MAPS RELATED TO POLAR SPACES PRESERVING AN EXTREMAL WEYL DISTANCE

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Abstract. Let Ω_i and Ω_j be the sets of elements of respective types i and j of a polar space Δ of rank at least 3. We show that a permutation ρ of $\Omega_i \cup \Omega_j$ with the property that, for each $I \in \Omega_i$ and $J \in \Omega_j$, I and J generate a maximal singular subspace in Δ if and only if $\rho(I)$ and $\rho(J)$ generate a maximal singular subspace in Δ, is induced by an automorphism of Δ. Building-theoretically, this means that if ρ preserves a certain Weyl distance in the Tits-building corresponding to Δ, then it preserves all Weyl-distances.

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1. Introduction

The following problem often occurs in geometry: for a given class $\mathbf{D}$ of geometric structures, suppose that a graph $\Gamma(\Delta)$ can be associated to every member Δ of $\mathbf{D}$ in a certain way, but such that each automorphism of Δ induces an automorphism of the graph $\Gamma(\Delta)$; can we recover Δ from its graph $\Gamma(\Delta)$? Less vaguely, is it true that, every automorphism of $\Gamma(\Delta)$ is induced by a unique automorphism of Δ? In short, are $\mathrm{Aut}(\Gamma(\Delta))$ and $\mathrm{Aut}(\Delta)$ isomorphic in the natural way? We can also be happy with less than this. Suppose that $\mathrm{Aut}(\Delta)$ admits a subgroup $\mathrm{Aut}_0(\Delta)$ such that only the automorphism of Δ which belong to $\mathrm{Aut}_0(\Delta)$ induce automorphisms of $\Gamma(\Delta)$ and, nevertheless, $\mathrm{Aut}_0(\Delta)$ is large enough for Δ to be uniquely determined by it, namely $\mathrm{Aut}_0(\Delta) \cong \mathrm{Aut}_0(\Delta')$ if and only if $\Delta \cong \Delta'$. Then having proved that $\mathrm{Aut}(\Gamma(\Delta))$ and $\mathrm{Aut}_0(\Delta)$ are isomorphic in the natural way is sufficient to claim that Δ can be recovered from $\Gamma(\Delta)$.

One of the earliest papers, if not the earliest at all, fully devoted to a problem of this kind is [Cho49], by W. L. Chow. The results obtained by Chow in his seminal paper [Cho49], freely

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rephrased according to the terminology currently used nowaday and generalized to the extent that the methods used by Chow allow, sound as follows: (1) every (finite dimensional) projective space can be recovered from any of its (non-trivial) Grassmann graphs; (2) every polar space of finite rank can be recovered from the collinearity graph of its dual; (3) every hyperbolic quadric can be recovered from the collinearity graph of any of its two associated half-spin geometries.

Chow's work did not get a great fall down at the time. However, three decades later, a strong interest blew up in problems strictly related to those that Chow studied, as finding characterizations of collinearity graphs of projective and polar Grassannians, dual polar spaces, half-spin geometries and certain point-line geometries associated to exceptional buildings of spherical type. A step in these characterizations is to show that the point-line geometry under consideration uniquely determines the whole building it arises from. Showing this often entails a proof of the fact that the building can be recovered from the collinearity graph of that point-line geometry, just like in Chow's paper. However, Chow's contribution was generally ignored. This research activity flourished for many years, with particular intensity from the late seventies to the early nineties of the last century. The original approach by Chow has been rediscovered years later, mainly by M. Pankov in his investigation of polar Grassmannians. Referring the reader to Pankov [Pan10] for the state of the art in this topic at 2010, we recall here the definition of the Grassmann graphs of a polar space, namely the collinearity graphs of its Grassmannians. Given a polar space Δ of rank $n < \infty$, for $s \in \{0, 1, \ldots, n-1\}$ the s-Grassmann graph $\Gamma_s(\Delta)$ of Δ is the graph the vertices of which are the s-dimensional singular subspaces of Δ, two such subspaces X and Y being adjacent in $\Gamma_s(\Delta)$ if and only if $\dim(X \cap Y) = s-1$ and, when $s < n-1$, a singular subspace exists which contains both X and Y.

A lively activity on polar Grassmannians has accompanied or followed Pankov's work on this topic (see e.g. [Hua10, Hua11, HH08, LMW12, LPW14, PPZ06, ZCFM13]). The notion of Grassmann graph is basic in all of these investigations. The adjacency relation of a Grassmann graph is 'being as close as possible' (collinear, indeed). In contrast, A. Kasikova and H. Van Maldeghem [KVM13] focus on the relation 'being as far as possible' (namely, being opposite in the building). These two different approaches can be subsumed as special cases in a more general setting based on *Weyl distances*, proposed in [DSVM18] by H. Van Maldeghem and the first author of this paper. In short (we shall be more precise in a few lines), given a polar space Δ of (finite) rank $n \geqslant 3$, some adjacency relation $\sim$ on the set Ω_s of singular subspaces of dimension $s \in \{0, 1, \ldots, n-1\}$ (more generally, type s, in the hyperbolic case) can be defined, with $\sim$ corresponding to a Weyl distance in the building Δ^b related to Δ between elements of type s. The main theorem of [DSVM18] states that, for almost all choices of s and $\sim$ (only requiring that $\sim$ is a given Weyl distance occurring between s-spaces), the building Δ^b can be recovered from the graph $(\Omega_s, \sim)$. The Weyl distances that were left out in [DSVM18] correspond to Weyl distances which are 'extremal' in some sense (with $|\ell| = n-1$, using the notation we introduce below). These wilder cases could not be dealt with using the techniques of [DSVM18]. In the current paper we partially fill this gap by treating a subcase of these remaining cases. In order to correctly phrase our main theorem and the main theorem of [DSVM18], we shall now explain in detail how the adjacency relation $\sim$ can be defined. Preliminaries and terminology regarding polar spaces can be found in Section 2.

The building Δ^b. If Δ is not a hyperbolic polar space, then Δ^b (viewed as a simplicial complex) has as vertex set the set of all singular subspaces of Δ, where the *type* of a subspace corresponds to its dimension. If Δ is a hyperbolic polar space of rank n (in which each singular subspace of dimension $n-2$ is contained in exactly two maximal singular subspaces, yielding two natural families of maximal singular subspaces), then the vertex set of Δ^b is the set of singular subspaces of the oriflamme complex of Δ, which consists of subspaces of types $\{0,\dots, n-3,(n-1)',(n-1)''\}$, where the subspaces of type d with $0 \leqslant d \leqslant n-3$ are the singular subspaces of Δ of dimension d and the subspaces of types $(n-1)'$ and $(n-1)''$ are the subspaces of the two families of maximal singular subspaces of Δ, respectively. To indicate the difference between type and dimension, we sometimes denote the dimension of a subspace of type t by $|t|$. We denote the type set of Δ^b by T.

Automorphisms of Δ and Δ^b. An automorphism ρ of Δ is a permutation of the point set of Δ that preserves collinearity and non-collinearity (ρ and ρ^{-1} preserve singular subspaces). An automorphism ρ of Δ^b is a permutation of all vertices of the simplicial complex Δ^b such that ρ and ρ^{-1} preserve incidence. Each automorphism ρ of Δ induces an automorphism of Δ^b. The latter is type-preserving unless Δ is a hyperbolic polar space and ρ is a duality (switching the two natural families of maximal singular subspaces). Only when Δ is a triality quadric, that is, a hyperbolic polar space of rank 4 (so Δ^b has Dynkin type D_4), the building Δ^b has automorphisms (trialities and trialities composed with dualities) which do not correspond to an automorphism of Δ (since points are mapped to 3-dimensional subspaces).

The Weyl distance in Δ^b. In [AVM04], Abramenko and Van Maldeghem study maps between buildings preserving a single Weyl distance between the chambers. In [DSVM18], the authors consider the induced Weyl distance on the vertices of Δ^b. Two such vertices of Δ^b are at a certain Weyl distance precisely if there are $i,j,k,\ell \in \mathsf{T}\cup\{-1\}$ such that the vertices correspond to singular subspaces U,V of Δ of types i and j, and such that the intersection $U\cap V$ has type k (with $k=-1$ if the intersection is empty) and the subspace generated by U and the projection of U on V (in symbols: $\langle U, U^\perp\cap V\rangle$) has type ℓ (cf. Lemma 2.1 of [DSVM18]).

$(i,j;k,\ell)$-Weyl graphs. Consider the graph which has as vertices all vertices of Δ^b with adjacency given by being at a certain Weyl distance, which we can label with $(i,j;k,\ell)$ in view of the above. Since two vertices that are adjacent correspond to vertices of type i and j, we do not loose any information by restricting ourselves to the graph which has as vertices the vertices of type i and j. If $i=j$ this is hence a graph on Ω_j, which we denote by $\Gamma^n_{j;k,\ell}(\Delta)$; if $i\neq j$ then we obtain a bipartite graph, which we denote by $\Gamma^n_{i,j;k,\ell}(\Delta)$.

