $M(G) \equiv 0 \pmod{27}$. Similarly, we will have $N_{2,2}(H) \equiv 0 \pmod{3}$, because the expansion at u can only add further edges between $\{a, b, c\}$; so if $m_{aj} \geq 3$ already, it will stay strictly above 2 even in $G_D \setminus \{v, w\}$, and therefore $N_{2,2}(H) \equiv 0 \pmod{3}$ by Lemma 6.26.

The only remaining cases then are those where all $m_{ij} \leq 2$. Up to permuting the vertex labels, there are only two isomorphism classes of such graphs: $(m_{ab}, m_{ac}, m_{bc}) = (0, 2, 2)$ and $(m_{ab}, m_{ac}, m_{bc}) = (1, 1, 2)$. The corresponding graphs G and G' are shown in Figure 6.14.

Lemma 6.36. *The Martin invariants of the two graphs in Figure 6.14 are*

$$M(G) = 84096 \equiv -9 \pmod{27} \quad \text{and} \quad M(G') = 97920 \equiv -9 \pmod{27}.$$

Proof. Apply the expansion (6.12) to the vertex u . This produces the terms in Table 6.1. Two of the arising 5-vertex graphs are totally decomposable due to 4-fold edges, so that

$$M\left(\text{graph with 4-fold edges}\right) = M\left(\text{graph with 4-fold edges}\right) = (4!)^2.$$

We evaluate the remaining two 5-vertex graphs with the expansion (6.12) for the bottom left vertex. Together with (6.14), this results in

$$M\left(\text{graph with 4-fold edges}\right) = \frac{2!3!3!}{1!1!2!} M\left(K_4^{[2,3,3]}\right) = 1296 \quad \text{and}$$

$$M\left(\text{graph with 4-fold edges}\right) = \frac{2!2!3!}{1} M\left(K_4^{[2,3,3]}\right) + \frac{2!2!3!}{2} M\left(K_4^{[2,2,4]}\right) + \frac{2!2!3!}{2} M\left(K_4^{[2,3,3]}\right) = 1584.$$

Inserting these values and $M(K_5^{[2]}) = 2016$ from (6.15) into Table 6.1, we obtain $M(G) = 84096$ and $M(G') = 97920$. \square

It remains to show that $N_{2,2}(H) \equiv -1 \pmod{3}$ for $H = G \setminus \{v, w\}$ and $H' = G' \setminus \{v, w\}$. By Lemma 6.26, we can compute $N_{2,2}(H) \pmod{3}$ by counting the number of transitions at u that produce $G_D \cong K_5^{[2]}$ and then multiplying by -1 . According to Table 6.1, in each case there is only one matrix D with this property, thus $N_{2,2} \equiv -d!/D! \pmod{3}$ gives

$$N_{2,2}(H) \equiv -4 \equiv -1 \pmod{3} \quad \text{and} \quad N_{2,2}(H') \equiv -16 \equiv -1 \pmod{3}.$$

This finishes the proof that (6.16) holds for all $H = G \setminus \{v, w\}$ with 4 vertices, that is, for all graphs G with 6 vertices.

Consequently, this completes the proof of Theorem 6.2 for all primes, especially $p = 3$.

7. Computer calculations

We implemented the recursion (6.12) in Maple™, using nauty [MP14] to identify isomorphic graphs.²² We use a cache table indexed by the canonical label of a graph, to avoid the recomputation of Martin invariants of isomorphic graphs. To test our implementation, we confirmed that the results for $M(G)$ and $M(G^{[2]})$ for all 4-regular graphs with ≤ 13 vertices, and $M(G^{[2]})$ for all 3-regular graphs with ≤ 18 vertices, agree with the values obtained by an independent, simple but much slower FORM [Ver00] program that computes the sum (2.12). We also checked in many cases that our results for $M(G^{[r]})$ are compatible with known permanents [Cru17b] and c_2 -invariants [HSSY22, BS13] via (4.3) and (6.3).

