

INTRANSITIVE DICE TOURNAMENT IS NOT QUASIRANDOM

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Abstract. We settle a version of the conjecture about intransitive dice posed by Conrey, Gabbard, Grant, Liu and Morrison in 2016 and Polymath in 2017. We consider generalized dice with n faces and we say that a die A beats B if a random face of A is more likely to show a higher number than an independently chosen random face of B . We study random dice with faces drawn iid from the uniform distribution on $[0, 1]$ and conditioned on the sum of the faces equal to $n/2$. Considering the “beats” relation for three such random dice, Polymath showed that each of eight possible tournaments between them is asymptotically equally likely. In particular, three dice form an intransitive cycle with probability converging to $1/4$. In this paper we prove that for *four* random dice *not* all tournaments are equally likely and the probability of a transitive tournament is strictly higher than $3/8$.

Keywords. Intransitive dice, central limit theorem

Mathematics Subject Classifications. 60C05

1. Introduction

Intransitive dice are an accessible example of counterintuitive behavior of combinatorial objects. For our purposes, a *die* is a vector of n real numbers $A = (a_1, \dots, a_n)$. The traditional cube dice have $n = 6$. Given two n -sided dice A and B , we say that A beats B , writing it as $A \succ B$, if

$$\left(\sum_{i,j=1}^n \mathbb{1}[a_i > b_j] - \mathbb{1}[a_i < b_j] \right) > 0 .$$

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In other words, A beats B if a random roll (uniformly chosen face) of A is more likely to display a larger number than a random roll of B . The term “intransitive dice” refers to the fact that the “beats” relation is not transitive. One well-known example due to Efron [Gar70] with $n = 6$ is

$$\begin{aligned} A &= (0, 0, 4, 4, 4, 4), & B &= (3, 3, 3, 3, 3, 3), \\ C &= (2, 2, 2, 2, 6, 6), & D &= (1, 1, 1, 5, 5, 5), \end{aligned}$$

where one checks that $A \succ B \succ C \succ D \succ A$. In particular, this set of dice allows for an amusing game: If two players sequentially choose one die each and make a throw, with the player rolling a higher number receiving a dollar from the other player, then it is the second player who has a strategy with positive expected payout.

The intransitivity of dice is a noted subject in popular mathematics [Gar70, Sch00, Gri17]. It has been studied under various guises for a considerable time [ST59, Try61, Try65, MM67, SJ94]. There is a multitude of constructions of intransitive dice with various properties [MM67, FT00, BB13, AD17, SS17, BGH18, AS21], as well as studies of game-theoretic aspects [Rum01, DSDMDB06, HW19]. In particular, it is known (e.g., [MM67, AD17]) that for every tournament on k vertices, there exists a set of k dice whose “beats” relation realizes the tournament.

We also note another related result by Buhler, Graham and Hales [BGH18] that constructs an unexpected example which they dub “maximally nontransitive dice”. Given a set of k dice, let $T_1, \dots, T_\ell, \dots$ be the sequence of tournaments on k vertices defined in the following way: There is an edge from i to j in the tournament T_ℓ if and only if the sum of ℓ independent rolls of die i is more likely to yield a higher value than the sum of ℓ independent rolls of die j . [BGH18] constructs, for every k , a set of k dice where every possible tournament on k vertices occurs infinitely often in the sequence $(T_\ell)_{\ell \in \mathbb{N}}$.

While most of these results concern interesting examples and constructions, there has been recent interest in understanding how common intransitive dice really are. In other words, is there a natural way of generating intransitive examples? Conrey, Gabbard, Grant, Liu and Morrison [CGG⁺16] proposed several models of random dice to study this question. A natural starting point is to pick a distribution of a single face, e.g., uniform over $[0, 1]$ or $[n] := \{1, \dots, n\}$ and sample dice with n iid faces from this distribution. Some thought reveals that this does not generate intransitivity (see, e.g., Theorem 6 in [HMRZ20] for details). The reason is that with high probability, in a small set of dice sampled from this model, A beats B if and only if $\sum_{i=1}^n a_i > \sum_{i=1}^n b_i$. That is, the *face-sum* $\sum_{i=1}^n a_i$ is a proxy of how strong a die is and the “beats” relation on the set of dice is transitive.

[CGG⁺16] suggested and presented experimental evidence that to “reveal” intransitivity one can sample *random balanced dice*. What we mean by that is the faces are iid and the die is conditioned on having the expected face-sum. They conjectured (in a related model where random die is a uniform balanced multiset of $[n]$ rather than iid sequence) that three dice in such model form an intransitive cycle with asymptotic probability $1/4$. A collaborative Polymath project [Pol17] was launched to investigate this conjecture and indeed proved it for dice with iid uniform faces:

Theorem 1.1 ([Pol22]). *If A, B, C are three random dice with n faces iid uniform in $[n]$ and conditioned on all face-sums equal to $n(n+1)/2$, then the probability that A, B, C are intransitive is $1/4 + o(1)$.*

For later reference we now give an equivalent statement of Theorem 1.1. What it says is that a random balanced die is unbiased with respect to the “beats” relation:

Theorem 1.2 ([Pol22]). *If A, B are two random dice with n faces iid uniform in $[n]$ and conditioned on all face-sums equal to $n(n + 1)/2$, then, except with probability $o(1)$, we have $\Pr[A \succ B \mid B] = 1/2 + o(1)$.*

The equivalence of the two theorems is proved using general properties of tournaments and establishing it is the first step in the Polymath proof of Theorem 1.1.

Indeed, it seems that the balanced face-sum for uniform faces is the “right” model for generating intransitive dice. Perhaps surprisingly, it turns out that both balancedness and uniformity are important. In particular, a follow-up work [HMRZ20] showed that if dice are balanced with faces iid from any (continuous) distribution that is *not* uniform (and also in some other models with correlated Gaussian faces), then the intransitivity does not show up and three dice are transitive with high probability.

Recall that there are eight tournaments on three vertices, six of them transitive and the other two cycles. By symmetry considerations, tournaments in each of those two groups are equally likely when sampling three random dice. Therefore, Theorem 1.1 means that each tournament on three random balanced dice is asymptotically equally likely. In that context, [CGG+16] made a natural further-reaching conjecture in the multiset model on $[n]$. They conjectured that, for any fixed k , all tournaments on k dice are equally likely. Using terminology from the paper by Chung and Graham [CG91], what they conjectured is that the balanced dice tournament is *quasirandom*. However, in another surprising turn, experiments done by Polymath [Gow17a] suggested that for four balanced dice with faces iid uniform in $[n]$ this conjecture is false, with the probability of a transitive outcome hovering around 0.39 rather than $3/8 = 0.375$ expected in a quasirandom tournament (cf. Figure 1.1). However, it seems that the question was ultimately left open by Polymath [Gow17b].

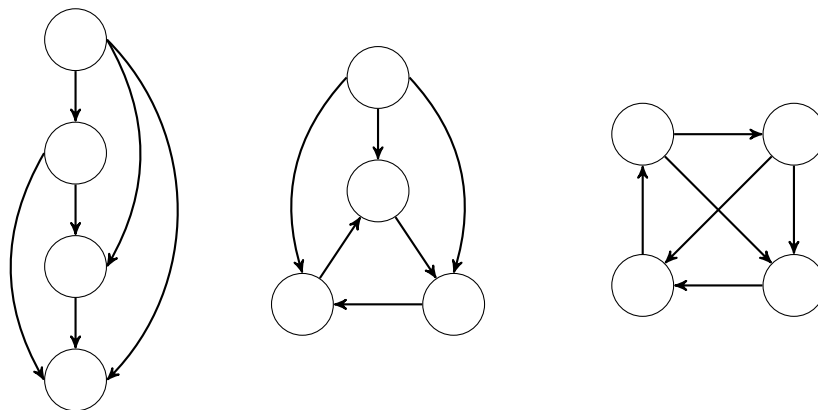


Figure 1.1: Three types of tournaments on four elements. From the left: A transitive ordering (there are 24 in total), an overall winner on top of a 3-cycle (16 in total together with a symmetric case of overall loser) and a 4-cycle (24 in total).

In this paper we settle a version of this conjecture in accordance with the Polymath experiments. We are working in the model of random balanced dice with faces uniform in $[0, 1]$: The

dice are drawn with faces iid uniform in $[0, 1]$ and conditioned on the face-sums equal to $n/2$. To be precise, we think of drawing balanced dice by rejection sampling: Let (a_1, \dots, a_{n-1}) be iid in $[0, 1]$, set $a_n := n/2 - \sum_{i=1}^{n-1} a_i$ and accept and output a sample if $0 \leq a_n \leq 1$, otherwise reject and repeat.

Theorem 1.3. *Let A, B, C, D be four random dice with n faces iid uniform in $[0, 1]$ conditioned on all face-sums equal to $n/2$. Then, there exists $\varepsilon > 0$ such that, for n large enough, the probability that the “beats” relation on A, B, C, D is transitive is larger than $3/8 + \varepsilon$.*

Theorem 1.3 implies that the dice tournament is not quasirandom, since there are $2^{\binom{4}{2}} = 64$ tournaments on four elements and $4! = 24$ of them, i.e., $3/8$ fraction, are transitive. We note that our main motivation for the choice of the continuous distribution on $[0, 1]$ was the ease of presentation (in particular we do not have to worry about ties). While we have not verified the details, we believe that our proof strategy can be adapted to the discrete case without conceptual difficulties.

The proof of Theorem 1.3 combines the main ideas present in two preceding works [Pol22] and [HMRZ20] and can roughly be divided into two parts inspired by these papers. However, there is a considerable number of items that need to be fleshed out and some care is required to get the details right.

The resulting ε in Theorem 1.3 seems tiny and we do not attempt to estimate it. We also note that in our model, for typical balanced dice A and B , the “one-roll probability”

$$P_1(n) := \Pr[A \text{ rolls higher than } B] = \frac{1}{n^2} \sum_{i,j=1}^n \left(\mathbb{1}(a_i > b_j) - \mathbb{1}(a_i < b_j) \right)$$

is rather close to half, satisfying $1/2 - c/n \leq P_1(n) \leq 1/2 + C/n$ for some universal constants $c, C > 0$. Therefore, in the game we described before it takes the second player order of n dice rolls to achieve one expected unit of payoff.

We leave open some intriguing questions. For example, is there a more complete characterization of the “dice tournament” except for the fact that it is not quasirandom?¹ Is there another natural tweak to the model that makes the tournament quasirandom or is there a reason to believe otherwise?

To explain how Theorem 1.3 comes about, it is worthwhile to state its equivalent version that is a close analogue of Theorem 1.2. Indeed, in Section 2.1 we show that the following statement implies Theorem 1.3:

Theorem 1.4. *Let A, B, C be three random dice with n faces iid uniform in $[0, 1]$ conditioned on all face-sums equal to $n/2$. Then, there exists $\varepsilon > 0$ such that, for large enough n , with probability at least ε we have $\left| \Pr[A \succ B, C \mid B, C] - 1/4 \right| > \varepsilon$.*

In the rest of this section we are going to give a sketch of the strategy that we use to prove Theorem 1.4.

¹We note that this question was extensively addressed and answered in a subsequent work by Sah and Sawney [SS24].

Proof strategy For the sake of clarity let us assume that we want to demonstrate a slightly stronger statement $\Pr[A \succ B, C \mid B, C] > 1/4 + \varepsilon$ with constant probability. Intuitively, we would like to show that, with constant probability, two random balanced dice B, C are, in some sense, close to each other and therefore there is positive correlation in the “beats” relation with another random die A . The equivalence between Theorems 1.3 and 1.4 has already been spotted by Polymath and [Gow17b] discusses some hypotheses on how the correct notion of closeness could look like.

To continue, we need to say a few more words about the ideas used by Polymath to prove Theorem 1.2. For a die A with faces in $[0, 1]$ and $0 \leq x \leq 1$, let

$$f_A(x) := \sum_{i=1}^n \mathbb{1}[a_i \leq x] = \left| \{i \in [n] : a_i \leq x\} \right|, \quad g_A(x) := f_A(x) - xn. \quad (1.1)$$

One checks that, as long as there are no ties between the faces, a die A beats B if and only if $\sum_{i=1}^n f_A(b_i) < n^2/2$. Furthermore, for balanced A and B it holds that

$$\sum_{i=1}^n g_A(b_i) = \sum_{i=1}^n f_A(b_i) - n^2/2,$$

thus A beats B if and only if $\sum_{i=1}^n g_A(b_i) < 0$. For a fixed balanced die A , let V be a uniform random variable in $[0, 1]$ and $U_A := g_A(V)$. Now the event “ A beats B ” for a random balanced die B can be written as

$$U_A^{*n} < 0 \quad \text{conditioned on} \quad V^{*n} = n/2,$$

where $(*n)$ denotes the n -wise convolution of random variables, i.e., $(U_A^{*n}, V^{*n}) = (\sum_{i=1}^n U_{A,i}, \sum_{i=1}^n V_i)$, and $(U_{A,i}, V_i)$ are n iid copies of the joint distribution of (U_A, V) . [Pol22] shows that, in their setting, a precise version of central limit theorem holds for (U_A, V) for a typical balanced A . Since $\mathbb{E}U_A = 0$ and we condition on V^{*n} being equal to its expectation, after the conditioning U_A^{*n} is still close to a centered Gaussian and Theorem 1.2 follows.

In order to prove Theorem 1.4, we follow suggestions from the Polymath discussion and extend this limit theorem to a triple of random variables (U_B, U_C, V) for typical random balanced dice B and C . Now a notion of closeness suggests itself: Since we are interested in finding B, C for which $\Pr[U_B^{*n}, U_C^{*n} > 0]$ is bounded away from $1/4$, it might be helpful to prove that, with constant probability, the covariance $\text{Cov}[U_B, U_C]$ is bounded away from zero. Since for balanced B and C it holds that

$$\int_0^1 (g_B(x) - g_C(x))^2 dx = \text{Var} U_B + \text{Var} U_C - 2 \text{Cov}[U_B, U_C],$$

a somewhat large value of the covariance is roughly equivalent to somewhat small L^2 distance between functions g_B and g_C . While we follow this intuitive picture, in the proofs we will be always dealing directly with the covariance.

As a matter of fact, the preceding discussion is not quite correct. Since we are conditioning on $V^{*n} = n/2$, we need to look at the *conditional covariance*, that is exclude the possibility that all the covariance between U_B and U_C is “mediated” through V . An elementary calculation

shows that the correct expression is $CV_{\text{cond}} := \text{Cov}[U_B, U_C] - 12 \text{Cov}[U_B, V] \text{Cov}[U_C, V]$. The event $|\Pr[A \succ B, C \mid B, C] - 1/4| > \varepsilon$ is essentially equivalent to $|CV_{\text{cond}}|$ being not too small compared to the variances of U_B and U_C .

We show the bound on $|CV_{\text{cond}}|$ using the second moment method. Thinking of CV_{cond} as a random variable over the choice of balanced B and C , we bound its second and fourth moments and apply the Paley–Zygmund inequality to conclude that indeed sometimes it is bounded away from zero. In the proof we employ a modification of the argument used in [HMRZ20]: If B and C are not balanced, estimating the moments is an elementary calculation. To account for the conditioning on balanced B and C , we employ yet another precise local central limit theorem to estimate changes in the relevant moments. This requires some care, but in the end we conclude that the conditioning does not fundamentally alter the behavior of CV_{cond} .

Organization The rest of the paper contains the proof of Theorem 1.3. Section 2 includes a more detailed outline, while Sections 3 and 4 the details, of, respectively, our central limit theorem and the moment calculations.

Notation All statements of theorems, lemmas etc. concerning dice are meant to hold for number of faces n large enough, even if not explicitly mentioned. We also assume for ease of presentation that the number of faces n is odd, but the proof can be adapted to the case of even n . All logarithms are natural. We use $\mathcal{O}(\cdot)$ for the Big-oh notation, as well as $f = \Omega(g)$ for $g = \mathcal{O}(f)$ and $f = \Theta(g)$ for $f = \mathcal{O}(g)$ and $f = \Omega(g)$. Furthermore, $\tilde{\mathcal{O}}(f(n))$ denotes $\mathcal{O}(f(n) \cdot \log^c n)$ for some fixed c . For $S \subseteq \mathbb{R}$, we let $d_S(x) := \inf \{|x - s| : s \in S\}$. For a random variable U , we use the notation $\|U\|_\infty$ for the essential supremum of $|U|$.

As we mentioned, by a “random balanced die” we mean a die with iid uniform faces constrained to have the expected face-sum by rejection sampling. Equivalently, if we consider the uniform measure μ over $a_1, \dots, a_{n-1} \in [0, 1]^{n-1}$ and adopt the convention that given a_1, \dots, a_{n-1} we let $a_n := n/2 - \sum_{i=1}^{n-1} a_i$, then we decree that for an integrable $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we have

$$\mathbb{E} \left[f(a_1, \dots, a_n) \left| \sum_{i=1}^n a_i = n/2 \right. \right] := \frac{\int f(a_1, \dots, a_n) \cdot \mathbb{1}(0 \leq a_n \leq 1) \, d\mu}{\int \mathbb{1}(0 \leq a_n \leq 1) \, d\mu}. \quad (1.2)$$

2. Proof outline

In this section we present the main steps in the proof of Theorem 1.3, leaving more technical details for the following sections. We present the proof divided into a number of stages, starting with reducing Theorem 1.3 to Theorem 1.4.

2.1. Reduction from four to three dice

In order to make a connection between Theorems 1.3 and 1.4, we also need a version of Theorem 1.2 adapted to our setting.

Theorem 2.1. *Let A, B be two random balanced dice with faces uniform in $[0, 1]$. Then, except with probability $1/n$ over the choice of A , we have*

$$\left| \Pr \left[A \succ B \mid A \right] - \frac{1}{2} \right| = \tilde{O} \left(\frac{1}{\sqrt{n}} \right).$$

While one could prove Theorem 2.1 directly by adapting the argument from [Pol22], we will obtain it as a byproduct in the proof of Theorem 1.4.

We will now show that Theorem 1.3 can be deduced from Theorems 1.4 and 2.1. We start with some preliminary observations. Note that in our model almost surely all faces of the sampled dice will have different values. In particular, assuming n odd, the draws between dice (i.e., situations where neither $A \succ B$ nor $B \succ A$) occur with probability zero. Therefore, almost surely, the “beats” relation on three random balanced dice forms one of the eight possible tournaments. As discussed, six of these tournaments are transitive and the other two are cycles, and by symmetry the tournaments inside each class occur with the same probability. Denoting the overall probabilities of the two classes as $P_{3\text{-line}}$ and P_{Δ} , we have, e.g.,

$$\begin{aligned} P_{3\text{-line}} &= 6 \Pr[A \succ B, C \text{ and } B \succ C], \\ P_{\Delta} &= 2 \Pr[A \succ B \succ C \succ A]. \end{aligned}$$

Similarly, a random tournament on four balanced dice is characterized by three probabilities of different types of tournaments on four elements, as seen in Figure 1.1. These are 24 transitive tournaments, 16 tournaments with a unanimous winner (or loser) above (respectively below) a 3-cycle, and 24 4-cycles. Again, tournaments inside each class are equally likely and we denote the respective overall probabilities as $P_{4\text{-line}}$, $P_{1,\Delta}$ and P_{\square} .

We will need a simple application of Markov’s inequality:

Claim 2.2. *Let a random variable X satisfy $|X - \mathbb{E} X| \leq 1$ almost surely.*

If $\Pr[|X - \mathbb{E} X| \geq \varepsilon_1] \leq \varepsilon_2$, then $\text{Var } X \leq \varepsilon_1^2 + \varepsilon_2$.

On the other hand, if $\Pr[|X - \mathbb{E} X| \geq c_1] \geq c_2$, then $\text{Var } X \geq c_1^2 c_2$.

We can now state and prove the lemma establishing the reduction:

Lemma 2.3. *Let A, B, C, D be random balanced dice and let $X := \Pr[A \succ B \mid A]$ and $Y := \Pr[A \succ B, C \mid B, C]$. Assume that for some constants $\varepsilon_1, \varepsilon_2, c_1, c_2 > 0$ such that $c_1 - \varepsilon_1^2 - \varepsilon_2 \geq 0$ the following two conditions hold:*

$$\Pr \left[\left| X - \frac{1}{2} \right| \geq \varepsilon_1 \right] \leq \varepsilon_2, \tag{2.1}$$

$$\Pr \left[\left| Y - \frac{1}{4} \right| \geq c_1 \right] \geq c_2. \tag{2.2}$$

Then,

$$P_{4\text{-line}} \geq \frac{3}{8} + 6c_2(c_1 - \varepsilon_1^2 - \varepsilon_2)^2.$$

Proof. Note that if $A, B \succ C, D$, then A, B, C, D form a transitive tournament. Hence, using symmetry considerations,

$$\mathbb{E} Y^2 = \mathbb{E} \Pr[A, B \succ C, D \mid C, D] = \Pr[A, B \succ C, D] = \frac{1}{6} P_{4\text{-line}} .$$

Accordingly, we have $P_{4\text{-line}} = 6 \mathbb{E} Y^2$. At the same time, by symmetry we have $\mathbb{E} X = 1/2$ and, applying Claim 2.2 to (2.1), $\text{Var} X \leq \varepsilon_1^2 + \varepsilon_2$.

Noting that $\mathbb{E} Y = \Pr[A \succ B, C] = \mathbb{E} X^2 = \mathbb{E}^2 X + \text{Var} X \leq 1/4 + \varepsilon_1^2 + \varepsilon_2$, we apply (2.2) and Claim 2.2 to conclude

$$P_{4\text{-line}} = 6 \mathbb{E} Y^2 = 6 \mathbb{E}^2 Y + 6 \text{Var} Y \geq 6 \mathbb{E}^4 X + 6 \text{Var} Y \geq \frac{3}{8} + 6c_2(c_1 - \varepsilon_1^2 - \varepsilon_2)^2 . \quad \square$$

The fact that Theorem 1.3 follows from Theorems 1.4 and 2.1 is now a straightforward application of Lemma 2.3.

Remark 2.4. For a somewhat broader picture of the dice tournament, note that a similar argument as in the proof of Lemma 2.3 gives

$$3 \mathbb{E} X^2 = 3 \Pr[A \succ B, C] = P_{3\text{-line}} .$$

Hence, as a result of Theorem 2.1 we have estimates

$$P_{3\text{-line}} = 3 \mathbb{E} X^2 = 3 \mathbb{E}^2 X + 3 \text{Var} X = \frac{3}{4} + \tilde{\mathcal{O}}(1/n)$$

and similarly $P_{\Delta} = 1/4 - \tilde{\mathcal{O}}(1/n)$, meaning that in the limit all eight tournaments are equally likely.