For reasons of uniformity, we also consider a bipartite version of $\Gamma^n_{j;k,\ell}(\Delta)$; denoted by $\Gamma^n_{j,j;k,\ell}(\Delta)$, whose bipartition classes are given by two copies of Ω_j, and adjacency between those biparts is given by the Weyl distance with parameters $(j,j;k,\ell)$ as before (so two vertices of distinct biparts are adjacent if they were adjacent in the non-bipartite version). Our aim is to show that, for certain parameters $(i,j;k,\ell)$, the graphs $\Gamma^n_{i,j;k,\ell}(\Delta)$ determine the building Δ^b (up to diagram automorphisms). In other words, we show that each automorphism of $\Gamma^n_{i,j;k,\ell}(\Delta)$ is essentially induced by an automorphism of Δ^b. In case $\Gamma^n_{i,j;k,\ell}(\Delta)$ or its bipartite complement (changing edges and non-edges between the biparts) is either empty or a matching, then these graphs do not contain information about Δ^b and hence we call it trivial. Note that each automorphism of $\Gamma^n_{j;k,\ell}(\Delta)$ induces an automorphism of $\Gamma^n_{j,j;k,\ell}(\Delta)$ (so the automorphism group can only get larger when taking a bipartite copy). This means that, if we can show that each automorphism of the $\Gamma^n_{j,j;k,\ell}(\Delta)$ is essentially induced by one of Δ^b, then the same holds for $\Gamma^n_{j;k,\ell}(\Delta)$. For these reasons, we will be working with bipartite graphs throughout.

A brief version of the main result of [DSVM18] reads as follows:

Theorem 1.1 (Theorem 3.5 of [DSVM18], informal statement)**.** *Let Δ be a polar space of rank $n \geqslant 3$ with type set T. Suppose $i, j, \ell \in \mathsf{T}$ and $k \in \mathsf{T} \cup \{-1\}$ are such that if $|\ell| = n-1$, then also $|i| = |j| = n-1$. If $\Gamma = \Gamma^n_{i,j;k,\ell}(\Delta)$ is non-trivial, then each automorphism of Γ is induced by a (not necessarily type-preserving) automorphism of the building Δ^b related to Δ, up to two classes of exceptions where there are also automorphisms of a building naturally containing Δ^b (cf. Examples 3.2 and 3.3 of [DSVM18]).*

Here, it is assumed that $n \geqslant 3$, as the case where $n = 2$ (in which case Δ is a generalized quadrangle) has been dealt with in [GVM02]. The condition 'if $|\ell| = n-1$, then also $|i| = |j| = n-1$', we believe, is merely there because when $|\ell| = n-1$ but at least one of $|i|$ and $|j|$ is less than $n-1$, the proof requires a different method. Small as this case might seem compared to the number of other cases, it is not trivial to get rid of it, even with alternative methods. In the current paper, we tackle one of the cases where $|\ell| = n-1$ but $\{|i|, |j|\} \neq \{n-1\}$, requiring additionally that $|k| = |i| + |j| - n + 1$. Geometrically, this means that $I \in \Omega_i$ and $J \in \Omega_j$ are adjacent if they generate a maximal singular subspace of Δ, where the latter is of a certain type in case Δ is hyperbolic. We call this an 'extremal' Weyl distance because the projection of I on J is maximal when $|\ell| = n-1$. We show that, also here, if $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$ is non-trivial, then each of its automorphisms is induced by an automorphism of Δ^b (without exceptions).

Denote by $\mathsf{Aut}_c(\Gamma)$ the automorphisms of Γ preserving the biparts of Γ. We show that:

Theorem 1.2. *Let Δ be a polar space of rank $n \geqslant 3$ with type set T. Suppose $i, j, \ell \in \mathsf{T}$ and $k \in \mathsf{T} \cup \{-1\}$ are such that $|i| + |j| - |k| = n - 1 = |\ell|$. Then $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$ is non-trivial if $\min\{|i|, |j|\} < n-1$ and in that case, each element ρ of $\mathrm{Aut}_c(\Gamma)$ is induced by an automorphism $\tilde{\rho}$ of the building Δ^b related to Δ, i.e., $\rho(X) = \tilde{\rho}(X)$ for each $X \in \Omega_i \cup \Omega_j$. More specifically,*

- (i) *If Δ is not hyperbolic, then $\tilde{\rho}$ is an automorphism of Δ, and conversely, each automorphism of Δ induces an element of $\mathrm{Aut}_c(\Gamma)$;*
- (ii) *If Δ is hyperbolic and $n > 4$, then $\tilde{\rho}$ is type-preserving, i.e., it is an automorphism of Δ which is not a duality. Conversely, each type-preserving automorphism of Δ^b induces an element of $\mathrm{Aut}_c(\Gamma)$;*
- (iii) *If Δ is the triality quadric (i.e., Δ is hyperbolic and $n = 4$), then either $\{|i|, |j|\} = \{0, 3\}$ or $\{|i|, |j|\} = \{1, 3\}$ and*
  - *(a) if $\{|i|, |j|\} = \{0, 3\}$, then $\tilde{\rho}$ is type-preserving;*
  - *(b) if $\{|i|, |j|\} = \{1, 3\}$, then $\tilde{\rho}$ is type-preserving, or it switches types 0 and t but preserves types i and j, with $\{0, t, i, j\} = \{0, 1, 3', 3''\}$.*

Conversely, each such $\tilde{\rho}$ induces an element of $\mathrm{Aut}_c(\Gamma)$.

If $i = j$, or, if $i \neq j$ but Δ^b has an automorphism switching the biparts (which is the case only if Δ is the triality quadric and $\{|i|, |j|\} = \{0, 3\}$), then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}_c(\Gamma) \times 2$.

The geometric nature of our arguments allows to also consider the graphs on $\Omega_i \cup \Omega_j$ in which $I \in \Omega_i$ and $J \in \Omega_j$ are adjacent if they generate just any maximal singular subspace in case Δ is hyperbolic, a relation which does not correspond to a certain Weyl distance but which fits in the same framework nonetheless. In this case we use $\mathsf{T} = \{0, 1, \ldots, n-2, n-1\}$ instead of $\mathsf{T} = \{0, 1, \ldots, n-3, (n-1)', (n-1)''\}$, and hence $\ell = n-1$ instead of $\ell \in \{(n-1)', (n-1)''\}$. This gives us the following theorem:

Theorem 1.3. *Let Δ be a hyperbolic polar space of rank $n \geqslant 3$. Suppose $i, j, \ell \in \{0, 1, \ldots, n-1\}$ and $k \in \mathsf{T} \cup \{-1\}$ are such that $i + j - k = \ell = n - 1$. Then $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$ is non-trivial if $\min\{i, j\} < n-1$ and in that case, each element ρ of $\mathrm{Aut}_c(\Gamma)$ is induced by an automorphism $\tilde{\rho}$ of the building Δ^b related to Δ, i.e., $\rho(X) = \tilde{\rho}(X)$ for each $X \in \Omega_i \cup \Omega_j$. More specifically, $\tilde{\rho}$ is an automorphism of Δ, including dualities. Conversely, each such $\tilde{\rho}$ induces an element of $\mathrm{Aut}_c(\Gamma)$. If $i = j$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}_c(\Gamma) \times 2$.*

Theorems 1.2 and 1.3 imply the following result for the non-bipartite counterparts of $\Gamma^n_{i,j;k,\ell}(\Delta)$ in the case where $i = j$ (where we also include $\ell = n - 1$ in the hyperbolic case, just like in the above theorem).

Corollary 1.4. *Let Δ be a polar space of rank $n \geqslant 3$ with type set T. Suppose $j, \ell \in \mathsf{T}$ and $k \in \mathsf{T} \cup \{-1\}$ are such that $2|j| - |k| = n - 1 = |\ell|$. Then $\Gamma := \Gamma^n_{j;k,\ell}(\Delta)$ is non-trivial if $|j| < n - 1$ and in that case, each element ρ of $\mathrm{Aut}(\Gamma)$ is induced by an automorphism $\tilde{\rho}$ of the building Δ^b related to Δ.*

Proof. Firstly, $|j| = n - 1$ then Γ is the empty graph, if $|j| < n - 1$ then each vertex has both neighbors and non-neighbors. Secondly, we can construct the bipartite graph $\Gamma^n_{j,j;k,\ell}(\Delta)$ from Γ by taking two isomorphic copies Ω^1_j and Ω^2_j of the vertex set Ω_j of Γ as the two biparts, where vertices of Ω^1_j and Ω^2_j are adjacent if and only if the corresponding vertices of Γ are adjacent. Let α be an automorphism of Γ. Then α induces an automorphism of $\Gamma^n_{j,j;k,\ell}(\Delta)$ that preserves the biparts by letting α act on both biparts. By Theorems 1.2 and 1.3, ρ is induced by an automorphism $\tilde{\rho}$ of the corresponding building Δ^b with $\rho(X) = \tilde{\rho}(X)$ for each $X \in \Omega_j$. We conclude that α is induced by $\tilde{\rho}$. $\square$

Remark 1.5. *Note that, if $|i| = |j| = n - 1$, then $|i| + |j| - |k| = n - 1 = |\ell|$ implies $|k| = n - 1$, so adjacency in $\Gamma^n_{i,j;k,\ell}(\Delta)$ is given by equality, and hence the graph is trivial. So the case where $|i| = |j| = |\ell|$, which was covered by Theorem 1.1 in more generality, is naturally excluded here. If $\max\{|i|, |j|\} = n - 1$, then $|i| + |j| - |k| = n - 1 = |\ell|$ implies that $I \in \Omega_i$ and $J \in \Omega_j$ are adjacent vertices of $\Gamma^n_{i,j;k,\ell}(\Delta)$ precisely if $I \cap J$ is either I or J, i.e., the adjacency relation is given by (symmetrized) containment. This case is covered by Proposition 7.4. of [DSVM18]; we recall that proposition here for further reference.*