Using this code, we computed the first few entries of the Martin sequences of all 3- and 4-regular graphs with small numbers of vertices. These results are provided in open data text files `Martin3.txt` and `Martin4.txt`, respectively, that form part of this paper [Pan24]. Below we give a summary of those computational results and highlight several observations.

7.1. 4-regular graphs

Due to Theorem 1.2 and the product identity (1.4), we only consider cyclically 6-connected 4-regular graphs without a 3-vertex cut. The isomorphism classes of such graphs have been enumerated in [Sch10] up to $n \leq 13$ vertices. They are labelled $P_{\ell,1}, P_{\ell,2}, \dots$ where $\ell = n - 2 \leq 11$ denotes the loop number of any decomposition. Explicit definitions of these graphs are given in a file provided with [PS17].

Table 7.1 summarizes our findings for the first two entries $M(G)$ and $M(G^{[2]})$ of the Martin sequences. Explicit values are given for up to $\ell \leq 8$ loops in Table 7.3. We note:

- Up to 7 loops, the Martin invariant $M(G)$ is a perfect period invariant: By this we mean that $M(G_1) = M(G_2)$ if and only if $\mathcal{P}(G_1 \setminus v) = \mathcal{P}(G_2 \setminus v)$. This follows from Table 7.3 because at 7 loops, all periods are known [PS17] and the twist $P_{7,4} \leftrightarrow P_{7,7}$ and the duality $P_{7,5} \leftrightarrow P_{7,10}$ provide the only two relations.

²²Maple is a trademark of Waterloo Maple Inc.

loop order ℓ	3	4	5	6	7	8	9	10	11
$ \{\text{graphs } G\} $	1	1	1	4	11	41	190	1182	8687
$ \{M(G)\} $	1	1	1	4	9	25	100	409	1622
$ \{M(G^{[2]})\} $	1	1	1	4	9	29	129	776	6030
$ \{\text{Hepp bounds } \mathcal{H}(G \setminus v)\} $	1	1	1	4	9	29	129	776	6030

Table 7.1: The number of graphs, Martin invariants, and Hepp bounds up to 11 loops.

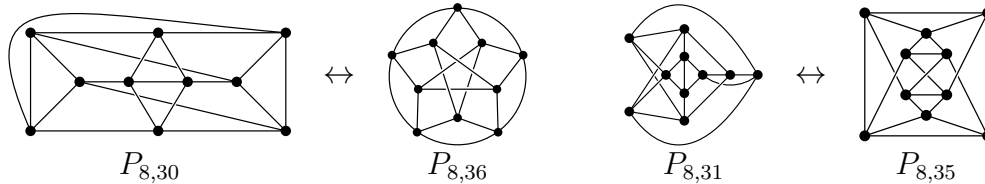


Figure 7.1: Two unexplained pairs of graphs with apparently equal Martin sequences.

- Starting at 8 loops, there are pairs of graphs with the same Martin invariant, but different period. At 8 loops, there are 4 such pairs (highlighted in Table 7.3).
- If we consider instead the Martin invariants of the doubled graphs, then we find that for all graphs with $\ell \leq 11$ loops, $M(G_1^{[2]}) = M(G_2^{[2]})$ holds if and only if $\mathcal{H}(G_1 \setminus v) = \mathcal{H}(G_2 \setminus v)$. Since the Hepp bound is expected to be a perfect period invariant for 4-regular graphs [Pan23, Conjecture 1.2], we thus expect that $M(G^{[2]})$ is a perfect period invariant up to 11 loops.
- For higher powers, we computed: $M(G^{[3]})$ up to $\ell \leq 11$, $M(G^{[4]})$ up to $\ell \leq 10$, $M(G^{[5]})$ and $M(G^{[6]})$ up to $\ell \leq 9$, $M(G^{[7]})$ and $M(G^{[8]})$ up to $\ell \leq 8$, and $M(G^{[9]})$ and $M(G^{[10]})$ up to $\ell \leq 7$. For all these higher powers we find, as for $M(G^{[2]})$, that they coincide precisely when the Hepp bounds (and hence conjecturally the periods) coincide. All this data constitutes our evidence for Conjecture 1.10.

