Differential equations for the series of hypermaps with control on their full degree profile

Houcine Ben Dali[1]

[1] *Université de Lorraine, CNRS, IECL, F-54000 Nancy,*
Department of Mathematics, Harvard University, Cambridge, MA 02138, U.S.A.
bendali@math.harvard.edu

Submitted: Apr 10, 2024; Accepted: Dec 10, 2024; Published: Mar 15, 2025

Abstract. We consider the generating series of oriented and non-oriented hypermaps with controlled degrees of vertices, hyperedges and faces. It is well known that these series have natural expansions in terms of Schur and Zonal symmetric functions, and with some particular specializations, they satisfy the celebrated KP and BKP equations.

We prove that the full generating series of hypermaps satisfy a family of differential equations. We give a first proof which works for an α deformation of these series related to Jack polynomials. This proof is based on a recent construction formula for Jack characters using differential operators. We also provide a combinatorial proof for the orientable case.

Our approach also applies to the series of k-constellations with control of the degrees of vertices of all colors. In other words, we obtain an equation for the generating function of Hurwitz numbers (and their α-deformations) with control of full ramification profiles above an arbitrary number of points. Such equations are new even in the orientable case.

Keywords. Hypermaps, differential equations, Jack characters

Mathematics Subject Classifications. 05E05

1. Introduction

1.1. Maps

A *connected map* is a cellular embedding of a connected graph into a closed surface without boundary, orientable or not. In this paper, a *map* is an unordered collection of connected maps. We say that a map is *orientable* if each one of its connected components is embedded on a orientable surface. We will use the word *non-oriented* for maps on general surfaces, orientable or not. Maps appear in various branches of algebraic combinatorics, probability and physics.

The study of maps involves various methods such as generating series, matrix integral techniques and bijective methods, see e.g [LZ04, Eyn16, BC86, Cha11, AL20, CD22].

A *hypermap* is a map whose faces are colored in two colors $(+)$ and $(-)$, and such that each edge is incident to two faces of different colors. Usually the faces of one color are called *hyperedges*, and the faces of the other color are the *faces* of the hypermap. Hypermaps were first introduced by Cori in [Cor75] and are in bijection with bipartite maps by duality [Wal75].

We consider generating series of hypermaps with three alphabets of variables controlling the degrees of vertices, faces of color $(+)$ and faces of color $(-)$, see Equations (1.1) and (1.2) below. Representation theory allows one to relate these generating series, in the orientable and the non-orientable cases to Schur and Zonal symmetric functions, respectively [JV90, GJ96a].

The generating series in which we keep one alphabet $\mathbf{p}$ and we replace the alphabets $\mathbf{q}$ and $\mathbf{r}$ by two variables u and v are well studied and are known to be functions of the KP hierarchy (resp. BKP hierarchy) in the orientable (resp. the non-orientable) case; see [KMM+91, vdL01].

When we keep two alphabets and we only specialize the third one (see Section 1.3), the hypermap series also satisfies differential equations related to the integrable 2-Toda hierarchy, see *e.g* [AvM01, BMS02, EO07]. Moreover, it has been recently proved in [CD22] that the two-alphabet series satisfies a family of decomposition equations, which are a sort of Tutte equations.

However, studying the full generating series of hypermaps *i.e.* without any specialization of the three alphabets of variables, is known to be a hard problem: the usual decomposition a la Tutte does not seem to exist and we are not aware of any differential equations satisfied by these series.

The main contribution of this paper is to prove that the generating series of orientable and non-orientable hypermaps satisfy a family of differential equations; see Theorem 1.5 below. We also prove that these equations characterize the generating series. These results are established for a more general series which depends on a deformation parameter α, and which gives the series of orientable hypermaps when $\alpha = 1$, and the series of the non-orientable hypermaps when $\alpha = 2$. The result is based on a recent result of Ben Dali and Dołęga about Jack characters [BDD23], see also Theorem 1.3.

1.2. Generating series of hypermaps

Throughout the paper, we use straight letters to denote series (H,G,...), and curved letters to denote operators ($\mathcal{B}_n^{(\alpha)}$, $\mathcal{C}_\ell^{(\alpha)}$, $\mathcal{G}^{(\alpha)}$,...) or linear spaces ($\mathcal{A}$, $\mathcal{S}_\alpha$,...).

We start by some definitions related to hypermaps.

Definition 1.1. The *size* of a map M is its number of edges, and will be denoted $|M|$. If M is a hypermap, then we associate to it three integer partitions of size $|M|$:

- its *vertex type*, denoted $\lambda^\bullet(M)$, is the partition obtained by reordering the vertex degrees divided[1] by 2.
- its $(+)$ *type*, denoted $\lambda^+(M)$, is the partition obtained by reordering the degrees of the $(+)$ faces.

[1]When we turn around a vertex in a hypermap, colors $(+)$ and $(-)$ alternate. By consequence, each vertex has necessarily even degree.

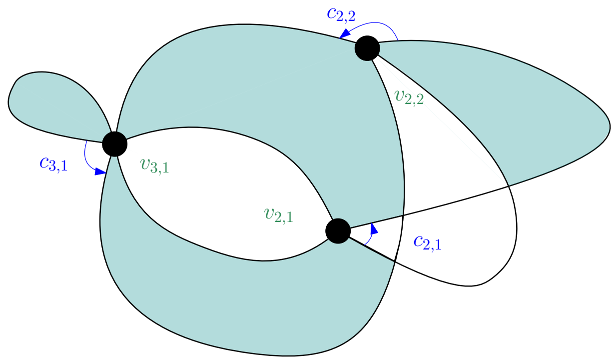


Figure 1.1: An example of a vertex labelled hypermap of profile $([3,2,2],[5,2],[6,1])$. Faces of color $(-)$ are represented in blue, the root of the vertex $v_{d,i}$ is denoted by $c_{d,i}$.

- its $(-)$ *type*, denoted $\lambda^-(M)$, is the partition obtained by reordering the degrees of the $(-)$ faces.

The *profile* of M, is then the tuple of partitions $(\lambda^\bullet(M),\lambda^+(M),\lambda^-(M))$. Finally, we say that a hypermap (oriented or not) is *vertex labelled* if:

1. for each $d \geqslant 1$, vertices of same degree $2d$ are labelled $v_{d,1}, v_{d,2},\dots$.
2. each vertex has a marked oriented corner in a face colored $(+)$. This corner is called the *vertex root*.

When a map is orientable, we can choose for each one of the connected surfaces an orientation, which will be called the *direct orientation of the surface*. Once this orientation is fixed, we say that the map is oriented. An oriented hypermap is said vertex labelled if vertices are numbered as in item *(1)* above, and if

(2′) each vertex has a marked $(+)$ *corner, oriented in the direct orientation, called the vertex root.*

We give in Figure 1.1 an example of an oriented vertex labelled hypermap. Notice that if we start from a hypermap whose faces of color $(-)$ have all degree 2, then we can glue the double edges forming $(-)$ faces in order to obtain a map with only $(+)$ faces. Hence, simple maps (maps with uncolored faces) can be seen as hypermaps for which $\lambda^- = [2,2,\dots,2]$.

Understanding hypermaps with controlled profile is a hard combinatorial problem. Indeed, usual techniques to enumerate maps do not allow to control the three partitions of the profile and the only known answer to this question is given by an algebraic approach, which provides an expression of the generating series of hypermaps using symmetric functions.

In order to define these generating series, we consider a variable t and three alphabets of variables $\mathbf{p} = (p_i)_{i\geqslant 1}$, $\mathbf{q} = (q_i)_{i\geqslant 1}$ and $\mathbf{r} = (r_i)_{i\geqslant 1}$. For any integer partition $\lambda := [\lambda_1,\lambda_2,\dots,\lambda_s]$, we define p_λ as follows;

$$p_\lambda := p_{\lambda_1}p_{\lambda_2}\dots p_{\lambda_s}.$$

We define q_λ and r_λ in a similar way. These three alphabets will be the respective weights of vertices, $(+)$ and $(-)$ faces. More precisely, we consider the two generating series

$$H^{(1)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_M \frac{t^{|M|}}{z_{\lambda^\bullet(M)}} p_{\lambda^\bullet(M)} q_{\lambda^+(M)} r_{\lambda^-(M)}, \tag{1.1}$$

$$H^{(2)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_M \frac{t^{|M|}}{2^{\ell(\lambda^\bullet(M))} z_{\lambda^\bullet(M)}} p_{\lambda^\bullet(M)} q_{\lambda^+(M)} r_{\lambda^-(M)}, \tag{1.2}$$

where the sum runs over vertex labelled oriented (resp. non-oriented) hypermaps in Equation (1.1) (resp. Equation (1.2)). Here, z_λ is a normalization numerical factor (see Section 2.1) related to the labelling of vertices. By the remark above, the generating series of simple maps are obtained from $H^{(1)}$ and $H^{(2)}$ by taking the specialization $r_2 = 1$ and $r_i = 0$ for $i \neq 2$. It is also worth mentioning that the series $H^{(1)}$ and $H^{(2)}$ are symmetric in the three alphabets $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$: the symmetry between $\mathbf{q}$ and $\mathbf{r}$ is clear from the definition and symmetry between $\mathbf{p}$ and $\mathbf{q}$ can be seen using a duality operation which exchanges vertices and $(+)$ faces (see *e.g* [CD22, Definition 2.4]).

Representation theory tools can be used to give an explicit expression of the series $H^{(1)}$ and $H^{(2)}$ in terms of symmetric functions (Schur and Zonal functions); see Theorem 3.2. These results use an encoding of hypermaps with permutations and matchings. The disadvantage of this approach is that it is quite rigid and it is hard to generalize to the case of generating series of maps with additional weights; an example of such a problem is given by the Matching-Jack conjecture of Goulden–Jackson [GJ96a] (see also Conjecture 3.3 below).

The main contribution of this paper is to establish a differential equation for the generating series of hypermaps (Theorem 1.5 below) and to solve this equation by giving a recursive formula for the number of hypermaps with fixed profile (π,μ,ν), see Equation (6.6). This recursive formula is given for a one parameter deformation related to Jack polynomials and also to the enumeration of maps with non-orientability weights.

In order to understand the recursive structure of a family of coefficients indexed by three partitions of the same size, it is sometimes convenient to start by introducing a generalized family of coefficients indexed by partitions of arbitrary size (see [AF17] for an example of application of this idea). It turns out that one way to make such a generalization in the case of hypermaps is by marking faces of degree 1. This operation will be justified algebraically in Section 1.4 and a natural combinatorial interpretation in terms of "partially constructed" hypermaps is given for orientable maps in Section 5.

Definition 1.2. Let π, μ and ν be three partitions. We denote by $\mathcal{OH}^\pi_{\mu,\nu}$ (resp. $\mathcal{H}^\pi_{\mu,\nu}$) the set of vertex labelled oriented hypermaps (resp. non-oriented hypermaps) of profile $(\pi, \mu \cup 1^{|\pi|-|\mu|}, \nu \cup 1^{|\pi|-|\nu|})$, with $|\pi|-|\mu|$ marked $(+)$ faces of degree 1, $|\pi|-|\nu|$ marked $(-)$ faces of degree 1, with the condition that in an isolated loop[2], we can not have both the $(+)$ face and the $(-)$ face marked. By definition, $\mathcal{OH}^\pi_{\mu,\nu}$ and $\mathcal{H}^\pi_{\mu,\nu}$ are empty when $|\pi| < \max(|\mu|,|\nu|)$.

[2]An isolated loop is a connected component of a hypermap with exactly one edge and one vertex. It has necessarily two faces, one colored $(+)$ and one colored $(-)$.

We then define the generating series of hypermaps with marked faces;

$$\widetilde{H}^{(1)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_{\pi,\mu,\nu} \frac{|\mathcal{OH}^{\pi}_{\mu,\nu}|}{z_\pi} t^{|\mu|+|\nu|-|\pi|} p_\pi q_\mu r_\nu,$$

$$\widetilde{H}^{(2)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_{\pi,\mu,\nu} \frac{|\mathcal{H}^{\pi}_{\mu,\nu}|}{2^{\ell(\pi)} z_\pi} t^{|\mu|+|\nu|-|\pi|} p_\pi q_\mu r_\nu,$$

The integer $|\mu| + |\nu| - |\pi|$ corresponds to the number of edges of the hypermap which are not incident to a marked face. It is straightforward from the definitions that the series $H^{(1)}$ and $H^{(2)}$ are respectively the homogeneous parts of $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$. Conversely, $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$ can be obtained from $H^{(1)}$ and $H^{(2)}$ by simple operations (see Equation (3.5)).

1.3. Jack characters

In order to define a deformed series which interpolates between the series $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$, we use *Jack polynomials*. Jack polynomials $J^{(\alpha)}_\lambda$ are symmetric functions introduced by Jack [Jac70]. They are indexed by an integer partition λ and depend on a deformation parameter α (See Theorem 2.1 for a precise definition). They can be seen as generalization of Schur functions which are obtained by setting $\alpha = 1$. Jack polynomials have a rich combinatorial structure [Sta89, KS97, CD22, Mol23] and are related to various models of statistical and quantum mechanics [LV95, DE02, For10].

Jack characters are an α-deformation of the characters of symmetric group introduced by Lassalle in [Las08]. They have been a useful tool to understand asymptotic behavior of large Young diagrams under a Jack deformation of the Plancherel measure [DF16, Śni19, CDM23]. The Jack character $\theta^{(\alpha)}_\mu$ is the function on Young diagrams defined by:

$$\theta^{(\alpha)}_\mu(\lambda) := \begin{cases} 0 & \text{if } |\lambda| < |\mu|, \\ \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} [p_{\mu,1^{|\lambda|-|\mu|}}] J^{(\alpha)}_\lambda & \text{if } |\lambda| \geqslant |\mu|, \end{cases}$$

where $m_1(\mu)$ is the number of parts equal to 1 in the partition μ, and p_μ is the power-sum symmetric function indexed by μ (see Section 2.2).

In [CD22], Chapuy and Dołęga have introduced a new family of differential operators $\mathcal{B}^{(\alpha)}_n$ which can be used to construct an α deformation of the generating series of hypermaps[3] with controlled vertex degrees, $(+)$ face degrees, and the *number* of $(-)$ faces. This corresponds to generating series in which we keep two alphabets $\mathbf{p}$ and $\mathbf{q}$ and replace each r_i by the same variable u for $i \geqslant 1$. The definition of operators $\mathcal{B}^{(\alpha)}_n(\mathbf{p}, u)$ uses catalytic variables and will be recalled in Section 4.1. These operators have also a combinatorial interpretation for general α which we recall in Section 5.1 for $\alpha = 1$. As an example, we give here the first two operators;

$$\mathcal{B}^{(\alpha)}_1(\mathbf{p}, u) = \frac{up_1}{\alpha} + \sum_{i \geqslant 1} p_{i+1} \frac{i\partial}{\partial p_i},$$

[3] Actually, they use bipartite maps which can be obtained from hypermaps by duality (a bijection which exchanges the vertices of the map with its faces).

$$\mathcal{B}_2^{(\alpha)}(\mathbf{p},u)=\sum_{i\geqslant 1}\left(\big(2\alpha u+(i+1)(\alpha-1)\big)p_{i+2}+\sum_{\substack{j+k=i+2\\ j,k\geqslant 1}}p_jp_k\right)\frac{i\partial}{\partial p_i}$$
$$+\frac{u^2p_2}{\alpha}+\frac{u}{\alpha}\big((\alpha-1)p_2+p_{1,1}\big)+\alpha\sum_{i,j\geqslant 1}p_{i+j+2}\frac{i\partial}{\partial p_i}\frac{j\partial}{\partial p_j}.$$

It turns out that Jack characters can be constructed using these differential operators, and consequently they have a combinatorial interpretation in terms of a family of weighted maps (see [BDD23, Theorem 1.4 and Proposition 4.6]).

Theorem 1.3 ([BDD23]). *For any partitions μ and $\lambda=[\lambda_1,\lambda_2,\dots,\lambda_s]$,*

$$\theta_\mu^{(\alpha)}(\lambda)=[t^{|\mu|}p_\mu]\exp\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{p},-\alpha\lambda_1)\right)\dots\exp\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{p},-\alpha\lambda_s)\right)\cdot 1.$$

where

$$\mathcal{B}_\infty^{(\alpha)}(t,\mathbf{p},u):=\sum_{n\geqslant 1}\frac{t^n}{n}\mathcal{B}_n^{(\alpha)}(\mathbf{p},u).$$

1.4. Structure coefficients

Jack characters form a linear basis of the space of shifted symmetric functions, see Section 2.3. Hence, their structure coefficients $g_{\mu,\nu}^{\pi}(\alpha)$ are well defined:

$$\theta_\mu^{(\alpha)}\theta_\nu^{(\alpha)}=\sum_{\pi}g_{\mu,\nu}^{\pi}(\alpha)\theta_\pi^{(\alpha)}. \tag{1.3}$$

We now explain the connection between these coefficients and the enumeration of hypermaps. We introduce the series of the structure coefficients;

$$G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}):=\sum_{\pi,\mu,\nu}\frac{g_{\mu,\nu}^{\pi}(\alpha)}{z_\pi\alpha^{\ell(\pi)}}t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu. \tag{1.4}$$

Moreover, the coefficients $g_{\mu,\nu}^{\pi}$ satisfy the following property (see Lemma 3.1),

$$g_{\mu,\nu}^{\pi}=0 \text{ if } |\pi|>|\mu|+|\nu|. \tag{1.5}$$

Hence, $G^{(\alpha)}$ is well defined in the algebra $\mathcal{A}:=\mathbb{Q}(\alpha)[t,\mathbf{p}][\![\mathbf{q},\mathbf{r}]\!]$. This is the space of formal power series in the variables q_i and r_i, whose coefficients are polynomial in t and p_i.

The series $G^{(\alpha)}$ is actually an α-deformation of the generating series of hypermaps with marked faces (see Section 3.4).

Proposition 1.4. *For $\alpha\in\{1,2\}$, we have*

$$G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})=\widetilde{H}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}).$$

The hope is to generalize the previous proposition for any α by considering maps counted with α-weights as in [LC09, CD22, BDD23]. However, such a question seems to be out of reach when we consider generating series with three alphabets $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. The main goal of this paper is to shed some light on the combinatorics of the series $G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})$ by giving a family of differential equations which characterize it.

1.5. Main theorem

Let $\left(\mathcal{B}_\infty^{(\alpha)}(t,\mathbf{p},u)\right)^\perp$ denote the adjoint operator of $\mathcal{B}_\infty^{(\alpha)}(t,\mathbf{p},u)$ with respect to the usual scalar product of $\mathbb{Q}(\alpha)[\mathbf{p}]$, see Section 2.2. In Section 7.2, we provide a differential expression for this dual operator.

We are interested in the operators $\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q},u)$, $\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{r},u)$ as well as $\mathcal{B}_\infty^{(\alpha)\perp}(-t,\mathbf{p},u)$. These three operators are well defined as operators from $\mathcal{A}$ to $\mathcal{A}[\![u]\!]$, see Remark 4.1. We now state the main theorem of the paper.