With four dice, observe that choosing three dice is equivalent to choosing four (unordered) dice and then randomly discarding one of those four and permuting the remaining three. Analyzing this procedure, we see that

$$\begin{aligned} P_{3\text{-line}} &= P_{4\text{-line}} + \frac{1}{2} P_{\square} + \frac{3}{4} P_{1,\Delta} , \\ P_{\Delta} &= \frac{1}{2} P_{\square} + \frac{1}{4} P_{1,\Delta} , \end{aligned}$$

from which it follows that in our case $P_{4\text{-line}} = P_{\square} + \tilde{\mathcal{O}}(1/n)$.

Therefore, in light of Theorem 1.3, the following picture emerges. Asymptotically, all transitive tournaments and all 4-cycles have the same probability² $1/64 + c$ for some constant $c > 0$, while all “winner/loser + 3-cycle” tournaments have probability $1/64 - 3c$.

²Technically we only prove that this probability is $1/64 + c_n$ for some $c_n > c > 0$ without excluding the possibility that c_n does not converge.

2.2. Function g_A

In order to prove Theorems 1.4 and 2.1, a crucial quantity is the g_A function given by (1.1) (and already defined by Polymath). For reasons of presentation we will consider dice with faces sampled from different uniform distributions on intervals, not necessarily $[0, 1]$. In that general case of an interval $[z_1, z_2]$, a die is called balanced if its face-sum is equal to $n(z_1 + z_2)/2$. We still consider sampling uniform balanced dice.

Given a die $A = (a_1, \dots, a_n)$ with faces sampled from such a uniform distribution on an interval with cdf F , we let for $z_1 \leq x \leq z_2$

$$f_A(x) := \sum_{i=1}^n \mathbb{1}[a_i \leq x] \ , \quad g_A(x) := f_A(x) - F(x) \cdot n \ .$$

Indeed, in our current setting of faces in $[0, 1]$ this formula matches (1.1). In general, the value of $g_A(x)$ indicates the difference between the number of faces of A not exceeding x and the expected number of faces not exceeding x if drawn iid from the distribution.

We now state a number of straightforward claims about the function g_A . The proofs are calculations using that $F(x) = \frac{x-z_1}{z_2-z_1}$ for $z_1 \leq x \leq z_2$.

Claim 2.5. *In the following let A and B be fixed dice with n faces from $[z_1, z_2]$ and V a random variable uniform in $[z_1, z_2]$ independent of everything else:*

1. $\mathbb{E} F(V) = 1/2$. Furthermore, if A is balanced, then $\sum_{i=1}^n F(a_i) = n/2$.
2. If A is balanced, then $\mathbb{E} g_A(V) = 0$.
3. If B is balanced and $a_i \neq b_j$ for every pair of faces, then B beats A if and only if $\sum_{i=1}^n g_A(b_i) > 0$.
4. If B has face-sum b , then $\sum_{i=1}^n g_A(b_i) \in \mathbb{Z} - n \frac{b-nz_1}{z_2-z_1}$. In particular, if B is balanced and n is odd, then $\sum_{i=1}^n g_A(b_i) \in \mathbb{Z} + 1/2$.

Let V be a random variable uniform in $[0, 1]$ and for a die A let $U_A := g_A(V)$. By the above, and using the notation $(U_A^{*n}, U_B^{*n}, U_C^{*n}, V^{*n})$ for the n -fold convolution of the random tuple (U_A, U_B, U_C, V) , for balanced dice A, B, C with faces uniform in $[0, 1]$ we have

$$\begin{aligned} \Pr[A \succ B, C \mid B, C] &= \Pr \left[U_B^{*n}, U_C^{*n} > 0 \mid B, C, V^{*n} = \frac{n}{2} \right] \ , \\ \Pr[A \succ B \mid A] &= \Pr \left[U_A^{*n} < 0 \mid A, V^{*n} = \frac{n}{2} \right] \ . \end{aligned}$$

Therefore, from now on we can focus on proving the following equivalent versions of Theorems 1.4 and 2.1. For clarity we omit the conditioning on dice A and B from the probabilities:

Theorem 2.6. *There exists $\varepsilon > 0$ such that with probability at least ε over the choice of two random balanced dice A and B with faces in $[0, 1]$ we have*

$$\left| \Pr \left[U_A^{*n}, U_B^{*n} > 0 \mid V^{*n} = \frac{n}{2} \right] - \frac{1}{4} \right| > \varepsilon \ .$$

Theorem 2.7. *Except with probability $1/n$ over the choice of a random balanced die A with faces in $[0, 1]$,*

$$\left| \Pr \left[U_A^{*n} > 0 \mid V^{*n} = \frac{n}{2} \right] - \frac{1}{2} \right| = \tilde{\mathcal{O}} \left(\frac{1}{\sqrt{n}} \right).$$

In the following sections we explain how to prove Theorems 2.6 and 2.7.

2.3. Central limit theorem for g_A

One method for understanding the distribution of the random variables $(U_A^{*n}, U_B^{*n}, V^{*n})$ is to establish a limit theorem for them. This is indeed the approach taken by [Pol22] and the one we adopt. To that end, let (G_A, G_B, H) be joint Gaussians with the same first and second moments as (U_A, U_B, V) . Recall that by Claim 2.5 we have $\mathbb{E} U_A = \mathbb{E} U_B = 0$ and $\mathbb{E} V = 1/2$. In addition, we introduce the notation for the respective variances

$$\begin{aligned} \text{Var}_A &:= \text{Var} U_A, & \text{Var}_B &:= \text{Var} U_B, \\ \text{CV}_{AB} &:= \text{Cov}[U_A, U_B], & \text{CV}_A &:= \text{Cov}[U_A, V], & \text{CV}_B &:= \text{Cov}[U_B, V]. \end{aligned}$$

In the end the main result we prove is

Theorem 2.8. *Fix $\varepsilon > 0$. Except with probability $1/n$ over the choice of two balanced dice A and B , we have the following: If the dice satisfy $\text{Var}_A - (\text{CV}_A^2 / \text{Var} V)$, $\text{Var}_B - (\text{CV}_B^2 / \text{Var} V) \geq \varepsilon n$, then*

$$\left| \Pr \left[U_A^{*n}, U_B^{*n} > 0 \mid V^{*n} = \mathbb{E} V^{*n} \right] - \Pr \left[G_A^{*n}, G_B^{*n} > 0 \mid H^{*n} = \mathbb{E} H^{*n} \right] \right| = \tilde{\mathcal{O}} \left(\frac{1}{\sqrt{n}} \right).$$

In Section 3 we also show that Theorem 2.7 is a direct consequence of intermediate results established on the way to proving Theorem 2.8. We note that the “error probability” $1/n$ in the statement of Theorem 2.8 can be decreased, at least to n^{-k} for arbitrary k . Before proceeding with the proof outline, let us explain the $\text{Var}_A - (\text{CV}_A^2 / \text{Var} V)$ factors from the statement of the theorem. The reason for them is the following claim, which is proved by a standard calculation in Section 3.1:

Claim 2.9. *Assume A and B are balanced dice with uniform faces in an interval $[z_1, z_2]$. Conditioned on $H^{*n} = \mathbb{E} H^{*n}$, random variables G_A^{*n} and G_B^{*n} are joint centered Gaussians and furthermore:*

$$\text{Var} \left[G_A^{*n} \mid H^{*n} = \mathbb{E} H^{*n} \right] = n \left(\text{Var}_A - \frac{\text{CV}_A^2}{\text{Var} V} \right), \quad (2.3)$$

$$\text{Var} \left[G_B^{*n} \mid H^{*n} = \mathbb{E} H^{*n} \right] = n \left(\text{Var}_B - \frac{\text{CV}_B^2}{\text{Var} V} \right), \quad (2.4)$$

$$\text{Cov} \left[G_A^{*n}, G_B^{*n} \mid H^{*n} = \mathbb{E} H^{*n} \right] = n \left(\text{CV}_{AB} - \frac{\text{CV}_A \text{CV}_B}{\text{Var} V} \right). \quad (2.5)$$

Proof strategy for Theorem 2.8 As a first step, for consistency with Polymath [Pol22] we find it convenient to consider balanced dice with faces uniform in $[0, n]$. That is, we assume that A and B are random dice with faces iid uniform in $[0, n]$ and conditioned on face-sums equal to $\mathbb{E} V^{*n} = \mathbb{E} H^{*n} = n^2/2$. Furthermore, we have $g_A(x) = f_A(x) - x$ and V is uniform in $[0, n]$.

Using the natural measure-preserving bijection between balanced dice $A \leftrightarrow nA$, it is easy to check that the distribution of random variable Var_A in the $[0, n]$ setting is the same as the distribution of Var_A in the $[0, 1]$ setting. Furthermore, since the covariance CV_A is multiplied by a factor n in the $[0, n]$ setting and the variance $\text{Var} V$ is n^2 times larger, also the random variable $\text{Var}_A - (CV_A^2 / \text{Var} V)$ has the same distribution. The upshot is that in the proof of Theorem 2.8 we can assume that the faces of A and B are drawn from $[0, n]$ and indeed this is what we will do from now on and throughout Section 3.

The proof strategy follows Polymath in most important respects. That is, we proceed by directly bounding the characteristic function of the triple of random variables $(U_A, U_B, V - n/2)$:

$$\widehat{f}(\alpha, \beta, \gamma) := \mathbb{E} e\left(\alpha U_A + \beta U_B + \gamma(V - n/2)\right),$$

where $e(x)$ is a shorthand for $\exp(2\pi i x)$. The main bound we achieve is given in the following lemma:

Lemma 2.10. *Except with probability n^{-2} over the choice of balanced dice A and B , the following holds: $\|U_A\|_\infty, \|U_B\|_\infty \leq 5\sqrt{n \log n}$ and furthermore, for every $\alpha, \beta, \gamma \in \mathbb{R}$ such that $|\alpha|, |\beta| \leq 1/2$ and either $|\alpha|$ or $|\beta|$ exceeds $10^{10} \log n/n$ or $|\gamma|$ exceeds $6 \log^2 n/n^{3/2}$,*

$$|\widehat{f}(\alpha, \beta, \gamma)| \leq 1 - \frac{10 \log n}{n},$$

and consequently

$$|\widehat{f}(\alpha, \beta, \gamma)|^n \leq n^{-10}.$$

Recall that (G_A, G_B, H) are joint Gaussians with the first and second moments matching those of (U_A, U_B, V) . We let

$$\begin{aligned} \widehat{g}(\alpha, \beta, \gamma) &:= \mathbb{E} e\left(\alpha G_A^{*n} + \beta G_B^{*n} + \gamma(H^{*n} - n^2/2)\right), \\ \widehat{u}(\alpha, \beta, \gamma) &:= \begin{cases} \widehat{f}(\alpha, \beta, \gamma)^n & \text{if } |\alpha|, |\beta| \leq 1/2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.6}$$

We then use Lemma 2.10 and a couple of other bounds to establish

Lemma 2.11. *Whenever the thesis of Lemma 2.10 holds, in particular except with probability at most n^{-2} , we have*

$$\|\widehat{g} - \widehat{u}\|_1 = \int_{\mathbb{R}^3} |\widehat{g}(\alpha, \beta, \gamma) - \widehat{u}(\alpha, \beta, \gamma)| \, d\alpha \beta \gamma \leq \frac{\log^{16} n}{n^4}. \tag{2.7}$$

This in turn is used in the proof of Theorem 2.8 by applying Fourier inversion formulas. Similar arguments allow to directly prove Theorem 2.7. Full proofs of Theorems 2.8 and 2.7 and Lemmas 2.10 and 2.11 are given in Section 3. While we presented a rough outline here, there are multiple technical details to take care of. Most (but not all) of them are handled in a way which is identical to or inspired by the CLT proof in [Pol22]. Except for adaptations of arguments to the continuous setting, among the main differences to Polymath are: using a grid of points and interpolating by Lemma 3.15; applying Poisson limit theorem in addition to Berry–Esseen in the proof of Lemma 3.20; and a more careful estimate required in the final proof of Theorem 2.8.

2.4. Bounding the covariance

If the random variables $(U_A^{*n}, U_B^{*n}, V^{*n})$ behave like Gaussians, the way to show that

$$\Pr[U_A^{*n}, U_B^{*n} > 0 \mid V^{*n} = \mathbb{E} V^{*n}]$$

is significantly different from $1/4$ is to bound away from zero the relevant (conditional) covariance between U_A and U_B . More precisely, by (2.5) we are interested in the expression $CV_{AB} - (CV_A CV_B / \text{Var} V)$. Let us make more concise notation

$$CV_{AB,\text{cond}} := CV_{AB} - \frac{CV_A CV_B}{\text{Var} V}, \quad (2.8)$$

$$\text{Var}_{A,\text{cond}} := \text{Var}_A - \frac{CV_A^2}{\text{Var} V}, \quad (2.9)$$

$$\text{Var}_{B,\text{cond}} := \text{Var}_B - \frac{CV_B^2}{\text{Var} V}. \quad (2.10)$$

We now give two main lemmas that we use in the proof of Theorem 2.6. In the following the probability is over the choice of random balanced dice A, B with faces in $[0, 1]$.

Lemma 2.12. *For every $\varepsilon > 0$ there exists $K > 0$ such that*

$$\Pr \left[\text{Var}_{A,\text{cond}} > Kn \right] < \varepsilon.$$

Lemma 2.13. *There exists $\varepsilon > 0$ such that*

$$\Pr \left[|CV_{AB,\text{cond}}| > \varepsilon n \right] > \varepsilon.$$

Proof strategy for Lemmas 2.12 and 2.13 To start with, as in Section 2.3 we will argue that throughout the whole proof we can assume faces come from a uniform distribution on an interval different from $[0, 1]$. This time, for consistency with [HMRZ20] and to simplify some calculations we take the faces to be uniform in $[-\sqrt{3}, \sqrt{3}]$. That is, the single-face distribution is centered with variance $\text{Var} V = \text{Var} H = 1$. As in Section 2.3, this does not change the joint distribution of the random variables $\text{Var}_{A,\text{cond}}$, $\text{Var}_{B,\text{cond}}$ and $CV_{AB,\text{cond}}$. Therefore, from now on we assume the faces and random variable V are uniform in $[-\sqrt{3}, \sqrt{3}]$ and that a balanced die has face-sum zero.

Lemmas 2.12 and 2.13 are proved by the first and second-moment methods. More precisely, in Section 4 we establish the following moment bounds:

Lemma 2.14. $\mathbb{E} \text{Var}_A^2 = \mathcal{O}(n^2)$.

Lemma 2.15. $\mathbb{E} CV_{AB,\text{cond}}^2 = \Omega(n^2)$.

These lemmas are proved through a careful calculation. If A and B are random (not balanced) dice, the expressions $\mathbb{E} \text{Var}_A^2$ and $\mathbb{E} CV_{AB,\text{cond}}^2$ are polynomials in n that we can explicitly compute. Adding conditioning on balanced dice changes the formulas, but if we use a precise version of the local central limit theorem for densities from [Pet75] we can sufficiently control the error terms and show that qualitatively the moment bounds remain the same. The technique we use is very similar to the one employed in [HMRZ20] (in particular in the proof of Proposition 1 therein), but we need to use more error terms than them. On the other hand, some expressions simplify since we consider a special case of the uniform distribution. The proofs of both lemmas are left to Section 4.

Given Lemmas 2.14 and 2.15 it is not difficult to establish Lemmas 2.12 and 2.13:

Proof of Lemma 2.12. Fix $\varepsilon > 0$. By Lemma 2.14 we have $\mathbb{E} \text{Var}_A^2 = \mathcal{O}(n^2)$. But now by Markov

$$\Pr [\text{Var}_{A,\text{cond}} > Kn] \leq \Pr [\text{Var}_A > Kn] = \Pr [\text{Var}_A^2 > K^2 n^2] = \mathcal{O} \left(\frac{1}{K^2} \right) < \varepsilon ,$$

where we can certainly satisfy the last inequality by choosing K large enough. □

Proof of Lemma 2.13. By Lemma 2.15 we have $\mathbb{E} CV_{AB,\text{cond}}^2 = \Omega(n^2)$. On the other hand, applying Lemma 2.14 and Cauchy–Schwarz multiple times, we also see that

$$\begin{aligned} \mathbb{E} CV_{AB}^4 &\leq \mathbb{E} [\text{Var}_A^2 \text{Var}_B^2] = \mathbb{E}^2 \text{Var}_A^2 = \mathcal{O}(n^4) , \\ \mathbb{E} (CV_A CV_B)^4 &= \mathbb{E}^2 CV_A^4 \leq \mathbb{E}^2 \text{Var}_A^2 = \mathcal{O}(n^4) , \end{aligned}$$

and therefore also

$$\mathbb{E} CV_{AB,\text{cond}}^4 = \mathbb{E} (CV_{AB} - CV_A CV_B)^4 \leq 16 (\mathbb{E} CV_{AB}^4 + \mathbb{E} (CV_A CV_B)^4) = \mathcal{O}(n^4) .$$

Recalling the Paley–Zygmund inequality $\Pr[Z > c] \geq (1 - c/\mathbb{E} Z)^2 \frac{(\mathbb{E} Z)^2}{\mathbb{E} Z^2}$ for $Z \geq 0$ and $0 \leq c \leq \mathbb{E} Z$, we apply it to $Z = CV_{AB,\text{cond}}^2$ and get

$$\begin{aligned} \Pr [|CV_{AB,\text{cond}}| > \varepsilon n] &= \Pr [CV_{AB,\text{cond}}^2 > \varepsilon^2 n^2] \\ &\geq \left(1 - \frac{\varepsilon^2 n^2}{\mathbb{E} CV_{AB,\text{cond}}^2} \right)^2 \cdot \frac{\mathbb{E}^2 CV_{AB,\text{cond}}^2}{\mathbb{E} CV_{AB,\text{cond}}^4} > \varepsilon > 0 , \end{aligned}$$

where the last inequality is true if ε is chosen to be a small enough absolute constant. □

2.5. Proof of Theorem 2.6

We present the remaining steps in the proof of Theorem 2.6. Recall the notation we defined in (2.8). From Lemmas 2.12 and 2.13 we know that for some constants $\varepsilon, K > 0$, with probability at least ε , random choice of balanced A and B gives the dice such that $\text{Var}_{A,\text{cond}}, \text{Var}_{B,\text{cond}} \leq Kn$ and $|CV_{AB,\text{cond}}| > \varepsilon n$ all hold. In fact, this means that at least one of $CV_{AB,\text{cond}} > \varepsilon n$ or $CV_{AB,\text{cond}} < -\varepsilon n$ holds with probability at least $\varepsilon/2$. Let us consider the former case $CV_{AB,\text{cond}} > \varepsilon n$, the latter being symmetric.

Note that by Claim 2.9 random variables G_A^{*n} and G_B^{*n} conditioned on $H^{*n} = n/2$ are joint centered Gaussians with variances respectively $n \text{Var}_{A,\text{cond}}$ and $n \text{Var}_{B,\text{cond}}$ and covariance $nCV_{AB,\text{cond}}$. Let

$$\Gamma_\rho(a, b) := \Pr \left[\mathcal{G}_1 < a \text{ and } \mathcal{G}_2 < b \right]$$

where \mathcal{G}_1 and \mathcal{G}_2 are joint, centered, unit variance Gaussians with covariance ρ . We will need a couple of standard properties of Γ_ρ summarized in the following:

Claim 2.16. *Using the notation above, we have*

$$\Gamma_\rho(0, 0) = \Pr[\mathcal{G}_1, \mathcal{G}_2 < 0] = \Pr[\mathcal{G}_1, \mathcal{G}_2 > 0].$$

Furthermore, for $-1 \leq \rho \leq 1$ the value of $\Gamma_\rho(0, 0)$ is a strictly increasing continuous function of ρ , such that $\Gamma_{-1}(0, 0) = 0$, $\Gamma_0(0, 0) = 1/4$ and $\Gamma_1(0, 0) = 1/2$.

By the discussion above, letting

$$\rho := \frac{nCV_{AB,\text{cond}}}{\sqrt{n^2 \text{Var}_{A,\text{cond}} \text{Var}_{B,\text{cond}}}} > \frac{\varepsilon}{K} > 0,$$

we have

$$\Pr \left[G_A^{*n}, G_B^{*n} > 0 \mid H^{*n} = \frac{n}{2} \right] = \Gamma_\rho(0, 0) > \frac{1}{4} + \delta$$

for some absolute constant $\delta > 0$. Furthermore, by Cauchy–Schwarz,

$$\text{Var}_{A,\text{cond}} \geq \frac{CV_{AB,\text{cond}}^2}{\text{Var}_{B,\text{cond}}} \geq \frac{\varepsilon^2}{K} \cdot n$$

and similarly $\text{Var}_{B,\text{cond}} \geq \varepsilon^2 n / K$. Consequently, except with probability $1/n$, the conclusion of Theorem 2.8 holds. Overall, with probability at least $\varepsilon/2 - 1/n > \varepsilon/4$ we have

$$\Pr \left[U_A^{*n}, U_B^{*n} > 0 \mid V^{*n} = \frac{n}{2} \right] > \frac{1}{4} + \delta - \tilde{\mathcal{O}} \left(\frac{1}{\sqrt{n}} \right) > \frac{1}{4} + \delta/2. \quad \square$$

3. CLT for two dice

In this section we prove the results concerned with our central limit theorem for dice, that is Theorems 2.7 and 2.8, Claim 2.9 and Lemmas 2.10 and 2.11. We remind that in this section we assume that the dice have faces which are iid uniform in $[0, n]$, a balanced die has face-sum $n^2/2$ and $g_A(x) = f_A(x) - x$. The method for the most part follows the Polymath draft [Pol22] with adaptations to the continuous setting.

The proof takes some space and we separate it into several modules. We start with some basic machinery of characteristic functions and Gaussians, as well as some useful tools and concentration bounds. Then, we give concentration bounds on U_A and U_B and related Lipschitz bounds on the characteristic function \hat{f} . Subsequently, we address the decay of \hat{f} in different regimes, e.g., small $|\alpha|$ and $|\beta|$ and/or large $|\gamma|$. Finally, we put everything together in the proofs of the aforementioned lemmas and theorems.

