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A REALIZATION OF POSET ASSOCIAHEDRA

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Abstract. Given any connected poset P, we provide a simple realization of Galashin's P-associahedron $\mathscr{A}(P)$ as a convex polytope in $\mathbb{R}^P$. This realization is inspired by the description of $\mathscr{A}(P)$ as a compactification of the configuration space of order-preserving maps $P \to \mathbb{R}$. Additionally, we provide an analogous realization for Galashin's affine poset cyclohedra.

Keywords. Poset, associahedron, cyclohedron, realization, configuration space, compactification

Mathematics Subject Classifications. 52B11, 06A07

1. Introduction

Given a finite connected poset P, the poset associahedron $\mathscr{A}(P)$ is a simple, convex polytope of dimension $|P|-2$ introduced by Galashin [Gal24]. Poset associahedra arise as a natural generalization of Stasheff's associahedra [Hai84, Pet15, Sta97, Tam54], and were originally discovered by considering compactifications of the configuration space of order-preserving maps $P \to \mathbb{R}$. These compactifications are generalizations of the Axelrod–Singer compactification of the configuration space of points on a line [AS94, LTV10, Sin04]. Galashin constructed the polar dual of a poset associahedron by performing iterated stellar subdivisions on the polar dual of Stanley's *order polytope* [Sta86], but did not provide an explicit realization. Various poset associahedra and cyclohedra have already been studied including *permutohedra*, *associahedra*, *operahedra* [LA22], *type B permutohedra* [FR07], and *cyclohedra* [BT94].

Poset associahedra bear resemblance to graph associahedra, where the face lattice of each is described by a *tubing criterion*. However, neither class is a subset of the other. When Carr and

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Devadoss introduced graph associahedra in [CD06], they distinguished between *bracketings* and *tubings* of a path, where the idea of bracketings does not naturally extend to any simple graph. In the case of poset associahedra, the idea of bracketings *does* extend to every connected poset.

Galashin [Gal24] also introduces *affine posets,* and analagous *affine order polytopes* and *affine poset cyclohedra*. In this paper, we provide a simple realization of poset associahedra and affine poset cyclohedra as an intersection of half spaces, inspired by the compactification description and by a similar realization of graph associahedra due to Devadoss [Dev09]. In independent work [MPP24], Mantovani, Padrol, and Pilaud found a realization of poset associahedra as sections of graph associahedra. The authors of [MPP24] also generalize from posets to oriented building sets (which combine a building set with an oriented matroid).

2. Background

2.1. Poset Associahedra

We start by defining the poset associahedron.

Definition 2.1. Let $(P, \preceq)$ be a finite poset. We make the following definitions:

- A subset $\tau \subseteq P$ is *connected* if it is connected as an induced subgraph of the Hasse diagram of P.
- $\tau \subseteq P$ is *convex* if whenever $a, c \in \tau$ and $b \in P$ such that $a \preceq b \preceq c$, then $b \in \tau$.
- A *tube* of P is a connected, convex subset $\tau \subseteq P$ such that $2 \leqslant |\tau|$.
- A tube τ is *proper* if $|\tau| \leqslant |P| - 1$.
- Two tubes $\sigma, \tau \subseteq P$ are *nested* if $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$. Tubes σ and τ are *disjoint* if $\tau \cap \sigma = \varnothing$.
- For disjoint tubes σ, τ we say $\sigma \prec \tau$ if there exists $a \in \sigma, b \in \tau$ such that $a \prec b$.
- A *proper tubing* T of P is a set of proper tubes of P such that any pair of tubes is nested or disjoint and the relation $\prec$ extends to a partial order on T. That is, whenever $\tau_1, \ldots, \tau_k \in T$ with $\tau_1 \prec \cdots \prec \tau_k$ then $\tau_k \not\prec \tau_1$. This is referred to as the *acyclic tubing condition.*
- A proper tubing T is *maximal* if it is maximal by inclusion on the set of all proper tubings.

Figure 2.1 shows examples and non-examples of proper tubings.

Definition 2.2. For a finite poset P, the *poset associahedron* $\mathscr{A}(P)$ is any simple, convex polytope of dimension $|P| - 2$ whose face lattice is isomorphic to the set of proper tubings ordered by reverse inclusion. That is, if F_T is the face corresponding to T, then $F_S \subset F_T$ if one can make S from T by adding tubes. Vertices of $\mathscr{A}(P)$ correspond to maximal tubings of P.

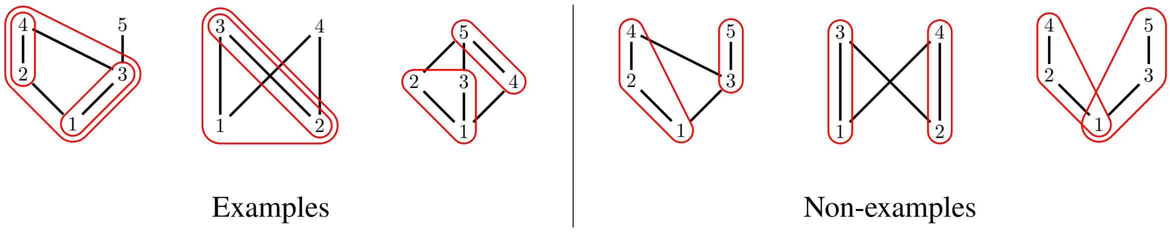


Figure 2.1: Examples and non-examples of proper tubings.

We realize poset associahedra as an intersection of half-spaces. Let P be a finite poset and let $n = |P|$. We work in the ambient space $\mathbb{R}^P_{\Sigma=0}$, the space of real-valued functions on P that sum to 0. For a subset $\tau \subseteq P$, define a linear function α_τ on $\mathbb{R}^P_{\Sigma=0}$ by

$$\alpha_\tau(p) := \sum_{\substack{i \prec\!\!\cdot\, j \\ i,j\in\tau}} (p_j - p_i).$$

Here the sum is taken over all covering relations contained in τ. We define the half-space h_τ and the hyperplane H_τ by

$$h_\tau := \{p \in \mathbb{R}^P_{\Sigma=0} \mid \alpha_\tau(p) \geqslant n^{2|\tau|}\} \quad \text{and}$$
$$H_\tau := \{p \in \mathbb{R}^P_{\Sigma=0} \mid \alpha_\tau(p) = n^{2|\tau|}\}.$$

The following is our main result in the finite case:

Theorem 2.3. *If P is a finite, connected poset, the intersection of H_P with h_τ for all proper tubes τ gives a realization of $\mathscr{A}(P)$.*

2.2. Affine Poset Cyclohedra

Now we describe affine poset cyclohedra.