Theorem 1.6. [DSVM18, Proposition 7.4] *Let* $\mathrm{C}_{i,j}$ *be the restriction of the incidence graph of* Δ *to the types* i *and* j*. For all* $i, j \in \mathrm{T}$ *with* $|i| < |j|$*, every automorphism of* $\mathrm{C}_{i,j}$ *is induced by an automorphism of* Δ^b*. Moreover, each automorphism of* Δ^b *inducing an automorphim of* $\mathrm{C}_{i,j}$ *is either type-preserving or a duality (if* Δ *is hyperbolic and* $|j| < n-1$*) or even a* t*-duality (if* Δ *has type* D_4 *and either* $\{i, j, t\} = \{0, 3', 3''\}$ *or* $(i, j) = (1, t)$*).*

For the sequel, we hence assume $\max\{|i|, |j|\} < n-1$*. Observe that this implies that* $|k| < \min\{|i|, |j|\}$*, and that we may omit the 'absolute value signs' for* $|k|, |i|, |j|$ *as dimension and type coincide now.*

Proof strategy. Our goal is to construct a graph $\widehat{\Gamma}$ from $\Gamma = \Gamma^n_{i,j;k,\ell}$, which turns out to be a subgraph of the incidence graph of Δ. Once this is achieved we can invoke Result 1.6 to finish the proof of our main result. The construction of $\widehat{\Gamma}$ relies on two graphs Γ'_i and Γ'_j derived from Γ. The definition of these two graphs (Definition 4.1) is the crucial step in the proof of our main result and it is motivated by the following observation: when two subspaces, say i-spaces, X_1 and X_2 are contained in a common singular subspace (i.e., they are collinear), then any i-space other than X_1 and X_2 is non-adjacent to some common neighbour of X_1 and X_2 in Γ. Put differently, we have $\mathsf{N}(\mathsf{N}(X_1, X_2)) = \{X_1, X_2\}$ (where N denotes the neighbours in Γ). This property is not shared by many other pairs of i-spaces. Given two i-spaces or two j-spaces X_1 and X_2, we show that $\mathsf{N}(\mathsf{N}(X_1, X_2)) = \{X_1, X_2\}$ holds if and only if X_1 and X_2 are contained in a common singular subspace or, for certain polar spaces (to be made precise later, see Lemma 3.2) if $\dim(X_1 \cap X_2) = i - 1$. If both options occur, then we need some additional work to separate these two cases. In all cases, we obtain a graph from which the automorphism group is known.

In this approach, but also in view of the statement that no automorphism of Γ can switch the biparts if $i \neq j$, it is important that we can tell the biparts apart. In most cases this can be done geometrically, except for the case where $s = 2$, where we need to rely on further calculations. A note ([Pas21]) on this has been written by the second author, concerning the graphs occurring in this paper, but containing the calculations for any finite order (s, t).

2. Preliminaries

Below we list all the notions and terminology used throughout the paper.

A pair $\Delta = (X, \mathcal{L})$ is a *point-line geometry* if X is a set and $\mathcal{L}$ is a set of subsets of X of size at least 2 covering X; the elements of X are called points and those of $\mathcal{L}$ lines. Two points p, q of X which are on a common line $\mathcal{L}$ are called *collinear*, denoted $p \perp q$; the set of points collinear to p is denoted by $p^\perp$. A *subspace* S of Δ is a subset of X such that the lines joining any two collinear points of S are contained in S. Moreover, if all pairs of points in S are collinear, the subspace is called *singular*.

A *polar space* is either a set of points X of size at least 3, or a point-line geometry $\Delta = (X, \mathcal{L})$ with the properties that, each line has at least three points, no point is collinear to all other points, every nested sequence of singular subspaces is finite, and, finally, for each point x and any line $L \in \mathcal{L}$ either one or all points of L are collinear to x. A singular subspace V of a polar space Δ is a projective space, with projective dimension $\dim V$. We warn that, throughout this paper, we adopt the following convention: given a subspace W of a projective space V, the codimension $\mathrm{codim}_V W$ of W in V is the dimension of a complement of W in V; accordingly, $\mathrm{codim}_V W = \dim V - \dim W - 1$. The polar space Δ has a well-defined finite *rank* $n \geqslant 1$, which is one more than the dimension of any maximal singular subspace of Δ (so it is 1 if the line set is empty). Since we assume $n \geqslant 3$, Δ has an *order* (s,t), which means that each line of Δ contains $s+1$ points (possibly $s=\infty$) and through each singular subspace of dimension $n-2$, there are $t+1$ singular subspaces of dimension $n-1$ (possibly $t=\infty$). We note that it follows from the classification of polar spaces of rank $n \geqslant 3$ that if $s<\infty$ then also $t<\infty$.

If $t=1$, then we say that Δ is *thin*. In this case there are two natural families of maximal singular subspaces, which we will refer to as the $(n-1)'$-spaces and the $(n-1)''$-spaces, as mentioned in the introduction. When $n \geqslant 4$, a thin polar space of rank n is isomorphic to a hyperbolic quadric $Q(2n-1,\mathbb{K})$ with $\mathbb{K}$ a (commutative) field, i.e., the null set of a quadratic equation $X_0X_1+\cdots X_{2n-2}X_{2n-1}$ in $\mathsf{PG}(2n-1,\mathbb{K})$. If $n=3$, then a thin polar space of rank 3 is isomorphic to the line Grassmannian of a projective space $\mathsf{PG}(3,\mathbb{L})$ over a skew field $\mathbb{L}$. If $\mathbb{L}$ is commutative, then again we obtain that this is isomorphic to a hyperbolic quadric $Q(5,\mathbb{L})$. Although we assume $n \geqslant 3$ throughout, we mention that a thin polar space of rank 2 is just a grid (i.e., the direct product of not necessarily isomorphic projective lines). In case the thin polar space Δ is a hyperbolic quadric, hence defined over a field $\mathbb{K}$, then there are automorphisms of Δ which switch the two families of maximal singular subspaces. Such an automorphism is called a duality of Δ (note that it gives rise to a non-type preserving automorphism of the associated building of type $\mathsf{D}_n(\mathbb{K})$). For ease of notation, we will also call the thin polar space of rank 3 which is associated to the line Grassmannian of $\mathsf{PG}(3,\mathbb{L})$ with $\mathbb{L}$ a non-commutative skew field as a hyperbolic polar space, even though it is not isomorphic to a hyperbolic quadric and it might admit no duality.

Two singular subspaces U,V of Δ are called *collinear* (denoted $U \perp V$) if they are contained in a common singular subspace and the smallest such subspace is denoted by $\langle U,V\rangle$ and referred to as the singular subspace generated or spanned by U and V (with obvious generalisation to a higher number of pairwise collinear singular subspaces). The set of all points collinear to (all points of) U is denoted by $U^\perp$. The *projection* $\mathsf{proj}_V(U)$ of a singular subspace U on a singular subspace V is $V \cap U^\perp$. We have $\mathsf{proj}_V(U) \supseteq U \cap V$ and $\dim U - \dim \mathsf{proj}_U(V) = \dim V - \dim \mathsf{proj}_V(U)$, as one can see by noticing that $U^\perp \cap V = X^\perp \cap V$ for a complement X of $U \cap V^\perp$ in U and $\mathrm{codim}_V(X^\perp \cap V) = \dim(X)$. We say that U and V are *opposite* if $\mathsf{proj}_V(U) = \mathsf{proj}_U(V) = \varnothing$, in which case $\dim U = \dim V$. In particular, two non-collinear points of X are opposite. The subspace spanned by U and $\mathsf{proj}_V(U)$ is denoted by U^V (note that $\dim(U^V) = \dim(V^U)$).

If p,q are opposite points, then the set of points in $(p^\perp \cap q^\perp)^\perp$ is called the *hyperbolic line* containing p and q. When $n \geqslant 3$ (as in the hypotheses of Theorem 1.2) all hyperbolic lines of Δ have the same size, say $h(\Delta)$, called the *hyperbolic order*. Clearly, $p,q \in (p^\perp \cap q^\perp)^\perp$ and hence $h(\Delta) \geqslant 2$. If $h(\Delta) > 2$, we call the hyperbolic line *proper*. Note that if Δ is a hyperbolic polar space, then $h(\Delta) = 2$.

Let Ω be the set of singular subspaces of Δ and consider a $K \in \Omega$ with $\dim(K) = k \leqslant n-2$. Put $X_K = \{U \in \Omega \mid K \subset U, \dim(U) = k+1\}$. First suppose $k < n-2$. Then, if M is an element of Ω of dimension $k+2$ containing K, we let M/K represent the elements of X_K contained in M. We define $\mathcal{L}_K$ as $\{M/K \mid K \subseteq M \in \Omega, \dim(M) = k+2\}$ and $\mathrm{Res}_\Delta(K) = (X_K, \mathcal{L}_K)$ is the *residue of* Δ *in* K. This is a polar space of rank $n-k-1$ of the same "kind" as Δ, e.g. the residue of a parabolic polar space is parabolic too, and likewise for hyperbolic, unitary, mixed and so on. If $\dim(K) = n-2$, then $\mathrm{Res}_\Delta(K)$ is just the set X_K, which is a polar space of rank 1. It contains at least 2 points and it contains precisely 2 if and only if Δ is hyperbolic.