3.1. Characteristic functions and joint Gaussians

In this section we gather some basic claims about characteristic functions and Gaussian random variables. Before we proceed, let us make a more systematic exposition of our notation. Recall from (2.6) that the characteristic function of the convoluted Gaussians $(G_A^{*n}, G_B^{*n}, H^{*n} - n^2/2)$ is denoted by $\hat{g}(\alpha, \beta, \gamma)$. Accordingly, we let $g(x, y, z)$ be the joint density function of these random variables. Furthermore, we let $g(z)$ to be the marginal density of $H^{*n} - n^2/2$. Analogously, recall that $\hat{u}(\alpha, \beta, \gamma)$ is the characteristic function of convoluted random variables $(U_A^{*n}, U_B^{*n}, V^{*n} - n^2/2)$ (as we will discuss shortly, since the first two random variables lie on a lattice, the relevant domain is $|\alpha|, |\beta| \leq 1/2$). Only the third of these random variables is fully continuous, so there is no joint density, but we denote by $u(z)$ the marginal density of $V^{*n} - n^2/2$.

First, we state a claim that follows from the definition of \hat{g} and the formula for multivariate Gaussian characteristic function $\mathbb{E} \exp(i\vec{t} \cdot \vec{G}) = \exp(-\frac{1}{2}\vec{t}^T \Sigma \vec{t})$, where Σ is the covariance matrix:

Claim 3.1. $\hat{g}(\alpha, \beta, \gamma) = \exp(-nQ)$, where

$$Q := Q(\alpha, \beta, \gamma) \tag{3.1}$$

$$= 2\pi^2 \left(\alpha^2 \text{Var}_A + \beta^2 \text{Var}_B + \gamma^2 \text{Var } V + 2\alpha\beta CV_{AB} + 2\alpha\gamma CV_A + 2\beta\gamma CV_B \right). \tag{3.2}$$

Turning to the other random variables, the distribution of $(U_A^{*n}, U_B^{*n}, V^{*n})$ is a particular kind of a discrete-continuous mix. More concretely, recall from Claim 2.5 that V^{*n} is continuous, and U_A^{*n}, U_B^{*n} are both supported on $\mathbb{Z} - V^{*n}$. In particular, since we assumed n is odd, if $V^{*n} = n^2/2$, then $U_A^{*n}, U_B^{*n} \in \mathbb{Z} + 1/2$. Therefore, their inverse Fourier transform can be checked to take a particular form that we state below:

Claim 3.2 (Inverse Fourier transform). *Let $u(z)$ denote the density function of $V^{*n} - n^2/2$. For*

fixed dice A and B , for every $a, b \in \mathbb{Z} + 1/2$ we have the formula

$$\Pr \left[U_A^{*n} = a, U_B^{*n} = b \mid V^{*n} = \frac{n^2}{2} \right] \cdot u(0) \quad (3.3)$$

$$\begin{aligned} &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{-\infty}^{\infty} \widehat{f}(\alpha, \beta, \gamma)^n e(-\alpha a - \beta b) d\gamma d\beta d\alpha \\ &= \int_{\mathbb{R}^3} \widehat{u}(\alpha, \beta, \gamma) e(-\alpha a - \beta b) d\alpha d\beta d\gamma. \end{aligned} \quad (3.4)$$

At the same time, if $g(x, y, z)$ denotes the joint density of $(G_A^{*n}, G_B^{*n}, H^{*n} - n^2/2)$, then we have

$$g(a, b, 0) = \int_{\mathbb{R}^3} \widehat{g}(\alpha, \beta, \gamma) e(-\alpha a - \beta b) d\alpha d\beta d\gamma. \quad (3.5)$$

Proof. The formula in (3.5) is just standard Fourier inversion for an integrable function \widehat{g} . As for (3.4), later on in Lemma 3.22 we will prove that $|\widehat{f}(\alpha, \beta, \gamma)| \leq 1/|\gamma - \alpha - \beta|$, in particular for $-1/2 \leq \alpha, \beta \leq 1/2$ we have $|\widehat{f}(\alpha, \beta, \gamma)| \leq \frac{2}{|\gamma|}$ whenever $|\gamma| \geq 2$. Therefore, we have $|\widehat{u}(\alpha, \beta, \gamma)| \leq (2/|\gamma|)^n$ and the characteristic function \widehat{u} is integrable for $n > 1$. Accordingly, the integral in (3.4) is well-defined and we are free to change the order of integration and apply Fourier inversion in what follows.

The rest of the justification for (3.4) is a calculation that we include for completeness. Since $U_A^{*n}, U_B^{*n} \in \mathbb{Z} - V^{*n}$, let $W_A := U_A^{*n} + V^{*n} - n^2/2$ and $W_B := U_B^{*n} + V^{*n} - n^2/2$. Since we assumed n odd, that gives $W_A, W_B \in \mathbb{Z} + 1/2$ and also

$$\begin{aligned} \widehat{u}(\alpha, \beta, \gamma) &= \mathbb{E} e\left(\alpha U_A^{*n} + \beta U_B^{*n} + \gamma(V^{*n} - n^2/2)\right) \\ &= \mathbb{E} e\left(\alpha W_A + \beta W_B + (\gamma - \alpha - \beta)(V^{*n} - n^2/2)\right) =: \widehat{u}'(\alpha, \beta, \gamma - \alpha - \beta), \end{aligned}$$

where \widehat{u}' is the characteristic function of random tuple $(W_A, W_B, V^{*n} - n^2/2) \in (\mathbb{Z} + 1/2) \times (\mathbb{Z} + 1/2) \times \mathbb{R}$. Let $w(x, y, z) := \Pr[W_A = x \text{ and } W_B = y \mid V^{*n} - n^2/2 = z] \cdot u(z)$. Now we calculate, starting from the right-hand side of (3.4), for $a, b \in \mathbb{Z} + 1/2$,

$$\begin{aligned} &\int_{[-1/2, 1/2]^2 \times \mathbb{R}} \widehat{u}(\alpha, \beta, \gamma) e(-\alpha a - \beta b) d\alpha d\beta d\gamma \\ &= \int_{[-1/2, 1/2]^2 \times \mathbb{R}} \widehat{u}'(\alpha, \beta, \gamma - \alpha - \beta) e(-\alpha a - \beta b) d\alpha d\beta d\gamma \\ &= \int_{[-1/2, 1/2]^2 \times \mathbb{R}} \sum_{x, y \in \mathbb{Z} + 1/2} \int_{\mathbb{R}} w(x, y, z) \\ &\quad e\left(-\alpha(a - x) - \beta(b - y) + (\gamma - \alpha - \beta)z\right) dz d\alpha d\beta d\gamma \\ &= \sum_{x, y \in \mathbb{Z} + 1/2} \int_{[-1/2, 1/2]^2} e\left(-\alpha(a - x) - \beta(b - y)\right) \\ &\quad \int_{\mathbb{R}} \int_{\mathbb{R}} w(x, y, z) e\left((\gamma - \alpha - \beta)z\right) d\gamma dz d\alpha d\beta \\ &= \sum_{x, y \in \mathbb{Z} + 1/2} \int_{[-1/2, 1/2]^2} e\left(-\alpha(a - x) - \beta(b - y)\right) \int_{\mathbb{R}} \int_{\mathbb{R}} w(x, y, z) e(\gamma z) d\gamma dz d\alpha d\beta. \end{aligned}$$

For fixed x, y , by the Fourier inversion formula we have

$$\int_{\mathbb{R}^2} w(x, y, z)e(\gamma z) d\gamma z = w(x, y, 0) .$$

Substituting and continuing,

$$\begin{aligned} & \int_{[-1/2, 1/2]^2 \times \mathbb{R}} \widehat{u}(\alpha, \beta, \gamma)e(-\alpha a - \beta b) d\alpha\beta\gamma \\ &= \sum_{x, y \in \mathbb{Z} + 1/2} w(x, y, 0) \int_{[-1/2, 1/2]^2} e\left(-\alpha(a-x) - \beta(b-y)\right) d\alpha\beta \\ &= w(a, b, 0) , \end{aligned}$$

where in the end we used the Fourier inversion for integers since $a-x, b-y \in \mathbb{Z}$. But now we are done, since

$$\begin{aligned} w(a, b, 0) &= \Pr[W_A = a \text{ and } W_B = b \mid V^{*n} = n^2/2]u(0) \\ &= \Pr[U_A^{*n} = a \text{ and } U_B^{*n} = b \mid V^{*n} = n^2/2]u(0) . \end{aligned} \quad \square$$

We also state the standard connection between characteristic function \widehat{f} and respective moments, which follows from an application of Taylor's theorem to $e(t)$ giving $|e(t) - (1 + 2\pi it - 2\pi^2 t^2)| \leq 4\pi^3 |t|^3/3$:

Claim 3.3. *We have $\widehat{f}(\alpha, \beta, \gamma) = 1 - Q + R$, where Q is defined in (3.1) and $R = R(\alpha, \beta, \gamma)$ is such that*

$$\begin{aligned} |R| &\leq \frac{4\pi^3}{3} \left(|\alpha| \|U_A\|_\infty + |\beta| \|U_B\|_\infty + |\gamma| \|V - n/2\|_\infty \right)^3 \\ &\leq 1200 \left(|\alpha|^3 \|U_A\|_\infty^3 + |\beta|^3 \|U_B\|_\infty^3 + |\gamma|^3 \frac{n^3}{8} \right) . \end{aligned}$$

We will also need some elementary bounds on the densities of H^{*n} and V^{*n} around their means. While the anticoncentration bound in the third point in Claim 3.4 is not optimal, it will be enough for our purposes.

Claim 3.4. *Let $g(z)$ be the density function of the random variable $H^{*n} - n^2/2$ and $u(z)$ the density of $V^{*n} - n^2/2$. Then, we have:*

1. $g(0), u(0) \geq \frac{1}{4n\sqrt{n}}$.
2. For $0 < \varepsilon \leq n\sqrt{n}$, it holds that $\Pr \left[|V^{*n} - n^2/2| \leq \varepsilon \right] \geq \varepsilon/4n\sqrt{n}$.
3. For any $\varepsilon_1 < \varepsilon_2$, it holds that $\Pr \left[\varepsilon_1 \leq V^{*n} \leq \varepsilon_2 \right] \leq (\varepsilon_2 - \varepsilon_1)/n$.

Proof. Since H^{*n} is Gaussian, clearly the density $g(z)$ achieves maximum at $z = 0$. Similarly, since the density of V^{*n} is a convolution of n symmetric, unimodal densities, $u(z)$ achieves

maximum at $z = 0$; in fact, $u(z)$ is unimodal and symmetric around zero. On the other hand, by standard concentration inequalities (cf. Claim 3.6, recall that $V \in [0, n]$ and $\text{Var } H^{*n} = n^3/12$)

$$\Pr \left[\left| H^{*n} - \frac{n^2}{2} \right| \geq n\sqrt{n} \right], \Pr \left[\left| V^{*n} - \frac{n^2}{2} \right| \geq n\sqrt{n} \right] \leq 2 \exp(-2) \leq \frac{1}{2} \quad (3.6)$$

and hence $\int_{-n\sqrt{n}}^{n\sqrt{n}} g(z) dz \geq 1/2$ and similarly for $u(z)$. Consequently, $g(0), u(0) \geq 1/4n\sqrt{n}$, establishing the first point.

As for the second point, we proceed by contradiction. If $\Pr \left[|V^{*n} - n^2/2| \leq \varepsilon \right] < \varepsilon/4n\sqrt{n}$, by unimodality and symmetry of u , $\int_{-n\sqrt{n}}^{n\sqrt{n}} u(z) dz \geq 1/2$ implies

$$u(\varepsilon) = u(-\varepsilon) \geq \frac{1/2 - \varepsilon/4n\sqrt{n}}{2n\sqrt{n}} \geq \frac{1}{8n\sqrt{n}},$$

which by the unimodality of $u(z)$ gives $\Pr \left[|V^{*n} - n^2/2| \leq \varepsilon \right] = \int_{-\varepsilon}^{\varepsilon} u(z) dz \geq \varepsilon/4n\sqrt{n}$ anyway.

Finally, for the third point the bound follows from the fact that the density of V is uniformly bounded by $1/n$ and the convolution cannot increase this bound, hence $\int_{\varepsilon_1}^{\varepsilon_2} u(z) dz \leq (\varepsilon_2 - \varepsilon_1)/n$. \square

We will also use a standard anti-concentration estimate for the Gaussians that follows from the formula for Gaussian density:

Claim 3.5. *Let G be a Gaussian random variable with variance at least σ^2 . Then, for all a, b , we have*

$$\Pr[a \leq G \leq b] \leq \frac{b - a}{\sqrt{2\pi\sigma}}.$$

Finally, we derive the formulas for conditional variances of $(G_A^{*n}, G_B^{*n}, H^{*n})$ that were stated in Claim 2.9:

Proof of Claim 2.9. Note that by considering the relevant variances and covariances and using the Hilbert space structure of joint Gaussians, random variables G_A and G_B can be written as

$$G_A = \sqrt{\text{Var}_A - \frac{CV_A^2}{\text{Var } V}} \cdot \mathcal{G}_A + \frac{CV_A}{\text{Var } V} \cdot \left(H - \frac{n}{2} \right),$$

$$G_B = \sqrt{\text{Var}_B - \frac{CV_B^2}{\text{Var } V}} \cdot \mathcal{G}_B + \frac{CV_B}{\text{Var } V} \cdot \left(H - \frac{n}{2} \right),$$

where \mathcal{G}_A and \mathcal{G}_B are joint centered Gaussians independent of H , each with variance one. Let $\alpha_A := \sqrt{\text{Var}_A - (CV_A^2/\text{Var } V)}$ and $\alpha_B := \sqrt{\text{Var}_B - (CV_B^2/\text{Var } V)}$. Accordingly, we also have

$$G_A^{*n} = \alpha_A \cdot \mathcal{G}_A^{*n} + \frac{CV_A}{\text{Var } V} \cdot \left(H^{*n} - \frac{n^2}{2} \right), \quad (3.7)$$

$$G_B^{*n} = \alpha_B \cdot \mathcal{G}_B^{*n} + \frac{CV_B}{\text{Var } V} \cdot \left(H^{*n} - \frac{n^2}{2} \right). \quad (3.8)$$

The fact that after conditioning G_A^{*n} and G_B^{*n} are centered joint Gaussians, as well as (2.3) and (2.4) all follow (note that $\text{Var } \mathcal{G}_A^{*n} = \text{Var } \mathcal{G}_B^{*n} = n$).

As for the covariance in (2.5), first by rearranging (3.7) and (3.8) we observe that

$$\text{Cov}[\mathcal{G}_A^{*n}, \mathcal{G}_B^{*n}] = \frac{n}{\alpha_A \alpha_B} \left(CV_{AB} - \frac{CV_A CV_B}{\text{Var } V} \right).$$

Finally, we invoke (3.7) and (3.8) again to see that

$$\text{Cov} \left[G_A^{*n}, G_B^{*n} \mid H^{*n} = \frac{n^2}{2} \right] = \alpha_A \alpha_B \cdot \text{Cov} [\mathcal{G}_A^{*n}, \mathcal{G}_B^{*n}] = n \left(CV_{AB} - \frac{CV_A CV_B}{\text{Var } V} \right). \quad \square$$

3.2. Tools

In this section we present a few tools, mostly imported from [Pol22], that we will use in various proofs in this section. Many of them will be used to prove Lemma 2.10. We start with referencing some standard concentration bounds:

Claim 3.6. *If X_1, \dots, X_n are independent random variables in $[0, 1]$ and $t > 0$, then*

$$\Pr \left[\left| \sum_{i=1}^n X_i - \sum_{i=1}^n \mathbb{E} X_i \right| \geq t \right] \leq 2 \exp(-2t^2/n).$$

On the other hand, if G is a centered Gaussian of variance σ^2 , then, for $t > 0$,

$$\Pr \left[|G| \geq t \right] \leq 2 \exp(-t^2/2\sigma^2).$$

We also need other estimates of sums of iid random variables. In particular, we will use explicit error bounds for both the central limit theorem and the Poisson limit theorem:

Lemma 3.7 (Berry–Esseen theorem). *Let X_1, \dots, X_n be iid random variables with $\mathbb{E} X_i = 0$, $\mathbb{E} X_i^2 = \sigma^2$ and $\mathbb{E} |X_i|^3 = \rho$. Let $X := \sum_{i=1}^n X_i$ and N be a standard normal. Then, for every real a, b ,*

$$\left| \Pr[a \leq X \leq b] - \Pr \left[\frac{a}{\sigma\sqrt{n}} \leq N \leq \frac{b}{\sigma\sqrt{n}} \right] \right| \leq \frac{\rho}{\sigma^3\sqrt{n}}.$$

Lemma 3.8 (Poisson limit theorem convergence rate, see (1.1) in [BH84]). *Let X_1, \dots, X_n be iid Bernoulli random variables with $\mathbb{E} X_i = \lambda/n$. Let $X := \sum_{i=1}^n X_i$ and let Z be a Poisson random variable with mean λ . Then, for every $S \subseteq \mathbb{N}$,*

$$\left| \Pr[X \in S] - \Pr[Z \in S] \right| \leq \frac{\lambda^2}{n}.$$

We use a technical lemma from the Polymath paper connecting the distance to integers to the values of $e(\cdot)$:

Lemma 3.9 (Lemma 5.9 in [Pol22]). *For real numbers θ_1 and θ_2 , we have*

$$\frac{1}{2} |e(\theta_1) + e(\theta_2)| \leq 1 - d_{\mathbb{Z}}(\theta_1 - \theta_2)^2 .$$

Similarly, for any $\theta_1, \theta_2, \theta_3$ and θ_4 it holds that

$$\frac{1}{4} |e(\theta_1) + e(\theta_2) + e(\theta_3) + e(\theta_4)| \leq 1 - d_{\mathbb{Z}}(\theta_1 - \theta_2 + \theta_3 - \theta_4)^2 / 4 .$$

We also employ a concentration bound analogous to Lemmas 5.4 and 5.8 in [Pol22]. Rather than prove it directly as Polymath does, we resort to the Azuma's inequality:

Lemma 3.10 (Azuma's inequality). *Let random variables Y_0, \dots, Y_n form a submartingale with respect to a filtration $\mathcal{F}_0, \dots, \mathcal{F}_n$ and with bounded differences, i.e., the random variable Y_i is measurable with respect to the σ -algebra \mathcal{F}_i and*

$$\mathbb{E}[Y_{i+1} - Y_i \mid \mathcal{F}_i] \geq 0 , \quad |Y_{i+1} - Y_i| \leq c$$

both hold almost surely. Then, for all n and for every $\varepsilon > 0$,

$$\Pr[Y_n \leq Y_0 - \varepsilon] \leq \exp\left(-\frac{\varepsilon^2}{2cn}\right) .$$

Lemma 3.11 (Azuma except with small probability). *As in Lemma 3.10, let Y_0, \dots, Y_n be a sequence of random variables adapted to a filtration $\mathcal{F}_0, \dots, \mathcal{F}_n$ and with bounded differences $|Y_{i+1} - Y_i| \leq 1$. Furthermore, let D_{i+1} denote the event*

$$\mathbb{E}[Y_{i+1} - Y_i \mid \mathcal{F}_i] < 0 .$$

Then, for every $\varepsilon > 0$,

$$\Pr[Y_n \leq Y_0 - \varepsilon] \leq \exp\left(-\frac{\varepsilon^2}{2n}\right) + \Pr\left[\bigcup_{i=1}^n D_i\right] . \quad (3.9)$$

Proof. The only difference to Lemma 3.10 is that random variables Y_0, \dots, Y_n do not always satisfy the submartingale property. To fix this, let Y'_0, \dots, Y'_n be a sequence of random variables given by

$$Y'_0 := Y_0 , \quad Y'_{i+1} := \begin{cases} Y_{i+1} & \text{if none of events } D_1, \dots, D_{i+1} \text{ happened,} \\ Y'_i & \text{otherwise.} \end{cases}$$

Keeping in mind the definitions of D_i and Y'_i , by induction we see that Y'_0, \dots, Y'_n form a bounded difference submartingale with respect to the filtration $\mathcal{F}_0, \dots, \mathcal{F}_n$. Consequently, by Lemma 3.10,

$$\Pr[Y'_n \leq Y'_0 - \varepsilon] \leq \exp\left(-\frac{\varepsilon^2}{2n}\right) .$$

On the other hand, it also holds that if none of D_1, \dots, D_n happened, then $Y'_n = Y_n$. Hence,

$$\Pr[Y_n \leq Y_0 - \varepsilon] \leq \Pr[Y'_n \leq Y'_0 - \varepsilon] + \Pr[Y'_n \neq Y_n] \leq \exp\left(-\frac{\varepsilon^2}{2n}\right) + \Pr\left[\bigcup_{i=1}^n D_i\right] . \quad \square$$

Finally, in the proof of Lemma 2.11 we are going to employ Lemma 6.2 from [Pol22] in order to approximate $(1 - Q + R)^n$ with $\exp(-nQ)$ for some $R \ll Q \ll 1$:

Lemma 3.12 (Lemma 6.2 in [Pol22]). *Let $Q, R \in \mathbb{R}$ such that $Q^2, |R| \leq 1/4n$. Then,*

$$\left| 1 - \frac{(1 - Q + R)^n}{\exp(-nQ)} \right|, \left| 1 - \frac{\exp(-nQ)}{(1 - Q + R)^n} \right| \leq 4n(Q^2 + |R|).$$

Actually, we need a complex version of Lemma 3.12 which uses the real version in the proof:

Lemma 3.13. *Let $Q \in \mathbb{R}, R \in \mathbb{C}$ such that $Q^2, |R| \leq 1/100n$. Then,*

$$\left| 1 - \frac{(1 - Q + R)^n}{\exp(-nQ)} \right|, \left| 1 - \frac{\exp(-nQ)}{(1 - Q + R)^n} \right| \leq 40n(Q^2 + |R|).$$

Proof. Let us write $R = \alpha + \beta i$ and denote $L := (1 - Q + R)^n, z := 1 - L/\exp(-nQ)$. We proceed to estimate the imaginary and real part of z :

$$\begin{aligned} |\operatorname{Im} L| &= \left| \operatorname{Im} \sum_{k=0}^n \binom{n}{k} (1 - Q)^{n-k} \sum_{j=0}^k \binom{k}{j} \alpha^{k-j} (\beta i)^j \right| \\ &\leq \sum_{k=1}^n \binom{n}{k} (1 - Q)^{n-k} \sum_{j=0}^k \binom{k}{j} |\alpha|^{k-j} |\beta|^j = (1 - Q + |\alpha| + |\beta|)^n - (1 - Q)^n, \end{aligned}$$

and consequently, applying Lemma 3.12 to $(1 - Q + |\alpha| + |\beta|)^n$ and $(1 - Q)^n$,

$$\begin{aligned} |\operatorname{Im} z| &= \frac{|\operatorname{Im} L|}{\exp(-nQ)} \leq \frac{(1 - Q + |\alpha| + |\beta|)^n}{\exp(-nQ)} - \frac{(1 - Q)^n}{\exp(-nQ)} \\ &\leq 1 + 4n(Q^2 + 2|R|) - 1 + 4nQ^2 = 8n(Q^2 + |R|). \end{aligned} \tag{3.10}$$