Definition 2.4. An *affine poset* of *order* $n \geqslant 1$ is a poset $\tilde{P} = (\mathbb{Z}, \preceq)$ such that:

1. For all $i \in \mathbb{Z}, i \preceq i + n$;
2. $\tilde{P}$ is n-periodic: For all $i, j \in \mathbb{Z}, i \preceq j \Leftrightarrow i + n \preceq j + n$;
3. $\tilde{P}$ is *strongly connected*: for all $i, j \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $i \preceq j + kn$.

The *order* of $\tilde{P}$ is denoted $|\tilde{P}| := n$.

Following Galashin [Gal24], we give analogous versions of Definition 2.1. We give them only where they differ from the finite case.

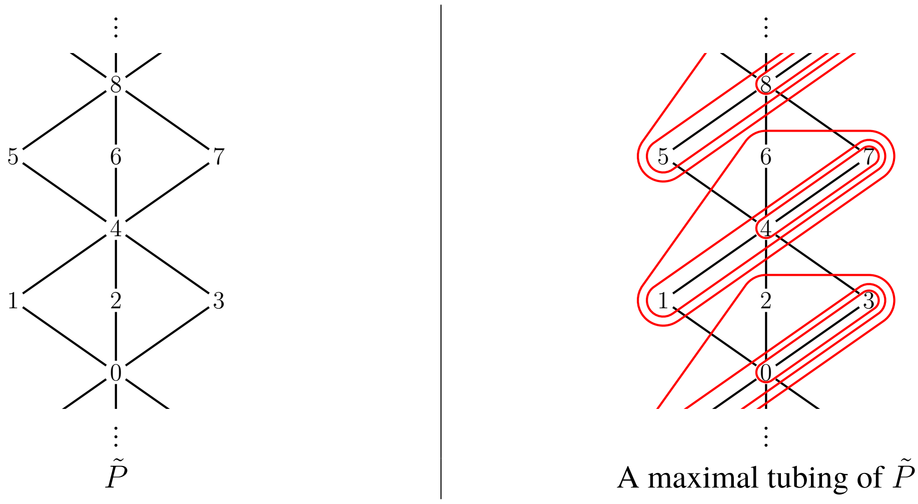


Figure 2.2: An affine poset of order 4 and a maximal tubing.

Definition 2.5. Let $\tilde{P} = (\mathbb{Z}, \preceq)$ be an affine poset.

- A *proper tube* of $\tilde{P}$ is a connected, convex subset $\tau \subseteq P$ such that $2 \leqslant |\tau|$ and either $\tau = \tilde{P}$ or τ has at most one element in each residue class modulo n.
- A collection of tubes T is n*-periodic* if for all $\tau \in T, k \in \mathbb{Z}$, $\tau + kn \in T$.
- A *proper tubing* T of $\tilde{P}$ is an n-periodic set of proper tubes of $\tilde{P}$ that satisfies the acyclic tubing condition and such that any pair of tubes is nested or disjoint.

Figure 2.2 gives an example of an affine poset of order 4 and a maximal tubing of that poset.

Definition 2.6. For an affine poset $\tilde{P}$, the *affine poset cyclohedron* $\mathscr{C}(\tilde{P})$ is a simple, convex polytope of dimension $|\tilde{P}| - 1$ whose face lattice is isomorphic to the set of proper tubings ordered by reverse inclusion. Vertices of $\mathscr{C}(\tilde{P})$ correspond to maximal tubings of $\tilde{P}$.

We also realize affine poset cyclohedra as an intersection of half-spaces. Let $\tilde{P}$ be an affine poset and let $n = |\tilde{P}|$. Define $\mathbb{R}^{\mathbb{Z}}$ to be the space of bi-infinite sequences $\tilde{\mathbf{x}} = (\tilde{x}_i)_{i\in\mathbb{Z}}$ with $x_i \in \mathbb{R}$. Let $\mathcal{C} \subseteq \mathbb{R}^{\mathbb{Z}}$ be the subspace of constant sequences and let $\pi : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}/\mathcal{C}$ be the canonical projection map onto the quotient space $\mathbb{R}^{\mathbb{Z}}/\mathcal{C}$.

Fix some constant $c \in \mathbb{R}^{+}$. We define the space of *affine maps* $\mathbb{R}^{\tilde{P}} \subset \mathbb{R}^{\mathbb{Z}}$ as the set of sequences $\tilde{\mathbf{x}} = (\tilde{x}_i)_{i\in\mathbb{Z}}$ such that $\tilde{x}_{i+n} = \tilde{x}_i + c$ for all $i \in \mathbb{Z}$. We work in the ambient space $\pi(\mathbb{R}^{\tilde{P}})$ where the constant c in the definition of affine maps is given by $c = n^{2(n+1)}$. One may observe that $\dim(\pi(\mathbb{R}^{\tilde{P}})) = n - 1$.

For a finite subset $\tau \subseteq P$, define a linear function α_τ on $\pi(\mathbb{R}^{\tilde{P}})$ by

$$\alpha_\tau(\tilde{\mathbf{x}}) := \sum_{\substack{i \prec j \\ i,j\in\tau}} (\tilde{x}_j - \tilde{x}_i).$$