3. Pairs $J_1, J_2 \in C_2$ with $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$

We now start with the proofs of Theorems 1.2 and 1.3. Throughout, Δ is a polar space of rank $n \geqslant 3$ with type set T, and its set of singular subspaces of type t is denoted by Ω_{t}. Take types $i, j, \ell \in \mathsf{T}$ and $k \in \mathsf{T} \cup \{-1\}$ such that $|i| + |j| - |k| = n-1 = |\ell|$. We consider the $(i,j;k,\ell)$-Weyl graph $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$, as defined in the introduction. We recall that Γ has bipartition classes Ω_i and Ω_j and that adjacent vertices correspond to singular subspaces I and J of Δ that generate a (maximal) singular subspace of Δ of type ℓ, i.e., $\langle I, J\rangle \in \Omega_\ell$. As mentioned in Remark 1.5, we may assume that $\max\{|i|,|j|\} < n-1$ and hence also $|k| < \min\{|i|,|j|\}$, and moreover this means that $|i| = i$, $|j| = j$ and $|k| = k$.

As explained in the proof strategy at the end of the introduction, we will consider pairs of vertices J_1 and J_2 of Γ of one of the biparts (say one with vertex set Ω_j) and consider the neighbors of their common neighbors, i.e., $\mathsf{N}(\mathsf{N}(J_1, J_2))$. In case $\mathsf{N}(J_1, J_2))$ is non-empty, then the set $\mathsf{N}(\mathsf{N}(J_1, J_2))$ is well defined and contains J_1, J_2. The goal of this section (and the crux of the main theorem) is to prove a criterion that captures the condition $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$. To state the criterion, we need the following definition:

Definition 3.1. *We say that J_1 and J_2 are ℓ-*collinear *(denoted $J_1 \perp_\ell J_2$) if there is a subspace N of type ℓ with $J_1 \cup J_2 \subseteq N$. If $\ell = n-1$, this coincides with ordinary collinearity between J_1 and J_2, but if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$ then there exist j-spaces J_1, J_2 with $J_1 \perp J_2$ which are not ℓ-collinear (if the singular subspace they generate has type ℓ', where $\{\ell, \ell'\} = \{(n-1)', (n-1)''\}$).*

Also, recall that $h(\Delta)$ is the *hyperbolic order* of Δ, which means that $h(\Delta)$ is the size of a hyperbolic line of Δ (we refer to the preliminaries for this); and, if Δ is hyperbolic, then $h(\Delta) = 2$. The criterion then reads as follows:

Proposition 3.2. *Let J_1 and J_2 be j-spaces of Δ.*

(i) *If $h(\Delta) > 2$ or, if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, then $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$ if and only if $J_1 \perp_\ell J_2$.*

(ii) *If $h(\Delta) = 2$ and, if Δ is hyperbolic then $\ell = n-1$, then $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$ if and only if $J_1 \perp_\ell J_2$ or if $\dim(J_1 \cap J_2) = j-1$.*

We start by investigating the set of neighbors $\mathsf{N}(J_1, J_2)$ with $J_1, J_2 \in \Omega_j$ distinct but arbitrary.

Notation. We introduce some notation that we will use throughout, regarding the mutual position of two arbitrary j-spaces J_1 and J_2 of Δ (see summary below and also the more general picture in Figure 3.1). We decompose J_1 and J_2 in three compartments as follows: firstly, their is their intersection $S := J_1 \cap J_2$, with dimension s. Secondly, there is the subspace $\mathsf{proj}_{J_1}(J_2) = J_1 \cap J_2^{\perp}$ of J_1 which is collinear to J_2 (the projection of J_2 on J_1), which has dimension $s + \alpha + 1$. Thirdly, we put $\beta := \operatorname{codim}_{J_1}(\mathsf{proj}_{J_1}(J_2)) = j - (\alpha + s + 1) - 1$, so β is the dimension of a subspace in J_1 complementary to $\mathsf{proj}_{J_1}(J_2)$. Since $\dim(J_1) = \dim(J_2)$, the projection $\mathsf{proj}_{J_2}(J_1)$ also has dimension $\alpha + s + 1$ and a subspace of J_2 complementary to $\mathsf{proj}_{J_2}(J_1)$ also has dimension β.

Next to this decomposition, we will also consider the subspace $J_1^{J_2} = \langle J_1, \mathsf{proj}_{J_2}(J_1)\rangle$ generated by J_1 and its projection on J_2, and likewise $J_2^{J_1} = \langle J_2, \mathsf{proj}_{J_1}(J_2)\rangle$. By the foregoing both have dimension $j + \alpha + 1$. In case they have maximal dimension $n - 1$, their type will become relevant, so we also introduce λ_1 as the type of $J_1^{J_2}$ and λ_2 as the type of $J_2^{J_1}$ (with $|\lambda_1| = |\lambda_2| = j + \alpha + 1$).

Lemma 3.3. *We have* $J_1^{J_2} \cap J_2^{J_1} = \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle$.

Proof. Clearly, $\langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle \subseteq J_1^{J_2} \cap J_2^{J_1}$. For the converse inclusion, let $x \in J_1^{J_2} \cap J_2^{J_1}$. Hence $x \in \langle x_1, y_2\rangle \cap \langle x_2, y_1\rangle$ for $x_i \in J_i$ and $y_j \in \mathsf{proj}_{J_j}(J_i) = J_j \cap J_i^{\perp}$, for $\{i, j\} = \{1, 2\}$. We shall show that $x \in \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle$. If $x \in \{y_1, y_2\}$ there is nothing to prove. So, let $y_1 \neq x \neq y_2$. Hence $x_i \neq y_j$ for $\{i, j\} = \{1, 2\}$. As $x, y_1 \in J_2^{J_1} \subseteq J_2^{\perp}$ and $x_2, y_2 \in J_2 \subseteq J_2^{\perp}$, we obtain that $x_1 \in J_2^{\perp} \cap J_1 = \mathsf{proj}_{J_1}(J_2)$. Hence $\langle x_1, y_2\rangle \subseteq \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle$. Consequently, $x \in \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle$. $\square$

We put $D := J_1^{J_2} \cap J_2^{J_1} = \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle$. With this notation, we claim that

$$2\beta + \dim D + 2 = 2j - s \tag{3.1}$$

Indeed, $j = \beta + \dim \mathsf{proj}_{J_1}(J_2) + 1$ and $j = \beta + \dim \mathsf{proj}_{J_2}(J_1) + 1$. By summing these two equations and recalling that $D = \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle$ we obtain

$$2j = 2\beta + \dim \mathsf{proj}_{J_1}(J_2) + \dim \mathsf{proj}_{J_2}(J_1) + 2 = 2\beta + \dim D + s + 2.$$

Hence $2\beta + \dim D + 2 = 2j - s$, as claimed.

Finally, noting that a common neighbor I of J_1 and J_2 results in ℓ-spaces $\langle I, J_1\rangle$ and $\langle I, J_2\rangle$ containing J_1 and J_2, respectively, we will also need to consider ℓ-spaces of Δ containing $J_1^{J_2}$ and $J_2^{J_1}$, respectively. We denote these by $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. In summary:

$$\begin{aligned}
&S := J_1 \cap J_2, && s := \dim S\\
&D := J_1^{J_2} \cap J_2^{J_1} = \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle, && \lambda_1 := \mathsf{t}(J_1^{J_2}) \text{ and } \lambda_2 := \mathsf{t}(J_2^{J_1})\\
&\alpha := |\lambda_1| - j - 1 = |\lambda_2| - j - 1, && \beta := j - (\alpha + s + 1) - 1\\
&\mathcal{N}_1 = \{N \in \Omega_\ell \mid J_1^{J_2} \subseteq N\}, && \mathcal{N}_2 = \{N \in \Omega_\ell \mid J_2^{J_1} \subseteq N\}
\end{aligned}$$

Before describing the set $\mathsf{N}(J_1, J_2)$, we need to say something more about $\mathcal{N}_1$ and $\mathcal{N}_2$. Note firstly that $\mathcal{N}_1$ and $\mathcal{N}_2$ are nonempty except possibly when Δ is hyperbolic, $\ell \in \{(n-1)', (n-1)''\}$ and $\dim J_c^{J_d} = n-1$ for $\{c, d\} = \{1, 2\}$. In this particular case $\mathcal{N}_c \neq \varnothing$ (in fact $\mathcal{N}_c = \{J_c^{J_d}\}$) if and only if $t(J_c^{J_d}) = \ell$.