By a similar argument, we also establish

$$|\operatorname{Re} L - (1 - Q)^n| \leq (1 - Q + |\alpha| + |\beta|)^n - (1 - Q)^n$$

and

$$\begin{aligned} |\operatorname{Re} z| &= \left| 1 - \frac{(1 - Q)^n}{\exp(-nQ)} + \frac{(1 - Q)^n - \operatorname{Re} L}{\exp(-nQ)} \right| \\ &\leq \left| 1 - \frac{(1 - Q)^n}{\exp(-nQ)} \right| + \frac{(1 - Q + |\alpha| + |\beta|)^n}{\exp(-nQ)} - \frac{(1 - Q)^n}{\exp(-nQ)} \leq 12n(Q^2 + |R|). \end{aligned} \tag{3.11}$$

Equations (3.10) and (3.11) together imply

$$\left| 1 - \frac{(1 - Q + R)^n}{\exp(-nQ)} \right| \leq 20n(Q^2 + |R|),$$

as we wanted. Finally, we obtain the other inequality with

$$\left| 1 - \frac{\exp(-nQ)}{(1 - Q + R)^n} \right| = |z| \frac{\exp(-nQ)}{|1 - Q + R|^n} \leq 40n(Q^2 + |R|). \quad \square$$

3.3. Interpolating \widehat{f} and concentration of U

We continue to introduce some more technical tools. We start with some bounds on the characteristic function \widehat{f} that we will later use to interpolate values of \widehat{f} by adjacent values on a discrete grid:

Claim 3.14. *Let A, A' and B be three dice such that A and A' are equal except for one face i with $|a_i - a'_i| \leq \varepsilon$. Denote the characteristic functions of $(U_A, U_B, V - n/2)$ and $(U_{A'}, U_B, V - n/2)$ as, respectively, \widehat{f} and \widehat{f}' . Then, for every $\alpha, \beta, \gamma \in \mathbb{R}$,*

$$\left| \widehat{f}(\alpha, \beta, \gamma) - \widehat{f}'(\alpha, \beta, \gamma) \right| \leq 2\varepsilon/n .$$

Proof. Let

$$\begin{aligned} z_A(v) &:= e(\alpha g_A(v) + \beta g_B(v) + \gamma(v - n^2/2)) \\ \text{and} \quad z_{A'}(v) &:= e(\alpha g_{A'}(v) + \beta g_B(v) + \gamma(v - n^2/2)) \end{aligned}$$

that is $\widehat{f}(\alpha, \beta, \gamma) = \mathbb{E} z_A(V)$ and $\widehat{f}'(\alpha, \beta, \gamma) = \mathbb{E} z_{A'}(V)$. Since $|a_i - a'_i| \leq \varepsilon$, we have $z_A(v) = z_{A'}(v)$ everywhere except on an interval of length at most ε , i.e., on a set of measure at most ε/n . Since $|z_A(v) - z_{A'}(v)| \leq 2$ always holds, the result follows by the triangle inequality. \square

Lemma 3.15. *For every $\alpha_0, \alpha, \beta_0, \beta, \gamma \in \mathbb{R}$, we have*

$$\left| \widehat{f}(\alpha, \beta, \gamma) - \widehat{f}(\alpha_0, \beta_0, \gamma) \right| \leq 2\pi \left(|\alpha - \alpha_0| \|U_A\|_\infty + |\beta - \beta_0| \|U_B\|_\infty \right) .$$

Proof. Since we have

$$|e(x) - 1|^2 = (\cos(2\pi x) - 1)^2 + \sin^2(2\pi x) = 2(1 - \cos(2\pi x)) \leq 4\pi^2 x^2 ,$$

where in the last step we used $\cos x \geq 1 - x^2/2$ for $x \in \mathbb{R}$, we can use it to write

$$\begin{aligned} & \left| \widehat{f}(\alpha, \beta, \gamma) - \widehat{f}(\alpha_0, \beta_0, \gamma) \right| \\ &= \left| \mathbb{E} \left[e(\alpha_0 U_A + \beta_0 U_B + \gamma(V - n/2)) \left(e((\alpha - \alpha_0)U_A + (\beta - \beta_0)U_B) - 1 \right) \right] \right| \\ &\leq \mathbb{E} \left[\left| e((\alpha - \alpha_0)U_A + (\beta - \beta_0)U_B) - 1 \right| \right] \\ &\leq 2\pi \left(|\alpha - \alpha_0| \|U_A\|_\infty + |\beta - \beta_0| \|U_B\|_\infty \right) . \end{aligned} \quad \square$$

We also show that, with high probability over the choice of balanced A , the random variable U_A is uniformly bounded by $\widetilde{\mathcal{O}}(\sqrt{n})$. As in other places, this is an adaptation of a similar Polymath argument to the continuous setting.

Lemma 3.16. *Let A be a random balanced die. Then, except with probability n^{-10} , we have that*

$$|g_A(t)| < 5\sqrt{n \log n} .$$

for every $0 \leq t \leq n$. In other words, $\|U_A\|_\infty \leq 5\sqrt{n \log n}$.

Proof. For the purposes of this proof, let A be a die with faces iid uniform in $[0, n]$ and let \mathcal{E} denote the event that A is balanced. If not for the balancing, we would be done by an application of Claim 3.6.

To deal with the balanced case, let $\alpha \geq 2$ be a threshold that we will choose later. Recall that one way of sampling A from the conditional distribution is to sample a_1, \dots, a_{n-1} iid from $[0, n]$ and reject unless $\frac{n^2}{2} - n \leq \sum_{i=1}^{n-1} a_i \leq \frac{n^2}{2}$, in which case we set $a_n := n^2/2 - \sum_{i=1}^{n-1} a_i$. Because of that, it might be helpful to reduce the analysis of the “bad event” $|g_A(t)| \geq \alpha$ to a related event depending only on the initial $n-1$ coordinates. More specifically, let $W := |\{i \in [n-1] : a_i \leq t\}|$ and observe that $W \in \{f_A(t), f_A(t) - 1\}$. By an elementary analysis of cases $f_A(t) \geq t + \alpha$ and $f_A(t) \leq t - \alpha$, one checks that

$$\mathbb{1}\left[|g_A(t)| \geq \alpha\right] = \mathbb{1}\left[|f_A(t) - t| \geq \alpha\right] \leq \mathbb{1}\left[\left|\frac{W}{n-1} - \frac{t}{n}\right| \geq \frac{\alpha-1}{n-1}\right]. \tag{3.12}$$

Let \mathcal{F} denote the event $\left|\frac{W}{n-1} - \frac{t}{n}\right| \geq \frac{\alpha-1}{n-1}$. Since W is a sum of $n-1$ iid binary random variables with expectation t/n each, by Claim 3.6 we have

$$\Pr[\mathcal{F}] \leq 2 \exp\left(-\frac{2(\alpha-1)^2}{n-1}\right) \leq 2 \exp\left(-\frac{\alpha^2}{2n}\right), \tag{3.13}$$

where we used that $\alpha \geq 2$ implies $(\alpha-1)^2 \geq \alpha^2/4$. Let $D := \{(a_1, \dots, a_{n-1}) \in [0, n]^{n-1} : \frac{n^2}{2} - n \leq \sum_{i=1}^{n-1} a_i \leq \frac{n^2}{2}\}$ and $a := (a_1, \dots, a_{n-1})$. Putting (3.12) and (3.13) together, we have, for random balanced A ,

$$\Pr\left[|g_A(t)| \geq \alpha \mid \mathcal{E}\right] \leq \frac{n^{-(n-1)} \int \mathbb{1}[\mathcal{F}] \cdot \mathbb{1}[a \in D] da}{n^{-(n-1)} \int \mathbb{1}[a \in D] da} \leq \frac{\Pr[\mathcal{F}]}{\Pr[a \in D]} \leq 5\sqrt{n} \cdot 2 \exp\left(-\frac{\alpha^2}{2n}\right),$$

where in the end we made the estimate $\Pr[a \in D] \geq 1/5\sqrt{n}$ by using Lemma 3.7 (for that purpose, one checks $\mathbb{E}(a_i - n/2)^2 = n^2/12$ and $\mathbb{E}|a_i - n/2|^3 = n^3/32$).

Letting $\alpha := \sqrt{24n \log n}$ we get that $\Pr[|g_A(t)| \geq \alpha \mid \mathcal{E}] \leq 10\sqrt{n} \cdot n^{-12}$ for any fixed t . Applying this for $t = 1, 2, \dots, n-1$ (note that $g_A(0) = g_A(n) = 0$ almost surely), by union bound we get that, except with probability n^{-10} , $|g_A(i)| < \alpha$ for every $i = 1, \dots, n$. Finally, if $|g_A(i)| < \alpha$ for each integer i , then also for any $i < t < i+1$ we have

$$t - \alpha - 1 < i - \alpha < f_A(i) \leq f_A(t) \leq f_A(i+1) < i + 1 + \alpha < t + \alpha + 1$$

implying $|g_A(t)| < \alpha + 1 < 5\sqrt{n \log n}$ for every $0 \leq t \leq n$. □

Finally, we make a similar conditional concentration bound for the convolutions U_A^{*n} and G_A^{*n} .

Lemma 3.17. *Let A be a balanced die such that $\|U_A\|_\infty \leq 5\sqrt{n \log n}$. Then,*

$$\Pr\left[|U_A^{*n}| \geq 50n \log n \mid V^{*n} = \frac{n^2}{2}\right] \leq n^{-10}. \tag{3.14}$$

Similarly, for such a die it holds that

$$\Pr\left[|G_A^{*n}| \geq 30n \log n \mid H^{*n} = \frac{n^2}{2}\right] \leq n^{-10}. \tag{3.15}$$

Proof. For (3.14) the argument is similar as in the proof of Lemma 3.16. Specifically, we consider sampling $v := (v_1, \dots, v_{n-1})$ iid from the uniform distribution on $[0, n]$, rejecting if $\sum_{i=1}^{n-1} v_i > n^2/2$ or $\sum_{i=1}^{n-1} v_i < n^2/2 - n$ and otherwise setting $D := \{(v_1, \dots, v_{n-1}) : n^2/2 - n \leq \sum_{i=1}^{n-1} v_i \leq n^2/2\}$, $v_n := n^2/2 - \sum_{i=1}^{n-1} v_i$ and $u_i := g_A(v_i)$. Then, we have

$$\begin{aligned} \Pr \left[|U_A^{*n}| \geq 50n \log n \mid V^{*n} = \frac{n^2}{2} \right] &= \frac{n^{-(n-1)} \int \mathbb{1} \left[\left| \sum_{i=1}^n u_i \right| \geq 50n \log n \right] \cdot \mathbb{1} [v \in D] \, dv}{n^{-(n-1)} \int \mathbb{1} [v \in D] \, dv} \\ &\leq \frac{n^{-(n-1)} \int \mathbb{1} \left[\left| \sum_{i=1}^{n-1} u_i \right| \geq 40(n-1) \log n \right] \, dv}{n^{-(n-1)} \int \mathbb{1} [v \in D] \, dv} \\ &= \frac{\Pr \left[\left| \sum_{i=1}^{n-1} u_i \right| / (n-1) \geq 40 \log n \right]}{\Pr[v \in D]} \\ &\leq 5\sqrt{n} \cdot 2 \exp \left(-2 \frac{(40 \log n)^2 (n-1)}{100n \log n} \right) \leq n^{-10}, \end{aligned}$$

where to arrive in the last line we used $\Pr[v \in D] \geq 1/5\sqrt{n}$ by the same argument as in the proof of Lemma 3.16 and to bound the numerator we used the fact that u_1, \dots, u_{n-1} are iid centered random variables in $[-5\sqrt{n \log n}, 5\sqrt{n \log n}]$ and standard Hoeffding inequality $\Pr[|\sum_{i=1}^n x_i/n| \geq C] \leq 2 \exp(-2(C/2M)^2 n)$ for iid centered random variables such that $|x_i| \leq M$ (cf. Claim 3.6).

In case of (3.15), the argument uses properties of joint Gaussians G_A^{*n} and H^{*n} . First, by Claim 2.9, random variable G_A^{*n} conditioned on $H^{*n} = n^2/2$ is a centered Gaussian such that

$$\text{Var} \left[G_A^{*n} \mid H^{*n} = \frac{n^2}{2} \right] \leq \text{Var} [G_A^{*n}] = n \text{Var} [U_A] \leq n \|U_A\|_\infty^2 \leq 25n^2 \log n.$$

Consequently, using Claim 3.6,

$$\Pr \left[|G_A^{*n}| \geq 30n \log n \mid H^{*n} = \frac{n^2}{2} \right] \leq 2 \exp \left(-\frac{30^2 n^2 \log^2 n}{50n^2 \log n} \right) \leq n^{-10}. \quad \square$$

3.4. Decay of \hat{f} with respect to α

Clearly, $\hat{f}(0, 0, 0) = 1$. Our whole argument relies on controlling how fast \hat{f} decays from 1 in the neighborhood of the origin. We now state the main lemma concerning a rate of decay of \hat{f} in the most demanding case: as the first argument α moves away from the origin. We will spend the following couple of sections establishing:

Lemma 3.18. *Let α, β be such that $1/n^3 \leq |\alpha| \leq 1/2$ and $\delta := 10^{-14} \cdot \min \left(1, \frac{\alpha^2 n}{\log n} \right)$. Then, except with probability at most $40n^{-7}$ over the choice of balanced A and B , for every $\gamma \in \mathbb{R}$ it holds that*

$$|\hat{f}(\alpha, \beta, \gamma)| \leq 1 - \delta/2.$$

We prove Lemma 3.18 via two intermediate lemmas. First, we prove its version where A is a random die *without* conditioning on A being balanced:

Lemma 3.19. *Let α, β be such that $|\alpha| \leq 1/2$ and $\delta := 10^{-14} \cdot \min\left(1, \frac{\alpha^2 n}{\log n}\right)$ and B be a fixed die. Then, except with probability at most $2n^{-15}$ over the choice of random die A with faces iid in $[0, n]$, for every $\gamma \in \mathbb{R}$ it holds that*

$$|\widehat{f}(\alpha, \beta, \gamma)| \leq 1 - \delta. \tag{3.16}$$

The setting we use to prove Lemma 3.19 is as follows: Consider fixed $\alpha, \beta \in \mathbb{R}$ and a die B . Ultimately, we want to show that, except with probability $2n^{-15}$ over random choice of unbalanced die A , the bound $|\widehat{f}(\alpha, \beta, \gamma)| \leq 1 - \delta$ holds for every $\gamma \in \mathbb{R}$.

To that end, we choose some $m > 0$ and $k \in \mathbb{N}$. We will give specific values later, but we will always have $k \geq 3 \cdot 10^5 \log n$ and $n/2 - 4m < 4mk \leq n/2$. We partition the interval $[0, 4mk]$ into k consecutive intervals S_1, \dots, S_k of length $4m$ each. In other words, we let

$$S_i := [s_i, s_{i+1}) := [4m(i - 1), 4mi).$$

We specify

$$\varepsilon := 10^{-5} \cdot \min\left(1, |\alpha| \sqrt{\frac{n}{\log n}}\right) \tag{3.17}$$

and

$$\begin{aligned} \theta(t) &:= \theta_{\alpha, \beta}(t) := \alpha f_A(t) + \beta f_B(t), \\ z(t) &:= z_{\alpha, \beta, \gamma}(t) := e\left(\alpha g_A(t) + \beta g_B(t) + \gamma(t - n/2)\right), \end{aligned}$$

so that $\widehat{f}(\alpha, \beta, \gamma) = \mathbb{E} z(V)$ holds. The strategy is to show that for every i , the bound

$$\mathbb{E} [|z(V)| \mid V \in S_i] < 1 - \Omega(\varepsilon^2)$$

holds with probability at least $1/20$ over the choice of A . Furthermore, we show that this is true even if conditioned on all the faces of A that lie in the preceding intervals S_1, \dots, S_{i-1} . Then, we use the tools from Section 3.2 to conclude that with high probability, the event holds simultaneously for a constant fraction of S_i intervals, resulting in the overall inequality $|\widehat{f}| < 1 - \delta$.

With all that in mind, we state the second intermediate lemma giving the bound for fixed $i \in [k]$:

Lemma 3.20. *In the setting above, i.e., with fixed α, β and B , as well as random A with faces iid in $[0, n]$, there exists a choice of k and m such that $k \geq 3 \cdot 10^5 \log n$, $n/2 - 4m < 4mk \leq n/2$ and the following holds:*

Let $i \in [k]$. Define the event $D_i := f_A(s_i) > 3n/5$. Furthermore, let

$$A_{\leq x} := (a'_1, \dots, a'_n), \quad a'_j := \begin{cases} a_j & \text{if } a_j \leq x, \\ ? & \text{otherwise.} \end{cases}$$

Then, for every $s_i \leq v < s_i + m$, it always holds that

$$\Pr_A \left[d_{\mathbb{Z}}(\theta(v) - \theta(v + m) - \theta(v + 2m) + \theta(v + 3m)) \geq \varepsilon \mid A_{\leq s_i}, \neg D_i \right] \geq \frac{1}{20}. \tag{3.18}$$

Before proving Lemma 3.20, let us see how it implies Lemma 3.19. In short, we will use the triangle inequality to divide contributions to $|\widehat{f}|$ into intervals S_i , Lemma 3.9 to connect distance to integers from $\theta(t)$ to $z(t)$ values, and Lemma 3.11 to amplify $1/20$ probability from Lemma 3.20 into high probability.

Proof of Lemma 3.19 assuming Lemma 3.20. Define random variables Z_1, \dots, Z_k as

$$Z_i := \Pr \left[d_{\mathbb{Z}} \left(\theta(V) - \theta(V+m) - \theta(V+2m) + \theta(V+3m) \right) \geq \varepsilon \mid A, s_i \leq V < s_i + m \right].$$

Note that the value of Z_i is fully determined by the die A . We have $0 \leq Z_i \leq 1$ and, by applying (3.18) pointwise, it almost surely holds that

$$\neg D_i \implies \mathbb{E}[Z_i \mid A_{\leq s_i}] \geq \frac{1}{20}.$$

By the first moment method applied to $\mathbb{E}[Z_i \mid A_{\leq s_i}]$ pointwise for all conditionings on $A_{\leq s_i}$, we get, again almost surely, that

$$\neg D_i \implies \Pr \left[Z_i \geq \frac{1}{40} \mid A_{\leq s_i} \right] \geq \frac{1}{40}.$$

Now let $X_i := \mathbb{1}[Z_i \geq 1/40]$ be the indicator of the event $Z_i \geq 1/40$. It should now be clear that X_i is measurable with respect to the σ -algebra \mathcal{F}_i generated by $A_{\leq s_{i+1}}$. Furthermore, if the event D_i does not happen, then $\mathbb{E}[X_i \mid \mathcal{F}_{i-1}] \geq 1/40$. Therefore, defining $Y_0 := 0$ and $Y_{i+1} := Y_i + X_{i+1} - 1/40$, we get a sequence of random variables Y_0, \dots, Y_k adapted to $\mathcal{F}_0, \dots, \mathcal{F}_k$ with $|Y_{i+1} - Y_i| \leq 1$ and $\mathbb{E}[Y_{i+1} - Y_i \mid \mathcal{F}_i] \geq 0$ whenever the event D_{i+1} does not happen. Therefore, we can apply Lemma 3.11 to obtain

$$\begin{aligned} \Pr \left[\sum_{i=1}^k X_i \leq \frac{k}{80} \right] &= \Pr \left[Y_k \leq -\frac{k}{80} \right] \leq \exp \left(-\frac{k}{2 \cdot 10^4} \right) + \Pr \left[\bigcup_{i=1}^k D_i \right] \\ &\leq n^{-15} + \Pr \left[f_A(n/2) > 3n/5 \right] \leq 2n^{-15}, \end{aligned}$$

where in the end we used the bound $k \geq 3 \cdot 10^5 \log n$ and Claim 3.6 for the deviation of $f_A(n/2)$.

Therefore, except with probability $2n^{-15}$, we get that for at least $k/80$ indices $i \in [k]$, for at least $1/40$ fraction of $v \in [s_i, s_i + m]$, we have

$$d_{\mathbb{Z}} \left(\theta(v) - \theta(v+m) - \theta(v+2m) + \theta(v+3m) \right) \geq \varepsilon. \quad (3.19)$$

Recall that $\theta(t) = \alpha f_A(t) + \beta f_B(t)$ and let $\theta'(t) := \alpha g_A(t) + \beta g_B(t) + \gamma(t - n/2)$ and again recall that $z(t) = e(\theta'(t))$. Note that

$$\theta(v) - \theta(v+m) - \theta(v+2m) + \theta(v+3m) \quad (3.20)$$

$$= \theta'(v) - \theta'(v+m) - \theta'(v+2m) + \theta'(v+3m), \quad (3.21)$$

and from (3.19) and (3.20), applying Lemma 3.9, we get that for all v such that (3.19) holds we have

$$\left| z(v) + z(v+m) + z(v+2m) + z(v+3m) \right| \leq 4 - \varepsilon^2.$$

Consequently, we partition the interval $[0, n]$ into S_1, \dots, S_k and the leftover interval of length $n - 4km$. Using the trivial bound of one for the modulus of the characteristic function in the leftover interval, we can write

$$\begin{aligned} |\widehat{f}(\alpha, \beta, \gamma)| &= \frac{1}{n} \left| \int_0^n z(t) dt \right| \leq 1 - \frac{4km}{n} + \frac{1}{n} \sum_{i=1}^k \left| \int_{S_i} z(t) dt \right| \\ &\leq 1 - \frac{4km}{n} + \frac{1}{n} \sum_{i=1}^k \int_{s_i}^{s_i+m} |z(t) + z(t+m) + z(t+2m) + z(t+3m)| dt \\ &\leq 1 - \frac{4km}{n} + \frac{4km}{n} - \frac{1}{n} \cdot \frac{k}{80} \cdot \frac{m}{40} \cdot \varepsilon^2 \leq 1 - \frac{\varepsilon^2}{10^4}. \end{aligned}$$

Recalling (3.16) and (3.17) for the definitions of δ and ε , we notice that (3.16) is now established and the proof is concluded. \square

We still need to show that Lemma 3.19 implies Lemma 3.18. The Polymath draft in the discrete setting can just use the union bound, but we are conditioning on an event of measure zero. Furthermore, a basic approach like in Lemmas 3.16 and 3.17 does not seem to work, since, for larger values of $|\alpha|$, in principle changing just one face in a die can significantly change the value of \widehat{f} . Instead, we use a little more precise coupling argument:

Proof of Lemma 3.18 assuming Lemma 3.19. Fix α, β and a die B and let $s := n^{-13/2}$. Consider random choice of an unbalanced die A with faces in $[0, n]$. We define two events: let \mathcal{E} denote that there exists γ such that $|\widehat{f}(\alpha, \beta, \gamma)| > 1 - \delta$, and \mathcal{S} that $|\sum_{i=1}^n a_i - n^2/2| \leq s$.