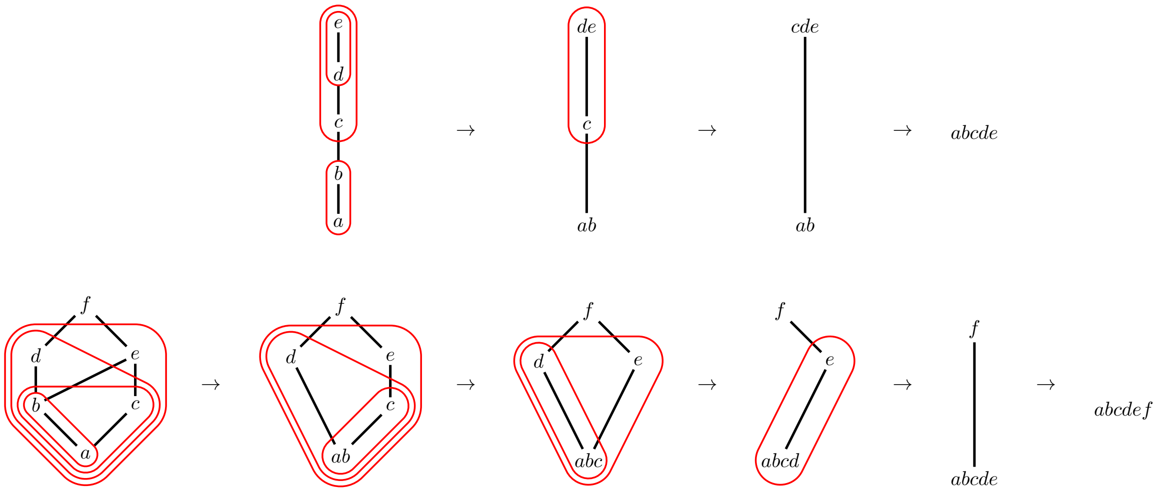


Figure 2.3: Multiplication of a word and of a generalized word.

Again, the sum is taken over all covering relations contained in τ. Note that α_τ is well-defined, regardless of the choice of representative in $\pi(\mathbb{R}^{\tilde{P}})$. We define the half-space h_τ and the hyperplane H_τ by

$$h_\tau := \{p \in \pi(\mathbb{R}^{\tilde{P}}) \mid \alpha_\tau(p) \geqslant n^{2|\tau|}\} \quad \text{and}$$
$$H_\tau := \{p \in \pi(\mathbb{R}^{\tilde{P}}) \mid \alpha_\tau(p) = n^{2|\tau|}\}.$$

Remark 2.7. Observe that for any tube τ and $k \in \mathbb{Z}$, $h_\tau = h_{\tau+kn}$.

The following is our main result in the affine case:

Theorem 2.8. *If $\tilde{P}$ is an affine poset, the intersection of h_τ for all proper tubes τ gives a realization of $\mathscr{C}(\tilde{P})$.*

2.3. An interpretation of tubings

When P is a chain, $\mathscr{A}(P)$ recovers the classical associahedron. There is a simple interpretation of proper tubings that explains all of the conditions above in terms of *generalized words.*

We can understand the classical associahedron as follows: Let $P = (\{1, \ldots, n\}, \leqslant)$ be a chain. We can think of the chain as a word we want to multiply together with the rule that two elements can be multiplied if they are connected by an edge. A maximal tubing of P is a way of disambiguating the order in which one performs the multiplication. If a pair of adjacent elements x and y have a pair of brackets around them, they contract along the edge connecting them and replace x and y by their product.

Similarly, we can understand the Hasse diagram of an arbitrary poset P as a *generalized word* we would like to multiply together. Again, we are allowed to multiply two elements if they are connected by an edge, but when multiplying elements, we contract along the edge connecting them and then take the transitive reduction of the resulting directed graph. That is, we identify the two elements and take the resulting quotient poset. A maximal tubing is again a way of disambiguating the order of the multiplication. See Figure 2.3 for an illustration of this multiplication. This perspective is discussed further in [Sto24, Remark 7.1.3] where it is shown that tubings form an operad governing properads and in [LA22, Section 2.1] in the case that the Hasse diagram of P is a rooted tree.

3. Configuration spaces and compactifications

We turn our attention to the relationship between poset associahedra and configuration spaces. For a poset P, the *order cone*

$$\mathscr{L}(P) := \{p \in \mathbb{R}^P_{\Sigma=0} \mid p_i \leqslant p_j \text{ for all } i \preceq j\}$$

is the set of order preserving maps $P \to \mathbb{R}$ whose values sum to 0.

Fix a constant $c \in \mathbb{R}^+$. The *order polytope,* first defined by Stanley [Sta86] and extended by Galashin [Gal24], is the $(|P| - 2)$-dimensional polytope

$$\mathscr{O}(P) := \{p \in \mathscr{L}(P) \mid \alpha_P(p) = c\}.$$

Remark 3.1. When P is *bounded*, that is, has a unique minimum $\hat{0}$ and maximum $\hat{1}$, this construction is projectively equivalent to Stanley's order polytope where we replace the conditions of the coordinates summing to 0 and $\alpha_P(p) = c$ with the conditions $p_{\hat{0}} = 0$ and $p_{\hat{1}} = 1$, for more details see [Gal24, Remark 2.5].

Galashin [Gal24] obtains the poset associahedra by an alternative compactification of $\mathscr{O}^\circ(P)$, the interior of $\mathscr{O}(P)$. We describe this compactification informally, as it serves as motivation for the realization in Theorem 2.3.

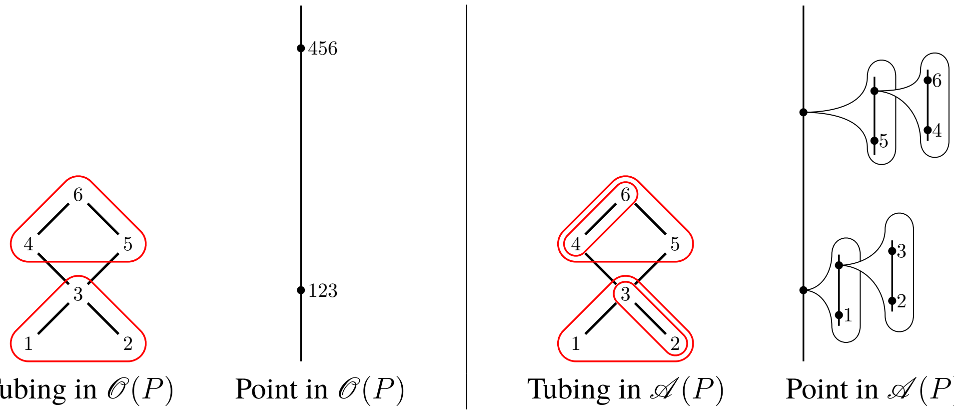
A point is on the boundary of $\mathscr{O}(P)$ when any of the inequalities in the order cone achieve equality. The faces of $\mathscr{O}(P)$ are in bijection with proper tubings of P such that all tubes are disjoint. Let T be such a tubing. If p is in the face corresponding to T and $\tau \in T$ then $p_i = p_j$ for $i, j \in \tau$.