The most interesting instances of Conjecture 1.10 are the graphs in Figure 7.1. These graphs come in two pairs that share the same Hepp bound [Pan23], hence we expect that

$$\mathcal{P}(P_{8,30} \setminus v) = \mathcal{P}(P_{8,36} \setminus v) \quad \text{and} \quad \mathcal{P}(P_{8,31} \setminus v) = \mathcal{P}(P_{8,35} \setminus v), \tag{7.1}$$

which does not follow from any of the known period identities. Correspondingly, the entire Martin sequences should agree, that is, we conjecture that

$$M(P_{8,30}^{[r]}) = M(P_{8,36}^{[r]}) \quad \text{and} \quad M(P_{8,31}^{[r]}) = M(P_{8,35}^{[r]}) \tag{7.2}$$

hold for all positive integers r . Our calculations confirm these identities explicitly in the range $1 \leq r \leq 8$. Modulo primes $p = 2r + 1$, (7.2) was checked much farther, up to $r \leq 50$ (via Theorem 4.1 from the permanents in [Cru17b, Appendix A]), and modulo the smaller

primes $p = r + 1$, (7.2) was checked up to $r = 100$ (via Theorem 6.2 from the c_2 invariants calculated in [BS13]). We have thus accumulated overwhelming evidence for (7.2). In our view, this strengthens considerably the credibility of the conjecture (7.1).

We also calculated examples that show that several of our results are best possible:

- Feynman period identities do not extend from the Martin invariant to the full Martin polynomial. For example, $P_{7,4}$ and $P_{7,7}$ are related by a twist, and also by planar duality (Figure 5.4), but their Martin polynomials

$$\begin{aligned} m(P_{7,4}, x) &= x(7x^3 + 210x^2 + 1054x + 1320) \quad \text{and} \\ m(P_{7,7}, x) &= x(13x^3 + 200x^2 + 1030x + 1320) \quad \text{differ.} \end{aligned}$$

- The divisibility of $M(G^{[r]})$ by $(r!)^{2(n-3)}$ for r -fold duplications of 4-regular graphs (Corollary 5.21) does not persist for other $4r$ -regular graphs. For example, the complete graph K_9 is an 8-regular graph on $n = 9$ vertices, but

$$M(K_9) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 167 \quad (7.3)$$

is not divisible by 2^{12} .

- The Martin invariant is determined by the cycle matroid, but it is not determined by the Tutte polynomial $T_G(x, y)$. For example, $P_{11,8403}$ and $P_{11,8404}$ have the same Tutte polynomial, but $M(P_{11,8403}) = 12438$ and $M(P_{11,8404}) = 12442$ differ.

Finally, we comment on the quantitative relation between the Martin invariant and the period. Using the known periods from [PS17], we find a negative approximate correlation (Figure 1.1): graphs with smaller $M(G)$ tend to have larger periods $\mathcal{P}(G \setminus v)$.

We find in particular that, at each loop order $\ell \leq 11$, the Martin invariant is minimized by a unique graph: the circulant $P_{\ell,1} = C_{1,2}^{\ell+2}$. Its decomposition is called zigzag graph, whose period is known and expected to be the largest [BS15]. While the period maximization is a conjecture, the Martin minimization was shown in [BG96, Theorem 6.2]:

Proposition 7.1. *If G is a cyclically 6-connected 4-regular graph with $n \geq 5$ vertices, then $M(G) \geq M(C_{1,2}^n)$. Furthermore, equality holds only if $G \cong C_{1,2}^n$, and the value is*

$$M(C_{1,2}^n) = \frac{(3n - 2) \cdot 2^{n-3} - 2 \cdot (-1)^n}{9}. \quad (7.4)$$

The expansion (1.2) gives the recurrence $M(C_{1,2}^n) = M(C_{1,2}^{n-1}) + 2M(C_{1,2}^{n-2}) + 2^{n-4}$ where 2^{n-4} comes from a totally decomposable graph, leading to the closed form (7.4). That this is the minimum among cyclically 6-connected graphs is related to the following properties.