By Claim 3.4.2, we have $\Pr[\mathcal{S}] \geq n^{-8}/4$. Applying Lemma 3.19, we get that

$$\Pr[\mathcal{E} \mid \mathcal{S}] \leq \frac{\Pr[\mathcal{E}]}{\Pr[\mathcal{S}]} \leq \frac{2n^{-15}}{n^{-8}/4} = 8n^{-7}.$$

Fix s' with $|s'| \leq s$. Recall that one way to choose A conditioned on face-sum equal to $n^2/2$ is by rejection sampling. That is, if the event \mathcal{T}_1 denoting $n^2/2 - n \leq \sum_{i=1}^{n-1} a_i \leq n^2/2$ happens, then set a_n to make the die balanced, otherwise reject and repeat. Similarly, to sample A conditioned on face-sum equal to $n^2/2 + s'$, one can utilize the rejection sampling using the event \mathcal{T}_2 standing for $n^2/2 + s' - n \leq \sum_{i=1}^{n-1} a_i \leq n^2/2 + s'$.

Note that the symmetric difference $\mathcal{T}_1 \Delta \mathcal{T}_2$ consists of $\sum_{i=1}^{n-1} a_i$ being in one of two intervals of width $|s'|$ and by Claim 3.4.3 it has probability at most

$$\Pr[\mathcal{T}_1 \Delta \mathcal{T}_2] \leq 2s/(n-1) \leq 3s/n = 3n^{-15/2}.$$

On the other hand, again by Claim 3.4.2, we have

$$\Pr[\mathcal{T}_1] = \Pr \left[\left| \sum_{i=1}^{n-1} a_i - (n-1)n/2 \right| \leq \frac{n}{2} \right] \geq \frac{n}{8(n-1)\sqrt{n-1}} \geq \frac{1}{8\sqrt{n}}.$$

Consider the procedure of rejection sampling of two dice using joint randomness. One die is balanced, using the event \mathcal{T}_1 and the other die has face-sum $n^2/2 + s'$ using the event \mathcal{T}_2 . Consider the round when one of the dice is successfully output, i.e, the first round when $\mathcal{T}_1 \cup \mathcal{T}_2$ occurs. The probability that the other die is not output in that round is at most

$$\frac{\Pr[\mathcal{T}_1 \Delta \mathcal{T}_2]}{\Pr[\mathcal{T}_1 \cup \mathcal{T}_2]} \leq \frac{3n^{-15/2}}{n^{-1/2}/8} = 24n^{-7} .$$

At the same time, if both dice are output in the same round of this joint rejection sampling, they will agree everywhere except for the last face. Consequently, there exists a coupling between a random balanced A on the one hand, and random A conditioned on face-sum equal to $n^2/2 + s'$ on the other hand, such that their first $n - 1$ faces are equal except with probability $24n^{-7}$. Averaging over s' , there also exists such a coupling between A conditioned on being balanced and A conditioned on \mathcal{S} .

Let A' be a die with face-sum $n^2/2 + s'$ such that $|s'| \leq s$ and let A be A' with the last face modified so that A is balanced. By Claim 3.14, if the event \mathcal{E} does not hold for A' , then for A it holds that for all $\gamma \in \mathbb{R}$ we have

$$\left| \widehat{f}(\alpha, \beta, \gamma) \right| \leq 1 - \delta + 2s/n = 1 - \delta + 2n^{-15/2} \leq 1 - \delta/2 , \quad (3.22)$$

where we used that $|\alpha| \geq 1/n^3$ implies $\delta \geq 1/n^6$. By union bound it follows that (3.22) holds for a random balanced A except with probability $24n^{-7} + 8n^{-7} < 40n^{-7}$. \square

3.5. Proof of Lemma 3.20

We delay choosing a specific m and for now assume that the unique integer k satisfying $n/2 - 4m < 4mk \leq n/2$ has value at least $3 \cdot 10^5 \log n$. Following the statement of the lemma, we fix v in the interval $s_i \leq v < s_i + m$ and a die B and condition on $A_{\leq s_i}$, i.e., on all faces of A that have values not exceeding s_i . Note that this determines the values of $f_A(x)$ for all $x \leq s_i$ and that we assume that the conditioning satisfies $f_A(s_i) \leq 3n/5$.

First, we observe that, after all the foregoing conditioning, letting $t := f_A(v + 2m)$, the difference $t - f_A(s_i)$ is a sum of $n - f_A(s_i)$ iid binary random variables, each with expectation $(v + 2m - s_i)/(n - s_i)$. Therefore, we have

$$\mathbb{E}[t \mid A_{\leq s_i}] = f_A(s_i) + \left(n - f_A(s_i) \right) \frac{v + 2m - s_i}{n - s_i} \leq 3n/5 + 6m ,$$

where we used that $n - s_i \geq n/2$. Since the conditional expectation of t can be written as a sum of a constant which is at most $3n/5$ and a nonnegative random variable with expectation at most $6m \leq n/100 \log n$ (due to bounds on k and m), by Markov's inequality, we have that $t \leq 2n/3$ with probability at least 0.99.

Consequently, let us further condition on $A_{\leq v+2m}$ such that $t \leq 2n/3$ and consider the distribution of $\theta(v) - \theta(v + m) - \theta(v + 2m) + \theta(v + 3m)$. After conditioning on $A_{\leq v+2m}$,

the only part of this expression that remains random is $f_A(v + 3m)$ featuring in $\theta(v + 3m) = \alpha f_A(v + 3m) + \beta f_B(v + 3m)$. What is more, this can be written as

$$f_A(v + 3m) = f_A(v + 2m) + \widetilde{X},$$

where $\widetilde{X} = \sum_{i=1}^{n-t} X_i$ is a sum of iid binary random variables with

$$m/n \leq \Pr[X_i = 1] = \frac{m}{n - (v + 2m)} \leq 2m/n.$$

All in all, omitting the conditioning from the notation, we can write the probability on the left-hand side of (3.18) as

$$\Pr \left[d_{\mathbb{Z}/|\alpha|}(\widetilde{X} + C) \geq \frac{\varepsilon}{|\alpha|} \right] \tag{3.23}$$

for some constant $C \in \mathbb{R}$. In order to lower bound (3.23), we proceed with two cases, using different limit theorems.

Case 1: $|\alpha| \leq 10^{-4}$ and Berry–Esseen theorem In this case we choose

$$m := \max \left(\frac{4\varepsilon^2}{\alpha^2}, 10^5 \right). \tag{3.24}$$

Considering two cases in (3.17) $|\alpha| < \sqrt{\log n/n}$ and $|\alpha| \geq \sqrt{\log n/n}$, we see that $m \leq 4 \cdot 10^{-10}n/\log n$. Accordingly, we verify that we have $k = \lfloor n/8m \rfloor \geq 3 \cdot 10^5 \log n$.

Recall that random variables X_i are binary iid with $m/n \leq \mathbb{E} X_i \leq 2m/n$. Consequently, also $0.99 \cdot m/n \leq \text{Var} X_i =: \sigma^2 \leq 2m/n$ and $\mathbb{E} |X_i - \mathbb{E} X_i|^3 \leq 2m/n$. Applying Lemma 3.7, we get that for any $a, b \in \mathbb{R}$,

$$\Pr \left[a \leq \widetilde{X} - \mathbb{E} \widetilde{X} \leq b \right] \geq \Pr \left[\frac{a}{\sigma\sqrt{n-t}} \leq N \leq \frac{b}{\sigma\sqrt{n-t}} \right] - \sqrt{13/m},$$

where N is a standard normal. Choosing $a := \sqrt{m}/2, b := 2\sqrt{m}, a' := -2\sqrt{m}, b' := -\sqrt{m}/2$, we get

$$\begin{aligned} \Pr \left[a \leq \widetilde{X} \leq b \right], \Pr \left[a' \leq \widetilde{X} \leq b' \right] &\geq \Pr \left[\frac{\sqrt{m}}{2\sigma\sqrt{n-t}} \leq N \leq \frac{2\sqrt{m}}{\sigma\sqrt{n-t}} \right] - \sqrt{13/m} \\ &\geq \Pr \left[1.01 \cdot \sqrt{3}/2 \leq N \leq \sqrt{2} \right] - \sqrt{13/m} \geq 1/10, \end{aligned}$$

where in the last step we used $\Pr[1.01 \cdot \sqrt{3}/2 \leq N \leq \sqrt{2}] \geq 0.112$ and $m \geq 10^5$. Hence, we have found two disjoint intervals such that \widetilde{X} lies within each of them with probability at least 1/10. To conclude the analysis of this case it is enough that we prove the following:

Claim 3.21. *At least one of the events $a' \leq \widetilde{X} \leq b'$ and $a \leq \widetilde{X} \leq b$ implies*

$$d_{\mathbb{Z}/|\alpha|}(\widetilde{X} + C) \geq \sqrt{m}/2 \geq \varepsilon/|\alpha|.$$

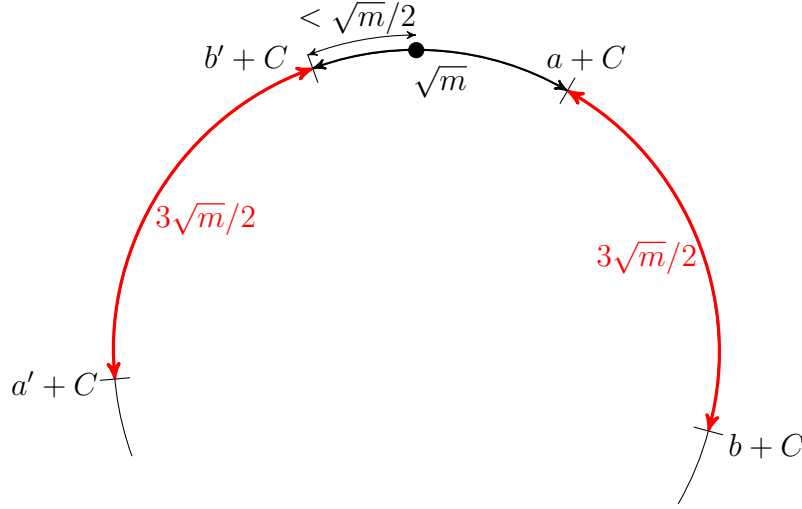


Figure 3.1: Illustration of the proof of Claim 3.21. The circle represents the set $\mathbb{R}/\mathbb{Z}/|\alpha|$ and its circumference is at least $6\sqrt{m}$. The black dot represents zero.

Proof (cf. Figure 3.1). The fact that $\sqrt{m}/2 \geq \varepsilon/|\alpha|$ follows directly from (3.24) and therefore it remains to show that the distance to $\mathbb{Z}/|\alpha|$ exceeds $\sqrt{m}/2$. Assume that the first event does not imply the conclusion, that is that there exists $a' \leq x' \leq b'$ such that $d_{\mathbb{Z}/|\alpha|}(x'+C) < \sqrt{m}/2$. In other words, there exists $k \in \mathbb{Z}$ such that

$$\left| x' + C - \frac{k}{|\alpha|} \right| < \sqrt{m}/2.$$

We will be done if we show that for every $a \leq x \leq b$ we have

$$k/|\alpha| + \sqrt{m}/2 \leq x + C \leq (k+1)/|\alpha| - \sqrt{m}/2.$$

Equivalently, we want to show

$$k/|\alpha| + \sqrt{m}/2 \leq a + C \quad \text{and} \quad b + C \leq (k+1)/|\alpha| - \sqrt{m}/2.$$

But indeed,

$$\begin{aligned} a + C - \frac{k}{|\alpha|} &= a - x' + \left(x' + C - \frac{k}{|\alpha|} \right) > a - b' - \sqrt{m}/2 = \sqrt{m}/2, \\ \frac{k+1}{|\alpha|} - (b + C) &= \frac{1}{|\alpha|} - (b - x') - \left(x' + C - \frac{k}{|\alpha|} \right) > \frac{1}{|\alpha|} - 4\sqrt{m} - \sqrt{m}/2 \geq \sqrt{m}, \end{aligned}$$

where in the last calculation one checks from (3.24) and (3.17) that $1/|\alpha| \geq 6\sqrt{m}$. \square

Case 2: $10^{-4} < |\alpha| \leq 1/2$ and Poisson limit theorem In the case of larger $|\alpha|$ we will use \widetilde{m} of the order of small constant. Here the Berry–Esseen theorem might be less helpful, since \widetilde{X} converges in distribution to a Poisson random variable. Note that also conceptually this case seems to be slightly different. For example, to establish $|\widehat{f}(1/2, 0, 1/2)| = |\mathbb{E} e(f_A(t)/2)| < 1$ we need to exclude the possibility that function f_A always takes even (or odd) values. In order to do that, we need to be able to home in on specific values of $\Pr[\widetilde{X} = k]$.

More concretely, in this case we fix $m := 1$. Again, it is easily checked that $k \geq 3 \cdot 10^5 \log n$. Let $\lambda := \mathbb{E} \widetilde{X} = (n - t) \Pr[X_i = 1]$ and note that we have $1/3 \leq \lambda \leq 2$. By Lemma 3.8,

$$\begin{aligned} \Pr[\widetilde{X} = 0] &\geq \Pr[Z_\lambda = 0] - 4/(n - t) \geq \exp(-\lambda) - 12/n \geq 1/10, \\ \Pr[\widetilde{X} = 1] &\geq \Pr[Z_\lambda = 1] - 4/(n - t) \geq \lambda \exp(-\lambda) - 12/n \geq 1/10. \end{aligned}$$

As in case 1, to finish the proof it is enough to argue that if $|C - k/|\alpha|| < \varepsilon/|\alpha|$, then $k/|\alpha| + \varepsilon/|\alpha| \leq 1 + C \leq (k + 1)/|\alpha| - \varepsilon/|\alpha|$ and therefore at least one of $\widetilde{X} = 0$ and $\widetilde{X} = 1$ must imply the event from (3.23). But this is again a straightforward verification using bounds on $|\alpha|$ and $\varepsilon \leq 10^{-5}$:

$$\begin{aligned} 1 + C - \frac{k}{|\alpha|} &> 1 - \frac{\varepsilon}{|\alpha|} \geq \frac{\varepsilon}{|\alpha|}, \\ \frac{k + 1}{|\alpha|} - (1 + C) &= \frac{1}{|\alpha|} - 1 + \left(\frac{k}{|\alpha|} - C \right) > \frac{1}{|\alpha|} - 1 - \frac{\varepsilon}{|\alpha|} \geq 1 - \frac{\varepsilon}{|\alpha|} \geq \frac{\varepsilon}{|\alpha|}, \end{aligned}$$

concluding the analysis of case 2.

To sum up, in both cases we obtained that, conditioned on $t \leq 2n/3$, which happens with probability at least 0.99, with probability at least 1/10,

$$d_{\mathbb{Z}}(\theta(v) - \theta(v + m) - \theta(v + 2m) + \theta(v + 3m)) \geq \varepsilon. \tag{3.25}$$

Therefore, (3.25) certainly holds with overall probability at least 1/20, as claimed in (3.18). \square

3.6. Bounding \widehat{f} with respect to γ

For the central limit theorem, we will need that $|\widehat{f}|$ is small everywhere outside a small box around the origin. We will use Lemma 3.18 to deal with the case of larger $|\alpha|$ and $|\beta|$. In this section we prove some similar, but simpler results to handle small $|\alpha|$ and $|\beta|$ and larger $|\gamma|$.

Lemma 3.22. *For any choice of dice A and B , we always have $|\widehat{f}(\alpha, \beta, \gamma)| \leq \frac{1}{|\gamma - \alpha - \beta|}$.*

Proof. We calculate, using the fact that the interval $[0, n]$ can be partitioned into $2n + 1$ intervals $[c_i, c_{i+1}]$ such that on each of the intervals the functions f_A and f_B are constant with $f_A(t) = d_i$ and $f_B(t) = e_i$:

$$\begin{aligned} \widehat{f}(\alpha, \beta, \gamma) &= \mathbb{E} e\left(\alpha g_A(V) + \beta g_B(V) + \gamma(V - n/2)\right) \\ &= \frac{1}{n} \int_0^n e\left(\alpha g_A(t) + \beta g_B(t) + \gamma(t - n/2)\right) dt \\ &= \frac{1}{n} \int_0^n e\left(\alpha f_A(t) + \beta f_B(t) + (\gamma - \alpha - \beta)t + \gamma n/2\right) dt \\ &= \frac{1}{n} \sum_{i=0}^{2n} e\left(\alpha d_i + \beta e_i + \gamma n/2\right) \int_{c_i}^{c_{i+1}} e\left((\gamma - \alpha - \beta)t\right) dt \\ &= \frac{1}{n} \sum_{i=0}^{2n} \frac{F_i \cdot \left[e\left((\gamma - \alpha - \beta)c_{i+1}\right) - e\left((\gamma - \alpha - \beta)c_i\right) \right]}{2\pi(\gamma - \alpha - \beta)}, \end{aligned}$$

where in the last step we denote the constant factor on the i -th interval by F_i with $|F_i| = 1$. Consequently, and using the triangle inequality and $|e(a) - e(b)| \leq 2$,

$$\left| \widehat{f}(\alpha, \beta, \gamma) \right| \leq \frac{2(2n+1)}{2\pi n |\gamma - \alpha - \beta|} \leq \frac{1}{|\gamma - \alpha - \beta|}. \quad \square$$

Lemma 3.22 handles the case where $|\gamma|$ is quite large. The remaining case is very small $|\alpha|, |\beta|$ and somewhat larger $|\gamma|$:

Lemma 3.23. *Let $|\alpha|, |\beta| \leq \frac{10^{10} \log n}{n}$. Then, provided that dice A and B satisfy $\|U_A\|_\infty, \|U_B\|_\infty \leq 5\sqrt{n \log n}$, for every γ with $|\gamma| \geq 6 \log^2 n / n^{3/2}$ it holds that*

$$\left| \widehat{f}(\alpha, \beta, \gamma) \right| \leq 1 - \frac{\log^4 n}{2n}.$$

Proof. Recall that we are bounding $\widehat{f}(\alpha, \beta, \gamma) = \mathbb{E} e(\alpha g_A(V) + \beta g_B(V) + \gamma(V - n/2))$. By our assumption we have $|g_A(t)|, |g_B(t)| \leq 5\sqrt{n \log n}$ for every t . Therefore, by the assumption on α and β , we can write

$$\widehat{f}(\alpha, \beta, \gamma) = e(-\gamma n/2) \mathbb{E} e(\gamma V + R(V)),$$

where it always holds that $|R(V)| \leq (\log^2 n / \sqrt{n})$. Following [Pol22], we adopt a simpler version of arguments from the proof of Lemma 3.19. Specifically, we will (later) choose some $0 < m \leq n/2$ and, letting $k := \lfloor n/2m \rfloor$, partition the interval $[0, n]$ into $2k$ intervals $S_1, T_1, \dots, S_k, T_k$ of length m each and a possible “leftover” interval S_0 of length at most $n/2$.

We focus on values of $z(t)$ and $z(t + m)$ inside the intervals. More specifically, we will choose m so that it always holds

$$\frac{3 \log^2 n}{\sqrt{n}} \leq |\gamma|m \leq \frac{1}{4}, \tag{3.26}$$

consequently giving

$$\frac{\log^2 n}{\sqrt{n}} \leq |\gamma|m - \frac{2 \log^2 n}{\sqrt{n}} \leq |\gamma m + R(t + m) - R(t)| \leq |\gamma|m + \frac{2 \log^2 n}{\sqrt{n}} \leq \frac{1}{2},$$

and

$$d_{\mathbb{Z}}(\gamma m + R(t + m) - R(t)) \geq \frac{\log^2 n}{\sqrt{n}}.$$

Applying triangle inequality and Lemma 3.9, we then get

$$\begin{aligned} |\widehat{f}(\alpha, \beta, \gamma)| &\leq \frac{1}{n} \left(\int_{S_0} dt + \sum_{i=1}^k \int_{S_i} |e(\gamma t + R(t)) + e(\gamma(t + m) + R(t + m))| dt \right) \\ &\leq \frac{1}{n} \left(\int_{S_0} dt + \sum_{i=1}^k \int_{S_i} 2 - 2 \frac{\log^4 n}{n} dt \right) = 1 - \frac{2km}{n} \cdot \frac{\log^4 n}{n} \\ &\leq 1 - \frac{\log^4 n}{2n}, \end{aligned}$$

as claimed. It remains to show that a value of $0 < m \leq n/2$ satisfying (3.26) can be chosen. This is done by considering two cases.

Case 1: $6 \log^2 n/n^{3/2} \leq |\gamma| \leq 1/2n$ In this case we just pick $m := n/2$. Indeed, we clearly have

$$\frac{3 \log^2 n}{\sqrt{n}} \leq |\gamma|m = \frac{|\gamma|n}{2} \leq \frac{1}{4}.$$

Case 2: $|\gamma| > 1/2n$ Here we choose $m := 1/4|\gamma|$ and easily check $\frac{3 \log^2 n}{\sqrt{n}} \leq m|\gamma| = 1/4$. \square

3.7. Proof of Lemma 2.10

By Lemma 3.16, $\|U_A\|_{\infty}, \|U_B\|_{\infty} < 5\sqrt{n \log n}$ holds except with probability $2n^{-10}$. Assume that it is so. If both $|\alpha|$ and $|\beta|$ are at most $10^{10} \log n/n$ and $|\gamma|$ exceeds $6 \log^2 n/n^{3/2}$, then we get $|\widehat{f}(\alpha, \beta, \gamma)| \leq 1 - 10 \log n/n$ directly by Lemma 3.23.