We can think of the point p in the face corresponding to T as being "what happens in $\mathscr{O}(P)$" when for each $\tau \in T$, the coordinates are infinitesimally close. However, by taking all coordinates in τ to be equal, we lose information about their relative ordering. In $\mathscr{A}(P)$, we still think of the coordinates in τ as being infinitesimally close, but we are still interested in their configuration. Upon zooming in, this is parameterized by the order polytope of the subposet $(\tau, \preceq)$. We iterate this process, allowing points in τ to be infinitesimally closer, and so on. We illustrate this in Figure 3.1. This idea is a common explanation of the Axelrod–Singer compactification of $\mathscr{O}^\circ(P)$ when P is a chain, see [AS94, LTV10, Sin04].

The idea of the realization in Theorem 2.3 is to replace the notions of *infinitesimally close* and *infinitesimally closer* with being *exponentially close* and *exponentially closer.* For $p \in \mathscr{L}(P)$ and $\tau \subsetneq P$ a tube, α_τ acts as a measure of how close the coordinates of $p|_\tau$ are. We can make this precise with the following definition and lemma.

Definition 3.2. For $S \subseteq P$ and $p \in \mathbb{R}^P$, define the *diameter* of p relative to S by

$$\operatorname{diam}_S(p) = \max_{i,j \in S} |p_i - p_j|.$$



Tubing in $\mathscr{O}(P)$ Point in $\mathscr{O}(P)$ Tubing in $\mathscr{A}(P)$ Point in $\mathscr{A}(P)$

Figure 3.1: A vertex in $\mathscr{O}(P)$ vs. $\mathscr{A}(P)$.

That is, $\operatorname{diam}_S(p)$ is the diameter of $\{p_i : i \in S\}$ as a subset of $\mathbb{R}$.

Lemma 3.3. *Let $\tau \subseteq P$ be a tube and let $p \in \mathscr{L}(P)$. Then*

$$\operatorname{diam}_\tau(p) \leqslant \alpha_\tau(p) \leqslant \frac{n^2}{4} \operatorname{diam}_\tau(p).$$

Proof. By the triangle inequality and the fact that τ is connected, $\operatorname{diam}_\tau(p) \leqslant \alpha_\tau(p)$. For the other inequality,

$$\begin{aligned}\alpha_\tau(p) &= \sum_{\substack{i \prec j \\ i,j \in \tau}} (p_j - p_i) \\ &\leqslant \sum_{\substack{i \prec j \\ i,j \in \tau}} \operatorname{diam}_\tau(p) \\ &\leqslant \frac{1}{4} n^2 \operatorname{diam}_\tau(p)\end{aligned}$$

The inequality in the last line comes from the fact that there are at most $\frac{n^2}{4}$ covering relations in P, which follows from Mantel's Theorem and the fact that Hasse diagrams are triangle-free. $\square$

In particular, let $\sigma \subsetneq \tau$ be tubes. For $p \in \mathscr{L}(P)$, if $p \in H_\sigma \cap h_\tau$, then $\{p_i \mid i \in \sigma\}$ is clustered tightly together compared to $\{p_i \mid i \in \tau\}$.

4. Realizing poset associahedra

We are now prepared to prove Theorem 2.3. Throughout this section, we fix a finite, connected poset P and let $n = |P|$. Furthermore, we assume $n \geqslant 3$. For $n = 2$, one may check that Theorem 2.3 holds. Define

$$\mathcal{I}(P) := \bigcap_{\sigma \subset P} h_\sigma \cap H_P$$

where the intersection is over all proper tubes of P. Note that $\mathcal{I}(P) \subseteq \mathscr{L}(P)$ as if $i \prec j$ is a covering relation, then for $p \in h_{\{i,j\}}$, $p_i \leqslant p_j$.

Theorem 2.3 follows as a result of three lemmas:

Lemma 4.1. *If T is a maximal tubing, then*

$$v^T := \bigcap_{\tau \in T \cup \{P\}} H_\tau$$

is a point.

Lemma 4.2. *If T is a collection of tubes that do not form a proper tubing, then*

$$\bigcap_{\tau \in T} H_\tau \cap \mathcal{I}(P) = \varnothing.$$

Lemma 4.3. *If T is a maximal tubing and $\tau \notin T$ is a proper tube, then $\alpha_\tau(v^T) > n^{2|\tau|}$. That is, v^T lies in the interior of h_τ.*

To see why these lemmas imply Theorem 2.3, recall that the face lattice of a polytope is determined by its vertex-facet incidences [KP02]. Lemma 4.1 and Lemma 4.3 establish that maximal tubings correspond to vertices of $\mathcal{I}(P)$. Lemma 4.2 establishes that any collection of hyperplanes H_τ that do not form a proper tubing do not form a face of $\mathcal{I}(P)$. Hence there are no additional vertices. Furthermore as each vertex is an intersection of exactly $(|P| - 1)$ hyperplanes, each H_τ corresponds to a facet of $\mathcal{I}(P)$ for a proper tube τ. Hence $\mathcal{I}(P)$ has the desired vertex-facet incidences and is indeed a realization of $\mathscr{A}(P)$.

Definition 4.4. A subset $A \subseteq P$ is *lower* if for all $x \in P$ and $y \in A$ such that $x \preceq y$, $x \in A$. Dually a set is *upper* if for all $x \in P$ and $y \in A$ such that $y \preceq x$, $x \in A$.

Lemma 4.5. *Let T be a maximal tubing of P and let $\tau \in T$ be maximal by inclusion. Then τ is upper or lower and $(P - \tau)$ is either a tube or a singleton. Furthermore, if τ is lower, then $(P - \tau)$ is upper and if τ is upper, then $(P - \tau)$ is lower.*

Proof. We restate an observation of [Gal24, Corollary 2.7]. Let T be a proper tubing of P. For each $\tau \in T \sqcup \{P\}$, consider the quotient $\tau / \sim_\tau$ where $i \sim_\tau j$ if there exists $\tau' \in T$ such that $\tau' \subsetneq \tau$ and $i, j \in \tau'$. Then the face of $\mathscr{A}(P)$ corresponding to T is combinatorially equivalent to the product

$$\prod_{\tau \in T \sqcup \{P\}} \mathscr{A}(\tau / \sim_\tau).$$

Let T be a maximal tubing and let $\tau_1, \ldots, \tau_k$ be the collection of maximal by inclusion tubes in T. By this observation, $\mathscr{A}(P/\sim_P)$ must be 0 dimensional, and hence $P/\sim_P$ contain only 2 elements. We can identify these two elements with τ_1 and $P - \tau_1$. As $\tau_1, \ldots, \tau_k$ are disjoint, $k \leqslant 2$. If $k = 1$, then $(P - \tau_1)$ is a singleton, and if $k = 2$ then $P - \tau_1 = \tau_2$. Furthermore, if $\tau_1 \prec (P - \tau_1)$ in $P/\sim_P$ then τ_1 is lower in P and $(P - \tau_1)$ is upper in P. Otherwise, $(P - \tau_1) \prec \tau_1$ in $P/\sim_P$, $(P - \tau_1)$ is lower in P, and τ_1 is upper in P. □

Definition 4.6. Let T be a tubing on P and let $S \subseteq P$. We define the *restriction* of T to S by

$$T|_S := \{\tau \in T \mid \tau \subsetneq S\}.$$

Observe that if T is a tubing on P and $\tau \in T$ then $T|_\tau$ is a tubing on τ.