Theorem 1.5. *The function $G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})$ satisfies the following equation*

$$\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q},u)+\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{r},u)\right)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \mathcal{B}_\infty^{(\alpha)\perp}(-t,\mathbf{p},u)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}). \quad (1.6)$$

The proof of this result is based on the differential construction of Jack characters given in Theorem 1.3. We also prove in Proposition 6.2 that Equation (1.6) characterizes the series $G^{(\alpha)}$.

By extracting coefficients in the variable u, Equation (1.6) can be alternatively written as a family of equations (independent of u) which are indexed by non-negative integers; see Section 4.5.

Furthermore, we solve this differential equation and give an explicit expression of coefficients $g^\pi_{\mu,\nu}(\alpha)$ using some coefficients a^λ_μ which are obtained from the operator $\mathcal{B}_\infty^{(\alpha)}$ and which are known to count maps (see Theorem 6.1).

It may be more convenient to think of the differential equation Equation (1.6) as a commutation relation that we now explain. Let $\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})$ be the operator defined by

$$\begin{array}{rccl}\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}): & \mathbb{Q}(\alpha)[\mathbf{p}] & \longrightarrow & \mathbb{Q}(\alpha)[\mathbf{q},\mathbf{r}][\![t]\!] \\ & p_\pi & \longmapsto & \displaystyle\sum_{\mu,\nu} t^{|\mu|+|\nu|-|\pi|} g^\pi_{\mu,\nu}(\alpha) q_\mu r_\nu.\end{array}$$

Actually, $\mathcal{G}^{(\alpha)}\cdot p_\pi$ is polynomial in t of degree at most $|\pi|$ (see Lemma 3.1). It turns out that for $\alpha=1$, $\mathcal{G}^{(\alpha)}$ is a hypermap construction operator, which acts on a map by adding edges and coloring faces (see Section 5.3). The following is a variant of the main theorem (the equivalence between the two results follows from the definitions, see also Lemma 4.3).

Theorem 1.6. *We have the following relation*

$$\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q},u)+\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{r},u)\right)\cdot \mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\cdot \mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{p},u) \quad (1.7)$$

between operators from $\mathbb{Q}(\alpha)[\mathbf{p}]$ to $\mathbb{Q}(\alpha)[\mathbf{q},\mathbf{r}][\![t,u]\!]$.

In Section 5, we give a combinatorial proof of this commutation relation for $\alpha=1$.

1.6. Goulden–Jackson and Śniady conjectures

The Matching-Jack conjecture, introduced by Goulden and Jackson in [GJ96a], suggests that when $|\pi| = |\mu| = |\nu|$, coefficients $g^{\pi}_{\mu,\nu}$ satisfy positivity and integrality properties, and that they count hypermaps with "non-orientability weights"; see also Conjecture 3.3. This conjecture is still open, despite many partial results [DF16, CD22, BD22, BD23].

The following conjecture, due to Śniady [Śni19], can be thought of as a generalization of the Matching-Jack conjecture to coefficients $g^{\pi}_{\mu,\nu}$ indexed by partitions of arbitrary sizes.

Conjecture 1.7. ([Śni19, Conjecture 2.2]) For any π, μ and ν partitions, $g^{\pi}_{\mu,\nu}$ is a polynomial in $b := \alpha - 1$ with non-negative integer coefficients.

In [DF16], Dołęga and Féray have proved that the coefficients $g^{\pi}_{\mu,\nu}$ are polynomial in the deformation parameter b. In Section 3, we deduce the integrality part in Conjecture 1.7 from a similar result for the coefficients of the Matchings-Jack conjecture, see Corollary 3.7.

Unfortunately, we have not been able to use the explicit expression of coefficients $g^{\pi}_{\mu,\nu}(\alpha)$ obtained in Theorem 6.1 to prove their positivity in b (the remaining part in Conjecture 1.7). It is however possible to use a variant of the main theorem to give a positive differential expression for the low degree terms of the operator $\mathcal{G}^{(\alpha)}$.

1.7. Low degree terms of $\mathcal{G}^{(\alpha)}$

We consider the homogeneous parts of the operators $\mathcal{G}^{(\alpha)}$ defined for any $k \geqslant 0$ by

$$\mathcal{G}^{(\alpha)}_k \cdot p_\pi = \sum_{|\mu|+|\nu|=|\pi|+k} g^{\pi}_{\mu,\nu}(\alpha) q_\mu r_\nu. \tag{1.8}$$

Note that from Equation (1.5), we have

$$\mathcal{G}^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r}) = \sum_{k\geqslant 0} t^k \mathcal{G}^{(\alpha)}_k(\mathbf{p}, \mathbf{q}, \mathbf{r}).$$

In Section 5.3 we prove that for $\alpha = 1$, the operator $\mathcal{G}^{(1)}_k$ is an operator which acts on a map by adding k edges satisfying some conditions.

We use Proposition 6.2 (a variant of the main theorem) to give a differential expression for the operators $\mathcal{G}^{(\alpha)}_0, \mathcal{G}^{(\alpha)}_1$ and $\mathcal{G}^{(\alpha)}_2$. First, we introduce the operator

$$\begin{aligned} \Psi(\mathbf{p}, \mathbf{q}, \mathbf{r}) : \mathbb{Q}(\alpha)[\mathbf{p}] &\longrightarrow \mathbb{Q}(\alpha)[\mathbf{q}, \mathbf{r}] \\ p_\pi &\longmapsto \prod_{1\leqslant i\leqslant \ell(\pi)} (q_{\pi_i} + r_{\pi_i}). \end{aligned}$$

Combinatorially, when we think of p_i (resp. q_i, r_i) as the weight of an uncolored face (reps $(+)$ face, $(-)$ face) of degree i in a map, Ψ is the operator which chooses a color $(+)$ or $(-)$ for each face.

Theorem 1.8. *We have the following differential expressions for* $\mathcal{G}_0^{(\alpha)}$, $\mathcal{G}_1^{(\alpha)}$ *and* $\mathcal{G}_2^{(\alpha)}$.

$$\mathcal{G}_0^{(\alpha)} = \Psi, \tag{1.9}$$

$$\mathcal{G}_1^{(\alpha)} = \sum_{m\geqslant 1} \sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geqslant 1}} q_{m_1} r_{m_2} \cdot \Psi \cdot \frac{m\partial}{\partial p_m}, \tag{1.10}$$

and

$$\begin{aligned}\mathcal{G}_2^{(\alpha)} =& \frac{1}{2} \sum_{m\geqslant 1} \sum_{\substack{m_1+m_2=m+2\\ m_1,m_2\geqslant 1}} b(m_1-1)(m_2-1) q_{m_1} r_{m_2} \Psi \frac{m\partial}{\partial p_m} \\ &+ \frac{1}{2} \sum_{m\geqslant 1} \sum_{\substack{m_1+m_2+m_3=m+2\\ m_1,m_2,m_3\geqslant 1}} (m_1-1)(q_{m_1} r_{m_2} r_{m_3} + r_{m_1} q_{m_2} q_{m_3}) \Psi \frac{m\partial}{\partial p_m} \\ &+ \frac{1}{2} \sum_{k,m\geqslant 1} \sum_{\substack{i_1+i_2=k+m+2\\ i_1,i_2\geqslant 1}} \alpha \min(m,k,i_1-1,i_2-1) q_{i_1} r_{i_2} \Psi \frac{m\partial}{\partial p_m} \frac{k\partial}{\partial p_k} \\ &+ \frac{1}{2} \sum_{m,k\geqslant 1} \sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geqslant 1}} \sum_{\substack{k_1+k_2=k+1\\ k_1,k_2\geqslant 1}} q_{m_1} q_{k_1} r_{m_2} r_{k_2} \Psi \frac{m\partial}{\partial p_m} \frac{k\partial}{\partial p_k}.\end{aligned} \tag{1.11}$$

This theorem can actually be obtained combinatorially from the cases $\alpha = 1$, $\alpha = 2$ and a polynomiality argument (all coefficients are polynomials of degree at most 1 in α). However, this argument works only for $k \leqslant 2$ while the approach based on Theorem 1.5 which we present here can be used, with more computations, to understand the operators $\mathcal{G}_k^{(\alpha)}$ for higher k.

Combining Theorem 1.8 and Corollary 3.7, we deduce the following special case of Conjecture 1.7.

Corollary 1.9. *Fix three* π, μ *and* ν *partitions such that* $|\pi| \geqslant |\mu| + |\nu| - 2$. *Then* $g^{\pi}_{\mu,\nu}$ *is a polynomial in* $b := \alpha - 1$ *with non-negative integer coefficients.*

We hope that a better understanding of the differential structure of the operator $\mathcal{B}_\infty^{(\alpha)}$ could allow one to generalize Theorem 1.8 in order to obtain a differential formula of $\mathcal{G}_k^{(\alpha)}$ for any k. This would eventually give the missing positivity part in Conjecture 1.7 and in Goulden–Jackson's Matching-Jack conjecture.

1.8. k-constellations and Hurwitz numbers

k-constellations represent a family of maps whose vertices are colored in $k+1$ colors and from which hypermaps can be obtained by setting $k = 1$. In the orientable case, k-constellations are related to the factorizations of the identity in the symmetric group into $k+2$ permutations [BMS00]. Recently, a model of constellations on non-orientable surfaces have been introduced in [CD22]. In these two cases, generating series of constellations have been a useful

tool to understand *Hurwitz numbers* by considering some particular specializations (see also [BCD23, BN23]).

Moreover, orientable constellations with control of all color types are in bijection with the ramified coverings of the sphere above an arbitrary number of points with the full ramification profiles; see [LZ04, Section 1.2] (see also [CD22, Section 2.2] for the non-orientable case).

The approach used here to prove the main theorem applies to the case of constellations and allows us to extend the differential equation to series with k alphabets; see Theorem 4.5.

1.9. Equations for connected series

In this paper, we are considering generating series of hypermaps not necessarily connected. Nevertheless, it is possible to obtain the generating series of connected hypermaps by taking a logarithm. More precisely, the series

$$\widehat{G}^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r}) = \alpha \cdot \log(G^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r}))$$

is an α-deformation of the generating series of connected hypermaps with marked faces. The homogeneous part of this series is the object of Goulden–Jackson's b-conjecture (known also as the hypermap-Jack conjecture) [GJ96a, Conjecture 6.3].

In Section 7, we derive from the main theorem a differential equation for $\widehat{G}^{(\alpha)}$, see Theorem 7.5.

1.10. Outline of the paper

The paper is organized as follows. In Section 2, we give some preliminaries related to partitions, symmetric and shifted symmetric functions. In Section 3, we establish some useful properties of structure coefficients $g^{\pi}_{\mu,\nu}$, we discuss their connection to the Matching-Jack conjecture and we prove Proposition 1.4. Section 4 is dedicated to the proof of the main theorem as well as its generalized version Theorem 4.5 related to Hurwitz numbers. In Section 5, we give a combinatorial proof of Theorem 1.6 for $\alpha = 1$ (this section is quite independent from the rest of the paper). We use the main theorem in Section 6 to give an explicit expression for coefficients $g^{\pi}_{\mu,\nu}$ and to prove Theorem 1.8. Finally, we prove Theorem 7.5 in Section 7.

2. Preliminaries

2.1. Partitions

A *partition* $\lambda=[\lambda_1, \dots, \lambda_s]$ is a weakly decreasing sequence of positive integers $\lambda_1 \geqslant \dots \geqslant \lambda_s > 0$. We denote by $\mathbb{Y}$ the set of all integer partitions. The integer s is called the *length* of λ and is denoted $\ell(\lambda)$. The size of λ is the integer $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_s$. If n is the *size* of λ, we say that λ is a partition of n and we write $\lambda \vdash n$. The integers $\lambda_1, \dots, \lambda_s$ are called the *parts* of λ. For $i \geqslant 1$, we denote $m_i(\lambda)$ the number of parts of size i in λ. We then set

$$z_\lambda := \prod_{i \geqslant 1} m_i(\lambda)! i^{m_i(\lambda)}.$$

We denote by $\leqslant$ the *dominance partial* order on partitions, defined by

$$\mu \leqslant \lambda \iff |\mu| = |\lambda| \text{ and } \quad \mu_1 + \cdots + \mu_i \leqslant \lambda_1 + \cdots + \lambda_i \text{ for } i \geqslant 1.$$

Finally, we identify a partition λ with its *Young diagram*, defined by

$$\lambda := \{(i,j), 1 \leqslant i \leqslant \ell(\lambda), 1 \leqslant j \leqslant \lambda_i\}.$$

2.2. Symmetric functions

We fix an alphabet $\mathbf{x} := (x_1, x_2, ..)$. We denote by $\mathcal{S}_\alpha$ the algebra of symmetric functions in $\mathbf{x}$ with coefficients in $\mathbb{Q}(\alpha)$. For every partition λ, we denote m_λ the monomial function

$$m_\lambda(\mathbf{x}) := \sum_{\beta=(\beta_1,\ldots,\beta_s)} \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_s} x_{i_1}^{\beta_1} \ldots x_{i_s}^{\beta_s},$$

where the sum is taken over all reorderings β of the partition λ. Moreover, let p_λ denote the power-sum symmetric function, defined as follows; if $n \geqslant 1$ then

$$p_n(\mathbf{x}) := \sum_{i \geqslant 1} x_i^n,$$

and if $\lambda = [\lambda_1, \ldots, \lambda_s]$ then

$$p_\lambda(\mathbf{x}) = p_{\lambda_1}(\mathbf{x}) \ldots p_{\lambda_s}(\mathbf{x}).$$

We consider the associated alphabet of power-sum functions $\mathbf{p} := (p_1, p_2, ..)$. It is well known that monomial functions and the power-sum functions both form bases of the symmetric function algebra; therefore $\mathcal{S}_\alpha$ can be identified with the polynomial algebra $\mathcal{P} := \mathrm{Span}_{\mathbb{Q}(\alpha)}\{p_\lambda\}_{\lambda \in \mathbb{Y}}$. If f is a symmetric function in the alphabet $\mathbf{x}$, it will be convenient to denote with the same letter the function and the associated polynomial in $\mathbf{p}$;

$$f(\mathbf{x}) \equiv f(\mathbf{p}).$$

We denote by $\langle ., . \rangle_\alpha$ the α-deformation of the Hall scalar product defined on $\mathcal{S}_\alpha$ by

$$\langle p_\lambda, p_\mu \rangle_\alpha = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda,\mu}, \text{ for any partitions } \lambda, \mu,$$

where $\delta_{\lambda,\mu}$ denotes the Kronecker delta.

Macdonald has established the following characterization for Jack polynomials which we take as a definition; see [Mac95, Chapter VI, Section 10].

Theorem 2.1 ([Mac95])**.** *Jack polynomials $(J^{(\alpha)}_\lambda)_{\lambda\in\mathbb{Y}}$ are the unique family of symmetric functions in $\mathcal{S}_\alpha$ indexed by partitions, satisfying the following properties:*

- *Orthogonality:*
$$\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\mu\rangle_\alpha = 0, \textit{ for } \lambda \neq \mu.$$
- *Triangularity:*
$$[m_\mu]J^{(\alpha)}_\lambda = 0, \textit{ unless } \mu \leqslant \lambda.$$
- *Normalization:*
$$[p_{1^n}]J^{(\alpha)}_\lambda = 1, \textit{ for } \lambda \vdash n, \tag{2.1}$$
where 1^n is the partition with n parts equal to 1.

Moreover, Jack polynomials form a basis of $\mathcal{S}_\alpha$.

We denote by $j^{(\alpha)}_\lambda$ the squared-norm of $J^{(\alpha)}_\lambda$;
$$j^{(\alpha)}_\lambda := \langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha. \tag{2.2}$$

Remark 2.2. Let $\theta^{(\alpha)}_\varnothing$ denote the Jack character indexed by the empty partition. It follows from Equation (2.1) that $\theta^{(\alpha)}_\varnothing(\lambda) = 1$ for any $\lambda \in \mathbb{Y}$. Hence, $g^{\varnothing}_{\varnothing,\varnothing}(\alpha) = 1$.

2.3. Jack characters and shifted symmetric functions

Definition 2.3 ([Las08])**.** We say that a polynomial of degree n in k variables $(s_1, \dots, s_k)$ with coefficients in $\mathbb{Q}(\alpha)$ is *α-shifted symmetric* if it is symmetric in the variables $s_i - i/\alpha$. An α-shifted symmetric function (or simply a shifted symmetric function) is a sequence $(f_k)_{k\geqslant 1}$ of shifted symmetric polynomials of bounded degrees, such that for every $k \geqslant 1$, the function f_k is an α-shifted symmetric polynomial in k variables and
$$f_{k+1}(s_1, \dots, s_k, 0) = f_k(s_1, \dots, s_k). \tag{2.3}$$
We denote by $\mathcal{S}^*_\alpha$ the algebra of shifted symmetric functions.

Let f be a shifted symmetric function and let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. Then we denote $f(\lambda) := f(\lambda_1, \dots, \lambda_k, 0, \dots)$.

Theorem 2.4 ([KS96])**.** *Let $n \geqslant 0$. And let g be a function on Young diagrams. There exists a unique shifted symmetric function f of degree less or equal than n such that $f(\lambda) = g(\lambda)$ for any $|\lambda| \leqslant n$.*

In particular, a shifted symmetric function is completely determined by its evaluation on Young diagrams $(f(\lambda))_{\lambda\in\mathbb{Y}}$. The following theorem due to Féray gives a characterization of Jack characters as shifted symmetric functions satisfying some properties (see [BDD23] for a proof).

Theorem 2.5 (Féray)**.** *Fix a partition μ. The Jack character $\theta^{(\alpha)}_\mu$ is the unique α-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p_\mu$, such that $\theta^{(\alpha)}_\mu(\lambda) = 0$ for any partition $|\lambda| < |\mu|$.*

3. Structure coefficients $g^{\pi}_{\mu,\nu}(\alpha)$ and proof of Proposition 1.4

The purpose of this section is to discuss some properties of coefficients $g^{\pi}_{\mu,\nu}$. In particular, we use their connection with the coefficients of the Matchings-Jack conjecture to establish integrality in Conjecture 1.7 and to prove Proposition 1.4.

3.1. Some properties

We start by proving some properties of the structure coefficients $g^{\pi}_{\mu,\nu}$.

Lemma 3.1. *The coefficient $g^{\pi}_{\mu,\nu}(\alpha)$ is 0 unless $\max(|\mu|,|\nu|) \leqslant |\pi| \leqslant |\mu|+|\nu|$.*

Proof. The upper bound is a direct consequence of the fact that $\theta^{(\alpha)}_{\mu}$ is a shifted symmetric function of degree $|\mu|$ and the fact that $(\theta^{(\alpha)}_{\pi})_{|\pi|\leqslant d}$ is a basis of shifted symmetric functions of degree less or equal than d, see Theorem 2.5. In order to obtain the lower bound we use the vanishing properties of $\theta^{(\alpha)}_{\mu}$. Fix two partitions μ and ν. Set $m := \max(|\mu|,|\nu|)$ and

$$F := \theta^{(\alpha)}_{\mu}\theta^{(\alpha)}_{\nu} - \sum_{m\leqslant|\pi|\leqslant|\mu|+|\nu|} g^{\pi}_{\mu,\nu}\theta^{(\alpha)}_{\pi} \tag{3.1}$$

$$= \sum_{|\pi|<m} g^{\pi}_{\mu,\nu}\theta^{(\alpha)}_{\pi}. \tag{3.2}$$

From Equation (3.2), the function F is shifted symmetric with degree at most $m-1$. Moreover, using Equation (3.1) and the definition of Jack characters, we get that $F(\lambda)=0$ for any $|\lambda|<m$. Applying Theorem 2.4, we deduce that $F=0$. By consequence $g^{\pi}_{\mu,\nu}=0$ for any $|\pi|<m$. □

As a consequence of this lemma, we get that the series $G^{(\alpha)}$ introduced in Equation (1.4) is well defined in $\mathbb{Q}(\alpha)[t,\mathbf{p}][\![\mathbf{q},\mathbf{r}]\!] \cap \mathbb{Q}(\alpha)[t,\mathbf{q},\mathbf{r}][\![\mathbf{p}]\!]$.