As for the case where either $|\alpha|$ or $|\beta|$ exceed $10^{10} \log n/n$, we cover the square $[-1/2, 1/2]^2$ with a grid of points (α_i, β_i) such that

$$\alpha_i, \beta_i \in \left\{ -\frac{1}{2}, -\frac{1}{2} + \frac{1}{n^2}, -\frac{1}{2} + \frac{2}{n^2}, \dots, -\frac{1}{2n^2}, \frac{1}{2n^2}, \dots, \frac{1}{2} \right\}.$$

This is a discrete grid of $(n^2 + 1)^2 \leq 2n^4$ points. We will proceed, for each applicable (α, β, γ) , to find a grid point (α_i, β_i) such that it holds both that

$$|\alpha - \alpha_i| + |\beta - \beta_i| \leq \frac{2}{n^2} \quad \text{and} \quad |\widehat{f}(\alpha_i, \beta_i, \gamma)| \leq 1 - 12 \log n/n. \quad (3.27)$$

Then we will be finished, since, applying Lemma 3.15, we will have

$$\begin{aligned} |\widehat{f}(\alpha, \beta, \gamma)| &\leq |\widehat{f}(\alpha_i, \beta_i, \gamma)| + |\widehat{f}(\alpha_i, \beta_i, \gamma) - \widehat{f}(\alpha, \beta, \gamma)| \\ &\leq 1 - \frac{12 \log n}{n} + \frac{20\pi\sqrt{n \log n}}{n^2} < 1 - \frac{10 \log n}{n}. \end{aligned}$$

To achieve that we apply Lemma 3.18. Given α and β , we take (α_i, β_i) to be the closest possible to (α, β) with $|\alpha_i| \geq |\alpha|$ and $|\beta_i| \geq |\beta|$. Clearly, the first condition in (3.27) holds and by Lemma 3.18 we get that $|\widehat{f}(\alpha_i, \beta_i, \gamma)| \leq 1 - 12 \log n/n$ for every $\gamma \in \mathbb{R}$ except with probability $40n^{-7}$.

Taking union bound over the whole grid and the events from Lemma 3.16, we see that $|\widehat{f}(\alpha, \beta, \gamma)| \leq 1 - 10 \log n/n$ for all applicable points, except with probability at most $2n^{-10} + 2n^4 \cdot 40n^{-7} < n^{-2}$. \square

3.8. Proof of Lemma 2.11

Assume that the thesis of Lemma 2.10 holds, in particular that $\|U_A\|_\infty, \|U_B\|_\infty \leq 5\sqrt{n \log n}$. By Claim 3.3, we have $\widehat{f}(\alpha, \beta, \gamma) = 1 - Q + R$, where Q is given in (3.1) and R is such that

$$|R| \leq 2 \cdot 10^5 (n \log n)^{3/2} (|\alpha|^3 + |\beta|^3) + 150n^3 |\gamma|^3. \quad (3.28)$$

Our task is to estimate the integral

$$\int_{\mathbb{R}^3} |\widehat{g}(\alpha, \beta, \gamma) - \widehat{u}(\alpha, \beta, \gamma)| \, d\alpha \beta \gamma.$$

To that end, we are going to divide \mathbb{R}^3 into various parts. More precisely, let

$$B := \left\{ (\alpha, \beta, \gamma) : |\alpha|, |\beta| \leq 10^{10} \log n/n, |\gamma| \leq 6 \log^2 n/n^{3/2} \right\}.$$

We make an estimate

$$\int_{\mathbb{R}^3} |\widehat{g}(\alpha, \beta, \gamma) - \widehat{u}(\alpha, \beta, \gamma)| \, d\alpha\beta\gamma \leq \int_B |\widehat{g}(\alpha, \beta, \gamma) - \widehat{u}(\alpha, \beta, \gamma)| \, d\alpha\beta\gamma + \int_B |\widehat{g}(\alpha, \beta, \gamma)| \, d\alpha\beta\gamma + \int_B |\widehat{u}(\alpha, \beta, \gamma)| \, d\alpha\beta\gamma$$

and proceed to bounding each of the three resulting terms.

Term 1: $\int_B |\widehat{g} - \widehat{u}|$ For this term, note that for every $(\alpha, \beta, \gamma) \in B$ we have

$$|Q| \leq 6\pi^2 (\alpha^2 \text{Var}_A + \beta^2 \text{Var}_B + \gamma^2 \text{Var } V) \tag{3.29}$$

$$\leq 6\pi^2 \left(2 \cdot \frac{10^{20} \log^2 n}{n^2} \cdot 25n \log n + \frac{36 \log^4 n}{n^3} \cdot \frac{n^2}{4} \right) \leq \frac{\log^5 n}{n}, \tag{3.30}$$

and accordingly $Q^2 \leq 1/100n$. Similarly, continuing from (3.28), we have

$$|R| \leq 2 \cdot 10^5 n^{3/2} \log^{3/2} n \cdot 2 \cdot \frac{10^{30} \log^3 n}{n^3} + 150n^3 \cdot \frac{6^3 \log^6 n}{n^{9/2}} \leq \frac{\log^7 n}{n^{3/2}}, \tag{3.31}$$

in particular again $|R| \leq 1/100n$. Therefore, Lemma 3.13 applies inside B and we can use it, together with definitions of \widehat{g} and \widehat{u} and Claim 3.1 to estimate

$$\begin{aligned} \int_B |\widehat{g} - \widehat{u}| &= \int_B |\exp(-nQ) - (1 - Q + R)^n| \leq \int_B \left| 1 - \frac{(1 - Q + R)^n}{\exp(-nQ)} \right| \\ &\leq \left(\frac{2 \cdot 10^{10} \log n}{n} \right)^2 \cdot \frac{12 \log^2 n}{n^{3/2}} \cdot 40n \cdot \left(\frac{\log^{10} n}{n^2} + \frac{\log^7 n}{n^{3/2}} \right) \leq \frac{\log^{15} n}{n^4}. \end{aligned}$$

Term 2: $\int_B |\widehat{g}|$ By Lemma 2.10, we have $|\widehat{u}(\alpha, \beta, \gamma)| \leq n^{-10}$ for all the points on the boundary of B . Furthermore, applying Lemma 3.13 as well as (3.30) and (3.31), we also get

$$|\widehat{g}| \leq |\widehat{u}| + |\widehat{u} - \widehat{g}| = |\widehat{u}| \left(1 + \left| 1 - \frac{\widehat{g}}{\widehat{u}} \right| \right) \leq |\widehat{u}| (1 + 40n(Q^2 + |R|)) \leq 2n^{-10} \tag{3.32}$$

everywhere on the boundary of B . We now proceed to estimating $\int_B |\widehat{g}|$ by a standard substitution of spherical coordinates $\alpha = t \sin \phi \cos \theta$, $\beta = t \sin \phi \sin \theta$, $\gamma = t \cos \phi$, where ϕ ranges from 0 to π and θ from 0 to 2π :

$$\int_B |\widehat{g}(\alpha, \beta, \gamma)| \, d\alpha\beta\gamma = \int_B t^2 \sin \phi \cdot \widehat{g}(t \sin \phi \cos \theta, t \sin \phi \sin \theta, t \cos \phi) \, dt\phi\theta, \tag{3.33}$$

For fixed ϕ and θ , we let $\widetilde{g}(t) := \widehat{g}(t \sin \phi \cos \theta, t \sin \phi \sin \theta, t \cos \phi)$ and note that for the corresponding iterated integral we have, for some appropriate $t_0 = t_0(\phi, \theta)$,

$$\int_B t^2 \sin \phi \cdot \widetilde{g}(t) \, dt = \int_{t_0}^\infty t^2 \sin \phi \cdot \widetilde{g}(t) \, dt \leq \int_{t_0}^\infty t^2 \cdot \widetilde{g}(t) \, dt.$$

In order to bound this last integral, note that by definition of $\hat{g} = \exp(-nQ)$ it must be that $\tilde{g}(t) = \exp(-kt^2)$ for some $k \geq 0$. Furthermore, by (3.32) we have $\exp(-kt_0^2) \leq 2n^{-10}$ and by definition of B also $1/n^2 \leq |t_0| \leq \log^2 n/n$. Letting $\delta := t_0$ and $\varepsilon := 2n^{-10}$, we estimate

$$\begin{aligned} \int_{t_0}^{\infty} t^2 \cdot \tilde{g}(t) dt &= \int_{t_0}^{\infty} t^2 \exp(-kt^2) dt \leq \delta \sum_{j=0}^{\infty} (t_0 + (j+1)\delta)^2 \exp(-k(t_0 + \delta j)^2) \\ &\leq t_0^3 \exp(-kt_0^2) \sum_{j=0}^{\infty} (j+2)^2 \exp(-kt_0^2 j) \\ &\leq t_0^3 \varepsilon \sum_{j=0}^{\infty} \left((j+2)(j+1) + (j+1) + 1 \right) \varepsilon^j \\ &= t_0^3 \varepsilon \left(\frac{1}{(1-\varepsilon)^3} + \frac{1}{(1-\varepsilon)^2} + \frac{1}{1-\varepsilon} \right) \leq n^{-12}. \end{aligned} \quad (3.34)$$

Chaining together (3.34) and (3.33) and applying Fubini's theorem, we finally get $\int_{\bar{B}} |\hat{g}| \leq 2\pi^2 n^{-12} \leq n^{-11}$, better than what we needed for (2.7).

Term 3: $\int_{\bar{B}} |\hat{u}|$ Here we divide the area of \bar{B} in two more subcases. First, for values $|\gamma| \leq 4$, by Lemma 2.10 we have that the total contribution to the integral is at most $8 \cdot n^{-10}$ (recall that \hat{u} is zero outside of $|\alpha|, |\beta| \leq 1/2$). For $|\alpha|, |\beta| \leq 1/2$ and $|\gamma| > 4$, by Lemma 3.22 we have $|\hat{f}(\alpha, \beta, \gamma)| \leq 1/|\gamma - \alpha - \beta| \leq 2/|\gamma|$, and consequently, for fixed α, β ,

$$\int_{|\gamma|>4} |\hat{u}(\alpha, \beta, \gamma)| d\gamma \leq 2^{n+1} \int_4^{\infty} \gamma^{-n} d\gamma = \frac{4}{(n-1)2^{n-1}} \leq 2^{-n}.$$

Therefore, we have $\int_{\bar{B}} |\hat{u}| \leq 8n^{-10} + 2^{-n} \leq n^{-9}$ and putting all the cases together

$$\int_{\mathbb{R}^3} |\hat{g} - \hat{u}| \leq \log^{16} n/n^4. \quad \square$$

3.9. Proof of Theorem 2.8

Finally we are ready to prove the main theorem of this section. As a preliminary point, by Claim 2.9 the condition $\text{Var}_A - (CV_A^2/\text{Var } V), \text{Var}_B - (CV_B^2/\text{Var } V) \geq \varepsilon n$ implies

$$\text{Var}[G_A^{*n} | H^{*n} = n^2/2], \text{Var}[G_B^{*n} | H^{*n} = n^2/2] \geq \varepsilon n^2.$$

This will be used when invoking Claim 3.5 later on. Throughout we use the notation \mathcal{E}_V for the event $V^{*n} = n^2/2$ and \mathcal{E}_H for $H^{*n} = n^2/2$.

Below we show a detailed calculation establishing $\Pr[U_A^{*n}, U_B^{*n} | \mathcal{E}_V] \geq \Pr[G_A^{*n}, G_B^{*n} | \mathcal{E}_H] - \tilde{O}(1/\sqrt{n})$. The justification for the reverse inequality is very similar and we skip it. We proceed by a string of inequalities applying the lemmas we proved before. Using union bound, we check that all events used in those lemmas happen except with probability $1/n$. Whenever we are

summing over a and b , the sum goes over the set $\mathbb{Z} + 1/2$:

$$\begin{aligned}
 & \Pr \left[U_A^{*n}, U_B^{*n} > 0 \mid \mathcal{E}_V \right] \\
 & \stackrel{\text{Lem 3.17}}{\geq} \frac{\Pr \left[50n \log n \geq U_A^{*n}, U_B^{*n} > 0 \mid \mathcal{E}_V \right]}{\Pr \left[|U_A^{*n}|, |U_B^{*n}| \leq 50n \log n \mid \mathcal{E}_V \right] + 2n^{-10}} \\
 & \stackrel{\text{Cl 3.2}}{=} \frac{\sum_{50n \log n \geq a, b > 0} \int_{\mathbb{R}^3} \hat{u}(\alpha, \beta, \gamma) e(-\alpha a - \beta b) \, d\alpha \beta \gamma}{\sum_{|a|, |b| \leq 50n \log n} \int_{\mathbb{R}^3} \hat{u}(\alpha, \beta, \gamma) e(-\alpha a - \beta b) \, d\alpha \beta \gamma + 2n^{-10}} \\
 & \stackrel{\text{Lem 2.11}}{\geq} \frac{\sum_{50n \log n \geq a, b > 0} \int_{\mathbb{R}^3} \hat{g}(\alpha, \beta, \gamma) e(-\alpha a - \beta b) \, d\alpha \beta \gamma - \log^{19} n/n^2}{\sum_{|a|, |b| \leq 50n \log n} \int_{\mathbb{R}^3} \hat{g}(\alpha, \beta, \gamma) e(-\alpha a - \beta b) \, d\alpha \beta \gamma + \log^{19} n/n^2} \\
 & \stackrel{\text{Cl 3.2}}{=} \frac{\sum_{50n \log n \geq a, b > 0} g(a, b, 0) - \log^{19} n/n^2}{\sum_{|a|, |b| \leq 50n \log n} g(a, b, 0) + \log^{19} n/n^2} \\
 & \stackrel{*}{\geq} \frac{g(0) \Pr \left[50n \log n > G_A^{*n}, G_B^{*n} > 1/2 \mid \mathcal{E}_H \right] - \log^{19} n/n^2}{g(0) + g(0)8 \Pr \left[|G_A^{*n}| \leq 1/2 \mid \mathcal{E}_H \right] + \log^{19} n/n^2} \\
 & \stackrel{\text{Cl 3.5}}{\geq} \frac{g(0) \left(\Pr \left[50n \log n > G_A^{*n}, G_B^{*n} > 0 \mid \mathcal{E}_H \right] - 2/\sqrt{\varepsilon n} \right) - \log^{19} n/n^2}{g(0) \left(1 + 8/\sqrt{\varepsilon n} \right) + \log^{19} n/n^2} \\
 & \stackrel{\text{Lem 3.17}}{\geq} \frac{g(0) \left(\Pr \left[G_A^{*n}, G_B^{*n} > 0 \mid \mathcal{E}_H \right] - 2/\sqrt{\varepsilon n} - 2n^{-10} \right) - \log^{19} n/n^2}{g(0) \left(1 + 8/\sqrt{\varepsilon n} \right) + \log^{19} n/n^2} \\
 & \stackrel{\text{Cl 3.4}}{\geq} \frac{\Pr \left[G_A^{*n}, G_B^{*n} > 0 \mid \mathcal{E}_H \right] - \tilde{O}(1/\sqrt{n})}{1 + \tilde{O}(1/\sqrt{n})} \geq \Pr \left[G_A^{*n}, G_B^{*n} > 0 \mid \mathcal{E}_H \right] - \tilde{O}(1/\sqrt{n}) .
 \end{aligned}$$

We need an additional explanation for the inequality marked with the $*$ sign. There, in the numerator we used the fact that $g(a, b, 0) \geq \int_B g(a', b', 0) \, da' b'$, where B is the 1×1 box

$$\begin{aligned}
 B = \left\{ (a', b') : |a| + 1 > |a'| > |a|, \text{sgn}(a') = \text{sgn}(a) \right. \\
 \left. \text{and } |b| + 1 > |b'| > |b|, \text{sgn}(b') = \text{sgn}(b) \right\} .
 \end{aligned}$$

Similarly, in the denominator we used that whenever $|a|, |b| > 1$, then

$$g(a, b, 0) \leq \int_B g(a', b', 0) \, da' b'$$

for

$$\begin{aligned}
 B = \left\{ (a', b') : |a| > |a'| > |a| - 1, \text{sgn}(a') = \text{sgn}(a) \right. \\
 \left. \text{and } |b| > |b'| > |b| - 1, \text{sgn}(b') = \text{sgn}(b) \right\} ,
 \end{aligned}$$

as well as $g(a, b, 0) \leq 4 \int_B g(a', b', 0) da'b'$ if $|a| = 1/2$ or $|b| = 1/2$ with

$$B = \left\{ (a', b') : |a| > |a'| > |a| - 1/2, \operatorname{sgn}(a') = \operatorname{sgn}(a) \right. \\ \left. \text{and } |b| > |b'| > |b| - 1/2, \operatorname{sgn}(b') = \operatorname{sgn}(b) \right\}.$$

An illustration of this argument is provided in Figure 3.2. □

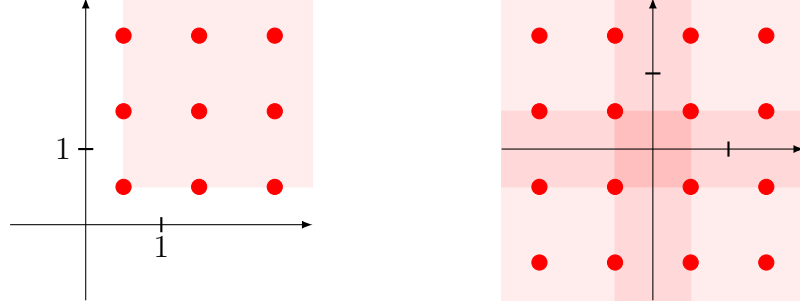


Figure 3.2: A graphical illustration of claims in the proof of Theorem 2.8. We are using the fact that $g(ta_0, tb_0, 0)$ is decreasing in $|t|$ for every direction (a_0, b_0) .

3.10. Proof of Theorem 2.7

For the same reasons as those given in Section 2.3, let us assume, as elsewhere in this section, that the dice have faces in $[0, n]$ and that the conditioning (denoted by \mathcal{E}_V) is given by the event $V^{*n} = n^2/2$. For purposes of this proof let $P(a, b) := \Pr[U_A^{*n} = a, U_B^{*n} = b \mid \mathcal{E}_V]$. To prove that $\Pr[U_A^{*n} > 0 \mid \mathcal{E}_V] \approx 1/2$, i.e., that a random balanced die A is approximately “fair”, we can employ a simplified argument using symmetry. Specifically, by Claim 3.2 and Lemma 2.11, except with probability n^{-2} we have that for every $a, b \in \mathbb{Z} + 1/2$ it holds that

$$\left| P(a, b) \cdot h(0) - g(a, b, 0) \right| \leq \int_{\mathbb{R}^3} \left| \hat{u}(\alpha, \beta, \gamma) - \hat{g}(\alpha, \beta, \gamma) \right| d\alpha\beta\gamma \leq \frac{\log^{16} n}{n^4}.$$

Since g is a density function of a centered Gaussian, we have $g(a, b, 0) = g(-a, -b, 0)$ which implies, using also Claim 3.4,

$$\left| P(a, b) - P(-a, -b) \right| \leq \frac{8 \log^{16} n}{n^{5/2}}. \quad (3.35)$$

Finally, we apply Lemma 3.17 (twice) and Lemma 3.16, as well as (3.35) to get that, except with probability $2n^{-10} + n^{-2} < 1/n$,

$$\begin{aligned} \left| \Pr \left[U_A^{*n} > 0 \mid V^{*n} = \frac{n^2}{2} \right] - \frac{1}{2} \right| &\leq \left| \sum_{\substack{0 < a \leq 50n \log n \\ |b| \leq 50n \log n}} P(a, b) - \frac{1}{2} \right| + 2n^{-10} \\ &\leq \frac{1}{2} \left| \sum_{|a|, |b| \leq 50n \log n} P(a, b) - 1 \right| + \tilde{\mathcal{O}} \left(\frac{1}{\sqrt{n}} \right) \\ &\leq \tilde{\mathcal{O}} \left(\frac{1}{\sqrt{n}} \right). \quad \square \end{aligned}$$

4. Bounding moments

This section contains proofs of Lemmas 2.14 and 2.15, following the method originally used in [HMRZ20]. As a reminder, in this section we consider faces drawn uniformly in $[-\sqrt{3}, \sqrt{3}]$, i.e., $\mathbb{E} V = 0$ and $\text{Var} V = 1$ and for $x \in [-\sqrt{3}, \sqrt{3}]$ we have

$$g_A(x) = \left| \{i : a_i \leq x\} \right| - n \frac{x + \sqrt{3}}{2\sqrt{3}}.$$

In this setting, a die $A = (a_i)_{i \in [n]}$ is said to be balanced if $\sum_{i=1}^n a_i = 0$ and the proper adaptation of the expectation formula (1.2) for random balanced die becomes

$$\mathbb{E} [f(a_1, \dots, a_n)] = \frac{\int_{\mathbb{R}^{n-1}} f(a_1, \dots, a_n) \cdot \bigwedge_{i=1}^n \mathbb{1} \left(-\sqrt{3} \leq a_i \leq \sqrt{3} \right) da_1 \dots a_{n-1}}{\int_{\mathbb{R}^{n-1}} \bigwedge_{i=1}^n \mathbb{1} \left(-\sqrt{3} \leq a_i \leq \sqrt{3} \right) da_1 \dots a_{n-1}}, \quad (4.1)$$

where we let $a_n := -\sum_{i=1}^{n-1} a_i$.

4.1. Warm-up

In this section we illustrate the methodology of the proofs of Lemmas 2.14 and 2.15 on a simpler example. Namely, we show

Lemma 4.1. $\mathbb{E}[\text{Var}_A] = \mathcal{O}(n)$.

Note that strictly speaking Lemma 4.1 is not necessary for our proof, since it follows from Lemma 2.14 by Cauchy–Schwarz inequality.