Proof of Lemma 4.1. Let $\tau \subseteq P$ be a tube and let $\vec{e}_i$ be the i-th coordinate vector in $\mathbb{R}^P$. Define

$$\vec{v}_\tau := \sum_{\substack{i,j\in\tau \\ i\prec j}} (\vec{e}_j - \vec{e}_i).$$

As each maximal tubing has $(|T|-1)$ proper tubes, it suffices to show that for every maximal tubing T,

$$V_P(T) := \{\vec{v}_\tau \mid \tau \in T \cup \{P\}\}$$

is linearly independent.

We proceed by induction on $|P|$. When $|P| \leqslant 2$ the result is trivial. Now suppose that $|P| > 2$ and let $\tau \in T$ be maximal by inclusion. By Lemma 4.5, τ and $(P-\tau)$ partition P into a lower and upper set. Call the lower set A and the upper set B.

By induction, $V_A(T|_A)$ and $V_B(T|_B)$ are linearly independent. Furthermore as A and B are disjoint, the vectors in each of $V_A(T|_A)$ and $V_B(T|_B)$ sum over disjoint sets of $\vec{e}_i$ and hence $V_A(T|_A) \cup V_B(T|_B)$ is linearly independent.

Finally, define

$$\mathbf{1}_A := \sum_{i\in A} \vec{e}_i \text{ and } \mathbf{1}_B := \sum_{i\in B} \vec{e}_i.$$

Consider the linear function $F : \mathbb{R}^P \to \mathbb{R}^2$ defined by

$$T(\vec{v}) = \begin{pmatrix} \langle \mathbf{1}_A, \vec{v}\rangle \\ \langle \mathbf{1}_B, \vec{v}\rangle \end{pmatrix}$$

and observe that

$$F(V_A(T|_A)) = F(V_B(T|_B)) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

However, we have that

$$F(\vec{v}_P) = \sum_{\substack{i\in A, j\in B \\ i\prec j}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which is non-zero as P is connected. Hence $\vec{v}_P$ is not in the span of $V_A(T|_A) \cup V_B(T|_B)$ and so $V_T(P) = V_A(T|_A) \cup V_B(T|_B) \cup \{\vec{v}_P\}$ is linearly independent and the claim is proven. □

Definition 4.7. Let $S \subseteq P$. The *convex hull* of S as

$$\operatorname{conv}(S) := \{b \in P \mid \text{there exist } a, c \in S \text{ such that } a \leqslant b \leqslant c\}.$$

Observe that if $p \in \mathscr{L}(P)$, then $\operatorname{diam}_S(p) \leqslant \operatorname{diam}_{\operatorname{conv}(S)}(p)$.

Definition 4.8. For a poset $(P, \preceq)$, the *comparability graph* $G(P)$ is an undirected graph with vertex set P and edge set $\{\{i, j\} \mid i, j \in P \text{ such that } i \prec j \text{ or } j \prec i\}$.

The following lemma is straightforward to prove:

Lemma 4.9. *Let $S \subseteq P$ such that $2 \leqslant |S|$. If $G(S)$ is connected, then $\operatorname{conv}(S)$ is a tube.*

Proof of Lemma 4.2. If T is not a collection of tubes that form a proper tubing, then at least one of the following two cases holds:

(1) There is a pair of non-nested and non-disjoint tubes τ_1, τ_2 in T.

(2) There is a sequence of disjoint tubes $\tau_1, \ldots, \tau_k$ such that $\tau_1 \prec \cdots \prec \tau_k \prec \tau_1$.

In either case, we define

$$\mathcal{J} = \left(\bigcap H_{\tau_i}\right) \cap \mathscr{L}(P).$$

Our goal is to show that there exists a tube $\sigma \subseteq P$ such that $\mathcal{J} \cap h_\sigma = \varnothing$.

The idea of the proof is as follows: In case (1), take $\sigma = \operatorname{conv}(\tau_1 \cup \tau_2)$ and in case (2), take $\sigma = \operatorname{conv}(\tau_1 \cup \cdots \cup \tau_k)$. By Lemma 4.9, we have that σ is a tube, so Lemma 3.3 tells us that for each τ_i, $\operatorname{diam}_{\tau_i}(p)$ is very small compared to $n^{2|\sigma|}$. As the tubes either intersect or are cyclic, one can show this forces $\operatorname{diam}_\sigma(p)$ to also be small, so $\alpha_\sigma(p) < n^{2|\sigma|}$.

We start with case (1). Let $p \in \mathcal{J}(P)$. Note that for both $i = 1$ and $i = 2$, we have

$$|\tau_i| + 1 \leqslant |\sigma| \text{ as } |\tau_1|, |\tau_2| < |\tau_1 \cup \tau_2|.$$

Hence $\operatorname{diam}_{\tau_i}(p) \leqslant n^{2(|\sigma|-1)}$.

Now let $a, b \in \sigma$. There exists some $x \in (\tau_1 \cap \tau_2)$, so

$$\begin{aligned} |p_a - p_b| &\leqslant |p_a - p_x| + |p_x - p_b| \\ &\leqslant 2\max\{\operatorname{diam}_{\tau_1}(p), \operatorname{diam}_{\tau_2}(p)\} \\ &\leqslant 2n^{2(|\sigma|-1)} \\ &< \frac{4}{n^2} n^{2|\sigma|}. \end{aligned}$$

Hence $\operatorname{diam}_\sigma(p) < \frac{4}{n^2} n^{2|\sigma|}$, so by Lemma 3.3, $\alpha_\sigma(p) < n^{2|\sigma|}$, and $p \notin h_\sigma$.