Lemma 3.4. *If $N_1 \in \mathcal{N}_1$, then for any $(n-1)$-dimensional subspace N_2 of Δ containing $J_2^{J_1}$, we have $\dim(N_1 \cap N_2) \leqslant n-\beta-2$, with equality precisely when $N_2 = J_2^{N_1}$. When Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, $J_2^{N_1} \in \mathcal{N}_2$ if and only β is odd; in all other cases, $J_2^{N_1} \in \mathcal{N}_2$.*

Proof. Take any $(n-1)$-space N_2 containing ${J_2}^{J_1}$. By definition of $\mathcal{N}_1$, N_1 also has dimension $n-1$ and $J_1^{J_2} \subseteq N_1$. For $\{c, d\} = \{1, 2\}$, let B_c be a β-space in J_c complementary to $\mathsf{proj}_{J_c}(J_d)$. Since $B_1 \cap B_2^{\perp} = \varnothing$, $B_1 \subseteq N_1$, $\dim N_1 = n-1$ and $\dim B_1 = \dim B_2 = \beta$, we have $\dim(\mathsf{proj}_{N_1}(B_2)) = n-\beta-2$. As

$$N_1 \cap N_2 \subseteq \mathsf{proj}_{N_1}(N_2) \subseteq \mathsf{proj}_{N_1}(J_2) \subseteq \mathsf{proj}_{N_1}(B_2) \tag{3.2}$$

we have $\dim(N_1 \cap N_2) \leqslant n-\beta-2$, with equality if and only if all the above inclusions are equalities, namely if and only if $N_1 \cap N_2 = \mathsf{proj}_{N_1}(B_2)$. We show these conditions are equivalent to $N_2 = J_2^{N_1}$. To that end, note that $\langle B_2, N_1 \cap N_2 \rangle$ is contained in both maximal singular subspaces N_2 and $J_2^{N_1} = \langle J_2, \mathsf{proj}_{N_1}(J_2) \rangle$. If $\dim(N_1 \cap N_2) = n-\beta-2$ it follows that B_2 and $N_1 \cap N_2$ are complimentary subspaces in N_2 (i.e., $\langle B_2, N_1 \cap N_2 \rangle = N_2$) and hence the maximal singular subspaces N_2 and $J_2^{N_1}$ coincide. Conversely, if $N_2 = J_2^{N_1}$, then it follows easily that $N_1 \cap N_2 = \mathsf{proj}_{N_1}(B_2)$ (see (3.2)).

Finally, let Δ be hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$. Then N_1 and $J_2^{N_1}$ have the same type, namely ℓ, if and only if $\mathrm{codim}_{N_1}(N_1 \cap J_2^{N_1})$ is odd. Also, $N_1 \cap J_2^{N_1} = \mathsf{proj}_{N_1}(J_2^{N_1})$. However $\mathsf{proj}_{N_1}(J_2^{N_1}) = \mathsf{proj}_{N_1}(B_2)$, since B_2 is a complement of $J_1^{\perp} \cap J_2$ in J_2 and $J_1^{\perp} \cap J_2 \subseteq J_1^{J_2} \subseteq N_1$. Therefore $\dim \mathsf{proj}_{N_1}(J_2^{J_1}) = n-\beta-2$ since, as previously noticed, $\dim \mathsf{proj}_{N_1}(B_2) = n-\beta-2$. Consequently $\dim(N_1 \cap J_2^{N_1}) = n-\beta-2$, namely $\mathrm{codim}_{N_1}(N_1 \cap J_2^{N_1}) = \beta$. Thus, $J_2^{N_1} \in \mathcal{N}_2$ if and only if β is odd. In all other cases, the type of N_1 and N_2 is just their dimension $n-1$ and hence equal, so no other conditions are needed. $\square$

Lemma 3.5. *Assuming that $\mathrm{N}(J_1, J_2) \neq \varnothing$, let $I \in \mathrm{N}(J_1, J_2)$. For $c = 1, 2$ let $K_c = I \cap J_c$ and $N_c = \langle I, J_c \rangle$ and let K_0 be a complement of $\langle K_1, K_2 \rangle$ in I. Then the following hold.*

(a) $k \leqslant s+\alpha+1$ *and* $K_1 \cap S = K_2 \cap S = I \cap S$.

(b) $K_c \subseteq \mathsf{proj}_{J_c}(J_d) \subseteq N_1 \cap N_2$ *for* $\{c, d\} = \{1, 2\}$. *Moreover,* $J_c^{J_d} \subseteq N_c$ *for* $\{c, d\} = \{1, 2\}$.

(c) $N_c = J_c^{N_d}$ *for* $\{c, d\} = \{1, 2\}$ *and* $\dim(N_1 \cap N_2) = n-\beta-2$. *In particular, if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, then β is odd and $t(J_c^{N_d}) = \ell$.*

(d) K_0 *is a complement of both* $\langle K_1, \mathsf{proj}_{J_2}(J_1) \rangle$ *and* $\langle K_2, \mathsf{proj}_{J_1}(J_2) \rangle$ *in* $N_1 \cap N_2$.

Proof. As $S = J_1 \cap J_2$, we have $K_1 \cap S = (I \cap J_1) \cap (J_1 \cap J_2) = I \cap S$. By symmetry, $K_2 \cap S = I \cap S$. Also, with $\{c, d\} = \{1, 2\}$, from $K_c = I \cap J_c$ and $I \subseteq J_d^{\perp}$ we get $K_c \subseteq \mathsf{proj}_{J_c}(J_d)$. Hence $k = \dim(K_c) \leqslant s+\alpha+1$. Claim (a) is proven.

Recall that $J_1^{J_2} = \langle J_1, \mathsf{proj}_{J_2}(J_1)\rangle$. As both J_1 and $\mathsf{proj}_{J_2}(J_1)$ are collinear with both I and J_1, it follows that $J_1^{J_2}$ is collinear with $N_1 = \langle J_1, I\rangle$. Hence $J_1^{J_2} \subseteq N_1$, as N_1 is a maximal singular subspace. By symmetry, $J_2^{J_1} \subseteq N_2$. We know from Lemma 3.3 that $J_1^{J_2} \cap J_2^{J_1} = \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle$. Hence $N_1 \cap N_2$ contains both $\mathsf{proj}_{J_1}(J_2)$ and $\mathsf{proj}_{J_2}(J_1)$. Moreover $K_c \subseteq \mathsf{proj}_{J_c}(J_d)$, for $\{c, d\} = \{1, 2\}$, as noticed in the previous paragraph. All parts of claim (b) are thus proved. We turn to claim (c).

Let $\{c, d\} = \{1, 2\}$. Since $I \subseteq N_c \cap J_d^\perp$ we have $N_d = \langle I, J_d\rangle \subseteq \langle N_c \cap J_d^\perp, J_d\rangle = J_d^{N_c}$ and hence $N_d = J_d^{N_c}$ because N_d is a maximal singular subspace. The first part of (c) is proved. From $N_c = J_c^{N_d}$ the equality $\dim(N_1 \cap N_2) = n - \beta - 2$ also follows, by Lemma 3.4, which also yields the second part of (c).

Claim (d) remains to be proved. Let K_c' be a complement to $K_1 \cap K_2$ in K_c for $c = 1, 2$, so that $\langle K_1, K_2\rangle = \langle K_1', K_1 \cap K_2, K_2'\rangle$. We recall that $\mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1), I, K_0, K_1, K_1', K_2, K_2'$ are all subspaces of $N_1 \cap N_2$.

We claim that $\langle \mathsf{proj}_{J_1}(J_2), K_2'\rangle \cap K_0 = \varnothing$. Suppose by contradiction that there is a point $p \in \langle \mathsf{proj}_{J_1}(J_2), K_2'\rangle \cap K_0$. Since $K_2' \subseteq \langle K_1, K_2\rangle$ and K_0 is a complement of $\langle K_1, K_2\rangle$ in I we have $p \notin K_2'$. Thus there is a point $p' \in \mathsf{proj}_{J_1}(J_2) \cap \langle p, K_2'\rangle \subseteq J_1 \cap I = K_1$. It follows that $p \in \langle K_1, K_2\rangle$, yielding a contradiction because $p \in K_0$ and K_0 is a complement of $\langle K_1, K_2\rangle$ in I. This finishes the proof of our claim.

Note that $\mathsf{proj}_{J_1}(J_2) \cap K_2' \subseteq J_1 \cap I = K_1$ and therefore $\mathsf{proj}_{J_1}(J_2) \cap K_2' \subseteq K_1 \cap K_2' = \varnothing$ by the definition of K_2'. Now we have (see below for the explanation of each equality)

$$\begin{aligned}
\dim\langle \mathsf{proj}_{J_1}(J_2), K_2', K_0\rangle &= \dim\langle \mathsf{proj}_{J_1}(J_2), K_2'\rangle + \dim K_0 + 1 \\
&= \dim \mathsf{proj}_{J_1}(J_2) + \dim K_2' + 1 + \dim K_0 + 1 \\
&= (s + \alpha + 1) + \dim\langle K_2', K_0\rangle + 1 \\
&= s + \alpha + 1 + (n - j - 2) + 1 \\
&= n - (j - (\alpha + s + 1) - 1) - 2 \\
&= n - \beta - 2 = \dim(N_1 \cap N_2),
\end{aligned}$$

The first equality follows from $\langle \mathsf{proj}_{J_1}(J_2), K_2'\rangle \cap K_0 = \varnothing$ as shown above, the second one from $\mathsf{proj}_{J_1}(J_2) \cap K_2' = \varnothing$ and the third from the definition of $\mathsf{proj}_{J_1}(J_2)$ and the fact that $K_0 \cap K_2' \subseteq K_0 \cap \langle K_1, K_2\rangle = \varnothing$. Note that $\langle K_0, K_2'\rangle$ is a complement of K_1 in I and $\dim I = i = n - 1 - j + k$. As $\dim K_1 = k$, it follows that $\dim\langle K_0, K_2'\rangle = n - j - 2$ which explains the fourth equality. The last equality follows from (c).