Proof. Let A be a fixed die, not necessarily balanced, with faces between $-\sqrt{3}$ and $\sqrt{3}$. Then, using $\Pr[a \leq V] = \Pr[a < V] = \frac{1}{2} - \frac{a}{2\sqrt{3}}$ and

$$\mathbb{E} \left[\mathbb{1}(a < V) V \right] = \int_a^{\sqrt{3}} \frac{1}{2\sqrt{3}} \cdot x dx = \frac{\sqrt{3}}{12} (3 - a^2), \quad (4.2)$$

we have

$$\begin{aligned}
\mathbb{E}[g_A(V)^2] &= \sum_{i,j=1}^n \mathbb{E} \left[\left(\mathbb{1}(a_i < V) - \frac{V + \sqrt{3}}{2\sqrt{3}} \right) \left(\mathbb{1}(a_j < V) - \frac{V + \sqrt{3}}{2\sqrt{3}} \right) \right] \\
&= \sum_{i,j=1}^n \Pr[\max\{a_i, a_j\} < V] - \frac{n}{\sqrt{3}} \sum_{i=1}^n \mathbb{E}[\mathbb{1}(a_i < V)(V + \sqrt{3})] \\
&\quad + \frac{n^2}{12} \mathbb{E}[(V + \sqrt{3})^2] \\
&= \frac{n^2}{2} - \frac{1}{2\sqrt{3}} \sum_{i,j=1}^n \max\{a_i, a_j\} - \frac{n^2}{4} + \frac{n}{12} \sum_{i=1}^n a_i^2 - \frac{n^2}{2} + \frac{n}{2\sqrt{3}} \sum_{i=1}^n a_i + \frac{n^2}{3} \\
&= \frac{n^2}{12} + \frac{n}{2\sqrt{3}} \sum_{i=1}^n a_i + \frac{n}{12} \sum_{i=1}^n a_i^2 - \frac{1}{2\sqrt{3}} \sum_{i,j=1}^n \max\{a_i, a_j\}. \tag{4.3}
\end{aligned}$$

Let us start with a heuristic argument. To that end, consider iid uniform a_1, \dots, a_n without balanced conditioning. In that case we can simply employ $\mathbb{E}[a_i] = 0$, $\mathbb{E}[a_i^2] = 1$ and

$$\mathbb{E}[\max\{a_i, a_j\}] = 2 \int_{-\sqrt{3}}^{\sqrt{3}} \int_x^{\sqrt{3}} \frac{1}{12} y \, dy \, dx = \frac{1}{12} \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 \, dx = \frac{\sqrt{3}}{3}$$

and substitute into (4.3), getting

$$\mathbb{E} \left[\mathbb{E}[g_A(V)^2 \mid A] \right] = \frac{n^2}{12} + \frac{n^2}{12} - \frac{n(n-1)}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{3} = \frac{n}{6}. \tag{4.4}$$

In other words, without conditioning, a typical value of the second moment $\mathbb{E}[g_A(V)^2]$ is $\Theta(n)$. Now recall from Claim 2.5 that if A is a balanced die, then $\mathbb{E}[g_A(V)] = 0$ and consequently $\text{Var}_A = \mathbb{E}[g_A(V)^2]$. We are counting on the estimate in (4.4) remaining of the same order of magnitude under the conditioning. To verify that, we substitute into (4.3) in the balanced case and using $\sum_{i=1}^n a_i = 0$, obtain

$$\mathbb{E}[\text{Var}_A] = \frac{n^2}{12} + \frac{n}{12} \mathbb{E} \left[\sum_{i=1}^n a_i^2 \right] - \frac{1}{2\sqrt{3}} \mathbb{E} \left[\sum_{i \neq j} \max\{a_i, a_j\} \right] \tag{4.5}$$

$$= \frac{n^2}{12} + \frac{n^2}{12} \mathbb{E}[a_1^2] - \frac{n(n-1)}{2\sqrt{3}} \mathbb{E}[\max\{a_1, a_2\}]. \tag{4.6}$$

What remains is to compute the conditional expectations featured in (4.6): For example, $\mathbb{E}[a_1^2]$ (which is really a function of n) is the expectation of a_1^2 for a random balanced die. In order to do that, we need to understand how the joint distribution of independent (a_1, a_2) changes upon balanced conditioning.

Let us start with $\mathbb{E}[a_1^2]$. Let $\tilde{\phi}_{n-1}$ be the density of $(n-1)$ -wise convolution of the uniform density on $[-\sqrt{3}, \sqrt{3}]$. Recall the convolution formula

$$\tilde{\phi}_{n-1}(s) = (2\sqrt{3})^{-(n-1)} \int_{\mathbb{R}^{n-1}} \bigwedge_{i=2}^n \mathbb{1}(-\sqrt{3} \leq a_i \leq \sqrt{3}) \, da_2 \dots a_{n-1},$$

where $a_n = s - \sum_{i=2}^{n-1} a_i$. Starting from (4.1), and this time letting $a_n := -\sum_{i=1}^{n-1} a_i$, we see that

$$\begin{aligned} \mathbb{E}[a_1^2] &= \frac{\int_{\mathbb{R}^{n-1}} a_1^2 \cdot \Lambda_{i=1}^n \mathbb{1} \left(-\sqrt{3} \leq a_i \leq \sqrt{3} \right) da_1 \dots a_{n-1}}{\int_{\mathbb{R}^{n-1}} \Lambda_{i=1}^n \mathbb{1} \left(-\sqrt{3} \leq a_i \leq \sqrt{3} \right) da_1 \dots a_{n-1}} \\ &= \frac{\int_{-\sqrt{3}}^{\sqrt{3}} a_1^2 \cdot \int_{\mathbb{R}^{n-2}} \Lambda_{i=2}^n \mathbb{1} \left(-\sqrt{3} \leq a_i \leq \sqrt{3} \right) da_2 \dots a_{n-1} da_1}{\int_{-\sqrt{3}}^{\sqrt{3}} \int_{\mathbb{R}^{n-2}} \Lambda_{i=2}^n \mathbb{1} \left(-\sqrt{3} \leq a_i \leq \sqrt{3} \right) da_2 \dots a_{n-1} da_1} \\ &= \frac{\int_{-\sqrt{3}}^{\sqrt{3}} t^2 \tilde{\phi}_{n-1}(-t) dt}{\int_{-\sqrt{3}}^{\sqrt{3}} \tilde{\phi}_{n-1}(-t) dt} = \frac{\int_{-\sqrt{3}}^{\sqrt{3}} t^2 \tilde{\phi}_{n-1}(t) dt}{\int_{-\sqrt{3}}^{\sqrt{3}} \tilde{\phi}_{n-1}(t) dt} = \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} t^2 p_{n-1}(t) dt = \mathbb{E}[V^2 p_{n-1}(V)] , \end{aligned}$$

where $p_{n-1}(x) = \frac{\tilde{\phi}_{n-1}(x)}{\frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \tilde{\phi}_{n-1}(s) ds}$ and $V \sim \mathcal{U}[-\sqrt{3}, \sqrt{3}]$. This way, we expressed the expectation of a_1^2 in the conditional distribution through an unconditional expectation of another deterministic function. Thus, our problem reduces to computing the density $\tilde{\phi}_{n-1}$ and the function p_{n-1} .

Similarly, for $\mathbb{E}[\max\{a_1, a_2\}]$ and letting $\tilde{\phi}_{n-2}$ be the density of $\sum_{i=3}^n a_i$, we have

$$\begin{aligned} \mathbb{E}[\max\{a_1, a_2\}] &= \frac{\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \max\{t_1, t_2\} \tilde{\phi}_{n-2}(t_1 + t_2) dt_1 dt_2}{\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \tilde{\phi}_{n-2}(t_1 + t_2) dt_1 dt_2} \\ &= \frac{1}{12} \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \max\{t_1, t_2\} p_{n-2}(t_1 + t_2) dt_1 dt_2 \\ &= \mathbb{E}[\max\{V_1, V_2\} p_{n-2}(V_1 + V_2)] , \end{aligned}$$

where $p_{n-2}(x) = \frac{\tilde{\phi}_{n-2}(x)}{\frac{1}{12} \iint_{[-\sqrt{3}, \sqrt{3}]^2} \tilde{\phi}_{n-2}(s_1 + s_2) ds_1 ds_2}$ and $V_1, V_2 \stackrel{iid}{\sim} \mathcal{U}[-\sqrt{3}, \sqrt{3}]$. Again, the problem reduces to the computation of the function p_{n-2} .

We defer estimating p_{n-1} and p_{n-2} to the following section. For now we finish the argument by applying (4.9) and (4.10) from Lemma 4.3 to get

$$p_{n-1}(t) = 1 + \mathcal{O}(n^{-1}) , \quad p_{n-2}(t) = 1 + \mathcal{O}(n^{-1}) ,$$

and consequently

$$\begin{aligned} \mathbb{E}[a_1^2] &= \mathbb{E}[V^2 p_{n-1}(V)] = 1 + \mathcal{O}(n^{-1}) , \\ \mathbb{E}[\max\{a_1, a_2\}] &= \mathbb{E} \left[\max\{V_1, V_2\} p_{n-2}(V_1 + V_2) \right] = \sqrt{3}/3 + \mathcal{O}(n^{-1}) , \\ \mathbb{E}[\text{Var}_A] &= \frac{n^2}{12} + \frac{n^2}{12} (1 + \mathcal{O}(n^{-1})) - \frac{n(n-1)}{2\sqrt{3}} \left(\frac{\sqrt{3}}{3} + \mathcal{O}(n^{-1}) \right) = \mathcal{O}(n) . \quad \square \end{aligned}$$

Remark 4.2. As a matter of fact, Lemma 4.3 yields more precise estimates

$$p_{n-1}(t) = 1 + \frac{1}{2n} - \frac{t^2}{2n} + \mathcal{O}(n^{-2}) , \quad p_{n-2}(t) = 1 + \frac{1}{n} - \frac{t^2}{2n} + \mathcal{O}(n^{-2}) , \quad (4.7)$$

which are then checked to lead to

$$\begin{aligned}
\mathbb{E}[V^2 p_{n-1}(V)] &= 1 + \frac{1}{2n} - \frac{\mathbb{E}[V^4]}{2n} + \mathcal{O}(n^{-2}) = 1 - \frac{2}{5n} + \mathcal{O}(n^{-2}), \\
\mathbb{E}[\max\{V_1, V_2\} p_{n-2}(V_1 + V_2)] &= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3n} - \frac{\mathbb{E}[\max\{V_1, V_2\}(V_1 + V_2)^2]}{2n} + \mathcal{O}(n^{-2}) \\
&= \frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{15n} + \mathcal{O}(n^{-2}), \\
\mathbb{E}[\text{Var}_A] &= \frac{n^2}{12} + \frac{n^2}{12} \left(1 - \frac{2}{5n}\right) - \frac{n(n-1)}{2\sqrt{3}} \left(\frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{15n}\right) + \mathcal{O}(1) \\
&= \frac{1}{15}n + \mathcal{O}(1).
\end{aligned}$$

Therefore, while conditioning on balanced A decreased the expected variance Var_A , its order of magnitude remained the same.

In the following sections we will continue with justifying (4.7). This is obtained by estimating $\tilde{\phi}_{n-k}$ using a precise local central limit theorem for densities. Heuristically, for $x = \mathcal{O}(1)$, we have

$$\tilde{\phi}_{n-k}(x) \propto \exp\left(\frac{x^2}{2(n-k)}\right) \approx 1 - \frac{x^2}{2n}.$$

However, there are several other error terms, including a correction coming from the denominator in the definition of $p_{n-k}(x)$. For our final objective, i.e., the proofs of Lemmas 2.14 and 2.15, we will need the approximations of $p_{n-1}(x)$ and $p_{n-2}(x)$ up to an $\mathcal{O}(n^{-3})$ error term.

4.2. Estimating p_{n-k}

In this section we estimate the ‘‘correction factors’’ p_{n-k} and derive general formulas for the balanced conditional expectations. The precise statement we will use later on is:

Lemma 4.3. *Let $A = (a_i)_{i \in [n]}$ and $B = (b_i)_{i \in [n]}$ be two random dice with faces drawn uniformly iid in $[-\sqrt{3}, \sqrt{3}]$, conditioned on both being balanced. Let $V_1, V_2, V_3, V_4 \stackrel{iid}{\sim} \mathcal{U}[-\sqrt{3}, \sqrt{3}]$. Let f denote a generic integrable function. Then,*

$$\begin{aligned}
\mathbb{E}[f(a_1)] &= \mathbb{E}[f(V_1)p_{n-1}(V_1)] \\
\mathbb{E}[f(a_1, a_2)] &= \mathbb{E}[f(V_1, V_2)p_{n-2}(V_1 + V_2)] \\
\mathbb{E}[f(a_1, a_2, a_3)] &= \mathbb{E}[f(V_1, V_2, V_3)p_{n-3}(V_1 + V_2 + V_3)] \\
\mathbb{E}[f(a_1, a_2, a_3, a_4)] &= \mathbb{E}[f(V_1, V_2, V_3, V_4)p_{n-4}(V_1 + V_2 + V_3 + V_4)] \\
\mathbb{E}[f(a_1, b_1)] &= \mathbb{E}[f(V_1, V_2)p_{n-1}(V_1)p_{n-1}(V_2)] \\
\mathbb{E}[f(a_1, a_2, b_1)] &= \mathbb{E}[f(V_1, V_2, V_3)p_{n-2}(V_1 + V_2)p_{n-1}(V_3)] \\
\mathbb{E}[f(a_1, a_2, b_1, b_2)] &= \mathbb{E}[f(V_1, V_2, V_3, V_4)p_{n-2}(V_1 + V_2)p_{n-2}(V_3 + V_4)], \tag{4.8}
\end{aligned}$$

where

$$p_{n-1}(x) = 1 + \frac{1}{2n} - \frac{x^2}{2n} + \frac{9}{40n^2} - \frac{9x^2}{20n^2} + \frac{x^4}{8n^2} + \mathcal{O}(n^{-3}), \tag{4.9}$$

$$p_{n-2}(x) = 1 + \frac{1}{n} - \frac{x^2}{2n} + \frac{6}{5n^2} - \frac{6x^2}{5n^2} + \frac{x^4}{8n^2} + \mathcal{O}(n^{-3}), \tag{4.10}$$

and

$$\begin{aligned} p_{n-3}(x) &= 1 + \frac{3}{2n} - \frac{x^2}{2n} + \mathcal{O}(n^{-2}), \\ p_{n-4}(x) &= 1 + \frac{2}{n} - \frac{x^2}{2n} + \mathcal{O}(n^{-2}). \end{aligned} \tag{4.11}$$

The main tool in the proof of Lemma 4.3 is a precise formula for $\tilde{\phi}_n(x)$. We state it as an application to the uniform distribution of a general local limit theorem for densities from [Pet75]:

Theorem 4.4 ([Pet75], Theorem 15, pp. 206-207). *Let (X_n) be a sequence of iid continuous random variables with bounded densities, zero mean, variance one, and finite absolute moment $\mathbb{E}|X_1|^k$ for some $k \geq 3$. Moreover, for $\nu \in \mathbb{N}$, let*

$$q_\nu(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum H_{\nu+2s}(x) \prod_{m=1}^{\nu} \frac{\Gamma_{m+2}^{k_m}}{k_m!}, \tag{4.12}$$

where the summation is over all non-negative integer solutions (k_1, k_2, \dots, k_ν) of the equalities $k_1 + 2k_2 + \dots + \nu k_\nu = \nu$, $s = k_1 + k_2 + \dots + k_\nu$, H_m is the m^{th} (probabilists') Hermite polynomial and $\Gamma_k = \gamma_k/k!$ with γ_k being the cumulant of order k of the random variable X_1 . Then, the density ϕ_n of the random variable $1/\sqrt{n} \sum_{i=1}^n X_i$ satisfies

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \sum_{\nu=1}^{k-2} \frac{q_\nu(x)}{n^{\nu/2}} + o\left(\frac{1}{n^{(k-2)/2}}\right),$$

uniformly in x .

Corollary 4.5. *Let (X_n) be a sequence of iid random variables uniform in $[-\sqrt{3}, \sqrt{3}]$. Then, the density ϕ_n of the random variable $1/\sqrt{n} \sum_{i=1}^n X_i$ satisfies*

$$\phi_n(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left[1 + \frac{1}{n} \Gamma_4 H_4(x) + \frac{1}{n^2} \left(\Gamma_6 H_6(x) + \frac{\Gamma_4^2}{2} H_8(x) \right) + \frac{P(x)}{n^3} \right] + o(n^{-3}),$$

where $P(x)$ is a fixed polynomial, $\Gamma_4 = -1/20$, $\Gamma_6 = 1/105$ and

$$H_4(x) = 3 - 6x^2 + \mathcal{O}(x^4), \quad H_6(x) = -15 + \mathcal{O}(x^2), \quad H_8(x) = 105 + \mathcal{O}(x^2).$$

Proof. It is a direct application of Theorem 4.4 to the uniform distribution on $[-\sqrt{3}, \sqrt{3}]$. For verification, it is useful to note that since the uniform distribution is symmetric around its mean zero, its odd order cumulants are zero. In particular, all q_ν polynomials for odd ν are identically zero (since in each term of the sum in (4.12) there is at least one factor $\Gamma_k = 0$ for odd k). On the other hand, we recall the formulas for Hermite polynomials and check that $\gamma_4 = -6/5$ and $\gamma_6 = 48/7$. □

We continue with a variable substitution to get:

Corollary 4.6. *Let $\tilde{\phi}_n(x)$ be the density of $\sum_{i=1}^n X_i$, where $(X_i)_{i \in [n]}$ are iid uniform in the interval $[-\sqrt{3}, \sqrt{3}]$. Then, for $|x| = \mathcal{O}(1)$, we have*

$$\tilde{\phi}_n(x) = \frac{1}{\sqrt{2\pi n}} \left[1 + \frac{A}{n} - \frac{x^2}{2n} + \frac{C}{n^2} - \frac{Ax^2}{2n^2} + \frac{Ex^2}{n^2} + \frac{x^4}{8n^2} \right] + \mathcal{O}(n^{-7/2}),$$

where we denoted $A := 3\Gamma_4$, $C := -15\Gamma_6 + \frac{105}{2}\Gamma_4^2$ and $E := -6\Gamma_4$.

Proof. Recall that ϕ_n is the density of $\frac{1}{\sqrt{n}} \sum_{i=1}^n a_i$ and $\tilde{\phi}_n$ the density of $\sum_{i=1}^n a_i$. These two are related by $\tilde{\phi}_n(x) = \frac{1}{\sqrt{n}} \phi_n\left(\frac{x}{\sqrt{n}}\right)$. Substituting into Corollary 4.5,

$$\begin{aligned} \tilde{\phi}_n(x) &= \frac{e^{-\frac{x^2}{2n}}}{\sqrt{2\pi n}} \left[1 + \frac{\Gamma_4}{n} H_4\left(\frac{x}{\sqrt{n}}\right) + \frac{\Gamma_6}{n^2} H_6\left(\frac{x}{\sqrt{n}}\right) + \frac{\Gamma_4^2}{2n^2} H_8\left(\frac{x}{\sqrt{n}}\right) \right] + \mathcal{O}(n^{-7/2}), \\ &= \frac{e^{-\frac{x^2}{2n}}}{\sqrt{2\pi n}} \left[1 + \frac{A}{n} + \frac{C}{n^2} + \frac{Ex^2}{n^2} \right] + \mathcal{O}(n^{-7/2}) \\ &= \frac{1}{\sqrt{2\pi n}} \left(1 - \frac{x^2}{2n} + \frac{x^4}{8n^2} \right) \left[1 + \frac{A}{n} + \frac{C}{n^2} + \frac{Ex^2}{n^2} \right] + \mathcal{O}(n^{-7/2}) \\ &= \frac{1}{\sqrt{2\pi n}} \left[1 + \frac{A}{n} - \frac{x^2}{2n} + \frac{C}{n^2} - \frac{Ax^2}{2n^2} + \frac{Ex^2}{n^2} + \frac{x^4}{8n^2} \right] + \mathcal{O}(n^{-7/2}). \quad \square \end{aligned}$$

We are now ready to complete the proof of Lemma 4.3.

Proof of Lemma 4.3. Let us start with approximating $\tilde{\phi}_{n-k}(x)$ for fixed k and $|x| = \mathcal{O}(1)$. Applying Corollary 4.6 and using $1/(n-k) = 1/n + k/n^2 + \mathcal{O}(n^{-3})$ and $1/(n-k)^2 = 1/n^2 + \mathcal{O}(n^{-3})$,

$$\begin{aligned} \tilde{\phi}_{n-k}(x) &= \frac{1}{\sqrt{2\pi(n-k)}} \left[1 + \frac{A}{n-k} - \frac{x^2}{2(n-k)} + \frac{C}{n^2} - \frac{Ax^2}{2n^2} + \frac{Ex^2}{n^2} + \frac{x^4}{8n^2} \right] + \mathcal{O}(n^{-7/2}) \\ &= \frac{1}{\sqrt{2\pi(n-k)}} \left[1 + \frac{A}{n} + \frac{kA}{n^2} - \frac{x^2}{2n} - \frac{kx^2}{2n^2} + \frac{C}{n^2} - \frac{Ax^2}{2n^2} + \frac{Ex^2}{n^2} + \frac{x^4}{8n^2} \right] \\ &\quad + \mathcal{O}(n^{-7/2}) \\ &= \frac{1}{\sqrt{2\pi(n-k)}} \left[1 + \frac{A}{n} - \frac{x^2}{2n} + \frac{kA+C}{n^2} + \frac{(2E-A-k)x^2}{2n^2} + \frac{x^4}{8n^2} \right] + \mathcal{O}(n^{-7/2}). \end{aligned} \tag{4.13}$$

As in the proof of Lemma 4.1, we let

$$p_{n-k}(x) := \frac{\tilde{\phi}_{n-k}(x)}{\frac{1}{(2\sqrt{3})^k} \int_{[-\sqrt{3}, \sqrt{3}]^k} \tilde{\phi}_{n-k}\left(\sum_{i=1}^k s_i\right) ds_1 \dots ds_k}.$$

Having already established (4.13), we turn to the denominator in the definition of p_{n-k} . Let V_1, \dots, V_k be iid uniform in $[-\sqrt{3}, \sqrt{3}]$. In the following we will use $\mathbb{E} \left(\sum_{i=1}^k V_i \right)^2 = k$ and $\mathbb{E} \left(\sum_{i=1}^k V_i \right)^4 = k \mathbb{E} V_1^4 + \binom{k}{2} \binom{4}{2} \mathbb{E}[V_1^2 V_2^2] = 9k/5 + 6 \binom{k}{2}$. Applying (4.13),

$$\begin{aligned} Z_{n-k} &:= \frac{1}{(2\sqrt{3})^k} \int_{[-\sqrt{3}, \sqrt{3}]^k} \tilde{\phi}_{n-k} \left(\sum_{i=1}^k t_i \right) dt_1 \dots t_k = \mathbb{E} \tilde{\phi}_{n-k} \left(\sum_{i=1}^k V_i \right) \\ &= \frac{1}{\sqrt{2\pi(n-k)}} \left[1 + \frac{A}{n} - \frac{k}{2n} + \frac{kA+C}{n^2} + \frac{k(2E-A-k)}{2n^2} + \frac{9k/5+6\binom{k}{2}}{8n^2} \right] \\ &\quad + \mathcal{O}(n^{-7/2}) \\ &= \frac{1}{\sqrt{2\pi(n-k)}} \left[1 + \frac{A-\frac{k}{2}}{n} + \frac{\frac{kA}{2}+C+kE-\frac{6k+5k^2}{40}}{n^2} \right] + \mathcal{O}(n^{-7/2}). \end{aligned}$$