3.2. Goulden–Jackson's Matchings-Jack conjecture

Goulden and Jackson have introduced in [GJ96a] the coefficients $c^{\pi}_{\mu,\nu}(\alpha)$ indexed by three partitions of the same size, and defined by the following expansion:

$$\tau^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) := \sum_{n\geqslant 0} t^n \sum_{\theta\vdash n} \frac{1}{j^{(\alpha)}_{\theta}} J^{(\alpha)}_{\theta}(\mathbf{p})J^{(\alpha)}_{\theta}(\mathbf{q})J^{(\alpha)}_{\theta}(\mathbf{r}) = \sum_{n\geqslant 0} t^n \sum_{\pi,\mu,\nu\vdash n} \frac{c^{\pi}_{\mu,\nu}(\alpha)}{z_{\pi}\alpha^{\ell(\pi)}} p_{\pi}q_{\mu}r_{\nu}. \tag{3.3}$$

For $\alpha\in\{1,2\}$, this series is known to be the generating series of hypermaps.

Theorem 3.2 ([GJ96b])**.** *Fix three partitions π, μ and ν of the same size. For $\alpha=1$ (resp. $\alpha=2$), the coefficient $c^{\pi}_{\mu,\nu}(1)$ (resp. $c^{\pi}_{\mu,\nu}(2)$) counts the number of oriented (resp. orientable or not) vertex labelled hypermaps of profile (π,μ,ν). Equivalently, for $\alpha\in\{1,2\}$ we have*

$$\tau^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = H^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}),$$

These two cases are obtained using representation theory tools (the case $\alpha = 1$ is a classical result, while the case $\alpha = 2$ is due to Goulden and Jackson [GJ96b]).

The Matching-Jack conjecture states that the coefficients $c^{\pi}_{\mu,\nu}(\alpha)$ still have a combinatorial interpretation in terms of maps for any α.

Conjecture 3.3 ([GJ96a])**.** For any partitions π, μ and ν of the same size, $c^{\pi}_{\mu,\nu}(\alpha)$ is a polynomial in the shifted parameter $b := \alpha - 1$ with non-negative integer coefficients. Equivalently, there exists a statistic ϑ on non-oriented hypermaps with non-negative integer values, such that

- $\vartheta(M) = 0$ if and only if M is oriented,
- for any partitions π, μ and ν of the same size

$$c^{\pi}_{\mu,\nu} = \sum_{M \in \mathcal{H}^{\pi}_{\mu,\nu}} b^{\vartheta(M)}.$$

In the next subsection, we use the following integrality result for coefficients $c^{\pi}_{\mu,\nu}$ in order to obtain a similar result for coefficients $g^{\pi}_{\mu,\nu}$.

Theorem 3.4 ([BD23])**.** *The coefficients $c^{\pi}_{\mu,\nu}$ are polynomials in $b = \alpha - 1$ with integer coefficients.*

3.3. Links between coefficients $c^{\pi}_{\mu,\nu}$ and $g^{\pi}_{\mu,\nu}$

The following proposition has been proved by Dołęga and Féray [DF16, Proposition B.1].

Proposition 3.5. *If π, μ and ν are of the same size then*

$$c^{\pi}_{\mu,\nu}(\alpha) = g^{\pi}_{\mu,\nu}(\alpha).$$

As a consequence, Conjecture 1.7 is a generalization of Conjecture 3.3.

If π is a partition, we denote by $\widetilde{\pi} := \pi \backslash 1^{m_1(\pi)}$ the partition obtained by deleting all parts of size 1. The following proposition is a generalization of [DF16, Equation (19)]. The proof is quite the same.

Proposition 3.6. *For every partitions π, μ and ν such that $\pi \vdash n \geqslant |\mu|, |\nu|$, we have*

$$\sum_{i=0}^{m_1(\pi)} \binom{m_1(\pi)}{i} g^{\widetilde{\pi} \cup 1^i}_{\mu,\nu} = \binom{m_1(\mu) + n - |\mu|}{m_1(\mu)} \binom{m_1(\nu) + n - |\nu|}{m_1(\nu)} c^{\pi}_{\mu \cup 1^{n-|\mu|}, \nu \cup 1^{n-|\nu|}}, \tag{3.4}$$

Equivalently,

$$\exp\left(\frac{p_1}{t\alpha}\right) G^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r}) = \exp\left(\frac{\partial}{t\partial q_1} + \frac{\partial}{t\partial r_1}\right) \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r}), \tag{3.5}$$

where the last equality holds in $\mathbb{Q}(\alpha)[t, 1/t, \mathbf{q}, \mathbf{r}][\![\mathbf{p}]\!]$.

Proof. To simplify expressions, we denote $\bar{\mu} := \mu \cup 1^{n-|\mu|}$ and $\bar{\nu} := \nu \cup 1^{n-|\nu|}$. We get from the definition of Jack characters and Lemma 3.1 that for any partition λ of size n

$$\begin{aligned}\theta^{(\alpha)}_\mu(\lambda)\theta^{(\alpha)}_\nu(\lambda) &= \binom{n-|\mu|+m_1(\mu)}{m_1(\mu)}\binom{n-|\nu|+m_1(\nu)}{m_1(\nu)}\theta^{(\alpha)}_{\bar{\mu}}(\lambda)\theta^{(\alpha)}_{\bar{\nu}}(\lambda)\\ &= \binom{n-|\mu|+m_1(\mu)}{m_1(\mu)}\binom{n-|\nu|+m_1(\nu)}{m_1(\nu)}\sum_{n\leqslant|\rho|\leqslant 2n} g^{\rho}_{\bar{\mu},\bar{\nu}}(\alpha)\theta^{(\alpha)}_\rho(\lambda).\end{aligned}$$

Terms corresponding to $|\rho| > n$ vanish since characters are evaluated at a partition of size n. Hence

$$\theta^{(\alpha)}_\mu(\lambda)\theta^{(\alpha)}_\nu(\lambda) = \binom{n-|\mu|+m_1(\mu)}{m_1(\mu)}\binom{n-|\nu|+m_1(\nu)}{m_1(\nu)}\sum_{\rho\vdash n} g^{\rho}_{\bar{\mu},\bar{\nu}}(\alpha)\theta^{(\alpha)}_\rho(\lambda).$$

Using Proposition 3.5 we get that,

$$\theta^{(\alpha)}_\mu(\lambda)\theta^{(\alpha)}_\nu(\lambda) = \binom{n-|\mu|+m_1(\mu)}{m_1(\mu)}\binom{n-|\nu|+m_1(\nu)}{m_1(\nu)}\sum_{\rho\vdash n} c^{\rho}_{\bar{\mu},\bar{\nu}}\theta^{(\alpha)}_\rho(\lambda). \tag{3.6}$$

On the other hand, for any $\lambda \vdash n$,

$$\begin{aligned}\theta^{(\alpha)}_\mu(\lambda)\theta^{(\alpha)}_\nu(\lambda) &= \sum_{|\kappa|\leqslant n} g^{\kappa}_{\mu,\nu}(\alpha)\theta^{(\alpha)}_\kappa(\lambda)\\ &= \sum_{|\kappa|\leqslant n} g^{\kappa}_{\mu,\nu}(\alpha)\binom{m_1(\kappa)+n-|\kappa|}{m_1(\kappa)}\theta^{(\alpha)}_{\kappa\cup 1^{n-|\kappa|}}(\lambda)\\ &= \sum_{\rho\vdash n}\theta^{(\alpha)}_\rho(\lambda)\left(\sum_{i=0}^{m_1(\rho)} g^{\tilde{\rho}\cup 1^i}_{\mu,\nu}(\alpha)\binom{m_1(\rho)}{i}\right).\end{aligned} \tag{3.7}$$

The last equation is obtained by regrouping terms with respect to $\rho := \kappa \cup 1^{n-|\kappa|}$. We obtain Equation (3.4) by comparing the coefficient of $\theta^{(\alpha)}_\pi$ in Eqs. (3.6) and (3.7).

Let us now prove Equation (3.5). Let π, μ and ν be three partitions. We want to prove that the coefficient of $t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu/(z_\pi \alpha^{\ell(\pi)})$ is the same in both sides of Equation (3.5). It is easy to check that this is given by Equation (3.4) if $|\pi| \geqslant \max(|\mu|, |\nu|)$. Otherwise, each one of these coefficients is 0; this is a consequence of Lemma 3.1 and the fact that $\tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r})$ is homogeneous in the three alphabets $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. □

We deduce the following corollary.

Corollary 3.7. *The coefficients $g^{\pi}_{\mu,\nu}$ are polynomials in b with integer coefficients.*

Proof. Inverting Equation (3.5), we get

$$G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \exp\left(-\frac{p_1}{t\alpha}\right)\exp\left(\frac{\partial}{t\partial q_1}+\frac{\partial}{t\partial r_1}\right)\tau^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}). \tag{3.8}$$

By extracting the coefficient of $t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu/(z_\pi \alpha^{\ell(\pi)})$, we get

$$g^{\pi}_{\mu,\nu} = \sum_{e\leqslant i\leqslant m_1(\pi)} (-1)^{m_1(\pi)-i}\binom{m_1(\pi)}{i}\binom{|\widetilde{\pi}|+i-|\mu|+m_1(\mu)}{m_1(\mu)} \binom{|\widetilde{\pi}|+i-|\nu|+m_1(\nu)}{m_1(\nu)} c^{\widetilde{\pi}\cup 1^i}_{\mu\cup 1^{|\widetilde{\pi}|+i-|\mu|},\nu\cup 1^{|\widetilde{\pi}|+i-|\nu|}},$$

where $\widetilde{\pi} := \pi\backslash 1^{m_1(\pi)}$ and $e := \max(0,|\mu|-|\widetilde{\pi}|,|\nu|-|\widetilde{\pi}|)$. We conclude using Theorem 3.4. □

3.4. Proof of Proposition 1.4

Proof of Proposition 1.4. We prove the proposition for $\alpha = 1$. The proof is exactly the same for $\alpha = 2$. Fix three partitions π, μ and ν. We want to prove that

$$g^{\pi}_{\mu,\nu}(1) = \left|\mathcal{OH}^{\pi}_{\mu,\nu}\right|, \tag{3.9}$$

where $\mathcal{OH}^{\pi}_{\mu,\nu}$ is the set of hypermaps defined in Definition 1.2. Since Equation (3.4) fully characterizes the coefficients $g^{\pi}_{\mu,\nu}$, it is enough to prove that

$$\sum_{i=0}^{m_1(\pi)}\binom{m_1(\pi)}{i}\left|\mathcal{OH}^{\widetilde{\pi}\cup 1^i}_{\mu,\nu}\right| = \binom{m_1(\mu)+n-|\mu|}{m_1(\mu)}\binom{m_1(\nu)+n-|\nu|}{m_1(\nu)} c^{\pi}_{\mu\cup 1^{n-|\mu|},\nu\cup 1^{n-|\nu|}}(1),$$

holds for any partitions π, μ and ν, with $n = |\pi|$. Using Theorem 3.2, we know that the right-hand side of the last equation counts vertex labelled oriented hypermaps of profile $(\pi, \mu\cup 1^{|\pi|-|\mu|}, \nu\cup 1^{|\pi|-|\nu|})$, with $|\pi|-|\mu|$ marked $(+)$ faces of degree 1 and $|\pi|-|\nu|$ marked $(-)$ faces of degree 1 (unlike in Definition 1.2, it is possible here to have both faces of isolated loops marked).

On the other hand, the set of such maps M with a fixed number j of isolated loops with both $(+)$ and $(-)$ faces marked, can be obtained as follows:

- choose the labels of black vertices of degree 2, which form the isolated loops with two marked faces; there are $\binom{m_1(\pi)}{j}$ such possible choices,
- choose a hypermap in $\mathcal{OH}^{\widetilde{\pi}\cup 1^{|\pi|-j}}_{\mu,\nu}$ and associate to black vertices of degree 2 the labels not chosen in the first step.

Summing over all $i := m_1(\pi) - j$ between 0 and $m_1(\pi)$, we obtain the left hand-side. This finishes the proof of the proposition. □

We conclude this section by the following table, which summarizes the different results proved or recalled in this section.

Series	$\tau^{(\alpha)}$ (homogeneous)	$G^{(\alpha)}$ (non-homogeneous)
Combinatorial interpretation for $\alpha \in \{1, 2\}$	[GJ96a] (see also Theorem 3.2)	Proposition 1.4 (see also Proposition 5.10 for $\alpha = 1$)
Relations between the series	Propositions 3.5 and 3.6	
Integrality of the coefficients (Conjecture 3.3)	[BD23] (see also Theorem 3.4)	Corollary 3.7

4. Proof of the main theorem

4.1. Differential operators

The purpose of this subsection is to recall the definition of operators $\mathcal{B}_n^{(\alpha)}$ from [CD22]. These operators are defined using an extra catalytic alphabet $Y := (y_i)_{i \geqslant 0}$. We recall that $\mathcal{P} := \mathrm{Span}_{\mathbb{Q}(\alpha)}\{p_\lambda\}_{\lambda \in \mathbb{Y}} = \mathbb{Q}(\alpha)[\mathbf{p}]$. We also consider the spaces

$$\mathcal{P}_Y := \mathrm{Span}_{\mathbb{Q}(\alpha)} \{y_i p_\lambda\}_{i \in \mathbb{N}, \lambda \in \mathbb{Y}} \quad \text{and} \quad \widetilde{\mathcal{P}}_Y := \mathcal{P} \oplus \mathcal{P}_Y.$$

We recall that $b := \alpha - 1$. Set the catalytic operators;

$$Y_+ = \sum_{i \geqslant 1} y_{i+1} \frac{\partial}{\partial y_i} : \mathcal{P}_Y \to \mathcal{P}_Y,$$

$$\Gamma_Y = (1+b) \cdot \sum_{i,j \geqslant 1} y_{i+j} \frac{j\partial^2}{\partial y_{i-1} \partial p_j} + \sum_{i,j \geqslant 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + b \cdot \sum_{i \geqslant 1} y_{i+1} \frac{i\partial}{\partial y_i} : \mathcal{P}_Y \to \mathcal{P}_Y,$$

$$\text{and} \quad \Theta_Y := \sum_{i \geqslant 1} p_i \frac{\partial}{\partial y_i} : \mathcal{P}_Y \to \mathcal{P}.$$

For $n \geqslant 0$, and u a variable, the operator $\mathcal{B}_n^{(\alpha)}(\mathbf{p}, u)$ is defined by

$$\mathcal{B}_n^{(\alpha)}(\mathbf{p}, u) := \Theta_Y \left(\Gamma_Y + uY_+\right)^n \frac{y_0}{1+b} : \mathcal{P} \to \mathcal{P}[u] \tag{4.1}$$

and the operator $\mathcal{B}_\infty^{(\alpha)}$ by

$$\mathcal{B}_\infty^{(\alpha)}(t, \mathbf{p}, u) := \sum_{n \geqslant 1} \frac{t^n}{n} \mathcal{B}_n^{(\alpha)}(\mathbf{p}, u) : \mathcal{P} \to \mathcal{P}[\![u, t]\!].$$

Remark 4.1. Note that the operator $\mathcal{B}_n^{(\alpha)}$ is homogeneous of degree n; namely for any λ, we have that $\mathcal{B}_n^{(\alpha)} \cdot p_\lambda$ is a linear combination of $u^\ell p_\mu$ for μ of size $|\lambda|+n$ and $\ell \leqslant n$. Similarly, if $n \leqslant |\lambda|$, then $\left(\mathcal{B}_n^{(\alpha)}\right)^\perp \cdot p_\lambda$ is a linear combination of $u^\ell p_\mu$ for μ of size $|\lambda| - n$ and $\ell \leqslant n$.

Hence, $\mathcal{B}_\infty^{(\alpha)}(t, \mathbf{p}, u) \cdot p_\lambda$ is a combination of $t^{|\mu|-|\lambda|} u^\ell p_\mu$ for $|\mu| \geqslant |\lambda|$ and $\ell \geqslant 0$. Moreover, ${\mathcal{B}_\infty^{(\alpha)}}^\perp(t, \mathbf{p}, u) \cdot p_\lambda$ is a linear combination of $t^{|\lambda|-|\mu|} p_\mu u^\ell$ for $|\mu| \leqslant |\lambda|$ and $\ell \geqslant 0$. Consequently, operators

$$\mathcal{B}_\infty^{(\alpha)}(t, \mathbf{p}, u) : \mathbb{Q}(\alpha)[t][\![\mathbf{p}]\!] \longrightarrow \mathbb{Q}(\alpha)[t][\![\mathbf{p}, u]\!]$$
$${\mathcal{B}_\infty^{(\alpha)}}^\perp(t, \mathbf{p}, u) : \mathbb{Q}(\alpha)[t, \mathbf{p}] \longrightarrow \mathbb{Q}(\alpha)[t, \mathbf{p}][\![u]\!]$$

are well defined. We deduce that operators $\mathcal{B}_\infty^{(\alpha)}(t, \mathbf{q}, u), \mathcal{B}_\infty^{(\alpha)}(t, \mathbf{r}, u)$ and ${\mathcal{B}_\infty^{(\alpha)}}^\perp(t, \mathbf{p}, u)$ are well defined from $\mathcal{A}$ to $\mathcal{A}[\![u]\!]$.

4.2. Skew Jack characters

Before proving Theorem 1.5, we introduce a skew[4] version of Jack characters.

Definition 4.2. We consider a sequence of variables $u_1, u_2 \dots$. For any partitions μ and ν satisfying $|\mu| \geqslant |\nu|$, we define the coefficient $\theta_{\mu/\nu}^{(\alpha)}$ which depends on one variable v, by

$$\theta_\mu^{(\alpha)}(v, u_1, u_2 \dots) = \sum_\nu \theta_{\mu/\nu}^{(\alpha)}(v) \theta_\nu^{(\alpha)}(u_1, u_2 \dots).$$

This expansion is well defined, since $\theta_\mu^{(\alpha)}(v, u_1, u_2 \dots)$ is a shifted symmetric function in the variables $u_1, u_2 \dots$, and $(\theta_\nu^{(\alpha)})_{\nu \in \mathbb{Y}}$ is a basis of $\mathcal{S}_\alpha^*$.

We then have the following lemma.