- If G has a 3-vertex cut, then $M(G) = M(G_1)M(G_2) \geq M(C_{1,2}^k)M(C_{1,2}^r) > M(C_{1,2}^n)$ cannot be minimal. Here $k + r = n + 3$ and $k, r \geq 5$.
- If G has no 3-vertex cut, then at each vertex, there is at least one transition such that G_τ is cyclically 6-connected [FGJ07, Lemma 3.2].

loops	max M(G)	G	HoG	Aut(G)
3	6	$P_{3,1} = K_5$	462	120
4	14	$P_{4,1} = C_{1,2}^6$	226	48
5	36	$P_{3,1}^2$	912	48
6	108	$P_{6,4} = K_{4,4} = C_{1,3}^8$	570	1152
7	256	$P_{7,8}$	50432	16
8	728	$P_{8,40} = C_{1,4}^{10} = C_{2,3}^{10}$	45705	320
9	1894	$P_{9,190} = C_{1,3}^{11} = C_{1,4}^{11} = C_{2,3}^{11}$	50433	22
10	5300	$P_{10,1182} = C_{2,3}^{12}$	33319	48
11	14376	$P_{11,8687} = C_{1,5}^{13} = C_{2,3}^{13}$	21065	52

Table 7.2: The unique primitive ϕ^4 graphs that maximize $M(G)$ at each loop order ≤ 11 . Complete (bipartite) and circulant graphs are highlighted, and the column HoG links to their entries in the House of Graphs [CDG23].

- If at each vertex there is only one transition such that G_τ is cyclically 6-connected, then $G \cong C_{1,2}^n$ by [BG96, Theorem 6.1].

At the other extreme, our data shows that at each loop order $\ell \leq 11$, the Martin invariant is also maximized by a unique graph, see Table 7.2. We did not identify a general pattern, but the non-trivial automorphism groups and the prevalence of circulants among these graphs suggest some underlying structure. The same graphs were recently noticed to maximize the number of connected sets [CGJ25].

7.2. 3-regular graphs

A 3-regular graph has an even number n of vertices, and we denote by $\ell = n/2 - 1$ the loop number of its decompletions. The cyclically 4-connected 3-regular graphs with $\ell \leq 9$ (that is, up to $n \leq 20$ vertices) have been enumerated in [BS22], where also many of their periods were computed. We use the notation $P_{\ell,k}$ from that paper; the definition of these graphs as edge lists can be found in the file PeriodsPhi3.txt from [BS22].

We computed $M(G^{[2]})$ and $M(G^{[4]})$ for all graphs $P_{\ell,k}$ from [BS22] with $\ell \leq 9$, and $M(G^{[6]})$ for $\ell \leq 8$. A summary of our findings is given in Table 7.4. We observe in particular:

- Up to $\ell \leq 5$, $M(G^{[2]})$ is a perfect period invariant: $\mathcal{P}(G \setminus v)$ is the same for two graphs if and only if $M(G^{[2]})$ is the same. This is not true for the Hepp bound: $P_{5,5}$ and $P_{5,9}$ have different periods but equal Hepp bound [BS22, §1.6].
- At $\ell = 6$ loops, there is a single pair of graphs $\{P_{6,59}, P_{6,68}\}$ with the property that their periods differ [BS22] while $M(P_{6,59}^{[2]}) = M(P_{6,68}^{[2]}) = 2^{29} \cdot 5^2$ agree. However, the next term $M(G^{[4]})$ in the Martin sequence takes different values on this pair, and we find that indeed $M(G^{[4]})$ is a perfect period invariant at $\ell = 6$.