Let us first show less precise formulas up to $\mathcal{O}(n^{-2})$ error. Those give

$$p_{n-k}(x) = \frac{\tilde{\phi}_{n-k}(x)}{Z_{n-k}} = \frac{1 + A/n - x^2/2n + \mathcal{O}(n^{-2})}{1 + A/n - k/2n + \mathcal{O}(n^{-2})} = 1 + \frac{k}{2n} - \frac{x^2}{2n} + \mathcal{O}(n^{-2}),$$

which indeed is consistent with (4.9)–(4.11). To establish more detailed (4.9) and (4.10), we apply the same method, also using $\frac{1}{1+\alpha/n+\beta/n^2} = 1 - \alpha/n - \beta/n^2 + \alpha^2/n^2 + \mathcal{O}(n^{-3})$ along the way. This time we find it easiest to treat $k = 1$ and $k = 2$ separately, recalling that $E = -6\Gamma_4 = 3/10$:

$$\begin{aligned} p_{n-1}(x) &= \frac{\tilde{\phi}_{n-1}(x)}{Z_{n-1}} \\ &= \sqrt{2\pi(n-1)} \tilde{\phi}_{n-1}(x) \\ &\quad \cdot \left(1 - \frac{A-\frac{1}{2}}{n} - \frac{\frac{A}{2}+C+E-\frac{11}{40}}{n^2} + \frac{A^2-A+\frac{1}{4}}{n^2} + \mathcal{O}(n^{-3}) \right) \\ &= \left(1 + \frac{A}{n} - \frac{x^2}{2n} + \frac{A+C}{n^2} + \frac{(2E-A-1)x^2}{2n^2} + \frac{x^4}{8n^2} \right) \\ &\quad \cdot \left(1 - \frac{A-\frac{1}{2}}{n} - \frac{\frac{3A}{2}-A^2+C+E-\frac{21}{40}}{n^2} \right) + \mathcal{O}(n^{-3}) \\ &= 1 + \frac{1}{2n} - \frac{x^2}{2n} + \frac{\frac{21}{40}-E}{n^2} + \frac{(E-\frac{3}{4})x^2}{n^2} + \frac{x^4}{8n^2} + \mathcal{O}(n^{-3}) \\ &= 1 + \frac{1}{2n} - \frac{x^2}{2n} + \frac{9}{40n^2} - \frac{9x^2}{20n^2} + \frac{x^4}{8n^2} + \mathcal{O}(n^{-3}). \end{aligned}$$

$$\begin{aligned}
p_{n-2}(x) &= \frac{\tilde{\phi}_{n-2}(x)}{Z_{n-2}} \\
&= \sqrt{2\pi(n-2)}\tilde{\phi}_{n-2}(x) \cdot \\
&\quad \cdot \left(1 - \frac{A-1}{n} - \frac{A+C+2E-\frac{4}{5}}{n^2} + \frac{A^2-2A+1}{n^2} + \mathcal{O}(n^{-3})\right) \\
&= \left(1 + \frac{A}{n} - \frac{x^2}{2n} + \frac{2A+C}{n^2} + \frac{(2E-A-2)x^2}{2n^2} + \frac{x^4}{8n^2}\right) \\
&\quad \cdot \left(1 - \frac{A-1}{n} - \frac{A+C+2E-\frac{4}{5}}{n^2} + \frac{A^2-2A+1}{n^2}\right) + \mathcal{O}(n^{-3}) \\
&= 1 + \frac{1}{n} - \frac{x^2}{2n} + \frac{\frac{9}{5}-2E}{n^2} + \frac{(E-\frac{3}{2})x^2}{n^2} + \frac{x^4}{8n^2} + \mathcal{O}(n^{-3}) \\
&= 1 + \frac{1}{n} - \frac{x^2}{2n} + \frac{6}{5n^2} - \frac{6x^2}{5n^2} + \frac{x^4}{8n^2} + \mathcal{O}(n^{-3}).
\end{aligned}$$

Finally, each of the formulas in (4.8) is justified in the same way as in the proof of Lemma 4.1. \square

4.3. Conditional expectations of terms

As in the proof of Lemma 4.1, the expectations considered in Lemmas 2.14 and 2.15 consist of several terms. In this section we apply Lemma 4.3 to estimate each of those terms. We start with a list of unconditional expectations that we need. Each of those can be verified by checking an elementary integral:

Lemma 4.7. For $V_1, V_2, V_3, V_4 \stackrel{iid}{\sim} \mathcal{U}[-\sqrt{3}, \sqrt{3}]$, we have

$$\begin{aligned}
\mathbb{E}[V_1^k] &= 0 \text{ for every } k \text{ odd}, & \mathbb{E}[V_1^2] &= 1, & \mathbb{E}[V_1^4] &= \frac{9}{5}, & \mathbb{E}[V_1^6] &= \frac{27}{7}, \\
\mathbb{E}[\max\{V_1, V_2\}] &= \frac{\sqrt{3}}{3}, & \mathbb{E}[\max\{V_1, V_2\}^2] &= 1, \\
\mathbb{E}[\max\{V_1, V_2\}V_1] &= \frac{1}{2}, & \mathbb{E}[\max\{V_1, V_2\}V_1^2] &= \frac{2\sqrt{3}}{5}, \\
\mathbb{E}[\max\{V_1, V_2\}V_1^3] &= \frac{9}{10}, & \mathbb{E}[\max\{V_1, V_2\}V_1^4] &= \frac{27\sqrt{3}}{35}, \\
\mathbb{E}[\max\{V_1, V_2\}V_1V_2] &= -\frac{\sqrt{3}}{5}, & \mathbb{E}[\max\{V_1, V_2\}V_1^2V_2] &= \frac{1}{2}, \\
\mathbb{E}[\max\{V_1, V_2\}V_1^2V_2^2] &= \frac{3\sqrt{3}}{7}, & \mathbb{E}[\max\{V_1, V_2\} \max\{V_1, V_3\}] &= \frac{3}{5}, \\
\mathbb{E}[\max\{V_1, V_2\} \max\{V_1, V_3\}V_1^2] &= \frac{33}{35}, & \mathbb{E}[\max\{V_1, V_2\} \max\{V_1, V_3\}V_2^2] &= \frac{23}{35}, \\
\mathbb{E}[\max\{V_1, V_2\} \max\{V_1, V_3\}V_2V_3] &= \frac{13}{35}.
\end{aligned}$$

We now give the list of conditional expectations for each term that we will need. Each of them is obtained by the substitution of appropriate values from Lemma 4.7 into one of the equations in (4.8) from Lemma 4.3:

Lemma 4.8. *Let $A = (a_i)_{i \in [n]}$ and $B = (b_i)_{i \in [n]}$ be two random balanced dice with faces uniform in $[-\sqrt{3}, \sqrt{3}]$. Then,*

$$\begin{aligned} \mathbb{E}[a_1^2] &= 1 - \frac{2}{5n} - \frac{18}{175n^2} + \mathcal{O}(n^{-3}), \\ \mathbb{E}[a_1^2 a_2^2] &= 1 - \frac{4}{5n} + \frac{48}{175n^2} + \mathcal{O}(n^{-3}), \\ \mathbb{E}[\max\{a_1, b_1\}] &= \frac{\sqrt{3}}{3} \left(1 - \frac{1}{5n} - \frac{2}{25n^2} \right) + \mathcal{O}(n^{-3}), \\ \mathbb{E}[\max\{a_1, b_1\} \max\{a_2, b_2\}] &= \frac{1}{3} \left(1 - \frac{19}{10n} - \frac{31}{50n^2} \right) + \mathcal{O}(n^{-3}), \\ \mathbb{E}[a_1^2 \max\{a_2, b_1\}] &= \frac{\sqrt{3}}{3} \left(1 - \frac{3}{5n} - \frac{4}{175n^2} \right) + \mathcal{O}(n^{-3}). \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathbb{E}[\max\{a_1, a_2\}] &= \frac{\sqrt{3}}{3} \left(1 + \frac{2}{5n} \right) + \mathcal{O}(n^{-2}), \\ \mathbb{E}[a_1^2 \max\{a_2, a_3\}] &= \frac{\sqrt{3}}{3} + \mathcal{O}(n^{-2}), \\ \mathbb{E}[\max\{a_1, a_2\} \max\{a_3, a_4\}] &= \frac{1}{3} \left(1 - \frac{11}{5n} \right) + \mathcal{O}(n^{-2}), \\ \mathbb{E}[a_1^4] &= \frac{9}{5} \left(1 - \frac{4}{7n} \right) + \mathcal{O}(n^{-2}), \\ \mathbb{E}[\max\{a_1, b_1\} \max\{a_1, b_2\}] &= \frac{3}{5} \left(1 - \frac{1}{n} \right) + \mathcal{O}(n^{-2}), \\ \mathbb{E}[a_1^2 \max\{a_1, b_1\}] &= \frac{2\sqrt{3}}{5} \left(1 - \frac{1}{2n} \right) + \mathcal{O}(n^{-2}), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[\max\{a_1, a_2\}^2] &= 1 + \mathcal{O}(n^{-1}), \\ \mathbb{E}[\max\{a_1, a_2\} \max\{a_1, a_3\}] &= \frac{3}{5} + \mathcal{O}(n^{-1}), \\ \mathbb{E}[a_1^2 \max\{a_1, a_2\}] &= \frac{2\sqrt{3}}{5} + \mathcal{O}(n^{-1}), \\ \mathbb{E}[\max\{a_1, b_1\}^2] &= 1 + \mathcal{O}(n^{-1}). \end{aligned}$$

Finally, we are ready to perform calculations needed to establish Lemmas 2.14 and 2.15.

4.4. Proof of Lemma 2.14

Our objective is to show that $\mathbb{E}[\text{Var}_A^2] = \mathcal{O}(n^2)$ for a random balanced A . We will use the foregoing lemmas to write this expression as a sum of conditional expectations of simple functions of a small number of faces of the random die. We will then substitute approximations from Lemma 4.8 and see that, up to $\mathcal{O}(n)$ error, it is a polynomial where the coefficients of degree more than two vanish. We now proceed with this plan.

Let A be a balanced die with faces in $[-\sqrt{3}, \sqrt{3}]$. Squaring the expression in (4.3) and using $\sum_{i=1}^n a_i = 0$, we get

$$\begin{aligned} \text{Var}_A^2 &= \frac{n^4}{144} + \frac{n^2}{144} \left(\sum_{i=1}^n a_i^2 \right)^2 + \frac{1}{12} \left(\sum_{i \neq j} \max\{a_i, a_j\} \right)^2 \\ &\quad + \frac{n^3}{72} \sum_{i=1}^n a_i^2 - \frac{n^2}{12\sqrt{3}} \sum_{i \neq j} \max\{a_i, a_j\} - \frac{n}{12\sqrt{3}} \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i \neq j} \max\{a_i, a_j\} \right). \end{aligned}$$

Taking expectation over balanced A , we have

$$\begin{aligned} \mathbb{E}[\text{Var}_A^2] &= \frac{n^4}{144} + \frac{n^2}{144} \left((n^2 - n) \mathbb{E}[a_1^2 a_2^2] + n \mathbb{E}[a_1^4] \right) \\ &\quad + \frac{n(n-1)}{12} \left(2 \mathbb{E}[\max\{a_1, a_2\}^2] + 4(n-2) \mathbb{E}[\max\{a_1, a_2\} \max\{a_1, a_3\}] \right. \\ &\quad \left. + (n-2)(n-3) \mathbb{E}[\max\{a_1, a_2\} \max\{a_3, a_4\}] \right) \\ &\quad + \frac{n^4}{72} \mathbb{E}[a_1^2] - \frac{n^3(n-1)}{12\sqrt{3}} \mathbb{E}[\max\{a_1, a_2\}] \\ &\quad - \frac{n^2(n-1)}{12\sqrt{3}} \left((n-2) \mathbb{E}[a_1^2 \max\{a_2, a_3\}] + 2 \mathbb{E}[a_1^2 \max\{a_1, a_2\}] \right) \\ &= \frac{n^4}{144} \left(1 + \mathbb{E}[a_1^2 a_2^2] + 12 \mathbb{E}[\max\{a_1, a_2\} \max\{a_3, a_4\}] + 2 \mathbb{E}[a_1^2] \right. \\ &\quad \left. - 4\sqrt{3} \mathbb{E}[\max\{a_1, a_2\}] - 4\sqrt{3} \mathbb{E}[a_1^2 \max\{a_2, a_3\}] \right) \\ &\quad + \frac{n^3}{144} \left(- \mathbb{E}[a_1^2 a_2^2] + \mathbb{E}[a_1^4] + 48 \mathbb{E}[\max\{a_1, a_2\} \max\{a_1, a_3\}] \right. \\ &\quad \left. - 72 \mathbb{E}[\max\{a_1, a_2\} \max\{a_3, a_4\}] + 4\sqrt{3} \mathbb{E}[\max\{a_1, a_2\}] \right. \\ &\quad \left. + 12\sqrt{3} \mathbb{E}[a_1^2 \max\{a_2, a_3\}] - 8\sqrt{3} \mathbb{E}[a_1^2 \max\{a_1, a_2\}] \right) \\ &\quad + \mathcal{O}(n^2). \end{aligned}$$

In order to show that the coefficients of n^4 and n^3 are zero we apply Lemma 4.8. To check the coefficient of n^4 we need to look at the leading terms of all summands grouped under n^4 .

These cancel out in the same way as in the unconditioned case. For the coefficient of n^3 we need to check second-order terms of summands grouped by n^4 and the most significant terms of summands grouped by n^3 . All in all we conclude that $\mathbb{E}[\text{Var}_A] = \mathcal{O}(n^2)$. \square

4.5. Proof of Lemma 2.15

In this section, we show that $\mathbb{E}[CV_{AB,\text{cond}}^2] = \Omega(n^2)$, where $CV_{AB,\text{cond}} = CV_{AB} - CV_A CV_B$. Note that by triangle inequality:

$$\begin{aligned} \mathbb{E}[(CV_{AB} - CV_A CV_B)^2] &\geq \left(\sqrt{\mathbb{E}[CV_{AB}^2]} - \sqrt{\mathbb{E}[CV_A^2] \mathbb{E}[CV_B^2]}\right)^2 \\ &= \left(\sqrt{\mathbb{E}[CV_{AB}^2]} - \mathbb{E}[CV_A^2]\right)^2. \end{aligned}$$

Therefore, we only need to show that

$$\sqrt{\mathbb{E}[CV_{AB}^2]} - \mathbb{E}[CV_A^2] = \Omega(n).$$

Indeed, this is established by two claims below. Their subsequent proofs complete the justification of Lemma 2.15.

Claim 4.9. $\mathbb{E}[CV_{AB}^2] = \frac{11}{12600}n^2 + \mathcal{O}(n)$, in particular $\sqrt{\mathbb{E}[CV_{AB}^2]} > 0.025n$.

Claim 4.10. $\mathbb{E}[CV_A^2] = \frac{n}{60} + \mathcal{O}(1) < 0.02n$.

Both claims are proved with the same method as we used for Lemma 2.14.

Proof of Claim 4.9. Let us calculate the expression for the covariance CV_{AB} for balanced dice A, B . For that, let $V \sim \mathcal{U}[-\sqrt{3}, \sqrt{3}]$. It will be useful to recall (4.2), as well as the formula $\Pr[a < V] = 1/2 - a/2\sqrt{3}$ and the fact that the face-sums of A and B are zero:

$$\begin{aligned} CV_{AB} &= \mathbb{E}[g_A(V)g_B(V)] = \sum_{i,j=1}^n \mathbb{E} \left[\left(\mathbb{1}(a_i < V) - \frac{V + \sqrt{3}}{2\sqrt{3}} \right) \left(\mathbb{1}(b_j < V) - \frac{V + \sqrt{3}}{2\sqrt{3}} \right) \right] \\ &= \sum_{i,j=1}^n \Pr[\max\{a_i, b_j\} < V] - \frac{n}{2\sqrt{3}} \sum_{i=1}^n \mathbb{E} [\mathbb{1}(a_i < V)(V + \sqrt{3})] \\ &\quad - \frac{n}{2\sqrt{3}} \sum_{j=1}^n \mathbb{E} [\mathbb{1}(b_j < V)(V + \sqrt{3})] + \frac{n^2}{12} \mathbb{E} [(V + \sqrt{3})^2] \\ &= \frac{n^2}{2} - \frac{1}{2\sqrt{3}} \sum_{i,j=1}^n \max\{a_i, b_j\} - \frac{n^2}{8} + \frac{n}{24} \sum_{i=1}^n a_i^2 - \frac{n^2}{4} - \frac{n^2}{8} + \frac{n}{24} \sum_{j=1}^n b_j^2 - \frac{n^2}{4} + \frac{n^2}{3} \\ &= \frac{n^2}{12} + \frac{n}{24} \sum_{i=1}^n (a_i^2 + b_i^2) - \frac{1}{2\sqrt{3}} \sum_{i,j=1}^n \max\{a_i, b_j\}. \end{aligned}$$

Taking the square we obtain

$$\begin{aligned}
CV_{AB}^2 &= \frac{n^4}{144} + \frac{n^2}{576} \left(\sum_{i=1}^n (a_i^2 + b_i^2) \right)^2 + \frac{1}{12} \left(\sum_{i,j} \max\{a_i, b_j\} \right)^2 \\
&\quad + \frac{n^3}{144} \sum_{i=1}^n (a_i^2 + b_i^2) - \frac{n^2}{12\sqrt{3}} \sum_{i,j} \max\{a_i, b_j\} \\
&\quad - \frac{n}{24\sqrt{3}} \left(\sum_{i=1}^n (a_i^2 + b_i^2) \right) \left(\sum_{i,j} \max\{a_i, b_j\} \right).
\end{aligned}$$

We can then compute the expectation over balanced A and B ,

$$\begin{aligned}
\mathbb{E}[CV_{AB}^2] &= \frac{n^4}{144} + \frac{n^3}{288} \left(\mathbb{E}[a_1^4] + (n-1) \mathbb{E}[a_1^2 a_2^2] + n \mathbb{E}[a_1^2]^2 \right) \\
&\quad + \frac{n^2}{12} \left(\mathbb{E}[\max\{a_1, b_1\}^2] + 2(n-1) \mathbb{E}[\max\{a_1, b_1\} \max\{a_1, b_2\}] \right. \\
&\quad \left. + (n-1)^2 \mathbb{E}[\max\{a_1, b_1\} \max\{a_2, b_2\}] \right) \\
&\quad + \frac{n^4}{72} \mathbb{E}[a_1^2] - \frac{n^4}{12\sqrt{3}} \mathbb{E}[\max\{a_1, b_1\}] \\
&\quad - \frac{n^3}{12\sqrt{3}} \left(\mathbb{E}[a_1^2 \max\{a_1, b_1\}] + (n-1) \mathbb{E}[a_1^2 \max\{a_2, b_1\}] \right) \\
&= \frac{n^4}{288} \left(2 + \mathbb{E}[a_1^2 a_2^2] + \mathbb{E}[a_1^2]^2 + 24 \mathbb{E}[\max\{a_1, b_1\} \max\{a_2, b_2\}] + 4 \mathbb{E}[a_1^2] \right. \\
&\quad \left. - 8\sqrt{3} \mathbb{E}[\max\{a_1, b_1\}] - 8\sqrt{3} \mathbb{E}[a_1^2 \max\{a_2, b_1\}] \right) \\
&\quad + \frac{n^3}{288} \left(\mathbb{E}[a_1^4] - \mathbb{E}[a_1^2 a_2^2] + 48 \mathbb{E}[\max\{a_1, b_1\} \max\{a_1, b_2\}] \right. \\
&\quad - 48 \mathbb{E}[\max\{a_1, b_1\} \max\{a_2, b_2\}] \\
&\quad \left. - 8\sqrt{3} \mathbb{E}[a_1^2 \max\{a_1, b_1\}] + 8\sqrt{3} \mathbb{E}[a_1^2 \max\{a_2, b_1\}] \right) \\
&\quad + \frac{n^2}{12} \left(\mathbb{E}[\max\{a_1, b_1\}^2] - 2 \mathbb{E}[\max\{a_1, b_1\} \max\{a_1, b_2\}] \right. \\
&\quad \left. + \mathbb{E}[\max\{a_1, b_1\} \max\{a_2, b_2\}] \right).
\end{aligned}$$

All that remains is a mechanical substitution of estimates from Lemma 4.8 verifying that indeed

$$\mathbb{E}[CV_{AB}^2] = \frac{11}{12600} n^2 + \mathcal{O}(n).$$

In fact, all we need for Lemma 2.15 is $\mathbb{E}[CV_{AB}^2] > 11n^2/12600 - \mathcal{O}(n)$ and to establish that one only has to check the coefficient of n^2 (since $CV_{AB}^2 \geq 0$, the leading coefficient must be positive, and the positive leading coefficient by n^3 or n^4 would only asymptotically increase $\mathbb{E}[CV_{AB}^2]$). \square

Proof of Claim 4.10. Let us now turn to the simpler case of $\mathbb{E}[CV_A^2]$. Again fixing a balanced A and taking $V \sim \mathcal{U}[-\sqrt{3}, \sqrt{3}]$, we get

$$\begin{aligned} CV_A &= \mathbb{E}[g_A(V)V] = \sum_{i=1}^n \mathbb{E} \left[\left(\mathbb{1}(a_i < V) - \frac{V + \sqrt{3}}{2\sqrt{3}} \right) \cdot V \right] = \frac{3n}{4\sqrt{3}} - \frac{1}{4\sqrt{3}} \sum_{i=1}^n a_i^2 - \frac{n}{2\sqrt{3}} \\ &= \frac{n}{4\sqrt{3}} - \frac{1}{4\sqrt{3}} \sum_{i=1}^n a_i^2, \end{aligned}$$

whose square is

$$CV_A^2 = \frac{1}{48} \left(n^2 + \left(\sum_{i=1}^n a_i^2 \right)^2 - 2n \sum_{i=1}^n a_i^2 \right).$$

Taking the expectation over balanced A and substituting from Lemma 4.8 as in the previous proofs,

$$\begin{aligned} \mathbb{E}[CV_A^2] &= \frac{1}{48} \left(n^2 + n \mathbb{E}[a_1^4] + n(n-1) \mathbb{E}[a_1^2 a_2^2] - 2n^2 \mathbb{E}[a_1^2] \right) \\ &= \frac{n^2}{48} \left(1 + \mathbb{E}[a_1^2 a_2^2] - 2 \mathbb{E}[a_1^2] \right) + \frac{n}{48} \left(\mathbb{E}[a_1^4] - \mathbb{E}[a_1^2 a_2^2] \right) \\ &= \frac{n}{60} + \mathcal{O}(1). \end{aligned} \quad \square$$

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