Lemma 4.3. *For any partitions μ, ν, π one has*

$$\sum_\kappa g_{\mu,\nu}^\kappa \theta_{\kappa/\pi}^{(\alpha)}(v) = \sum_{\rho,\xi} g_{\rho,\xi}^\pi \theta_{\mu/\rho}^{(\alpha)}(v) \theta_{\nu/\xi}^{(\alpha)}(v).$$

Proof. We have

$$\begin{aligned}
\sum_\pi \theta_\pi^{(\alpha)}(u_1, u_2 \dots) \left(\sum_\kappa g_{\mu,\nu}^\kappa \theta_{\kappa/\pi}^{(\alpha)}(v) \right) &= \sum_\kappa g_{\mu,\nu}^\kappa \theta_\kappa^{(\alpha)}(v, u_1, u_2 \dots) \\
&= \theta_\mu^{(\alpha)}(v, u_1, u_2 \dots) \theta_\nu^{(\alpha)}(v, u_1, u_2 \dots) \\
&= \sum_{\rho,\xi} \theta_{\mu/\rho}^{(\alpha)}(v) \theta_\rho^{(\alpha)}(u_1, u_2 \dots) \theta_{\nu/\xi}^{(\alpha)}(v) \theta_\xi^{(\alpha)}(u_1, u_2 \dots) \\
&= \sum_\pi \theta_\pi^{(\alpha)}(u_1, u_2 \dots) \left(\sum_{\rho,\xi} g_{\rho,\xi}^\pi \theta_{\mu/\rho}^{(\alpha)}(v) \theta_{\nu/\xi}^{(\alpha)}(v) \right).
\end{aligned}$$

We conclude by extracting the coefficient of $\theta_\pi^{(\alpha)}(u_1, u_2, \dots)$. □

[4]This is not the usual definition of skew characters in which we consider skew diagrams in the argument of the character; $\theta_\mu^{(\alpha)}(\lambda/\rho)$.

The following proposition gives a differential construction for skew characters.

Proposition 4.4. *For any partitions μ and ν one has*

$$\theta^{(\alpha)}_{\mu/\nu}(v) = [t^{|\mu|-|\nu|}p_\mu]\exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{p},-\alpha v)\right)\cdot p_\nu, \tag{4.2}$$

$$\text{and}\quad \frac{\alpha^{\ell(\mu)}z_\mu}{\alpha^{\ell(\nu)}z_\nu}\theta^{(\alpha)}_{\mu/\nu}(v) = [t^{|\mu|-|\nu|}p_\nu]\exp\left(\mathcal{B}^{(\alpha)\perp}_\infty(-t,\mathbf{p},-\alpha v)\right)\cdot p_\mu. \tag{4.3}$$

Proof. Fix $k \geqslant 0$. From Theorem 1.3, we have

$$\sum_\nu t^{|\nu|}\theta^{(\alpha)}_\nu(u_1,\dots,u_k)p_\nu = \exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{p},-\alpha u_1)\right)\dots\exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{p},-\alpha u_k)\right)\cdot 1.$$

By applying $\exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{p},-\alpha v)\right)$ and using Theorem 1.3 for $k+1$, we get

$$\sum_\nu\left(\exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{p},-\alpha v)\right)\cdot t^{|\nu|}p_\nu\right)\theta^{(\alpha)}_\nu(u_1,\dots,u_k) = \sum_\mu t^{|\mu|}\theta^{(\alpha)}_\mu(v,u_1,\dots,u_k)p_\mu.$$

Taking the limit over k and extracting the coefficient of p_μ, we get

$$[p_\mu]\sum_\nu\left(\exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{p},-\alpha v)\right)\cdot t^{|\nu|}p_\nu\right)\theta^{(\alpha)}_\nu(u_1,u_2,\dots) = t^{|\mu|}\theta^{(\alpha)}_\mu(v,u_1,u_2,\dots).$$

We obtain Equation (4.2) by extracting the coefficient of $\theta^{(\alpha)}_\nu(u_1,u_2\dots)$. This equation can be rewritten as follows

$$\begin{aligned} t^{|\mu|-|\nu|}\theta^{(\alpha)}_{\mu/\nu}(v) &= \left\langle \exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{p},-\alpha v)\right)\cdot p_\nu, \frac{p_\mu}{z_\mu\alpha^{\ell(\mu)}}\right\rangle \\ &= \left\langle p_\nu, \exp\left(\mathcal{B}^{(\alpha)\perp}_\infty(-t,\mathbf{p},-\alpha v)\right)\cdot\frac{p_\mu}{z_\mu\alpha^{\ell(\mu)}}\right\rangle \\ &= [p_\nu]\exp\left(\mathcal{B}^{(\alpha)\perp}_\infty(-t,\mathbf{p},-\alpha v)\right)\cdot\frac{z_\nu\alpha^{\ell(\nu)}}{z_\mu\alpha^{\ell(\mu)}}p_\mu. \end{aligned}$$

□

4.3. Proof of the main theorem

Proof of Theorem 1.5. We have from Equation (4.2)

$$\begin{aligned} &\exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{q},-\alpha v)\right)\exp\left(\mathcal{B}^{(\alpha)}_\infty(-t,\mathbf{r},-\alpha v)\right)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) \\ &\qquad = \sum_{\pi,\mu,\nu}\left(\sum_{\rho,\xi}\frac{g^\pi_{\rho,\xi}(\alpha)}{z_\pi\alpha^{\ell(\pi)}}\theta^{(\alpha)}_{\mu/\rho}(v)\theta^{(\alpha)}_{\nu/\xi}(v)\right)t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu. \end{aligned}$$

On the other hand, Equation (4.3) gives

$$\exp\left(\mathcal{B}^{(\alpha)\perp}_\infty(-t,\mathbf{p},-\alpha v)\right)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_{\pi,\mu,\nu}\left(\sum_\kappa\frac{g^\kappa_{\mu,\nu}}{z_\kappa\alpha^{\ell(\kappa)}}\theta^{(\alpha)}_{\kappa/\pi}(v)\right)t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu.$$

Combining these two equations with Lemma 4.3, we deduce that

$$\exp\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q},-\alpha v)+\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{r},-\alpha v)\right)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) \\ =\exp\left(\mathcal{B}_\infty^{(\alpha)\perp}(-t,\mathbf{p},-\alpha v)\right)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}). \quad (4.4)$$

Since $\mathcal{B}_\infty^{(\alpha)\perp}(-t,\mathbf{p},-\alpha v)$ commutes with each one of the operators $\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q},-\alpha v)$ and $\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{r},-\alpha v)$ as operators in $\mathcal{O}(\mathcal{A})[\![v]\!]$, we obtain by induction that for any $\ell\geqslant 1$

$$\exp\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q},-\alpha v)+\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{r},-\alpha v)\right)^\ell\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) \\ =\exp\left(\mathcal{B}_\infty^{(\alpha)\perp}(-t,\mathbf{p},-\alpha v)\right)^\ell\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}).$$

This allows to take the logarithm of the operators in Equation (4.4). We get that

$$\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q},-\alpha v)+\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{r},-\alpha v)\right)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})=\mathcal{B}_\infty^{(\alpha)\perp}(-t,\mathbf{p},-\alpha v)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}).$$

We conclude by replacing v by $-u/\alpha$. □

Let us now prove Theorem 1.6.

Proof of Theorem 1.6. Using Equation (4.2), Lemma 4.3 can be rewritten as the following commutation relation

$$\exp\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q},-\alpha v)+\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{r},-\alpha v)\right)\cdot \mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) \\ =\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\cdot\exp\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{p},-\alpha v)\right).$$

Finally, we "take the logarithm" of operators as in the proof of Theorem 1.5. □

4.4. Generalization to constellations

The purpose of this subsection is to briefly explain how to generalize the differential equation of the main theorem to series with finitely many alphabets, and how these are related to constellations.

Fix an integer $k\geqslant 1$. We consider $k+2$ alphabets $\mathbf{p}$, $\mathbf{q}^{(0)}$, $\mathbf{q}^{(1)}$, $\dots\mathbf{q}^{(k)}$ and we introduce the following generalization of the series $G^{(\alpha)}$;

$$G_k^{(\alpha)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)}):=\sum_{\pi,\mu^{(0)},\dots,\mu^{(k)}}\frac{g^\pi_{\mu^{(0)},\dots,\mu^{(k)}}(\alpha)}{z_\pi\alpha^{\ell(\pi)}}t^{|\mu^{(0)}|+\dots|\mu^{(k)}|-|\pi|}p_\pi q^{(0)}_{\mu^{(0)}}\dots q^{(k)}_{\mu^{(k)}},$$

where $g^\pi_{\mu^{(0)},\dots,\mu^{(k)}}$ are defined as the structure coefficients;

$$\theta^{(\alpha)}_{\mu^{(0)}}\dots\theta^{(\alpha)}_{\mu^{(k)}}=\sum_\pi g^\pi_{\mu^{(0)},\dots,\mu^{(k)}}(\alpha)\theta^{(\alpha)}_\pi.$$

We emphasize that the series $G_k^{(\alpha)}$ (or more precisely the closely related series $\tau_k^{(\alpha)}$ below) gives access to all Hurwitz numbers (and their b-deformation in the sense of [CD22]) with control on their full ramification profile.

Using the same arguments as in the case of three alphabets, we obtain the following theorem.

Theorem 4.5. *For any $k \geqslant 1$, we have,*

$$\left(\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q}^{(0)},u)+\cdots+\mathcal{B}_\infty^{(\alpha)}(-t,\mathbf{q}^{(k)},u)\right)\cdot G_k^{(\alpha)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)}) \\ =\mathcal{B}_\infty^{(\alpha)\perp}(-t,\mathbf{p},u)\cdot G_k^{(\alpha)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)}).$$

As mentioned in the introduction, such an equation seems to be new even in the classical case $b=0$. We now explain the connection of $G_k^{(\alpha)}$ to the series of k-constellations. As in Section 3.2, we consider the series

$$\tau_k^{(\alpha)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)}):=\sum_{n\geqslant 0}t^n\sum_{\theta\vdash n}\frac{1}{j_\theta^{(\alpha)}}J_\theta^{(\alpha)}(\mathbf{p})J_\theta^{(\alpha)}(\mathbf{q}^{(0)})\dots J_\theta^{(\alpha)}(\mathbf{q}^{(k)}).$$

It turns out that $\tau_k^{(\alpha)}$ corresponds to the series of orientable k-constellations when $\alpha=1$ [JV90], and to the series of all k-constellations (orientable or not) introduced in [CD22] when $\alpha=2$; see [BD22]. On the other hand, extending the proof of Proposition 3.6 to $k+2$ alphabets, we get

$$\exp\left(\frac{p_1}{t\alpha}\right)G_k^{(\alpha)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)})=\exp\left(\frac{\partial}{t\partial q_1^{(0)}}+\cdots+\frac{\partial}{t\partial q_1^{(k)}}\right)\tau_k^{(\alpha)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)}).$$

4.5. Operators $\mathcal{C}_\ell^{(\alpha)}$

For $\ell\geqslant 0$ we consider the operator $\mathcal{C}_\ell^{(\alpha)}(t,\mathbf{p})$ given by

$$\mathcal{C}_\ell^{(\alpha)}(t,\mathbf{p}):=[u^\ell]\mathcal{B}_\infty^{(\alpha)}(t,\mathbf{p},u):\mathcal{P}\to\mathcal{P}[\![t]\!].$$

The differential equation Equation (1.6) of the main theorem is then equivalent to the equations

$$\left(\mathcal{C}_\ell^{(\alpha)}(-t,\mathbf{q})+\mathcal{C}_\ell^{(\alpha)}(-t,\mathbf{r})\right)\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})=\mathcal{C}_\ell^{(\alpha)\perp}(-t,\mathbf{p})\cdot G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}),\quad \text{for } \ell\geqslant 0.$$

Similarly, Theorem 1.6 is equivalent to

$$\left(\mathcal{C}_\ell^{(\alpha)}(-t,\mathbf{q})+\mathcal{C}_\ell^{(\alpha)}(-t,\mathbf{r})\right)\cdot \mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})=\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\cdot\mathcal{C}_\ell^{(\alpha)}(-t,\mathbf{p})\ \text{ for } \ell\geqslant 0. \tag{4.5}$$

We deduce the following corollary which will be useful in the solution of the differential equation of the main theorem in Section 6.1.

Corollary 4.6. *For any partition $\lambda=[\lambda_1,\dots,\lambda_s]$, we have*

$$\prod_{1\leqslant i\leqslant s}\left(\mathcal{C}_{\lambda_i}^{(\alpha)}(-t,\mathbf{q})+\mathcal{C}_{\lambda_i}^{(\alpha)}(-t,\mathbf{r})\right)\cdot\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})=\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\cdot\prod_{1\leqslant i\leqslant s}\mathcal{C}_{\lambda_i}^{(\alpha)}(-t,\mathbf{p}). \tag{4.6}$$

Actually, the product in Equation (4.6) can be taken in any order, since the operators $\mathcal{C}_\ell^{(\alpha)}$ satisfy the following commutation relations; see [BDD23, Theorem 6.6].

Theorem 4.7. *[BDD23] Let $m>0$. Then*

$$\left[\mathcal{C}_\ell^{(\alpha)},\mathcal{C}_m^{(\alpha)}\right]=\begin{cases}0 & \textit{if } \ell>0,\\ (m+1)\mathcal{C}_{m+1}^{(\alpha)} & \textit{if } \ell=0.\end{cases}$$

5. Combinatorial proof of the differential equation for $\alpha = 1$

The purpose of this section is to give a combinatorial proof of the commutation relation Equation (4.5) (equivalently the differential equation Equation (1.6)) for $\alpha = 1$. To this purpose, we start by recalling the combinatorial interpretation of the operators $\mathcal{C}_\ell^{(1)}$ given in [BDD23], see Proposition 5.2. We then use Proposition 1.4 to obtain a combinatorial interpretation of the operator $\mathcal{G}^{(1)}$, see Corollary 5.13.

We believe that the combinatorial constructions of Subsections 5.2 and 5.3 are of independent interest and might be useful to shed some light on the combinatorics of hypermaps with controlled profile.

All maps considered in this section are orientable.

5.1. Interpretation of the operator $\mathcal{C}_\ell^{(\alpha)}$ for $\alpha = 1$

It will be more convenient at some steps of the proof to work with maps with labelled edges rather than labelled vertices.

Definition 5.1. We say that a map M is *labelled* if its edges are numbered $1, 2, \dots, |M|$. We say that a map is *bipartite* if its vertices are colored in white and black and each edge connects two vertices of different colors. If M is a bipartite map of size n, then its *face-type* is the partition of n obtained by reordering the face degrees divided by 2.

The following proposition is a special case of [BDD23, Proposition 4.5]. Since we consider here a different convention of map labelling, we briefly explain the main ideas of the proof.

Proposition 5.2 ([BDD23, Proposition 4.5])**.** *Fix an integer $\ell \geqslant 0$, a partition π, and let N be a labelled orientable bipartite map of face-type π. Then,*

$$\ell! \mathcal{C}_\ell^{(1)}(-t, \mathbf{p}) \cdot \frac{p_\pi}{|\pi|!} = \sum_{n \geqslant \ell} (-t)^n \sum_M \frac{p_{\text{face-type}(M)}}{|M|!}, \tag{5.1}$$

where the second sum is taken over labelled orientable maps M obtained from N as follows:

- *we add a black vertex v and ℓ new white vertices w_1, w_2,…,w_ℓ.*
- *we add n edges all incident to v and such that each new white vertex w_i is connected to v by at least one edge (we do not put any restriction on the degrees of the new vertices w_i).*
- *we relabel the edges of M in any way.*

An example of maps N and M is given in Figure 5.1.

Proof. We recall that the operator $\mathcal{C}_\ell^{(\alpha)}$ is defined by

$$\mathcal{C}_\ell^{(1)}(-t, \mathbf{p}) = [u^\ell] \sum_{n \geqslant 1} (-t)^n \frac{\mathcal{B}_n^{(1)}(\mathbf{p}, u)}{n}. \tag{5.2}$$

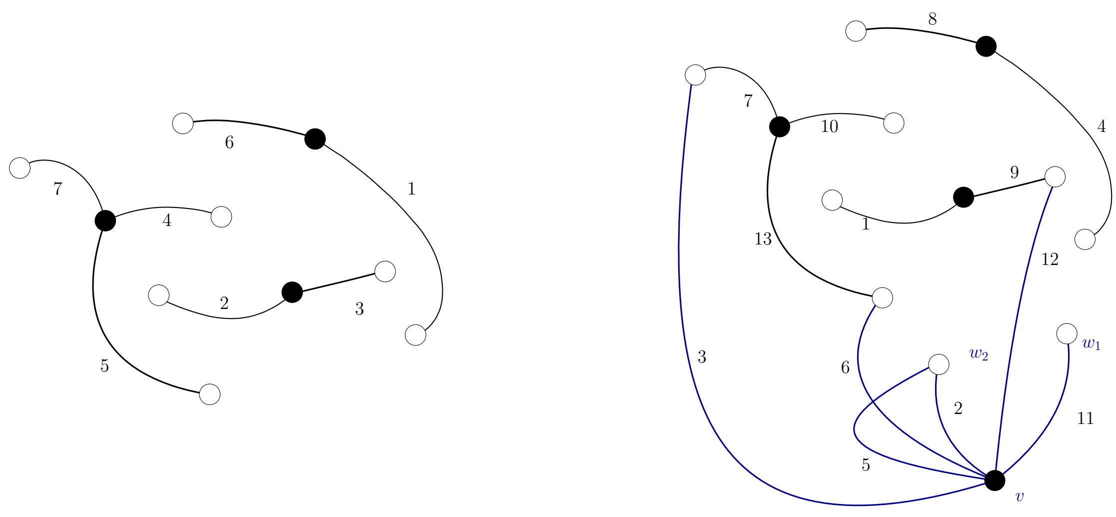


Figure 5.1: Example of the action of $\mathcal{C}_2^{(1)}$ on a map N. On the left the map N, and on the right a map M obtained by adding one black vertex v, two white vertices w_1 and w_2 and 6 edges (represented in blue).

where

$$\mathcal{B}_n^{(1)}(\mathbf{p}, u) := \Theta_Y \left(\Gamma_Y^{(\alpha=1)} + uY_+\right)^n y_0$$

see Section 4.1 for the definitions of the operators Θ_Y, Γ_Y and Y_+.

In order to understand the combinatorics of these operators, we consider *rooted maps*. A map is rooted if it has a marked corner c called the root corner. We associate to a rooted map the weight

$$\text{weight}(M, c) := y_{\deg(f_c)} \prod_{f \neq f_c} p_{\deg(f)} \in \mathcal{P}_Y,$$

where $\deg(f_c)$ denotes the degree of the root face, and the the product runs over all faces of M different from f_c. When $\alpha = 1$, the operator $\mathcal{B}^{(1)}$ can be interpreted as follows:

$$\mathcal{B}_n^{(1)}(\mathbf{p}, u) \cdot p_{\text{face-type}(N)} u^{|\mathcal{V}_\circ(N)|} = \sum_M p_{\text{face-type}(M)} u^{|\mathcal{V}_\circ(M)|},$$

where $|\mathcal{V}_\circ(.)|$ denotes the number of white vertices and where the sum is taken over orientable maps M obtained from N by adding a black vertex v of degree n with a rooted corner.