Graph G	$M(G)$	$M(G^{[2]})/4^{\ell-1}$	Graph G	$M(G)$	$M(G^{[2]})/4^{\ell-1}$
			$P_{8,1}$	398	13881906
			$P_{8,2}$	470	19288170
			$P_{8,3}$	470	19560330
			$P_{8,4}$	494	21875634
			$P_{8,6}, P_{8,9}$	510	23224770
			$P_{8,5}$	516	24331644
			$P_{8,7}, P_{8,8}$	518	24330906
			$P_{8,11}, P_{8,15}$	524	25080084
			$P_{8,14}$	534	26486154
			$P_{8,13}, P_{8,21}$	542	26900226
			$P_{8,10}, P_{8,22}$	548	27340956
			$P_{8,12}$	548	28399356
			$P_{8,18}, P_{8,25}$	564	30075084
			$P_{8,20}$	566	30153834
			$P_{8,19}, P_{8,27}$	572	31573476
			$P_{8,17}, P_{8,23}$	582	32301306
			$P_{8,16}$	584	31092984
			$P_{8,29}$	584	33515064
			$P_{8,30}, P_{8,36}$	602	36055206
			$P_{8,26}, P_{8,28}$	608	36570816
			$P_{8,33}$	618	38238966
			$P_{8,32}, P_{8,34}$	620	38026260
			$P_{8,31}, P_{8,35}$	624	38998224
			$P_{8,37}$	638	41602626
			$P_{8,24}$	656	42769584
			$P_{8,38}$	656	44586864
			$P_{8,39}$	660	45058860
			$P_{8,41}$	684	50848884
			$P_{8,40}$	728	54288936

Table 7.3: The first two Martin invariants of ϕ^4 primitives $P_{\ell,i}$ with $n = \ell + 2 \leq 13$ vertices. All identities (two graphs in the same row) are explained by twists or dualities, except for the two unexplained identities $P_{8,30} \leftrightarrow P_{8,36}$ and $P_{8,31} \leftrightarrow P_{8,35}$.

loop order ℓ	1	2	3	4	5	6	7	8	9
$ \{\text{graphs } G\} $	1	1	2	5	18	84	607	6100	78824
$ \{M(G^{[2]})\} $	1	1	2	5	17	72	441	4015	47074
$ \{M(G^{[4]})\} $	1	1	2	5	17	73	472	4534	58432
$ \{\text{periods } \mathcal{P}(G \setminus v)\} $	1	1	2	5	17	73	?	?	?
$ \{\text{Hepp bounds } \mathcal{H}(G \setminus v)\} $	1	1	2	5	16	72	470	4522	58409

Table 7.4: The number of Martin invariants, periods and Hepp bounds among cyclically 4-connected 3-regular graphs with $2\ell + 2$ vertices.

- We conjecture that $M(G^{[4]})$ remains a perfect period invariant also for $7 \leq \ell \leq 9$. In this range, however, not all periods are known, and the Hepp bounds are only expected to give a lower bound on the number of periods.²³

Because the Martin invariant is defined only for graphs with even degree, we define the Martin sequence of a 3-regular graph as

$$M(G^\bullet) = (M(G^{[2]}), M(G^{[4]}), M(G^{[6]}), \dots).$$

All available data is compatible with the following variant of Conjecture 1.10, which gives a purely combinatorial characterization of 3-regular graphs with equal periods:

Conjecture 7.2. Two cyclically 4-connected 3-regular graphs G_1 and G_2 with the same number of vertices have equal period $\mathcal{P}(G_1 \setminus v_1) = \mathcal{P}(G_2 \setminus v_2)$ if and only if they have equal Martin sequences $M(G_1^\bullet) = M(G_2^\bullet)$.