Thus we have proved that K_0 is a complement of $\langle \mathsf{proj}_{J_1}(J_2), K_2'\rangle$ in $N_1 \cap N_2$. However $\langle \mathsf{proj}_{J_1}(J_2), K_2'\rangle = \langle \mathsf{proj}_{J_1}(J_2), K_2\rangle$, since $K_1 \cap K_2 \subseteq \mathsf{proj}_{J_1}(J_2)$ and $\langle K_1 \cap K_2, K_2'\rangle = K_2$ by the choice of K_2'. So, K_0 is a complement of $\langle \mathsf{proj}_{J_1}(J_2), K_2\rangle$ in $N_1 \cap N_2$. By symmetry, K_0 is also a complement of $\langle \mathsf{proj}_{J_2}(J_1), K_1\rangle$ in $N_1 \cap N_2$. Claim (d) is proven. □

Corollary 3.6. *Let Δ be hyperbolic with $\ell \in \{(n-1)', (n-1)''\}$ and suppose that $\mathrm{N}(J_1, J_2) \neq \varnothing$. Then either $t(J_1^{J_2}) = t(J_2^{J_1}) = \ell$ or $\dim J_2^{J_1} = \dim J_1^{J_2} < n - 1$.*

Proof. From claim (c) of Lemma 3.5, recalling that $J_c^{J_d} \subseteq J_c^{N_d}$ and noting that, if $\dim J_c^{J_d} = n - 1$, then $J_c^{J_d} = J_c^{N_d}$. □

Lemma 3.7. *Suppose that $k \leqslant s+\alpha+1$. Then the following hold.*

(a) There exist singular k-spaces $K_1 \subseteq \mathsf{proj}_{J_1}(J_2)$ and $K_2 \subseteq \mathsf{proj}_{J_2}(J_1)$ such that $K_1 \cap S = K_2 \cap S = K_1 \cap K_2$.

(b) For $\{c,d\} = \{1,2\}$ and with K_1 and K_2 as in (a), $K_c \perp \mathsf{proj}_{J_d}(J_c)$. Let $L_c = \langle \mathsf{proj}_{J_d}(J_c), K_c\rangle$, then $L_1 \perp L_2$ and $\dim L_1 = \dim L_2$. In particular, if M is a singular subspace containing L_1 and L_2 then L_1 and L_2 admit a common complement in M.

Proof. We recall that $S = J_1 \cap J_2 \subseteq \mathsf{proj}_{J_1}(J_2) \cap \mathsf{proj}_{J_2}(J_1)$ and $s = \dim S$. As $k \leqslant s+\alpha+1 = \dim \mathsf{proj}_{J_1}(J_2)$ there is a k-subspace $K_1 \subseteq \mathsf{proj}_{J_1}(J_2)$. Let K_1' be a complement of $K_1 \cap S$ in K_1. So, $K_1' \cap S = \varnothing$ and $K_1' \subseteq \mathsf{proj}_{J_1}(J_2)$. However $\dim \mathsf{proj}_{J_1}(J_2) = s+\alpha+1$. Hence $\dim K_1' \leqslant \alpha$. Similarly, there exists a subspace K_2' of $\mathsf{proj}_{J_2}(J_1)$ such that $K_2' \cap S = \varnothing$ and $\dim K_2' = \dim K_1'$. Now $K_2 := \langle K_2', K_1 \cap S\rangle$ is a k-subspace of $\mathsf{proj}_{J_2}(J_1)$ such that $K_2 \cap S = K_1 \cap S$. Claim (a) has been established.

We shall now prove (b). Since K_1, K_2, $\mathsf{proj}_{J_1}(J_2)$ and $\mathsf{proj}_{J_1}(J_2)$ are contained in the singular subspace $D = \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle$, it follows that L_1 and L_2 are collinear singular subspaces indeed. Next, note that $\mathsf{proj}_{J_1}(J_2) \cap K_2 \subseteq J_1 \cap J_2 \subseteq S$ and therefore $\mathsf{proj}_{J_1}(J_2) \cap K_2 \subseteq S \cap K_2$. On the other hand, $S \subseteq \mathsf{proj}_{J_1}(J_2)$, yielding $S \cap K_2 \subseteq \mathsf{proj}_{J_1}(J_2) \cap K_2$. Thus $S \cap K_2 = \mathsf{proj}_{J_1}(J_2) \cap K_2$. By symmetry, we also have $S \cap K_1 = \mathsf{proj}_{J_2}(J_1) \cap K_1$. Hence $\dim(\mathsf{proj}_{J_2}(J_1) \cap K_1) = \dim(S \cap K_1) = \dim(S \cap K_2) = \dim(\mathsf{proj}_{J_1}(J_2) \cap K_2)$. As $\dim \mathsf{proj}_{J_2}(J_1) = \dim \mathsf{proj}_{J_1}(J_2) = s+\alpha+1$ and $\dim K_1 = \dim K_2 = k$ the equality $\dim L_1 = \dim L_2$ follows. □

Lemma 3.8. *Let $k \leqslant s+\alpha+1$ and, when Δ is hyperbolic with $\ell \in \{(n-1)', (n-1)''\}$, suppose that β is odd and $\mathcal{N}_1 \neq \varnothing$. Let $N_1 \in \mathcal{N}_1$ and put $N_2 := J_2^{N_1}$. Furthermore, let $K_1 \subseteq \mathsf{proj}_{J_1}(J_2)$ and $K_2 \subseteq \mathsf{proj}_{J_2}(J_1)$ be k-spaces such that $K_1 \cap S = K_2 \cap S$. For $\{c,d\} = \{1,2\}$ put $L_c := \langle \mathsf{proj}_{J_d}(J_c), K_c\rangle$. Then the following hold.*

(a) $N_2 \in \mathcal{N}_2$ and $L_c \subseteq N_1 \cap N_2$ for $c = 1,2$.

(b) If K_0 is a complement to L_1 and L_2 in $N_1 \cap N_2$, then $I := \langle K_0, K_1, K_2\rangle \in \mathsf{N}(J_1, J_2)$.

Proof. The fact that $N_2 \in \mathcal{N}_2$ follows from Lemma 3.4. Turning to L_c, we have $L_c = \langle \mathsf{proj}_{J_d}(J_c), K_c\rangle$ by definition. However $K_c \subseteq \mathsf{proj}_{J_c}(J_d)$ by choice of K_c. Hence $L_c \subseteq \langle \mathsf{proj}_{J_d} J_c, \mathsf{proj}_{J_c}(J_d)\rangle$. The latter equals $J_c^{J_d} \cap J_d^{J_c}$ by Lemma 3.3. Moreover $J_1^{J_2} \subseteq N_1$ since $N_1 \in \mathcal{N}_1$ and by definition of $\mathcal{N}_1$, while $J_2^{J_1} \subseteq J_2^{N_1} = N_2$ (by definition of N_2). Ultimately, $L_c \subseteq N_1 \cap N_2$. Claim (a) is proven.

Claim (b) remains to be proved. Let K_0 and I be as stated. Clearly $I \cap J_c = K_c$ for $c = 1,2$. Hence $\dim(I \cap J_c) = k$. We claim that $I \perp J_c$. Indeed $K_0 \subseteq N_1 \cap N_2$ by choice and, still by choice, $K_c \subseteq \mathsf{proj}_{J_c}(J_d)$ for $\{c,d\} = \{1,2\}$. Hence $\langle K_1, K_2\rangle \subseteq \langle \mathsf{proj}_{J_1}(J_2), \mathsf{proj}_{J_2}(J_1)\rangle = J_1^{J_2} \cap J_2^{J_1}$ (by Lemma 3.3). Since $J_c^{J_d} \subseteq N_c$, we obtain that $I \subseteq N_1 \cap N_2$. However $N_c \supseteq J_c$ for $c = 1,2$. Therefore $I \perp J_c$, as claimed.

In order to prove that $I \in \mathrm{N}(J_1, J_2)$ it remains to show that $\dim(I) = i$. This can be obtained by a straightforward computation. Explicitly, $\dim(I) = k_0 + 2k - s' + 1$ where $k_0 := \dim(K_0)$ and $s' := \dim(K_c \cap S)$. We must prove that

$$k_0 + 2k - s' + 2 = i. \tag{3.3}$$

As K_0 is a complement of L_1 (and L_2) in $N_1 \cap N_2$, we have $k_0 = \dim(N_1 \cap N_2) - \dim(L_1) - 1$. Also $\dim(N_1 \cap N_2) = n - \beta - 2$ by Lemma 3.4 and since $N_2 = J_2^{N_1}$ by definition. On the other hand, $\dim(L_1) = s + \alpha + k - s' + 1$. It follows that $k_0 = n - \beta - s - \alpha - k + s' - 4$. Hence equation (3.3) is equivalent to the following:

$$n - \beta - s - \alpha + k - 3 = i. \tag{3.4}$$

However $j = s + \alpha + \beta + 2$. Accordingly, equation (3.4) is in turn equivalent to $n - j + k - 1 = i$, namely $i + j - k = n - 1$, which is indeed the hypothesis we assume on i, j and k throughout this paper. □

Note 3.9. *Recall that, when Δ is hyperbolic with $\ell \in \{(n-1)', (n-1)''\}$, we have $\mathcal{N}_1 \neq \varnothing$ if and only if either $\dim(J_1^{J_2}) < n-1$ or $t(J_1^{J_2}) = \ell$. We can rephrase this disjunctive condition as follows: $\mathcal{N}_1 \neq \varnothing$ if and only if $\{\ell, t(J_1^{J_2})\} \neq \{(n-1)', (n-1)''\}$, with the convention that $t(J_1^{J_2})$ stands for $\dim(J_1^{J_2})$ when $\dim(J_1^{J_2}) = n-2$ (which is not a type of Δ^b).*

The following proposition describes the composition of an element in $\mathsf{N}(J_1, J_2)$, if non-empty, see also Figure 3.1.