Let us briefly explain this equation. First, we add an isolated root vertex, this corresponds to multiplication by y_0. We then add consecutively n edges to the root corner: the added edge can be connected to a new white vertex (this corresponds to the term uY_+) or to an existing white vertex in the map (ensured by the operator Γ_Y). Finally we apply Θ_Y to obtain the weight of an unrooted map; $\Theta_Y \cdot \text{weight}(M, c) = p_{\text{face-type(M)}}$. We refer to [BDD23, Section 4.2] for more details about the combinatorics of these operators.

If N and M are now labelled maps then the previous equation becomes

$$\frac{\mathcal{B}_n^{(1)}(\mathbf{p}, u)}{n} \cdot \frac{p_{\mathcal{V}_\diamond(N)}}{|N|!} u^{|\mathcal{V}_\circ(N)|} = \sum_M \frac{p_{\mathcal{V}_\diamond(M)}}{|M|!} u^{|\mathcal{V}_\circ(M)|}.$$

Indeed, we have $|M|!$ ways to choose new labels for the edges of M, and then we divide by $|N|!$ to “forget” the old labels in N, and by n to forget the root of the added black vertex v.

Since in Equation (5.2) we take the coefficient of $[u^\ell]$, then the operator $\mathcal{C}_\ell(-t, \mathbf{p})$ acts on a map by adding a black vertex using ℓ new white vertices with an extra weight $-t$ for each added edge. Finally, multiplying by $\ell!$ in Equation (5.1) corresponds to having a total order on the new added white vertices $w_1, w_2, \dots w_\ell$. □

5.2. BFC maps and pre-hypermaps

We start by introducing a family of maps which allows to encode the hypermaps with marked faces defined in Definition 1.2.

Definition 5.3. We say that a map is *bipartite face-colored* (BFC map) if its vertices are colored in black and white, its faces are colored in two colors $(+)$ and $(-)$, and such that each edge connects two vertices of different colors (but it does not necessarily separate two faces of different colors).

Moreover, a BFC map will be called a *pre-hypermap* if it satisfies the following additional conditions:

1. white vertices have degree at most 2.

2. all white vertices of degree 2 are incident to two faces of different colors.

Remark 5.4. Notice that a hypermap can be seen as a pre-hypermap; we color all the vertices of the hypermap in black and add in the middle of each edge a white vertex of degree 2. Hence, hypermaps are pre-hypermaps maps with only white vertices of degree 2. Conversely, if we delete all white vertices of degree 1 in a pre-hypermap we obtain a hypermap.

We distinguish two types of edges in a BFC map.

Definition 5.5. An edge is said *bicolor* if it is incident to two faces of different colors. We have two types of bicolor edges in a BFC map (see Figure 5.2):

- Type 1: on the $(+)$ side-face, we see the white vertex and then the black one when we travel along the edge-side in the direct orientation.

- Type 2: on the $(+)$ side-face, we see the black vertex and then the white one when we travel along the edge-side in the direct orientation.

By definition, all non bicolor edges will be considered of type 1.

Remark 5.6. Note that the types can be equivalently defined by conditions around vertices. For instance, when we turn around a black vertex in the direct orientation, then edges of type 2 are exactly those who separate faces $(-)/(+)$ in this order. Consequently, a vertex which is not monochromatic (*i.e.* incident at least to one $(+)$ and one $(-)$ face) is necessarily incident to a type 2 edge.

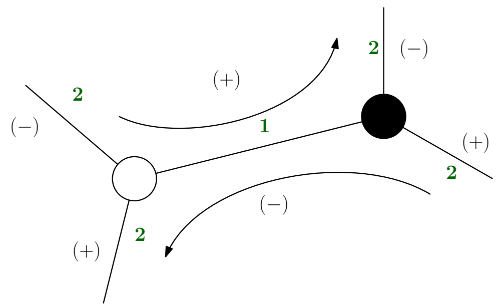


Figure 5.2: Types of bicolor edges in a BFC map.

It is easy to check that each black vertex of a pre-hypermap has even number of white neighbors of degree 2 (see *e.g* Remark 5.4). This allows to define the degree of a black vertex v in a pre-hypermap as follows

$$\deg(v) = \frac{1}{2}|\{\text{neighbors of } v \text{ of degree } 2\}| + |\{\text{neighbors of } v \text{ of degree } 1\}| \quad (5.3)$$
$$= |\{\text{incident edges to } v \text{ of type } 1\}|. \quad (5.4)$$

We now extend the notion of profile to pre-hypermaps.

Definition 5.7. Let M be a pre-hypermap. We denote by $\lambda^\bullet(M)$ the partition given by black vertices degrees (as defined in Equation (5.3)). We also denote by $\lambda^+(M)$ (resp. $\lambda^-(M)$ the partition given by the $(+)$ face (resp. the $(-)$ face) degrees divided by 2. We call *the profile* of M the triplet of partitions $(\lambda^\bullet(M), \lambda^+(M), \lambda^-(M))$.

One can check that the profile of a hypermap as defined in Definition 1.1 coincides with its pre-hypermap profile as defined in Definition 5.7.

Definition 5.8. We say that a pre-hypermap is *vertex labelled* if:

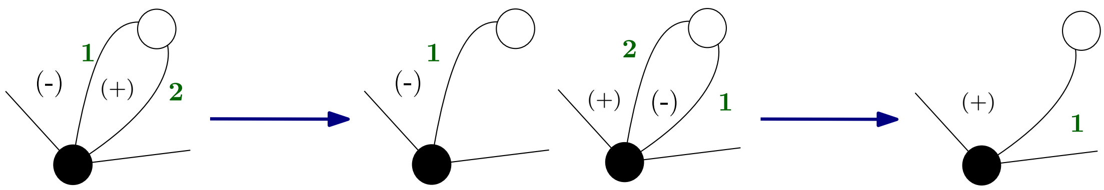
- for each $d \geqslant 1$, vertices of same degree d are numbered 1,2,$\dots$.
- each black vertex has a marked corner, oriented in the direct orientation and followed by an edge of type 1.

Fix three partitions π, μ and ν. We define $\mathcal{OPH}^{\pi}_{\mu,\nu}$ as the set of vertex-labelled oriented pre-hypermaps of profile (π, μ, ν).

The following lemma connects pre-hypermaps to hypermaps with marked faces.

Lemma 5.9. *Fix three partitions π, μ and ν. There is a bijection between $\mathcal{OH}^{\pi}_{\mu,\nu}$ (defined in Definition 1.2) and $\mathcal{OPH}^{\pi}_{\mu,\nu}$.*

Proof. First, notice that each degree 2 face of a pre-hypermap (degree 1 face in the associated hypermap) is incident exactly to one edge of type 1 and one edge of type 2. Notice also that the only case in which an edge is incident to two faces of degree 2 is the case of an isolated loop.



(a) The marked face is colored $(+)$ (b) The marked face is colored $(-)$

Figure 5.3: Deleting edges of a hypermap with marked faces to obtain a pre-hypermap.

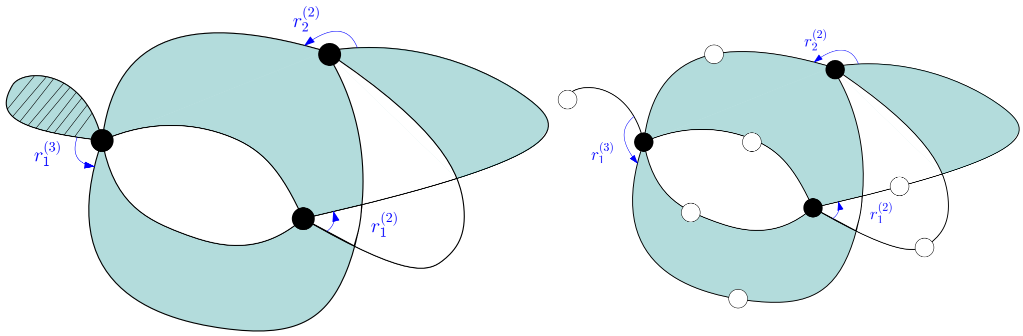


Figure 5.4: On the left a labelled hypermap with one marked face; $(-)$ faces are represented in blue and the marked face is crossed. On the right the associated pre-hypermap.

As in Remark 5.4, we can think of M as a pre-hypermap which we denote M'. First, notice that each degree 2 face in M' (degree 1 face in M) is incident exactly to one edge of type 1 and one edge of type 2. Notice also that the only case in which an edge of type 2 is incident to two faces of degree 2 is the case of an isolated loop.

By deleting in M' all edges of type 2 incident to a marked face and forgetting the colors of these faces, we obtain a map N in $\mathcal{OPH}^{\pi}_{\mu,\nu}$, see Figure 5.3. Indeed, in this procedure the degree of a black vertex (as defined in Equation (5.3)) is unchanged. Moreover, each one of the faces of N inherits a color from M' and its degree is unchanged.

Conversely, from $N \in \mathcal{OPH}^{\pi}_{\mu,\nu}$ we obtain a map $M \in \mathcal{OH}^{\pi}_{\mu,\nu}$ as follows; first we transform each white leaf into a loop, we mark the formed 2-degree face and we color it so that the added edge is bicolor. □

An example of the correspondence described above is given in Figure 5.4.

Proposition 5.10. *For any partitions π,μ and ν, we have*

$$g^{\pi}_{\mu,\nu}(1) = |\mathcal{OPH}^{\pi}_{\mu,\nu}|.$$

Equivalently, $(|\mu| + |\nu|)!/z_\pi g^{\pi}_{\mu,\nu}(1)$ is the number of labelled orientable pre-hypermaps of profile (π, μ, ν) (see Definition 5.1).

Proof. We know from Lemma 5.9 that $|\mathcal{OPH}^{\pi}_{\mu,\nu}| = |\mathcal{OH}^{\pi}_{\mu,\nu}|$. On the other hand $g^{\pi}_{\mu,\nu}(1) = |\mathcal{OH}^{\pi}_{\mu,\nu}|$ by Equation (3.9), this gives the first part of the proposition.

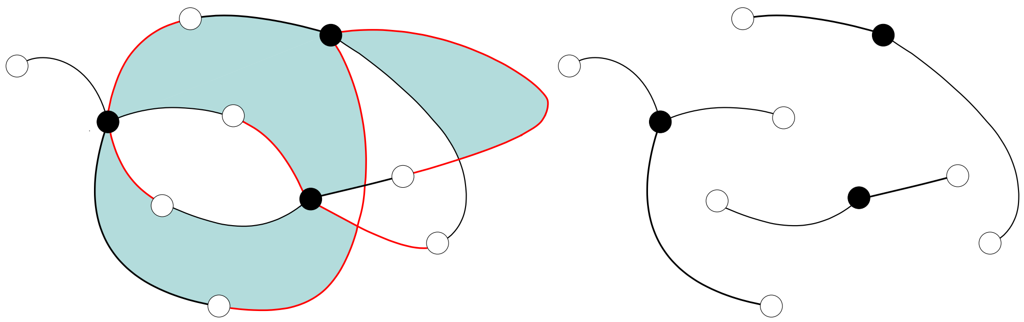


Figure 5.5: On the left an oriented pre-hypermap, faces colored in $(-)$ are represented in blue. Bicolor edges of type 2 are represented in red. On the right the $[3,2,2]$-star map obtained by deleting edges of type 2.

To obtain the second part, we start by noticing that a pre-hypermap of profile (π,μ,ν) has $|\mu|+|\nu|$ edges. Moreover, to pass from a vertex-labelled hypermap of vertex type π to a labelled hypermap, we start by labelling edges and then we forget vertex labels which corresponds to multiplying by $(|\mu|+|\nu|)!/z_\pi$. □

5.3. Combinatorial interpretation of $\mathcal{G}^{(1)}$

In this subsection, we give a second combinatorial interpretation for $g^{\pi}_{\mu,\nu}(1)$ which generalizes Proposition 5.10. This interpretation consists in seeing the series of hypermaps as an operator (see Corollary 5.13) rather than a "static" generating series.

Fix a partition π. We call *π-star* map the unique bipartite map with only white vertices of degree 1 and black vertices of type π. Notice that labelled π-star maps are in bijection with permutations of cyclic type π. In particular, there are $|\pi|!/z_\pi$ such maps.

Lemma 5.11. *Let M be a pre-hypermap of profile (π,μ,ν). Then the map obtained by deleting all edges of type 2 is the π-star map.*

Conversely, let M be a BFC *map such that $\lambda^+(M)=\mu$ and $\lambda^-(M)=\nu$. Assume that M is obtained by adding $|\mu|+|\nu|-|\pi|$ edges to the π-star map and by coloring the faces, such that the added edges are bicolor of type 2. Then all white vertices of M have degree 1 or 2, white vertices of degree 2 are incident to two faces of different colors, and $\lambda^\bullet(M)=\pi$. In other terms, M is a pre-hypermap of profile (π,μ,ν).*

Proof. We start by proving the first assertion. Let M be a pre-hypermap of profile (π,μ,ν) and let N be the map obtained by deleting all edges of type 2 (an example is given in Figure 5.5). By definition all white vertices of M have degree 1 or 2. Moreover, each white vertex of degree two is incident to one edge of type 1 and one edge of type 2. Hence, N is a star map. Furthermore, the degree of a black vertex in a pre-hypermap is given by the number of incident edges of type 1 (see Equation (5.4)), and is then unchanged by deleting edges of type 2. By consequence the type of black vertices in N is the same as in M, that is π.

Let us now prove the second assertion. Let M be a BFC map obtained from the π-star map as above. Since all added edges are of type 2, we can not add two edges incident to the same white corner. Hence, all white vertices in M have degree 1 or 2. Moreover, white vertices of degree 2 are incident to two faces of different colors since the added edges are bicolor. This proves that M is a pre-hypermap. The type of black vertices of M is π by the same argument as above. This finishes the proof of the lemma. □

We deduce a combinatorial interpretation of $g^{\pi}_{\mu,\nu}(1)$ which generalizes Proposition 5.10.

Proposition 5.12. *Fix three partitions π, μ and ν, and set $m := |\pi|$ and $n := |\mu| + |\nu| - |\pi|$. Let N be a labelled orientable bipartite map of face-type π. Then $\frac{n!}{m!} g^{\pi}_{\mu,\nu}(1)$ is the number of ways of obtaining a labelled* BFC *map M from N by:*

1. *adding $|\mu| + |\nu| - |\pi|$ edges to N to obtain a map of face type $\mu \cup \nu$,*
2. *coloring the faces such that the added edges are bicolor of type 2, and such that the obtained* BFC *map M satisfies $\lambda^+(M) = \mu$ and $\lambda^-(M) = \nu$,*
3. *relabelling all the edges.*

Note that the number of ways of obtaining a BFC M by the three operations described above from a bipartite map N depends only on the face-type of N but does not depend on its structure as will be shown in the proof.

Proof. We start by proving the result when N is a labelled π-star map. We know from Proposition 5.10 that the number of labelled pre-hypermaps of profile (π, μ, ν) is $n!/z_\pi g^{\pi}_{\mu,\nu}(1)$. We now count in a different way the number of labelled pre-hypermaps of profile (π, μ, ν).

Let $f^{\pi}_{\mu,\nu}$ be the number of ways of realizing the steps (1), (2) and (3) described above starting from a fixed labelled star map of face-type π.

In order to obtain a pre-hypermap of profile (π, μ, ν), we start by choosing a labelled π-star map (we have $m!/z_\pi$ choices), we then have $f^{\pi}_{\mu,\nu}$ ways to add edges to obtain a labelled pre-hypermap of profile (π, μ, ν) (we use here Lemma 5.11). Hence

$$n!/z_\pi g^{\pi}_{\mu,\nu}(1) = m!/z_\pi f^{\pi}_{\mu,\nu}.$$

We deduce that $f^{\pi}_{\mu,\nu} = \frac{n!}{m!} g^{\pi}_{\mu,\nu}(1)$. This finishes the proof for star maps.

In order to obtain the assertion for any bipartite labelled map N of face-type π, we prove that the number of ways to realize the steps *(1), (2)* and *(3)* on a map N depends only on the face-type of the map and not on its structure. Indeed, when we have two maps of the same face-type, one can always find a bijection between the corners of the two maps which preserves the face structures; two corners are consecutive when we travel along a face (in the direct orientation) in the first map, if and only if their images in the second map satisfy the same property. Once such bijection is fixed, each way of adding edges and coloring faces on one map can be copied on the second map in a unique way which respects the bijection of the corners. The two maps obtained have necessarily the same $(+)$ and $(-)$ types (but not necessarily the same vertex degrees). We give an example in Figure 5.6. □

Given Proposition 5.12, it is possible to think of pre-hypermaps as partially constructed hypermaps. Indeed, hypermaps are obtained by adding a maximal number of edges on π-star maps. We now deduce a combinatorial interpretation of $\mathcal{G}^{(1)}$.

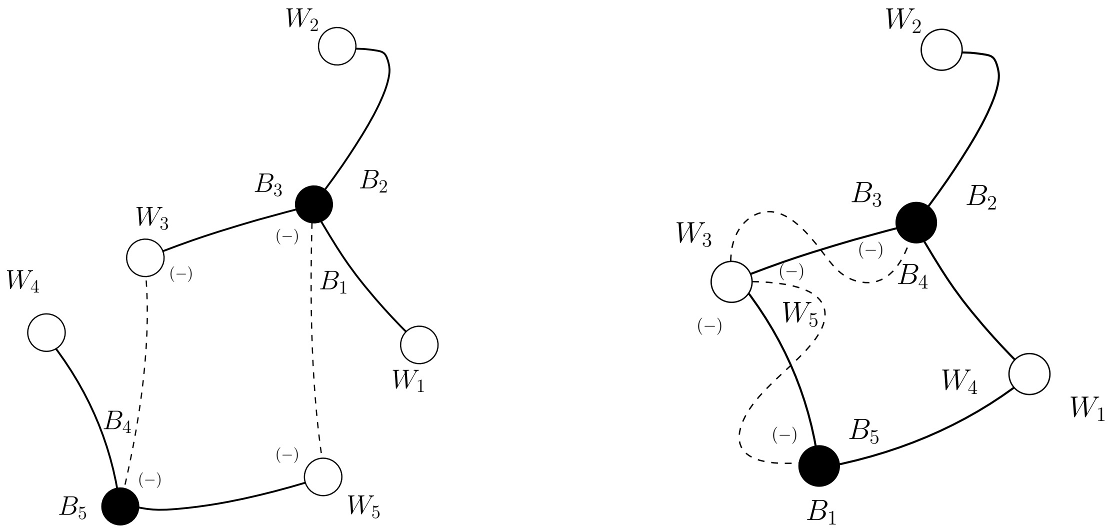


Figure 5.6: The plain edges represent two initial maps with the same face-type. The labels W_i and B_i give a bijection between the corners of the two maps which preserves the face structure. The two dashed edges are added and faces colored with respect to this bijection; an edge between corners (B_4, W_3) and an edge between (B_1, W_5). Corners incident to a face of color $(-)$ are marked with a sign $(-)$.