Now we move to case (2). Again let $p \in \mathcal{J}(P)$. Suppose there is a sequence of disjoint tubes $\tau_1, \ldots, \tau_k$ such that for each i there exists $x_i, y_i \in \tau_i$ where $x_i \prec y_{i+1}$ where we take the indices mod k. Observe that for each τ_i there exists τ_j disjoint from τ_i. Hence, $|\tau_i| + 2 \leqslant |\sigma|$, so $\operatorname{diam}_{\tau_i}(p) \leqslant n^{2(|\sigma|-2)}$. Then:

$$\begin{aligned} p_{y_i} - p_{y_{i+1}} &\leqslant p_{y_i} - p_{x_i} \\ &\leqslant \operatorname{diam}_{\tau_i}(p) \\ &\leqslant n^{2(|\sigma|-2)}. \end{aligned}$$

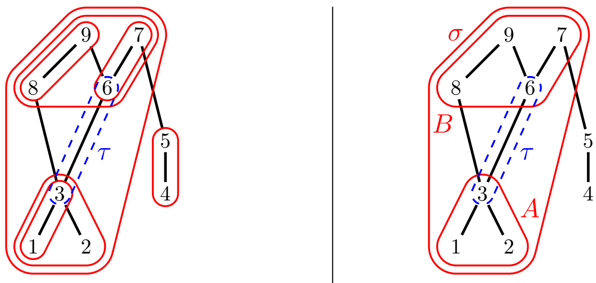


Figure 4.1: An example illustrating the proof of Lemma 4.3.

Then we have:

$$\begin{aligned} p_{y_1} - p_{y_i} &= (p_{y_1} - p_{y_2}) + \cdots + (p_{y_{i-1}} - p_{y_i}) \\ &\leqslant (i-1)n^{2(|\sigma|-2)} \qquad \text{and} \\ p_{y_i} - p_{y_1} &= (p_{y_i} - p_{y_{i+1}}) + \cdots + (p_{y_{k-1}} - p_{y_k}) + (p_{y_k} - p_{y_1}) \\ &\leqslant (k-i+1)n^{2(|\sigma|-2)}. \end{aligned}$$

As $(i-1), (k-i+1) \leqslant n$, we have $|p_{y_1} - p_{y_i}| \leqslant n^{2|\sigma|-3}$. Finally, if $z_i \in \tau_i, z_j \in \tau_j$, then

$$\begin{aligned} |p_{z_i} - p_{z_j}| &\leqslant |p_{z_i} - p_{y_i}| + |p_{y_i} - p_{y_1}| + |p_{y_1} - p_{y_j}| + |p_{y_j} - p_{z_j}| \\ &\leqslant 4n^{2|\sigma|-3} \\ &< \frac{4}{n^2} n^{2|\sigma|}. \end{aligned}$$

Hence $\mathrm{diam}_\sigma(p) < \frac{4}{n^2} n^{2|\sigma|}$, so by Lemma 3.3, $\alpha_\sigma(p) < n^{2|\sigma|}$, and $p \notin h_\sigma$. □

Proof of Lemma 4.3. Let T be a maximal tubing of P and let $\tau \notin T$ be a tube. Define the *convex hull* of τ *relative* to T by

$$\mathrm{conv}_T(\tau) := \min\{\sigma \in T \mid \tau \subset \sigma\}.$$

Let $\sigma = \mathrm{conv}_T(\tau)$. Consider the tubing on σ given by $T|_\sigma$. By Lemma 4.5, $T|_\sigma$ partitions σ into a lower set A and an upper set B where A and B are each either tubes or singletons. Furthermore, A and B both intersect τ by construction. See Figure 4.1 for an example illustrating this.

The idea of the proof is as follows: Let $p = v^T$. By Lemma 3.3, $\mathrm{diam}_A(p)$ and $\mathrm{diam}_B(p)$ are both very small compared to $\mathrm{diam}_\sigma(p)$. Then for any $a \in A, b \in B$, $|p_a - p_b|$ must be large. As τ intersects both A and B, $\mathrm{diam}_\tau(p)$ must be large and hence $p \in h_\tau$. See Figure 4.2 for an illustration of this. More precisely, we show that for any $i \in A, j \in B$, $(p_j - p_i) > (n^2)^{|\tau|}$, which implies that p lies in the interior of h_τ.

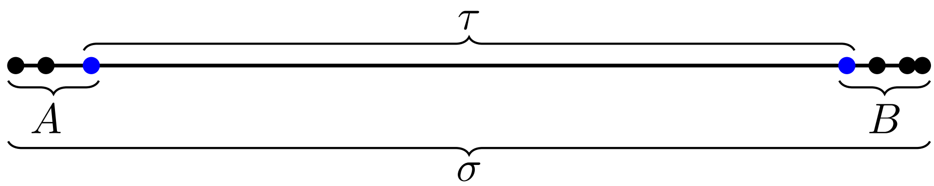


Figure 4.2: If $\operatorname{diam}_A(p)$ and $\operatorname{diam}_B(p)$ are small and $\operatorname{diam}_\sigma(p)$ is large, then $\operatorname{diam}_\tau(p)$ is large.

Observe that:

$$\sum_{\substack{x \prec y \\ x,y\in\sigma}} (p_y - p_x) = \underbrace{\sum_{\substack{x \prec y \\ x,y\in A}} (p_y - p_x)}_{\substack{\leqslant (n^2)^{|\sigma|-1} \\ < \frac{1}{8}(n^2)^{|\sigma|}}} + \underbrace{\sum_{\substack{x \prec y \\ x,y\in B}} (p_y - p_x)}_{\substack{\leqslant (n^2)^{|\sigma|-1} \\ < \frac{1}{8}(n^2)^{|\sigma|}}} + \sum_{\substack{x \prec y \\ x\in A, y\in B}} (p_y - p_x).$$

Fix $i \in (A \cap \tau)$ and $j \in (B \cap \tau)$. By Lemma 3.3, for any $x \in A, y \in B$,

$$\begin{aligned} p_y - p_x &\leqslant (p_j + \operatorname{diam}_A(p)) - (p_i - \operatorname{diam}_B(p)) \\ &\leqslant p_j - p_i + 2n^{2(|\sigma|-1)}. \end{aligned}$$