Proposition 3.10. *The set $\mathsf{N}(J_1, J_2)$ is non-empty if and only if*

(i) $k \leqslant s + \alpha + 1$, β odd and $\mathsf{t}(J_1^{J_2}) \neq \ell'$ (in case Δ is hyperbolic and $\{\ell, \ell'\} = \{(n-1)', (n-1)''\}$)

(ii) $k \leqslant s + \alpha + 1$ (otherwise)

Moreover, if $\mathsf{N}(J_1, J_2)$ is non-empty, then each of its elements can be constructed as follows:

1. *select a k-space K_1 in $\mathsf{proj}_{J_1}(J_2)$;*
2. *select a k-space K_2 in $\mathsf{proj}_{J_2}(J_1)$ such that $K_2 \cap S = K_1 \cap S$;*
3. *select an ℓ-space $N_1 \in \mathcal{N}_1$;*
4. *put $N_2 := J_2^{N_1}$ (where $J_2^{N_1} \in \mathcal{N}_2$);*
5. *select a subspace K_0 in $N_1 \cap N_2$ complementary to both $\langle \mathsf{proj}_{J_1}(J_2), K_2\rangle$ and $\langle \mathsf{proj}_{J_2}(J_1), K_1\rangle$.*

Then $I := \langle K_0, K_1, K_2\rangle \in \mathsf{N}(J_1, J_2)$ and each member of $\mathsf{N}(J_1, J_2)$ can be constructed in the above described way.

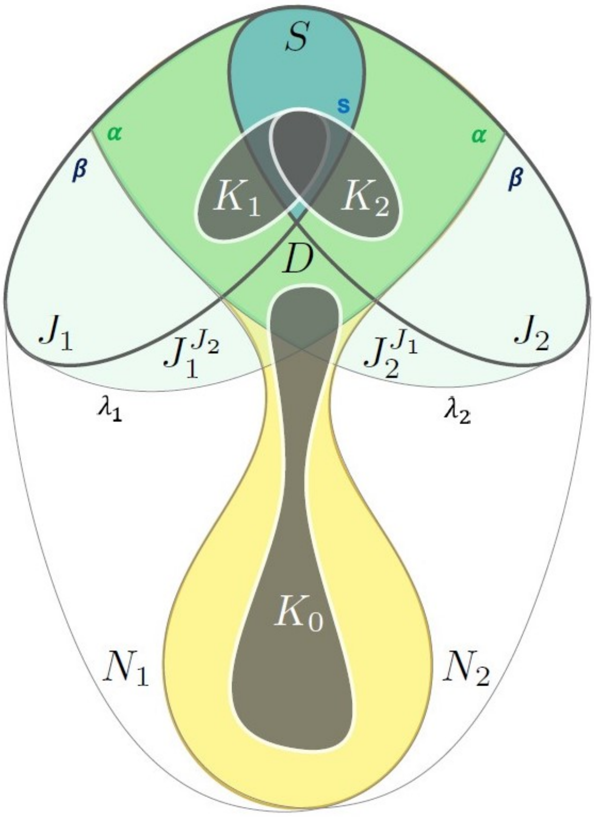

Figure 3.1: An element $\langle K_0, K_1, K_2\rangle$ of $\mathsf{N}(J_1, J_2)$.

Proof. Suppose $I \in \mathsf{N}(J_1, J_2)$. We show that statements (i) and (ii) are true and that I can be constructed as explained in steps 1 to 5. For both purposes, we make use of Lemma 3.5 as follows. For $c \in \{1, 2\}$, we put $K_c = I \cap J_c$, $N_c = \langle I, J_c\rangle$ and let K_0 be a complement of $\langle K_1, K_2\rangle$ in I and show that these choices comply with the 5 steps given above. Before doing so, we first note that $k \leqslant s + \alpha + 1$ already follows from Lemma 3.5(a). If Δ is hyperbolic and $\{\ell, \ell'\} = \{(n-1)', (n-1)''\}$, then β odd follows from (c) of the same lemma and $\mathsf{t}(J_1^{J_2}) \neq \ell'$ follows from Corollary 3.6. This asserts statements (i) and (ii). Item (b) of Lemma 3.5 shows that $K_c \subseteq \mathsf{proj}_{J_c}(J_d)$ (cf. steps 1 and 2), and that $J_c^{J_d} \subseteq N_c$. So $N_1 \in \mathcal{N}_1$ provided that $\mathsf{t}(N_1) = \ell$, which is the case since $N_1 = \langle I, J_1\rangle$ (by definition of $I \sim J_1$), confirming step 3. Now Lemma 3.5(c) tells us that $N_2 = J_2^{N_1} \in \mathcal{N}_2$, so also N_2 is as mentioned in step 4. Finally, item (d) of the same lemma shows that K_0 is chosen as in step 5.

Finally, suppose the assumptions of (i) and (ii) are met. Then Lemma 3.7(a) and $k \leqslant s+\alpha+1$ assures us that we can apply steps 1 and 2 of the construction. Step 3 is possible provided that $\mathcal{N}_1$ is non-empty, which is guaranteed by statement (i) (see Remark 3.9). Step 4 is possible by Lemma 3.8(a). Finally, a subspace K_0 as in step 5 exists by Lemma 3.7(b) and the resulting subspace I is indeed a member of $\mathsf{N}(J_1, J_2)$ by Lemma 3.8(b). This in particular shows that $\mathsf{N}(J_1, J_2)$ is non-empty. $\square$

Our first step toward a proof of Proposition 3.2 is to show the following proposition.

Proposition 3.11. *Let J_1, J_2 be distinct j-spaces with $J_1 \perp_\ell J_2$. Then* $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$.

We show this proposition in a series of technical lemmas, in most of which we suppose for a contradiction that J is a j-space in $\mathsf{N}(\mathsf{N}(J_1, J_2))$ distinct from J_1 and J_2. Our aim then is to find an i-space $I \in \mathsf{N}(J_1, J_2)$ with $I \not\sim J$, contradicting the fact that J has to be adjacent to every element of $\mathsf{N}(J_1, J_2)$.

We use the construction of $I = \langle K_1, K_2, K_0\rangle \in \mathsf{N}(J_1, J_2)$ as given in Proposition 3.10, which using the following observations, simplifies quite a bit (see below).

Suppose that $J_1 \perp_\ell J_2$. Then in particular $J_1 \perp J_2$, so $\mathsf{proj}_{J_1}(J_2) = J_1$ and $\mathsf{proj}_{J_2}(J_1) = J_2$. This means that $\beta = -1$ and also that $j = s + \alpha + 1$. Hence $k \leqslant s + \alpha + 1$. Moreover, we also have $J_1^{J_2} = J_2^{J_1} = \langle J_1, J_2\rangle$ with either $\dim\langle J_1, J_2\rangle < n - 1$ or $\mathsf{t}(\langle J_1, J_2\rangle) = \ell$, and hence $\mathcal{N}_1$ and $\mathcal{N}_2$ coincide and are non-empty.

Lemma 3.12. *Let J_1, J_2 be distinct j-spaces with $J_1 \perp_\ell J_2$. Then $\mathsf{N}(J_1, J_2)$ is non-empty and each of its elements can be constructed as follows:*

1. *select a k-space K_1 in J_1;*
2. *select a k-space K_2 in J_2 such that $K_2 \cap S = K_1 \cap S$;*
3. *select an ℓ-space $N_1 \in \mathcal{N}_1$;*
4. *put $N_2 := N_1$;*
5. *select a subspace K_0 in $N := N_1 = N_2$ complementary to both $\langle J_1, K_2\rangle$ and $\langle J_2, K_1\rangle$.*

We can choose K_0 by first choosing a subspace K_0' in $\langle J_1, J_2\rangle$ complementary to $\langle J_1, K_2\rangle$ and $\langle J_2, K_1\rangle$ and then a subspace K_0'' in N complementary to $\langle J_1, J_2\rangle$. Moreover, for each $I \in \mathsf{N}(J_1, J_2)$, the intersection $\overline{I} := I \cap \langle J_1, J_2\rangle$ satisfies $\dim(\overline{I} \cap J_c) = k$ and $\langle \overline{I}, J_c\rangle = \langle J_1, J_2\rangle$.