Corollary 5.13. *Fix a partition π of size $m \geqslant 0$. Let N be a labelled orientable bipartite map of face-type π. Then,*

$$\mathcal{G}^{(1)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r}) \cdot \frac{p_\pi}{m!} = \sum_M t^{|M|-|N|} \frac{q_{\lambda^+(M)} r_{\lambda^-(M)}}{|M|!}$$

where the sum is taken over all ways to add edges to N and to color faces, in order to obtain a BFC *orientable map M such that the added edges are of type 2.*

5.4. End of the combinatorial proof

Through this subsection, we fix once and for all an integer $\ell \geqslant 0$, a partition π and a labelled bipartite map N of face-type π. Our goal is to use Proposition 5.2 and Corollary 5.13 to prove that, for $\alpha = 1$, both sides of Equation (4.5) act in the same way on the weight of N given by $\frac{p_\pi}{|\pi|!} = \frac{p_{face-type(N)}}{|N|!}$. This would imply the commutation relation of Equation (4.5) since power-sum functions are a basis of $\mathcal{S}_\alpha$.

The following definition will be useful in the combinatorial proof of Equation (4.5); all along this subsection, $\mathcal{M}$ will be the (infinite) set of labelled BFC maps which are be obtained from N by

- adding one black vertex v and ℓ white vertex $w_1, w_2, \ldots, w_\ell$,
- adding some edges, such that each one of the new vertices is incident at least to one edge,
- choosing a color for each face,
- relabelling edges.

Moreover, in such a map we the added edges are marked.

Fix a BFC map M in $\mathcal{M}$. We denote by $\mathcal{E}_v(M)$ the set of edges incident to v in M, and we denote $\mathcal{T}_2(M\backslash N)$ the set of edges of type 2 in M not contained in N.

On one hand, we have

$$\ell!\left(\mathcal{C}_\ell^{(1)}(-t,\mathbf{q})+\mathcal{C}_\ell^{(1)}(-t,\mathbf{r})\right)\cdot\mathcal{G}^{(1)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\cdot\frac{p_\pi}{|N|!}$$
$$=\sum_{n\geqslant\ell}(-1)^n\sum_M t^{|M|-|N|}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!},\quad(5.5)$$

where the second sum is taken over BFC maps in $\mathcal{M}$ obtained from N as follows

- **Step 1:** We add edges to N and we color the faces of the obtained map, such that the added edges are of type 2.
- **Step 2:** We start by choosing either the $(+)$ or the $(-)$ part of the map, and we add a black vertex v of degree n and ℓ white vertices only connected to v, such that all added edges are incident to faces of the chosen color.

After each one of these operations we relabel all the edges.

Note that in this construction we can not obtain a map $M\in\mathcal{M}$ in two different ways, since all edges added in **Step 1** are bicolor while those added in **Step 2** are not. More precisely, the right-hand side of Equation (5.5) can be rewritten as follows

$$\sum_{M\in\mathcal{M}^{(1)}}(-1)^{|\mathcal{E}_v(M)|}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!},$$

where

$$\mathcal{M}^{(1)}:=\{M\in\mathcal{M}\text{ such that an added white vertex }w_i\text{ is only connected to }v$$
$$\text{and such that }\mathcal{E}(M\backslash N)=\mathcal{T}_2(M\backslash N)\uplus\mathcal{E}_v(M),\ \text{the union being disjoint.}\}$$

Actually, the following lemma allows us to simplify the definition of $\mathcal{M}^{(1)}$.

Lemma 5.14. *Fix a* BFC *map $M\in\mathcal{M}$. If*

$$\mathcal{E}(M\backslash N)=\mathcal{T}_2(M\backslash N)\uplus\mathcal{E}_v(M)\quad(5.6)$$

then the new white vertices w_i are all only connected to v.

Proof. Let us suppose that there exists an added white vertex w_i which is incident to a black vertex different from v. Since added edges are either incident to v or of type 2 (see Equation (5.6)), this implies that w_i is incident to a bicolor edge e, and by consequence is incident to faces of different colors. As e is of type 2, w_i can not have degree 1. Let e_1 and e_2 denote the edges forming a corner in w_i with e (e_1 and e_2 are not necessarily distinct). But since we can not have two consecutive edges of type 2 around a vertex, then e_1 and e_2 are both incident to v, and are besides incident to faces of different colors. We deduce that v is also incident to faces of different colors. By Remark 5.6 we get that v is incident to an edge of type 2, this contradicts the fact that the union is disjoint in Equation (5.6). □

We deduce that

$$\mathcal{M}^{(1)} := \{M \in \mathcal{M} \text{ such that } \mathcal{E}(M\backslash N) = \mathcal{T}_2(M\backslash N) \uplus \mathcal{E}_v(M)\}\,.$$

On the other hand, we have

$$\ell!\mathcal{G}^{(1)}(t,\mathbf{p},\mathbf{q},\mathbf{r}) \cdot \mathcal{C}_\ell^{(1)}(-t,\mathbf{p}) \cdot \frac{p_\pi}{|N|!} = \sum_{n\geqslant \ell}(-1)^n \sum_M t^{|M|-|N|}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!} \tag{5.7}$$

where the second sum runs over the set of labelled BFC maps obtained from N as follows;

- **Step 1'**: we add a black vertex v of degree n and ℓ white vertices only connected to v.
- **Step 2'**: we add edges and color faces such that the edges added in this step are bicolor and of type 2.

Note that all these maps are in the set

$$\mathcal{M}^{(2)} := \{M \in \mathcal{M} \text{ such that } \mathcal{E}(M\backslash N) = \mathcal{E}_v(M) \cup \mathcal{T}_2(M\backslash N), \\ \text{the union not necessarily disjoint}\}\,.$$

It is straightforward that $\mathcal{M}^{(1)} \subseteq \mathcal{M}^{(2)}$. Our goal is to prove that the total contribution of maps in $\mathcal{M}^{(2)}\backslash\mathcal{M}^{(1)}$ in Equation (5.7) is 0. Indeed, any map in $M \in \mathcal{M}^{(2)}$ contributes in Equation (5.7) with a coefficient

$$\sum_{\mathcal{E}(M\backslash N)=\mathcal{I}^{(1)}\uplus\mathcal{I}^{(2)}} (-1)^{|\mathcal{I}^{(1)}|}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!}, \tag{5.8}$$

where the sum runs over all possible ways to decompose the set edges of $M\backslash N$ into $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ such that $\mathcal{I}^{(1)} \subseteq \mathcal{E}_v(M)$ and $\mathcal{I}^{(2)} \subseteq \mathcal{T}_2(M)$. The only edges for which we have a choice (they can be either in $\mathcal{I}^{(1)}$ or in $\mathcal{I}^{(2)}$) are exactly the edges in $\mathcal{T}_2(M\backslash N) \cap \mathcal{E}_v(M)$. Let $r(M) := |\mathcal{T}_2(M\backslash N) \cap \mathcal{E}_v(M)|$. Then, Equation (5.8) can be rewritten as follows

$$\sum_{i=0}^{r(M)}(-1)^{i+(|M|-|N|-\mathcal{T}_2(M\backslash N))}\binom{r(M)}{i}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!} \\ = \begin{cases} 0 & \text{if } r(M) > 0 \\ (-1)^{|\mathcal{E}_v(M)|}\frac{q_{\lambda^+(M)}r_{\lambda^-(M)}}{|M|!} & \text{if } r(M) = 0. \end{cases} \tag{5.9}$$

This finishes the combinatorial proof of Theorem 1.6 for $\alpha = 1$.

6. Solution of the differential equation

The main purpose of this section is to solve the differential equation of the main theorem to give an explicit expression of the structure coefficients $g^{\pi}_{\mu,\nu}(\alpha)$, see Theorem 6.1. As a byproduct of this result we construct an algebra isomorphism between the space of symmetric functions and space of shifted symmetric functions (Corollary 6.3). Finally, we prove Theorem 1.8 in Section 6.3.

6.1. Explicit expression of coefficients $g^{\pi}_{\mu,\nu}$

We define the coefficients a^{λ}_{ξ} for any partitions λ and ξ by

$$a^{\lambda}_{\xi} := [t^{|\xi|}p_{\xi}]\left(\prod_{i=1}^{\ell(\lambda)}\alpha\lambda_i\mathcal{C}^{(\alpha)}_{\lambda_i}(t,\mathbf{p})\right)\cdot 1.$$

Note that by Theorem 4.7, the product in the last equation can be taken in any order. Using the combinatorial interpretation of the operators $\mathcal{C}^{(\alpha)}_{\ell}$ given in [BDD23] (see Section 5.1 for the case $\alpha = 1$), it is possible to give a combinatorial interpretation for the coefficients a^{λ}_{ξ} in terms of layered maps introduced in [BDD23].

We also consider the coefficients $d^{\lambda}_{\mu,\nu}$ defined by

$$d^{\lambda}_{\mu,\nu} := \sum_{\xi\cup\pi=\lambda} a^{\xi}_{\mu}a^{\pi}_{\nu} = [t^{|\mu|+|\nu|}q_{\mu}r_{\nu}]\prod_{1\leqslant i\leqslant \ell(\mu)}\alpha\lambda_i\left(\mathcal{C}^{(\alpha)}_{\lambda_i}(t,\mathbf{q})+\mathcal{C}^{(\alpha)}_{\lambda_i}(t,\mathbf{r})\right)\cdot 1,$$

where the sum is taken over all possible ways of grouping the parts of λ into two partitions ξ and π.

It follows from the definition of operators $\mathcal{C}^{(\alpha)}_{\ell}$ that

$$[t^k]\mathcal{C}^{(\alpha)}_{\ell}(t,\mathbf{p}) = \begin{cases} 0 & \text{if } k<\ell \\ p_{\ell}/(\alpha\ell) & \text{if } k=\ell, \end{cases} \tag{6.1}$$

see also Equation (6.10a) for details. We deduce that,

$$a^{\lambda}_{\xi} = \begin{cases} 0 & \text{if } |\xi|<|\lambda| \\ \delta_{\lambda,\xi} & \text{if } |\xi|=|\lambda|. \end{cases} \tag{6.2}$$

Similarly, $d^{\lambda}_{\mu,\nu} = 0$ if $|\mu|+|\nu|-|\lambda|<0$. We now state the main result of this section.

Theorem 6.1. *For any partitions λ, μ and ν we have*

$$g^{\lambda}_{\mu,\nu} = (-1)^{|\mu|+|\nu|-|\lambda|}\sum_{m\geqslant 0}(-1)^m\sum_{\substack{\pi_1,\dots,\pi_m\\|\lambda|<|\pi_1|<\dots<|\pi_m|}} a^{\lambda}_{\pi_1}a^{\pi_1}_{\pi_2}a^{\pi_2}_{\pi_3}\dots a^{\pi_{m-1}}_{\pi_m}d^{\pi_m}_{\mu,\nu}, \tag{6.3}$$

The term $m=0$ in the second sum is interpreted as $d^{\lambda}_{\mu,\nu}$.

Proof. We fix μ and ν, and we proceed by induction on $|\mu|+|\nu|-|\lambda|$. If $|\mu|+|\nu|-|\lambda|<0$ then the equality holds since $g^{\lambda}_{\mu,\nu}=0$ (see Lemma 3.1). Using Corollary 4.6, we write

$$\prod_{1\leqslant i\leqslant s}\left(\mathcal{C}^{(\alpha)}_{\lambda_i}(-t,\mathbf{q})+\mathcal{C}^{(\alpha)}_{\lambda_i}(-t,\mathbf{r})\right)\cdot\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\cdot 1=\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\prod_{1\leqslant i\leqslant s}\mathcal{C}^{(\alpha)}_{\lambda_i}(-t,\mathbf{p})\cdot 1.$$

But from Remark 2.2 and Lemma 3.1, we know that

$$\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\cdot 1=1. \tag{6.4}$$

Hence,

$$\prod_{1\leqslant i\leqslant s}\left(\mathcal{C}^{(\alpha)}_{\lambda_i}(-t,\mathbf{q})+\mathcal{C}^{(\alpha)}_{\lambda_i}(-t,\mathbf{r})\right)\cdot 1=\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\prod_{1\leqslant i\leqslant s}\mathcal{C}^{(\alpha)}_{\lambda_i}(-t,\mathbf{p})\cdot 1.$$

We multiply by $\prod_{1\leqslant i\leqslant 1}\alpha\lambda_i$ and we extract the coefficient of $[t^{|\mu|+|\nu|}q_\mu r_\nu]$. Using Equation (6.2) and Lemma 3.1 we obtain

$$\begin{aligned}(-1)^{|\mu|+|\nu|}d^{\lambda}_{\mu,\nu}&=\sum_{\substack{\kappa\\|\lambda|\leqslant|\kappa|\leqslant|\mu|+|\nu|}}(-1)^{|\kappa|}a^{\lambda}_{\kappa}g^{\kappa}_{\mu,\nu}\\&=(-1)^{|\lambda|}g^{\lambda}_{\mu,\nu}+\sum_{\substack{\kappa\\|\lambda|<|\kappa|\leqslant|\mu|+|\nu|}}(-1)^{|\kappa|}a^{\lambda}_{\kappa}g^{\kappa}_{\mu,\nu}.\end{aligned} \tag{6.5}$$

Using the induction hypothesis we get

$$\begin{aligned}g^{\lambda}_{\mu,\nu}=&(-1)^{|\mu|+|\nu|-|\lambda|}d^{\lambda}_{\mu,\nu}-\sum_{\substack{\kappa\\|\lambda|<|\kappa|\leqslant|\mu|+|\nu|}}(-1)^{|\kappa|-|\lambda|}a^{\lambda}_{\kappa}g^{\kappa}_{\mu,\nu}\\=&(-1)^{|\mu|+|\nu|-|\lambda|}d^{\lambda}_{\mu,\nu}-\\&\sum_{\substack{\kappa\\|\lambda|<|\kappa|\leqslant|\mu|+|\nu|}}(-1)^{|\mu|+|\nu|-|\lambda|}a^{\lambda}_{\kappa}\sum_{m\geqslant 0}(-1)^m\sum_{\substack{\pi_1,\dots,\pi_m\\|\kappa|<|\pi_1|<\dots<|\pi_m|}}a^{\kappa}_{\pi_1}a^{\pi_1}_{\pi_2}a^{\pi_2}_{\pi_3}\dots a^{\pi_{m-1}}_{\pi_m}d^{\pi_m}_{\mu,\nu}\\=&(-1)^{|\mu|+|\nu|-|\lambda|}\sum_{m\geqslant 0}(-1)^m\sum_{\substack{\pi_1,\dots,\pi_m\\|\lambda|<|\pi_1|<\dots<|\pi_m|}}a^{\lambda}_{\pi_1}a^{\pi_1}_{\pi_2}a^{\pi_2}_{\pi_3}\dots a^{\pi_{m-1}}_{\pi_m}d^{\pi_m}_{\mu,\nu}.\end{aligned} \tag{6.6}$$

□

We recall that we define $\mathcal{G}^{(\alpha)}_i$ is the homogeneous part of degree i in $\mathcal{G}^{(\alpha)}$ (see also Equation (1.8)). We deduce from the last proof the following proposition.

Proposition 6.2. *Fix $n\geqslant 0$. Then, Equation (6.4) and equations*

$$[t^k]\left(\mathcal{C}^{(\alpha)}_{\ell}(-t,\mathbf{q})+\mathcal{C}^{(\alpha)}_{\ell}(-t,\mathbf{r})\right)\cdot\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})=[t^k]\mathcal{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})\cdot\mathcal{C}^{(\alpha)}_{\ell}(-t,\mathbf{p}), \tag{6.7}$$

for $\ell\geqslant 1$ and $\ell\leqslant k\leqslant n+\ell$, characterize the operators $\mathcal{G}^{(\alpha)}_i$ for $i\leqslant n$.

Note that by definition, the lowest term in $\mathcal{C}^{(\alpha)}_{\ell}$ as a formal power-series in t has degree ℓ. Hence, the previous equations involve only operators $\mathcal{G}^{(\alpha)}_i$ for $i\leqslant n$.

Proof. Fix $n \geqslant 0$. First notice that Eqs. (6.7) imply by induction that for any partition $\lambda = [\lambda_1, \dots, \lambda_s]$ for any $|\lambda| \leqslant k \leqslant |\lambda| + n$, we have

$$[t^k] \prod_{1 \leqslant i \leqslant s} \left(\mathcal{C}^{(\alpha)}_{\lambda_i}(-t, \mathbf{q}) + \mathcal{C}^{(\alpha)}_{\lambda_i}(-t, \mathbf{r}) \right) \cdot \mathcal{G}^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r}) = [t^k] \mathcal{G}^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \mathbf{r}) \cdot \prod_{1 \leqslant i \leqslant s} \mathcal{C}^{(\alpha)}_{\lambda_i}(-t, \mathbf{p}).$$

In the proof of Theorem 6.1 these equations are used to obtain the explicit formula (6.3) of $g^{\lambda}_{\mu,\nu}$ for $0 \leqslant |\mu| + |\nu| - |\lambda| \leqslant n$. In particular, they characterize operators $\mathcal{G}^{(\alpha)}_0, \dots, \mathcal{G}^{(\alpha)}_n$. □

Actually, the operators $\mathcal{C}^{(\alpha)}_\ell$ for $\ell \geqslant 1$, can be generated using only operators $\mathcal{C}^{(\alpha)}_0$ and $\mathcal{C}^{(\alpha)}_1$ (see Theorem 4.7). One can use this result to show that Equations (6.7) for $\ell \in \{0, 1\}$ and $1 \leqslant k \leqslant n + 1$ also characterize the operators $\mathcal{G}^{(\alpha)}_i$ for $i \leqslant n$. In particular, each operator $\mathcal{G}^{(\alpha)}_i$ is characterized by finitely many equations.

6.2. $g^{\pi}_{\mu,\nu}$ as structure coefficients of symmetric functions

We denote for every μ

$$A^{(\alpha)}_\mu := \sum_{\lambda} (-1)^{|\mu|} a^{\lambda}_{\mu} p_\lambda.$$

With this definition, multiplying Equation (6.5) by p_λ and summing over all λ gives

$$A^{(\alpha)}_\mu \cdot A^{(\alpha)}_\nu = \sum_{\kappa} g^{\kappa}_{\mu,\nu} A^{(\alpha)}_\kappa.$$

When p_λ is thought of as a power-sum symmetric function in an underlying alphabet $\mathbf{x}$ (see Section 2.2), $A^{(\alpha)}_\mu$ becomes a symmetric function. We then have the following corollary.

Corollary 6.3. *The map*

$$\begin{array}{ccc} \mathcal{S}_\alpha & \longrightarrow & \mathcal{S}^*_\alpha \\ A^{(\alpha)}_\mu & \longmapsto & \theta^{(\alpha)}_\mu \end{array}$$

is an algebra isomorphism between $\mathcal{S}_\alpha$ and $\mathcal{S}^*_\alpha$.

For $\alpha = 1$, such an isomorphism has been obtained in [CGS04] using a different approach.