Lemma 7.3. For a cyclically 4-edge connected, 3-regular graph with $2\ell + 2$ vertices, the Martin invariant $M(G^{[2r]})$ is divisible by $r! \cdot [(2r)!]^{3\ell-2}$.

Proof. This follows similarly to Corollary 5.21, by adapting Lemma 5.9 to the case of half-integer $k = 3/2$. □

Turning to the quantitative relation between periods and Martin invariants, we find the same kind of behaviour as for 4-regular graphs; a plot of the known periods from [BS22] versus $M(G^{[2]})$ shows an approximate power law correlation, similar to Figure 1.1.

We also find that at any fixed number of vertices $8 \leq 2\ell + 2 \leq 20$, there is a unique graph that minimizes $M(G^{[2]})$: the prism $Y_{\ell+1} = K_2 \times C_{\ell+1}$ over a base polygon $C_{\ell+1}$ with $\ell + 1$ sides (see Figure 7.2). We expect that this persists for higher ℓ :

Conjecture 7.4. If G is a cyclically 4-connected 3-regular graph with $2\ell + 2 \geq 8$ vertices, then $M(G^{[2]}) \geq M(Y_{\ell+1}^{[2]})$ and equality holds only if $G \cong Y_{\ell+1}$.

²³We do not know that the Hepp bounds give a lower bound on the number of periods in this context, but we do expect it. Specifically, we expect that $\mathcal{P}(G_1) = \mathcal{P}(G_2)$ implies $\mathcal{H}(G_1) = \mathcal{H}(G_2)$.

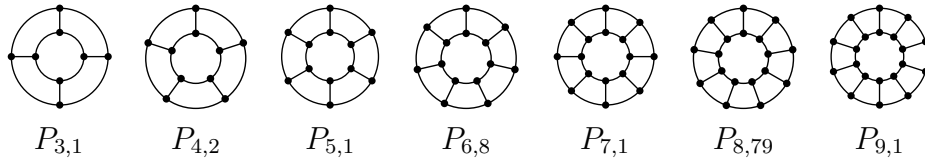


Figure 7.2: Prism graphs $Y_{\ell+1} = K_2 \times C_{\ell+1}$ and their labels in [BS22].

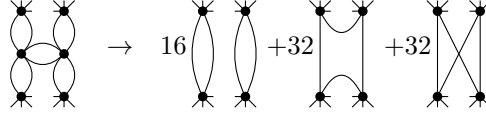


Figure 7.3: The result of applying the Martin recurrence to two vertices of a doubled rung. The remainder of the graph connects at top and bottom where indicated and is identical in all the terms.

We can give a closed formula for these minimal Martin invariants: For all $\ell \geq 2$,

$$M(Y_{\ell+1}^{[2]}) = 4^{2\ell-1}[3^{\ell-2}(4\ell-1) - 1]. \quad (7.5)$$

Our proof uses the approach of Lemma 6.9, namely we obtain a transfer matrix of rank 3 to derive a recurrence relation in the following manner: Beginning with one rung of prism visualized as a cyclic ladder, apply the Martin recurrence to the two vertices of the rung. This results in the three terms illustrated in Figure 7.3. For each of the graphs on the right-hand side, consider how the next rung below interacts with the part drawn. For the first term on the right-hand side we have simply returned to the left-hand side, but with the entire graph having one rung fewer. For the second and third terms on the right-hand side, we have different graphs to consider but in both cases we can, as before, reduce the two vertices where the next rung connected. The result in both cases will include some graphs we've already seen on the right-hand side and some new ones. Ultimately, we need four more equations, as illustrated in Figure 7.4. Letting a_n represent the Martin invariant of the original ladder with n rungs, and b_n , c_n , d_n , and e_n likewise for the graphs of Figure 7.4, the calculations of the figures can be rephrased as

$$\begin{aligned} a_n &= 16a_{n-1} + 32c_{n-1} + 32d_{n-1} \\ b_n &= 16b_{n-1} + 32c_{n-1} + 32d_{n-1} \\ c_n &= 12e_{n-1} + 48c_{n-1} \\ d_n &= 8a_{n-1} + 48c_{n-1} + 4e_{n-1} + 32d_{n-1} + 8b_{n-1} \\ e_n &= 0 \end{aligned}$$