Again, noting that the number of covering relations in σ is at most $\frac{n^2}{4}$ we obtain:

$$\begin{aligned} \sum_{\substack{x \prec y \\ x\in A, y\in B}} (p_y - p_x) &\leqslant \sum_{\substack{x \prec y \\ x\in A, y\in B}} (p_j - p_i + 2(n^2)^{|\sigma|-1}) \\ &\leqslant \frac{n^2}{4}\left(p_j - p_i + 2(n^2)^{|\sigma|-1}\right) \\ &= \frac{n^2}{4}(p_j - p_i) + \frac{1}{2}(n^2)^{|\sigma|}. \end{aligned}$$

Combining all of this we get:

$$\begin{aligned} \sum_{\substack{x \prec y \\ x,y\in\sigma}} (p_y - p_x) &= (n^2)^{|\sigma|} \\ &< \frac{1}{8}(n^2)^{|\sigma|} + \frac{1}{8}(n^2)^{|\sigma|} + \frac{1}{2}(n^2)^{|\sigma|} + \frac{n^2}{4}(p_j - p_i) \\ &= \frac{3}{4}(n^2)^{|\sigma|} + \frac{n^2}{4}(p_j - p_i) \end{aligned}$$

Then $(n^2)^{|\sigma|-1} < (p_j - p_i)$. By Lemma 3.3 and the fact that $|\tau| \leqslant |\sigma| - 1$, we have

$$(n^2)^{|\tau|} \leqslant (n^2)^{|\sigma|-1} < (p_i - p_j) \leqslant \operatorname{diam}_\tau(p) \leqslant \alpha_\tau(p).$$

Hence p is in the interior of h_τ. □

Remark 4.10. A similar approach for realizing graph associahedra is taken by Devadoss [Dev09]. For a graph $G = (V, E)$, Devadoss realizes the graph associahedron of G by taking the supporting hyperplane for a graph tube τ to be

$$\left\{ p \in \mathbb{R}^V \mid \sum_{i \in \tau} p_i = 3^{|\tau|} \right\}.$$

One difference is that Devadoss realizes graph associahedra by cutting off slices of a simplex whereas we cut off slices of an order polytope. As explained in detail in [War22, Section 3.4], when the Hasse diagram of P is a tree, the poset associahedron is combinatorially equivalent to the graph associahedron of the line graph of the Hasse diagram. In this case, we claim the two realizations have linearly equivalent normal fans. Indeed, let

$$C = \{(i, j) \in P \times P \mid i \prec j\}$$

be the set of covering relations in P. The equivalence is given by the map $\Psi : \mathscr{L}(P) \to \mathbb{R}^C$ which maps $p \mapsto (p_j - p_i)_{(i,j) \in C}$. If the Hasse diagram of P is a path graph, then both realizations have linearly equivalent normal fans to the realization of the associahedron due to Shnider and Sternberg [Sta97].

5. Realizing affine poset cyclohedra

The proofs in the affine case are nearly identical to the finite case with some additional technical components. The similarity comes from the fact that Lemma 3.3 still applies. We highlight where the proofs are different. In this section, let $\tilde{P}$ be an affine poset of order $n \geqslant 3$. One may verify that Theorem 2.8 holds when $n \leqslant 2$.

Define

$$\mathcal{I}(\tilde{P}) := \bigcap_{\sigma \subset P} h_\sigma$$
$$\mathscr{L}(\tilde{P}) := \{p \in \pi(\mathbb{R}^{\tilde{P}}) \mid p_i \leqslant p_j \text{ for all } i \preceq j\}.$$

where the intersection is over all tubes of $\tilde{P}$. Note that $\mathcal{I}(\tilde{P}) \subseteq \mathscr{L}(\tilde{P})$ as if $i \prec j$ is a covering relation with $j \neq i + n$, then for $p \in h_{\{i,j\}}$, $p_i \leqslant p_j$. If $j = i + n$, then $p_i < p_i + n^{2n} = p_{i+n}$. Theorem 2.8 follows as a result of 3 lemmas:

Lemma 5.1. *If T is a maximal tubing, then*

$$v^T := \bigcap_{\tau \in T} H_\tau$$

is a point.

Lemma 5.2. *If T is a collection of tubes that do not form a proper tubing, then*

$$\bigcap_{\tau \in T} H_\tau \cap \mathcal{I}(\tilde{P}) = \varnothing.$$

Lemma 5.3. *If T is a maximal tubing and $\tau \notin T$ is a proper tube, then $\alpha_\tau(v^T) > n^{2|\tau|}$. That is, v^T lies in the interior of h_τ.*

Proof of Lemma 5.1. Let T be a maximal tubing and take any $\sigma \in T$ such that $|\tau| = n$. Then restricting to $\tilde{P}|_\sigma$, Lemma 4.1 implies that

$$\bigcap_{\substack{\tau \in T \\ \tau \subseteq \sigma}} H_\tau$$

is a point. However, as T is n-periodic,

$$\bigcap_{\substack{\tau \in T \\ \tau \subseteq \sigma}} H_\tau = \bigcap_{\tau \in T} H_\tau.$$

□

Proof of Lemma 5.2. By Remark 2.7, we can assume T is n-periodic. The proof is almost identical to the proof of Lemma 4.2. Define

$$\mathscr{L}(\tilde{P}) := \{p \in \pi(\mathbb{R}^{\tilde{P}}) \mid p_i \leqslant p_j \text{ for all } i \preceq j\}.$$

and note that

$$\mathscr{L}(\tilde{P}) \subseteq \pi(\mathbb{R}^{\tilde{P}}) \bigcap_{\substack{i,j \in \tilde{P} \\ i \prec j}} h_{\{i,j\}}.$$

Let

$$p \in \bigcap H_{\tau_i} \cap \mathscr{L}(\tilde{P}).$$

We again break into two cases:

(1) There is a pair of non-nested and non-disjoint tubes τ_1, τ_2 in T.

(2) All tubes in T are pairwise nested or disjoint and there is a sequence of disjoint tubes $\tau_1, \dots, \tau_k$ such that $\tau_1 \prec \cdots \prec \tau_k \prec \tau_1$.