6.3. A differential expression for the lower terms of $\mathcal{G}^{(\alpha)}$

In this section we prove Theorem 1.8. This proof represents no difficulty but involves some lengthy computation. For any $\ell, k \geqslant 0$, we consider the operator

$$\mathcal{C}^{(\alpha)}_{\ell,k}(\mathbf{p}) = (\ell + k)[t^{k+\ell}] \mathcal{C}^{(\alpha)}_\ell(t, \mathbf{p}).$$

If $\mathcal{X}$ and $\mathcal{X}'$ are two vector spaces, we denote by $\mathcal{O}(\mathcal{X}, \mathcal{X}')$ the space of linear operators from $\mathcal{X}$ to $\mathcal{X}'$. We also set $\mathcal{O}(\mathcal{X}) := \mathcal{O}(\mathcal{X}, \mathcal{X})$. Let O be an operator in one alphabet such that $O(\mathbf{p}) \in \mathcal{O}\left(\mathbb{Q}(\alpha)[\mathbf{p}]\right)$, and let $P(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathcal{O}\left(\mathbb{Q}(\alpha)[\mathbf{p}], \mathbb{Q}(\alpha)[\mathbf{q}, \mathbf{r}]\right)$. We introduce their *three alphabet commutator* $[O, P]^{\mathbf{q},\mathbf{r}}_{\mathbf{p}} \in \mathcal{O}\left(\mathbb{Q}(\alpha)[\mathbf{p}], \mathbb{Q}(\alpha)[\mathbf{q}, \mathbf{r}]\right)$ defined by

$$[P, O]^{\mathbf{q},\mathbf{r}}_{\mathbf{p}} := P(\mathbf{p}, \mathbf{q}, \mathbf{r}) \cdot O(\mathbf{p}) - \left(O(\mathbf{q}) + O(\mathbf{r})\right) \cdot P(\mathbf{p}, \mathbf{q}, \mathbf{r}).$$

If $Q_1 \in \mathcal{O}(\mathbb{Q}(\alpha)[\mathbf{p}])$ and $Q_2 \in \mathcal{O}(\mathbb{Q}(\alpha)[\mathbf{q},\mathbf{r}])$, it is easy to check that this commutator satisfies the relation

$$[Q_2 \cdot P \cdot Q_1, O]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = [Q_2(\mathbf{q},\mathbf{r}), O(\mathbf{q})] \cdot P \cdot Q_1 + [Q_2(\mathbf{q},\mathbf{r}), O(\mathbf{r})] \cdot P \cdot Q_1 \\ + Q_2(\mathbf{q},\mathbf{r}) \cdot [P, O]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} \cdot Q_1(\mathbf{p}) + Q_2(\mathbf{q},\mathbf{r}) \cdot P \cdot [Q_1(\mathbf{p}), O(\mathbf{p})]\,. \tag{6.8}$$

We denote by $\widetilde{\mathcal{G}}_0^{(\alpha)}, \widetilde{\mathcal{G}}_1^{(\alpha)}, \widetilde{\mathcal{G}}_2^{(\alpha)}$ the differential operators given respectively by the right-hand sides of Equations (1.9) to (1.11). Our goal is to prove that $\mathcal{G}_i^{(\alpha)} = \widetilde{\mathcal{G}}_i^{(\alpha)}$ for $0 \leqslant i \leqslant 2$. Applying Proposition 6.2, it is enough to show that for any $\ell \geqslant 1$ and $0 \leqslant i \leqslant 2$

$$\sum_{0 \leqslant j \leqslant i} \frac{(-1)^{\ell+j}}{\ell+j} \left[\widetilde{\mathcal{G}}_{i-j}^{(\alpha)}, \mathcal{C}_{\ell,j}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = 0 \tag{6.9}$$

The following lemma gives a differential expression for the operators $\mathcal{C}_{\ell,j}^{(\alpha)}$ for $j \leqslant 2$, which does not involve the alphabet Y. The proof is given in Appendix A.

Lemma 6.4. *For any $\ell \geqslant 1$, we have*

$$\mathcal{C}_{\ell,0}^{(\alpha)} = p_\ell/\alpha, \tag{6.10a}$$

$$\mathcal{C}_{\ell,1}^{(\alpha)} = \binom{\ell+1}{2}\frac{b}{\alpha} \cdot p_{\ell+1} + (\ell+1)\sum_{m\geqslant 1} p_{m+\ell+1}\frac{m\partial}{\partial p_m} + \frac{\ell+1}{2\alpha}\sum_{1\leqslant i\leqslant \ell} p_i p_{\ell+1-i}, \tag{6.10b}$$

$$\begin{aligned}\mathcal{C}_{\ell,2}^{(\alpha)} = {}& \frac{1}{3\alpha}\binom{\ell+2}{2}\sum_{\substack{i_1+i_2+i_3=\ell+2\\ i_1,i_2,i_3\geqslant 1}} p_{i_1}p_{i_2}p_{i_3} + \binom{\ell+2}{3}\frac{(3\ell+5)b^2}{4\alpha}p_{\ell+2} \\ &+ \alpha\binom{\ell+2}{2}\sum_{k,m\geqslant 1} p_{\ell+k+m+2}\frac{m\partial}{\partial p_m}\frac{k\partial}{\partial p_k} + \binom{\ell+2}{2}\sum_{m\geqslant 1} b(\ell+m+1)p_{m+\ell+2}\frac{m\partial}{\partial p_m} \\ &+ \sum_{\substack{i_1+i_2=\ell+2\\ i_1,i_2\geqslant 1}} \frac{b\cdot(\ell+2)\left((\ell+1)^2 - i_1 i_2\right)}{2\alpha}p_{i_1}p_{i_2} + \binom{\ell+3}{4}p_{\ell+2} \\ &+ \binom{\ell+2}{2}\sum_{m\geqslant 1}\sum_{\substack{i_1+i_2=\ell+m+2\\ i_1,i_2\geqslant 1}} p_{i_1}p_{i_2}\frac{m\partial}{\partial p_m}.\end{aligned} \tag{6.10c}$$

In the following lemma we establish some useful commutation relations for the operator Ψ.

Lemma 6.5. *We have the following relations between operators in $\mathcal{O}(\mathbb{Q}(\alpha)[\mathbf{p}], \mathbb{Q}(\alpha)[\mathbf{q},\mathbf{r}])$*

$$\Psi \cdot p_\ell = (q_\ell + r_\ell)\cdot \Psi, \quad \textit{i.e.} \quad [\Psi, p_\ell]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = 0$$

$$\Psi \cdot \frac{\partial}{\partial p_\ell} = \frac{\partial}{\partial q_\ell}\cdot\Psi = \frac{\partial}{\partial r_\ell}\cdot\Psi.$$

Moreover,

$$\left[\Psi, p_i\frac{\partial}{\partial p_j}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = \left[\Psi, p_i\frac{\partial}{\partial p_{j_1}}\frac{\partial}{\partial p_{j_2}}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = 0,$$

$$[\Psi, p_{i_1}p_{i_2}]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = (q_{i_1}r_{i_2} + r_{i_1}q_{i_2})\Psi,$$

$$\left[\Psi, p_{i_1}p_{i_2}\frac{\partial}{\partial p_j}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = q_{i_1}r_{i_2}\Psi\frac{\partial}{\partial p_j} + r_{i_1}q_{i_2}\Psi\frac{\partial}{\partial p_j}.$$

Proof. The first equation is immediate from the definition of the operator Ψ. Let us show the second equation. Fix a partition λ. If $m_\ell(\lambda) = 0$ then

$$\Psi \cdot \frac{\partial}{\partial p_\ell}p_\lambda = \frac{\partial}{\partial q_\ell} \cdot \Psi p_\lambda = 0.$$

Otherwise, we denote by μ the partition obtained from λ by erasing a part of size ℓ. Then

$$\Psi \cdot \frac{\partial}{\partial p_\ell}p_\lambda = m_\ell(\lambda)\Psi p_\mu = m_\ell(\lambda)\prod_{i\in\mu}(q_i + r_i).$$

On the other hand,

$$\frac{\partial}{\partial q_\ell} \cdot \Psi p_\lambda = \frac{\partial}{\partial q_\ell}\prod_{i\in\lambda}(q_i + r_i) = m_\ell(\lambda)\prod_{i\in\mu}(q_i + r_i).$$

The last three equations follow from the first ones. □

We now prove Theorem 1.8.

Proof of Theorem 1.8. From Lemma 6.5 and Equations (6.10a) and (6.10b), we have

$$\frac{1}{\ell}\left[\widetilde{\mathcal{G}}_0^{(\alpha)}, \mathcal{C}_{\ell,0}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = 0,$$

$$\frac{1}{\ell+1}\left[\widetilde{\mathcal{G}}_0^{(\alpha)}, \mathcal{C}_{\ell,1}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = \frac{1}{\alpha}\sum_{\substack{m_1+m_2=\ell+1\\ m_1,m_2\geqslant 1}} q_{m_1}r_{m_2}\Psi = \frac{1}{\ell}\left[\widetilde{\mathcal{G}}_1^{(\alpha)}, \mathcal{C}_{\ell,0}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}}.$$

These two equations together with Proposition 6.2 give Equations (1.9) and (1.10). Let us now prove Equation (1.11). Using Lemma 6.5 and Equation (6.10c), we get

$$\begin{aligned}\frac{1}{\ell+2}\left[\widetilde{\mathcal{G}}_0^{(\alpha)}, \mathcal{C}_{\ell,2}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} =& \frac{\ell+1}{2\alpha}\sum_{\substack{i_1+i_2+i_3=\ell+2\\ i_1,i_2,i_3\geqslant 1}}(q_{i_1}q_{i_2}r_{i_3} + q_{i_1}r_{i_2}r_{i_3})\cdot\Psi\\ &+ \sum_{\substack{i_1+i_2=\ell+2\\ i_1,i_3\geqslant 1}}\frac{b\cdot((\ell+1)^2 - i_1i_2)}{2\alpha}q_{i_1}r_{i_2}\cdot\Psi\\ &+ (\ell+1)\sum_{m\geqslant 1}\sum_{\substack{i_1+i_2=\ell+m+2\\ i_1,i_2\geqslant 1}} q_{i_1}r_{i_2}\cdot\Psi\cdot\frac{m\partial}{\partial p_m}.\end{aligned}$$

Moreover,

$$\frac{1}{\ell}\left[\widetilde{\mathcal{G}}_2^{(\alpha)},\mathcal{C}_{\ell,0}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = \frac{1}{2\alpha}\sum_{\substack{m_1+m_2=\ell+2\\ m_1,m_2\geqslant 1}} b(m_1-1)(m_2-1)q_{m_1}r_{m_2}\Psi$$
$$+\frac{1}{2\alpha}\sum_{\substack{m_1+m_2+m_3=\ell+2\\ m_1,m_2,m_3\geqslant 1}}(m_1-1)(q_{m_1}r_{m_2}r_{m_3}+r_{m_1}q_{m_2}q_{m_3})\Psi$$
$$+\sum_{m\geqslant 1}\sum_{\substack{i_1+i_2=m+\ell+2\\ i_1,i_2\geqslant 1}}\min(\ell,m,i_1-1,i_2-1)q_{i_1}r_{i_2}\Psi\frac{m\partial}{\partial p_m}$$
$$+\frac{1}{\alpha}\sum_{m\geqslant 1}\sum_{\substack{k_1+k_2=\ell+1\\ k_1,k_2\geqslant 1}}\sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geqslant 1}} q_{m_1}q_{k_1}r_{m_2}r_{k_2}\Psi\frac{m\partial}{\partial p_m}.$$

Applying Equation (6.8), we get

$$\left[\widetilde{\mathcal{G}}_1^{(\alpha)},\mathcal{C}_{\ell,1}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = \sum_{m\geqslant 1}\sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geqslant 1}}\Bigg([q_{m_1}r_{m_2},\mathcal{C}_{\ell,1}^{(\alpha)}(\mathbf{q})]\cdot\Psi\cdot\frac{m\partial}{\partial p_m}+[q_{m_1}r_{m_2},\mathcal{C}_{\ell,1}^{(\alpha)}(\mathbf{r})]\cdot\Psi\cdot\frac{m\partial}{\partial p_m}$$
$$+q_{m_1}r_{m_2}\cdot\left[\Psi,\mathcal{C}_{\ell,1}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}}\cdot\frac{m\partial}{\partial p_m}+q_{m_1}r_{m_2}\cdot\Psi\cdot\left[\frac{m\partial}{\partial p_m},\mathcal{C}_{\ell,1}^{(\alpha)}(\mathbf{p})\right]\Bigg).$$

Hence,

$$\frac{1}{\ell+1}\left[\widetilde{\mathcal{G}}_1^{(\alpha)},\mathcal{C}_{\ell,1}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}} = \sum_{m\geqslant 1}\sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geqslant 1}}\Bigg(-(m_1q_{m_1+\ell+1}r_{m_2}+m_2q_{m_1}r_{m_2+\ell+1})\cdot\Psi\cdot\frac{m\partial}{\partial p_m}$$
$$+\frac{1}{\alpha}q_{m_1}r_{m_2}\sum_{\substack{k_1+k_2=\ell+1\\ k_1,k_2\geqslant 1}}q_{k_1}r_{k_2}\Psi\frac{m\partial}{\partial p_m}\Bigg)+\frac{b(\ell+1)\ell}{2\alpha}\sum_{\substack{m_1+m_2=\ell+2\\ m_1,m_2\geqslant 1}}q_{m_1}r_{m_2}\cdot\Psi$$
$$+\sum_{m\geqslant 1}\sum_{\substack{m_1+m_2=m+\ell+2\\ m_1,m_2\geqslant 1}}(m+\ell+1)q_{m_1}r_{m_2}\cdot\Psi\cdot\frac{m\partial}{\partial p_m}$$
$$+\frac{1}{\alpha}\sum_{\substack{m_1+m_2+m_3=\ell+2\\ m_1,m_2,m_3\geqslant 1}}(m_1+m_2-1)(q_{m_1}r_{m_2}q_{m_3}+q_{m_1}r_{m_2}r_{m_3})\cdot\Psi.$$

The last sum can be symmetrized as follows

$$\frac{1}{2\alpha}\sum_{\substack{m_1+m_2+m_3=\ell+2\\ m_1,m_2,m_3\geqslant 1}}(2m_1+m_2+m_3-2)(r_{m_1}q_{m_2}q_{m_3}+q_{m_1}r_{m_2}r_{m_3})\cdot\Psi.$$

One may also notice that for any $m,\ell,i_1,i_2\geqslant 1$, such that $i_1+i_2=m+\ell+2$ we have

$$m-\mathbb{1}_{i_1\geqslant\ell+2}(i_1-\ell-1)-\mathbb{1}_{i_2\geqslant\ell+2}(i_2-\ell-1)=\min(m,\ell,i_1-1,i_2-1).$$

Using these two remarks and combining the three equations above, we get

$$\frac{1}{\ell+2}\left[\widetilde{\mathcal{G}}_0^{(\alpha)},\mathcal{C}_{\ell,2}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}}-\frac{1}{\ell+1}\left[\widetilde{\mathcal{G}}_1^{(\alpha)},\mathcal{C}_{\ell,1}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}}+\frac{1}{\ell}\left[\widetilde{\mathcal{G}}_2^{(\alpha)},\mathcal{C}_{\ell,0}^{(\alpha)}\right]_{\mathbf{p}}^{\mathbf{q},\mathbf{r}}=0.$$

which gives Equation (6.9) for $n=2$ and finishes the proof of the theorem. □

7. Equations for connected series

In this section we consider a connected version $\widehat{G}^{(\alpha)}$ of the series $G^{(\alpha)}$. We establish some general properties about the series $\widehat{G}^{(\alpha)}$ and we derive from the main theorem a family of differential equation for this series.

7.1. Connected series

We introduce the two series

$$\widehat{G}\equiv\widehat{G}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}):=\alpha\cdot\log(G^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})),$$

and

$$\widehat{\tau}\equiv\widehat{\tau}^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r}):=\alpha\cdot\log(\tau^{(\alpha)}(t,\mathbf{p},\mathbf{q},\mathbf{r})),$$

where τ^{α} is the series defined in Equation (3.3). The series $\widehat{G}$ and $\widehat{\tau}$ are well defined in the space $\mathbb{Q}(\alpha)[t,\mathbf{p}][\![\mathbf{q},\mathbf{r}]\!]\cap\mathbb{Q}(\alpha)[t,\mathbf{q},\mathbf{r}][\![\mathbf{p}]\!]$, since

$$[p_\varnothing]G^{(\alpha)}=[p_\varnothing]\,\widehat{\tau}^{(\alpha)}=[q_\varnothing r_\varnothing]G^{(\alpha)}=[q_\varnothing r_\varnothing]\tau^{(\alpha)}=1,$$

where $\varnothing$ denotes the empty integer partition (see Lemma 3.1).

By Proposition 1.4 and Theorem 3.2, the series $\tau^{(\alpha)}$ (resp. $\widehat{G}^{(\alpha)}$) is a generating series of connected hypermaps (resp. connected hypermaps with marked faces) when $\alpha\in\{1,2\}$. These two series are related by a variant of Equation (3.8).

Lemma 7.1. *We have,*

$$\widehat{G}(t,\mathbf{p},\mathbf{q},\mathbf{r})=-p_1/t+\exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\widehat{\tau}(t,\mathbf{p},\mathbf{q},\mathbf{r}).$$

Proof. First, notice that the operator $\exp\left(\frac{\partial}{t\partial q_1}\right)$ is well defined on $\mathbb{Q}(\alpha)[\mathbf{q},\mathbf{r},t,1/t][\![\mathbf{p}]\!]$, and for any $A,B\in\mathbb{Q}(\alpha)[\mathbf{q},\mathbf{r},t,1/t][\![\mathbf{p}]\!]$,

$$\exp\left(\frac{\partial}{t\partial q_1}\right)\cdot(AB)=\left(\exp\left(\frac{\partial}{t\partial q_1}\right)\cdot A\right)\left(\exp\left(\frac{\partial}{t\partial q_1}\right)\cdot B\right).$$

Since $\widehat{\tau}(t,\mathbf{p},\mathbf{q},\mathbf{r}) \in \mathbb{Q}(\alpha)[\mathbf{q},\mathbf{r},t][\![\mathbf{p}]\!] \subset \mathbb{Q}(\alpha)[\mathbf{q},\mathbf{r},t,1/t][\![\mathbf{p}]\!]$, we get

$$\begin{aligned}\exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\cdot\tau(t,\mathbf{p},\mathbf{q},\mathbf{r}) &= \exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\cdot\sum_{k\geqslant 0}\frac{\widehat{\tau}(t,\mathbf{p},\mathbf{q},\mathbf{r})^k}{\alpha^k k!}\\ &= \sum_{k\geqslant 0}\frac{1}{k!}\left(\exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\cdot\frac{\widehat{\tau}(t,\mathbf{p},\mathbf{q},\mathbf{r})}{\alpha}\right)^k\\ &= \exp\left(\exp\left(\frac{\partial}{t\partial q_1}\right)\exp\left(\frac{\partial}{t\partial r_1}\right)\cdot\frac{\widehat{\tau}(t,\mathbf{p},\mathbf{q},\mathbf{r})}{\alpha}\right).\end{aligned}$$

We conclude by substituting this formula in Equation (3.8). □

The coefficients $h^{\pi}_{\mu,\nu}$, introduced in [GJ96a] are defined by

$$\widehat{\tau}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_{n\geqslant 1} t^n \sum_{\pi,\mu,\nu\vdash n}\frac{h^{\pi}_{\mu,\nu}(\alpha)}{n}p_\pi q_\mu r_\nu.$$

The coefficients $h^{\pi}_{\mu,\nu}$ are the object of the hypermaps-Jack conjecture (known also as the b-conjecture); see [GJ96a, Conjecture 6.3]. Similarly, we consider the coefficients $\hat{g}^{\pi}_{\mu,\nu}(\alpha)$

$$\widehat{G}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = \sum_{\pi,\mu,\nu}\frac{\hat{g}^{\pi}_{\mu,\nu}(\alpha)}{|\pi|}t^{|\mu|+|\nu|-|\pi|}p_\pi q_\mu r_\nu.$$

The following polynomiality result is due to Féray and Dołęga.