Observe that $d_n = \frac{1}{2}a_n + \frac{1}{2}b_n + \frac{1}{3}c_n$ (this is most apparent after reducing just one of the rung edges, but can also be verified from the equations above). Using this fact along with $e_n = 0$ we can simplify to a system of three recurrences

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} 32 & 16 & \frac{128}{3} \\ 16 & 32 & \frac{128}{3} \\ 0 & 0 & 48 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{pmatrix}.$$

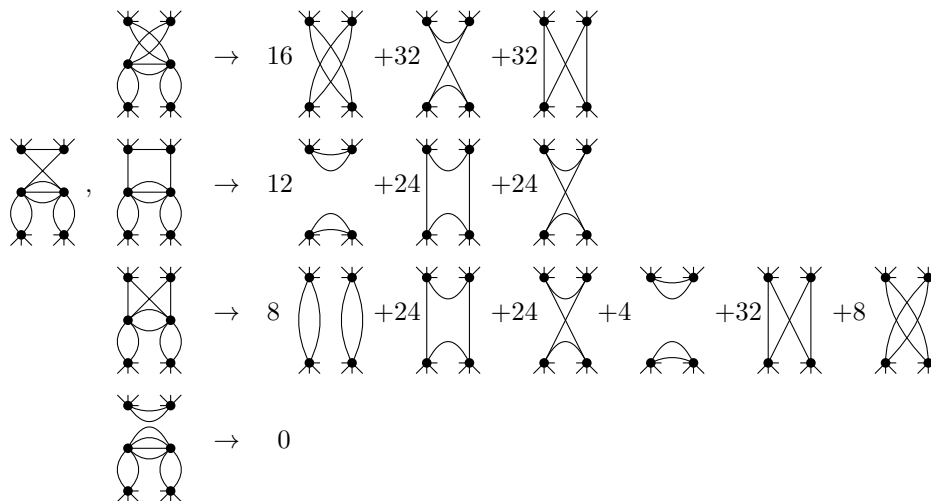


Figure 7.4: The result of applying the Martin recurrence to the other graphs needed to get a system of recurrences.

Solving the system gives the claimed formula.

Remark 7.5. As for ϕ^4 , our data shows that also that at each loop order $\ell \leq 9$, there is a unique ϕ^3 primitive graph that *maximizes* the Martin invariant $M(G^{[2]})$. Those graphs are collected in Table 7.5. Again, the same graphs were recently noticed to maximize the number of connected sets [CGJ25].

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Erik Panzer dedicates this work to his father, Michael Erhard Panzer, 1955–2022. In admiration, sorrow, gratitude, and loving memory.

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loops	$\max M(G^{[2]})$	G	HoG	$ \text{Aut}(G) $
1	$1 \cdot 2^3$	$P_{1,1}$	74	24
2	$4 \cdot 2^7$	$P_{2,1}$	84	72
3	$17 \cdot 2^{11}$	$P_{3,2}$	640	16
4	$92 \cdot 2^{15}$	$P_{4,5}$	660	120
5	$460 \cdot 2^{19}$	$P_{5,7}$	27409	16
6	$2760 \cdot 2^{23}$	$P_{6,7}$	1154	336
7	$14820 \cdot 2^{27}$	$P_{7,21}$	1229	96
8	$88536 \cdot 2^{31}$	$P_{8,3165}$	50424	12
9	$530093 \cdot 2^{35}$	$P_{9,54494}$	50425	8

Table 7.5: The unique graphs that maximize $M(G^{[2]})$ among ϕ^3 primitives at a fixed loop order. The column HoG links to their entries in the House of Graphs [CDG23].

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