The only difference in the proof occurs in case (1). Here, it is possible that there exists $x \in \tau_1 \cup \tau_2$ such that $x + n \in \tau_1 \cup \tau_2$ as well. In this case, the proof of Lemma 4.2 still implies that $\mathrm{diam}_{\tau_1 \cup \tau_2}(p) \leqslant \mathrm{diam}_{\tau_1}(p) + \mathrm{diam}_{\tau_2}(p) \leqslant 2n^{2n}$. However, $|p_{x+n} - p_x| = n^{2(n+1)}$. □

Proof of Lemma 5.3. Let T be a maximal tubing and $\tau \notin T$ be a proper tube. Let $p = v^T$. We claim that $\alpha_\tau(p) > n^{2|\tau|}$.

The only difference from the proof of Lemma 4.3 is that τ may not be contained by any tube in τ so $\mathrm{conv}_T(\tau)$ may not be well-defined. In this case, there exists $A \in T$ such that $|A| = n$, $A \cap \tau \neq \varnothing$, and $(A + n) \cap \tau \neq \varnothing$. Here, $(A + n)$ acts the same as B in the finite case, except the argument is much simpler.

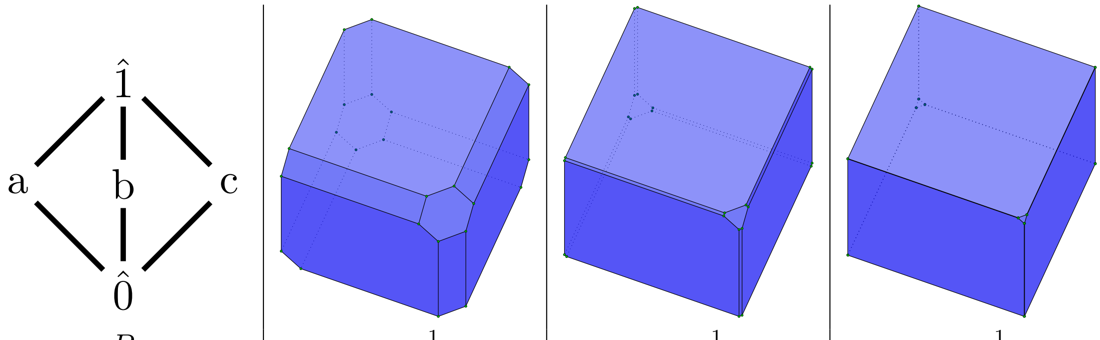


Figure 6.1: $\mathscr{O}(P)$ as a limit of $\mathcal{I}(P)$.

Let $i \in A \cap \tau, j \in (A+n) \cap \tau$. Observe that $\operatorname{diam}_A(p), \operatorname{diam}_{(A+n)}(p) \leqslant n^{2n}$ and that $i+n \in (A+n)$. Then

$$\begin{aligned}|p_j - p_i| &\geqslant (p_{i+n} - n^{2n}) - p_i \\ &\geqslant (p_{i+n} - p_i) - n^{2n} \\ &= n^{2(n+1)} - n^{2n} \\ &= n^{2n}(n^2-1) \\ &> n^{2n}\end{aligned}$$

Hence $\operatorname{diam}_\tau(p) > n^{2|\tau|}$ and by Lemma 3.3, $\alpha_\tau(p) > n^{2|\tau|}$. □

6. Remarks and Questions

Remark 6.1. Let $(P, \preceq)$ be a bounded poset. In Remark 3.1, we discuss how $\mathscr{O}(P)$ can be realized as the set of all $p \in \mathbb{R}^P$ such that $p_{\hat{0}} = 0$, $p_{\hat{1}} = 1$, and $p_i \leqslant p_j$ whenever $i \preceq j$. We can similarly realize $\mathscr{A}(P)$ as follows: Fix $0 < \varepsilon < \frac{1}{n^2}$.

For a proper tube $\tau \subset P$, let

$$h'_\tau = \{p \in \mathbb{R}^P \mid \alpha_\tau(p) \leqslant \varepsilon^{n-|\tau|}\}.$$

Then $\mathscr{A}(P)$ is realized as the intersection over all h'_τ with the hyperplanes

$$\{p_{\hat{0}} = 0\} \text{ and } \{p_{\hat{1}} = 1\}.$$

Letting $\varepsilon \to 0$, we obtain $\mathscr{O}(P)$ as a limit of $\mathcal{I}(P)$ as shown in Figure 6.1.

Remark 6.2. The key piece to the realizations in Theorems 2.3 and 2.8 is the linear form α_τ, where α_τ acts as an approximation of diam_τ.

However, there are many other options for choice of α_τ that could fill this role. Some other options include:

1. Sum over all pairs $i \prec j$ in τ.

$$\alpha_\tau(p) = \sum_{\substack{i \prec j \\ i,j \in \tau}} (p_j - p_i).$$

2. Let A and B be the set of minima and maxima of the restriction $P|_\tau$ respectively.

$$\alpha_\tau(p) = \sum_{\substack{i \prec j \\ i \in A, j \in B}} (p_j - p_i).$$

3. Fix a spanning tree T in the Hasse diagram of τ. Let $E = \{(i,j) \mid i \prec j\}$ be the set of edges in T.

$$\alpha_\tau(p) = \sum_{(i,j) \in E} (p_j - p_i).$$

An advantage of this option is that we would have

$$\operatorname{diam}_\tau(p) \leqslant \alpha_\tau(p) \leqslant (n-1) \operatorname{diam}_\tau(p).$$

One might expect that a similar realization can be obtained for each choice of α_τ.

Question 6.3. Recall that for a simple d-dimensional polytope P, the f-vector and h-vector of P are given by $(f_0, \ldots, f_d)$ and $(h_0, \ldots, h_d)$ where f_i is the number of i-dimensional faces and

$$\sum_{i=0}^{d} f_i t^i = \sum_{i=0}^{d} h_i (t+1)^i.$$

Postnikov, Reiner, and Williams [PRW08] found a statistic on maximal tubings of graph associahedra of chordal graphs where

$$\sum_T t^{\operatorname{stat}(T)} = \sum h_i t^i.$$

In particular, they define a map $T \mapsto w_T$ from maximal tubings of a graph on n vertices to the set of permutations S_n such that $\operatorname{stat}(T) = \operatorname{des}(w_T)$, the number of descents of w_T. It would be interesting to find a similar statistic on maximal tubings of poset associahedra. For a simple polytope P, one can orient the edges of P according to a generic linear form and take $\operatorname{stat}(v) = \operatorname{outdegree}(v)$ [Zie95, §8.2]. It may be possible to use our realization to find the desired statistic.

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