Theorem 7.2 ([DF17])**.** *For any partitions $\pi,\mu,\nu \vdash n \geqslant 1$, the coefficient $h^{\pi}_{\mu,\nu}$ is polynomial in b and*

$$\deg(h^{\pi}_{\mu,\nu}) \leqslant n + 2 - (\ell(\pi)+\ell(\mu)+\ell(\nu)).$$

We deduce the following corollary.

Corollary 7.3. *For any partitions π, μ and ν, the coefficients $\hat{g}^{\pi}_{\mu,\nu}$ is polynomial in b, and*

$$\deg(\hat{g}^{\pi}_{\mu,\nu}) \leqslant 2 + |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| + \ell(\pi)).$$

Proof. From Lemma 7.1, we get that for any $|\pi| \geqslant \max(|\mu|,|\nu|)$, the coefficient $\hat{g}^{\pi}_{\mu,\nu}$ is given by

$$\hat{g}^{\pi}_{\mu,\nu} = \begin{cases} 0 & \text{if } (\pi,\mu,\nu) = ([1],[0],[0])\\ \binom{m_1(\mu)+|\pi|-|\mu|}{m_1(\mu)}\binom{m_1(\nu)+|\pi|-|\nu|}{m_1(\nu)}h^{\pi}_{\mu\cup 1^{|\pi|-|\mu|},\nu\cup 1^{|\pi|-|\nu|}}, & \text{otherwise.}\end{cases}$$

From Theorem 7.2, we get that $\hat{g}^{\pi}_{\mu,\nu}$ is polynomial and

$$\begin{aligned}\deg\left(\hat{g}^{\pi}_{\mu,\nu}\right) &= \deg(h^{\pi}_{\mu\cup 1^{|\pi|-|\mu|},\nu\cup 1^{|\pi|-|\pi|}})\\ &\leqslant |\pi| + 2 - \left(\ell(\pi) + \ell\left(\mu\cup 1^{|\pi|-|\mu|}\right) + \ell\left(\nu\cup 1^{|\pi|-|\nu|}\right)\right)\\ &= 2 + |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| + \ell(\pi)).\end{aligned}$$
□

7.2. Dual operators

The purpose of this subsection is to give a differential expression of dual operators $\mathcal{B}_n^{(\alpha)}$. For this, we introduce the scalar product $\langle,\rangle_Y$ on $\widetilde{\mathcal{P}}_Y$ (see Section 4.1), defined by

$$\begin{cases} \langle p_\lambda, p_\mu\rangle_Y = \delta_{\lambda,\mu}\alpha^{\ell(\lambda)}z_\lambda = \langle p_\lambda, p_\mu\rangle; \\ \langle p_\lambda, y_i p_\mu\rangle_Y = 0; \\ \langle y_i p_\lambda, y_j p_\mu\rangle_Y = \delta_{i,j}\delta_{\lambda,\mu}\alpha^{\ell(\lambda)+1}z_\lambda, \end{cases}$$

for any $\lambda, \mu \in \mathbb{Y}$ and $i, j \geqslant 0$.

If A_Y is an operator on $\mathcal{P}_Y$, then we denote by $A_Y^\perp$ its dual operator with respect to $\langle,\rangle_Y$. We deduce from the definitions the following differential expressions for the catalytic operators;

$$(y_i)^\perp = \frac{\alpha\partial}{\partial y_i}, \text{ for any } i \geqslant 0.$$

$$Y_+^\perp = \sum_{i\geqslant 2} y_{i-1}\frac{\partial}{\partial y_i},$$

$$\Theta_Y^\perp(\mathbf{p}) = \sum_{i\geqslant 1} y_i \frac{i\partial}{\partial p_i},$$

$$\Gamma_Y^\perp(\mathbf{p}) = \sum_{i,j\geqslant 1} y_{i-1}p_j\frac{\partial}{\partial y_{i+j}} + (1+b)\cdot\sum_{i,j\geqslant 1} y_{i+j-1}\frac{j\partial^2}{\partial y_i \partial p_j} + b\cdot\sum_{i\geqslant 2} y_{i-1}\frac{(i-1)\partial}{\partial y_i},$$

$$\mathcal{B}_n^\perp(\mathbf{p}, u) = \frac{\partial}{\partial y_0}\left(\Gamma_Y^\perp + uY_+^\perp\right)^n \Theta_Y^\perp. \tag{7.1}$$

We recall that $b := \alpha - 1$.

7.3. Differential equation for the series of connected maps

We denote for each $m \geqslant 1$

$$\widehat{G}_{\mathbf{p}}^{[m]} = \frac{m\partial}{\partial p_m}\widehat{G}, \qquad \widehat{G}_{\mathbf{q}}^{[m]} = \frac{m\partial}{\partial q_m}\widehat{G}, \qquad \widehat{G}_{\mathbf{r}}^{[m]} = \frac{m\partial}{\partial r_m}\widehat{G}.$$

Proposition 7.4. *Fix $n \geqslant 1$. Then, we have the equality between operators in $\mathcal{O}(\mathcal{A})$*

$$\mathcal{B}_n^{(\alpha)}(\mathbf{q}, u)\cdot G^{(\alpha)} = G^{(\alpha)}.\Theta_Y(\mathbf{q})\left(\Gamma_Y(\mathbf{q}) + uY_+ + \sum_{i,j\geqslant 1} y_{i+j}\frac{\partial}{\partial y_{i-1}}\widehat{G}_{\mathbf{q}}^{[j]}\right)^n \frac{y_0}{1+b}.$$

Here, $G^{(\alpha)}$ acts on $\mathcal{A}$ by multiplication. Similarly,

$$\mathcal{B}_n^{(\alpha)\perp}(\mathbf{p}, u)\cdot G^{(\alpha)} \\ = G^{(\alpha)}\cdot\frac{\partial}{\partial y_0}\left(\Gamma_Y^\perp(\mathbf{p}) + uY_+^\perp + \sum_{i,j\geqslant 1} y_{i+j-1}\frac{\partial}{\partial y_i}\widehat{G}_{\mathbf{p}}^{[j]}\right)^n\left(\Theta_Y^\perp(\mathbf{p}) + \sum_{i\geqslant 1} y_i\,\widehat{G}_{\mathbf{p}}^{[i]}\right).$$

Proof. This is a consequence of the catalytic differential expressions of operators $\mathcal{B}_n^{(\alpha)}$ and $\mathcal{B}_n^{(\alpha)\perp}$ given resp. in Equation (4.1) and Equation (7.1). We also use the fact that

$$\left[(1+b)\frac{j\partial}{\partial p_j}, G^{(\alpha)}\right] = (1+b)\frac{j\partial G^{(\alpha)}}{\partial p_j} = G^{(\alpha)}.\, \widehat{G}^{[j]}_{\mathbf{p}} \quad \text{and} \quad \left[\frac{\partial}{\partial y_i}, G^{(\alpha)}\right] = 0. \qquad \square$$

We deduce from Theorem 1.5 and Proposition 7.4 the following theorem.

Theorem 7.5. *The series $\widehat{G}$ satisfies the following differential equation:*

$$\sum_{n\geqslant 1}\frac{(-1)^n}{n}\frac{\partial}{\partial y_0}\left(\Gamma_Y^{\perp}(\mathbf{p}) + uY_+^{\perp} + \sum_{i,j\geqslant 1} y_{i+j-1}\frac{\partial}{\partial y_i}\,\widehat{G}^{[j]}_{\mathbf{p}}\right)^n\left(\sum_{i\geqslant 1} y_i\,\widehat{G}^{[i]}_{\mathbf{p}}\right)\cdot 1$$
$$= \sum_{n\geqslant 1}\frac{(-1)^n}{n}\Theta_Y(\mathbf{q})\left(\Gamma_Y(\mathbf{q}) + uY_+ + \sum_{i,j\geqslant 1} y_{i+j}\frac{\partial}{\partial y_{i-1}}\,\widehat{G}^{[j]}_{\mathbf{q}}\right)^n\frac{y_0}{1+b}\cdot 1$$
$$+ \sum_{n\geqslant 1}\frac{(-1)^n}{n}\Theta_Y(\mathbf{r})\left(\Gamma_Y(\mathbf{r}) + uY_+ + \sum_{i,j\geqslant 1} y_{i+j}\frac{\partial}{\partial y_{i-1}}\,\widehat{G}^{[j]}_{\mathbf{r}}\right)^n\frac{y_0}{1+b}\cdot 1.$$

A. Proof of Lemma 6.4

We prove in this appendix Lemma 6.4. In order to obtain an explicit differential expression for operators $\mathcal{C}_0^{(\alpha)}$, $\mathcal{C}_1^{(\alpha)}$ and $\mathcal{C}_2^{(\alpha)}$ we develop the catalytic expressions given in Section 4.1 for these operators. Since the computations are lengthy for the operator $\mathcal{C}_2^{(\alpha)}$, we explain the important steps of the proof without giving all the details.

It is direct from the definitions that for any $i \geqslant 0$ we have

$$Y_+^i\frac{y_0}{\alpha} = \frac{y_i}{\alpha}, \quad \text{as operators on } \mathcal{P}_Y.$$

We apply Θ_Y to obtain Equation (6.10a). By applying Γ_Y on the last equation we get;

$$\Gamma_Y Y_+^i\frac{y_0}{\alpha} = \frac{ib}{\alpha}y_{i+1} + \sum_{m\geqslant 1} y_{i+m+1}\frac{m\partial}{\partial p_m} + \frac{1}{\alpha}\sum_{1\leqslant j\leqslant i} y_{i-j+1}p_j,$$

Hence, for any $i_1, i_2 \geqslant 0$, we have

$$Y_+^{i_2}\Gamma_Y Y_+^{i_1}\frac{y_0}{\alpha} = \frac{i_1 b}{\alpha}y_{i_1+i_2+1} + \sum_{m\geqslant 1} y_{i_1+i_2+m+1}\frac{m\partial}{\partial p_m} + \frac{1}{\alpha}\sum_{1\leqslant j\leqslant i_1} y_{i_1+i_2-j+1}p_j. \tag{A.1}$$

We apply Θ_Y and we sum over all tuples (i_1, i_2) such that $i_1 + i_2 = \ell$ to obtain

$$\mathcal{C}_{\ell,1}^{(\alpha)} = \frac{b}{\alpha}p_{\ell+1}\sum_{\substack{i_1+i_2=\ell\\ i_1,i_2\geqslant 0}} i_1 + (\ell+1)\sum_{m\geqslant 1} p_{\ell+m+1}\frac{m\partial}{\partial p_m} + \frac{1}{\alpha}\sum_{\substack{j_2+j_2=\ell+1\\ j_1,j_2\geqslant 1}} p_{j_1}p_{j_2}\sum_{\substack{i_1+i_2=\ell\\ i_1,i_2\geqslant 0}}\mathbb{1}_{i_1\geqslant j_1}$$
$$= \frac{b}{\alpha}\binom{\ell+2}{2}p_{\ell+1} + (\ell+1)\sum_{m\geqslant 1} p_{\ell+m+1}\frac{m\partial}{\partial p_m} + \frac{1}{\alpha}\sum_{\substack{j_2+j_2=\ell+1\\ j_1,j_2\geqslant 1}} p_{j_1}p_{j_2}j_2.$$

In order to obtain Equation (6.10b), we *symmetrize* the last sum with respect to (j_1, j_2);

$$\sum_{\substack{j_2+j_2=\ell+1\\ j_1,j_2\geqslant 1}} p_{j_1}p_{j_2}j_2 = \frac{1}{2}\left(\sum_{\substack{j_2+j_2=\ell+1\\ j_1,j_2\geqslant 1}} p_{j_1}p_{j_2}j_2 + \sum_{\substack{j_2+j_2=\ell+1\\ j_1,j_2\geqslant 1}} p_{j_1}p_{j_2}j_1\right)$$
$$= \frac{1}{2}\sum_{\substack{j_2+j_2=\ell+1\\ j_1,j_2\geqslant 1}} p_{j_1}p_{j_2}(\ell+1).$$

This idea will be used repeatedly in the proof of Equation (6.10c) which we now explain. We start by applying Γ_Y on Equation (A.1);

$$\begin{aligned}\Gamma_Y Y_+^{i_2}\Gamma_Y Y_+^{i_1}\frac{y_0}{\alpha} =& \frac{b^2}{\alpha}i_1(i_1+i_2+1)y_{i_1+i_2+2} + i_1 b\sum_{k\geqslant 1} y_{i_1+i_2+k+2}\frac{k\partial}{\partial p_k}\\
&+\frac{i_1 b}{\alpha}\sum_{j=1}^{i_1+i_2+1} y_{i_1+i_2-j+2}p_j + b(i_1+i_2+m+1)\sum_{m\geqslant 1} y_{i_1+i_2+m+2}\frac{m\partial}{\partial p_m}\\
&+\alpha\sum_{m,k\geqslant 1} y_{i_1+i_2+m+k+2}\frac{m\partial}{\partial p_m}\frac{k\partial}{\partial p_k} + \sum_{m\geqslant 1}\sum_{j=1}^{i_1+i_2+m+1} y_{i_1+i_2+m+2-j}p_j\frac{m\partial}{\partial p_m}\\
&+\frac{b}{\alpha}(i_1+i_2-j+1)\sum_{1\leqslant j\leqslant i_1} y_{i_1+i_2-j+2}p_j + \sum_{1\leqslant j\leqslant i_1}\sum_{k\geqslant 1} y_{i_1+i_2-j+k+1}\frac{k\partial}{\partial p_k}p_j\\
&+\sum_{j=1}^{i_1}\sum_{j'=1}^{i_1+i_2-j+1} y_{i_1+i_2-j+1}p_{j'}p_j.\end{aligned}$$

Fix three integers $i_1, i_2, i_3 \geqslant 0$ such that $i_1+i_2+i_3 = \ell$. We apply $Y_+^{i_3}$ on the last equation and we regroup the terms of the same type. We get

$$\begin{aligned}Y_+^{i_3}\Gamma_Y Y_+^{i_2}\Gamma_Y Y_+^{i_1}\frac{y_0}{\alpha} =& \frac{b^2}{\alpha}i_1(i_1+i_2+1)y_{\ell+2} + (2i_1+i_2+m+1)b\sum_{m\geqslant 1} y_{\ell+m+2}\frac{m\partial}{\partial p_m}\\
&+\frac{b}{\alpha}\sum_{j=1}^{\ell+1}(i_1\mathbb{1}_{j\leqslant i_1+i_2+1} + (i_1+i_2-j+1)\mathbb{1}_{j\leqslant i_1})y_{\ell-j+2}p_j\\
&+\alpha\sum_{m,k\geqslant 1} y_{\ell+m+k+2}\frac{m\partial}{\partial p_m}\frac{k\partial}{\partial p_k} + \sum_{m\geqslant 1}\sum_{j=1}^{\ell+m+1}\mathbb{1}_{j\leqslant i_1+i_2+m+1}y_{\ell+m+2-j}p_j\frac{m\partial}{\partial p_m}\\
&+\sum_{1\leqslant j\leqslant \ell}\sum_{k\geqslant 1} y_{\ell-j+k+2}\mathbb{1}_{j\leqslant i_1}\left(p_j\frac{k\partial}{\partial p_k} + k\delta_{k,j}\right)\\
&+\sum_{\substack{j,j'\geqslant 1\\ j+j'\leqslant \ell+1}}\mathbb{1}_{j\leqslant i_1}\mathbb{1}_{j'\leqslant i_1+i_2-j+1}y_{\ell-j+2}p_{j'}p_j.\end{aligned}$$

By applying Θ_Y and taking the sum over all tuples (i_1, i_2, i_3), we get[5]

$$\mathcal{C}^{(\alpha)}_{\ell,2} = \binom{\ell+2}{3}\frac{(3\ell+5)b^2}{4\alpha}p_{\ell+2} + b\binom{\ell+2}{2}\sum_{m\geqslant 1} p_{m+\ell+2}(m+\ell+1)\frac{m\partial}{\partial p_m}$$
$$+ \frac{b}{2\alpha}\sum_{\substack{j_1+j_2=\ell+2\\ j_1,j_2\geqslant 1}} j_2((\ell+1)^2 - j_1j_2)p_{j_1}p_{j_2} + \alpha\binom{\ell+2}{2}\sum_{m,k\geqslant 1} y_{\ell+m+k+2}\frac{m\partial}{\partial p_m}\frac{k\partial}{\partial p_k}$$
$$+ \binom{\ell+2}{2}\sum_{m\geqslant 1}\sum_{\substack{j_1+j_2=\ell+m+2\\ j_1,j_2\geqslant 1}} p_{j_1}p_{j_2}\frac{m\partial}{\partial p_m} + \sum_{\substack{j_1,j_2,j_3\geqslant 1\\ j_1+j_2+j_3=\ell+2}} \frac{j_3(2j_2+j_3-1)}{2}p_{j_1}p_{j_2}p_{j_3}$$
$$+ \binom{\ell+3}{4}p_\ell.$$

In order to conclude, we first symmetrize the third sum; we use the fact that $(\ell+1)^2 - j_1j_2)p_{j_1}p_{j_2}$ is symmetric in j_1 and j_2. Moreover, we symmetrize the last sum in two steps as follows; $j_3p_{j_1}p_{j_2}p_{j_3}$ is symmetric in j_1 and j_2, hence

$$\frac{1}{2}\sum_{\substack{j_1,j_2,j_3\geqslant 1\\ j_1+j_2+j_3=\ell+2}} j_3(2j_2+j_3-1)p_{j_1}p_{j_2}p_{j_3}$$
$$= \frac{1}{4}\sum_{\substack{j_1,j_2,j_3\geqslant 1\\ j_1+j_2+j_3=\ell+2}} j_3p_{j_1}p_{j_2}p_{j_3}(2j_2+j_3-1+2j_1+j_3-1)$$
$$= \frac{1}{2}\sum_{\substack{j_1,j_2,j_3\geqslant 1\\ j_1+j_2+j_3=\ell+2}} j_3p_{j_1}p_{j_2}p_{j_3}(\ell+1).$$

Finally, notice that $p_{j_1}p_{j_2}p_{j_3}$ is symmetric in j_1, j_2 and j_3. We then take the average between the three indexing tuples (j_1, j_2, j_3), (j_2, j_3, j_1), and (j_3, j_1, j_2). This finishes the proof of Equation (6.10c) and hence the proof of the lemma.

Acknowledgments

The author is very grateful to his advisors Valentin Féray and Guillaume Chapuy for many interesting discussions and for reviewing this paper several times. He also thanks the anonymous referees whose comments made this paper clearer.

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