Proof. This follows directly from Proposition 3.10, using the observations in the paragraph preceding this lemma. □

Lemma 3.13. *Let J_1, J_2 be distinct j-spaces with $J_1 \perp_\ell J_2$. Then each $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$ is contained in the intersection of all ℓ-spaces containing $\langle J_1, J_2\rangle$. This intersection coincides with $\langle J_1, J_2\rangle$, except in the following case:*

($*$) *Δ is hyperbolic, $\ell \in \{(n-1)', (n-1)''\}$, $\dim(\langle J_1, J_2\rangle) = n - 2$ and, in case J is not contained in $\langle J_1, J_2\rangle$, then $\langle J, J_1, J_2\rangle$ is the unique ℓ-space containing $\langle J_1, J_2\rangle$ and, with $J' := J \cap \langle J_1, J_2\rangle$, we have $\dim(J' \cap I) = k - 1$ for every $I \in \mathsf{N}(J_1, J_2)$.*

Proof. In case $J_1 \perp_\ell J_2$, $\mathcal{N}_1$ and $\mathcal{N}_2$ coincide as they are the non-empty set of all ℓ-spaces containing $\langle J_1, J_2\rangle = J_1^{J_2} = J_2^{J_1}$. Take any $N \in \mathcal{N}_1$ and let p be a point contained in $N \backslash (J_1 \cup J_2)$. As one can verify, Proposition 3.10 implies that there is an element $I \in \mathsf{N}(J_1, J_2)$ with $p \in I$ (either by choosing K_1 and K_2 such that $p \in \langle K_1, K_2\rangle$ or by choosing $K_0 \subseteq N$ such that $p \in K_0$; recall also that $k < \min(|i|, |j|) \leqslant \max(|i|, |j|) < n - 1$ in view of the conventions stated at the end of Remark 1.5). So, if $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$, then $J \perp I$, in particular $J \perp p$.

Therefore, J is collinear to all points of $N \setminus (J_1 \cup J_2)$ and hence also to the subspace N generated by these points. Since N is a maximal singular subspace, we obtain $J \subseteq N$.

As $N \in \mathcal{N}_1$ was arbitrary, when either $\dim(\langle J_1, J_2 \rangle) \neq n-2$ or $\dim(\langle J_1, J_2 \rangle) = n-2$ but $\ell = |\ell|$ $(= n-1)$, the space $\langle J_1, J_2 \rangle$ is the intersection of the ℓ-spaces which contain it. In this case J, being contained in all of those ℓ-spaces, is contained in $\langle J_1, J_2 \rangle$, as claimed.

Suppose that Δ is hyperbolic, $\ell \in \{(n-1)', (n-1)''\}$ and $\dim(\langle J_1, J_2 \rangle) = n-2$. Then $\langle J_1, J_2 \rangle$ is contained as a hyperplane in a unique ℓ-space N and, by the above, N contains every j-space $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$. Given $K_1 \subset J_1$ and $K_2 \subset J_2$ with $K_1 \cap S = K_2 \cap S$, a common complement K_0' of $\langle J_1, K_2 \rangle$ and $\langle J_2, K_1 \rangle$ in $\langle J_1, J_2 \rangle$ and a point $p \in N \setminus \langle J_1, J_2 \rangle$, the subspace $K_0 = \langle p, K_0' \rangle$ is a common complement of $\langle J_1, K_2 \rangle$ and $\langle J_2, K_1 \rangle$ in N and $\langle p, K_0', K_1, K_2 \rangle$ is a member of $\mathsf{N}(J_1, J_2)$. All members of $\mathsf{N}(J_1, J_2)$ can be obtained in this way. In the sequel we denote by $\mathcal{K}$ the set of triples (K_1, K_2, K_0') as above.

Let $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$. Then $\dim(J \cap \langle p, K_0', K_1, K_2 \rangle) = k$ for every choice of $(K_1, K_2, K_0') \in \mathcal{K}$ and $p \in N \setminus \langle J_1, J_2 \rangle$. Accordingly, $k-1 \leqslant \dim(J \cap \langle K_0', K_1, K_2 \rangle) \leqslant k$ for every choice of $(K_1, K_2, K_0') \in \mathcal{K}$. By way of contradiction, suppose that $J \not\subseteq \langle J_1, J_2 \rangle$ but $\dim(J \cap \langle K_0', K_1, K_2 \rangle) = k$ for at least one choice of $(K_1, K_2, K_0') \in \mathcal{K}$. Then, with $p \in J \setminus \langle J_1, J_2 \rangle$ we obtain $\dim(J \cap \langle p, K_0', K_1, K_2 \rangle) = k+1$, a contradiction. Therefore, keeping the hypothesis that $J \not\subseteq \langle J_1, J_2 \rangle$, we have $\dim(J \cap \langle K_0', K_1, K_2 \rangle) = k-1$ for every choice of $(K_1, K_2, K_0') \in \mathcal{K}$, namely $\dim(J \cap I \cap \langle J_1, J_2 \rangle) = k-1$ for every $I \in \mathsf{N}(J_1, J_2)$, as claimed in $(*)$. $\square$

We are now going to further limit the possibilities for a j-space J in $\mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$. Eventually, we shall prove that $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$, as claimed in Proposition 3.11, thus also ruling out case $(*)$ of Lemma 3.13. However, for a while, we must take that case into account.

Lemma 3.14. *Let J_1, J_2 be distinct j-spaces with $J_1 \perp_\ell J_2$. Then each $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$ strictly contains $\langle J \cap J_1, J \cap J_2 \rangle$. Moreover, in case $(*)$ of Lemma 3.13 the intersection $J \cap \langle J_1, J_2 \rangle$ strictly contains $\langle J \cap J_1, J \cap J_2 \rangle$.*

Proof. Suppose first we are not in case $(*)$ of Lemma 3.13 and suppose for a contradiction that there is a j-space $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$ with $J = \langle J \cap J_1, J \cap J_2 \rangle$. Put $S_c := J \cap J_c$, $c \in \{1, 2\}$. Possibly by switching the roles of J_1 and J_2, we have $\dim S_1 \leqslant \dim S_2$.

We will use Proposition 3.10 to construct an i-space $I = \langle K_1, K_2, K_0 \rangle \in \mathsf{N}(J_1, J_2)$ with J not adjacent to I, hence contradicting $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$. We distinguish three cases:

Case 1: $\dim S_2 \leqslant k$**.** Choose K_c such that it contains S_c, $c = 1, 2$. Then $J = \langle S_1, S_2 \rangle \subseteq \langle K_1, K_2 \rangle$ and hence regardless of the choice of K_0, $I \cap J$ contains J. We conclude that $\dim(I \cap J) = j > k$ and therefore $I \nsim J$.

Case 2: $\dim S_2 > k \geqslant 0$**.** Note that $S_1 \cap S_2$ is a strict subspace of S_c for $c = 1, 2$, for otherwise, if $S_1 \cap S_2 = S_1$ for instance, then $S_1 \subseteq J_2$ and hence $J = \langle S_1, S_2 \rangle \subseteq J_2$, contradicting $J \neq J_2$. This allows us to select the k-space K_2 such that it is contained in S_2 but not contained in S and select the k-space K_1 such that it shares at least a point p with $S_1 \setminus S$ (note that we also rely on $k \geqslant 0$ here). Then $I \cap J$ contains $\langle K_2, p \rangle$, which has dimension $k+1$, and hence $I \nsim J$ as above.

Case 3: $\dim S_2 > k = -1$. Since $k = -1$, $\dim I = \dim K_0 \geqslant 0$. So we can select K_0 such that it shares at least a point with $J \setminus (J_1 \cup J_2) = \langle S_1, S_2\rangle \setminus (S_1 \cup S_2)$. Then again, $\dim(I \cap J) \geqslant 0 > k$.

Since $\dim S_1 + \dim S_2 + 1 = j$, we certainly have $\dim S_2 > -1$, so the above case distinction is exhaustive. In each case, we obtained an $I \in \mathsf{N}(J_1, J_2)$ with $I \notin \mathsf{N}(J)$, so the hypothesis $J = \langle S_1, S_2\rangle \in \mathsf{N}(\mathsf{N}(J_1, J_2))$ must be false.

Suppose now that J_1, J_2 and J are as in $(*)$ of Lemma 3.13 and, for a contradiction, suppose that $J' = \langle S_1, S_2\rangle$, where $J' = J \cap \langle J_1, J_2\rangle$ and $S_c = J \cap J_c$, as above. We can assume that $\dim S_1 \leqslant \dim S_2$. Recall that, according to $(*)$, $\dim(J' \cap I) = k - 1$ for every $I \in \mathsf{N}(J_1, J_2)$. Hence $k \geqslant 0$ and, consequently, $j > 0$. Hence only Case 1 and 2 survive of the above case distinction. In Case 2 we can choose $K_2 \subseteq S_2$, thus obtaining that $I \cap J' \supseteq K_2$, hence $\dim(I \cap J') \geqslant k$, a contradiction to the condition $\dim(I \cap J') = k - 1$. Turning to Case 1, with the same choice of K_1 and K_2 as above, we obtain that $I \supseteq J'$. Hence $k - 1 = \dim(I \cap J') = \dim J' = j - 1$, namely $k = j$. This contradicts the standing hypothesis $k < j$. □

Lemma 3.15. *Let J_1 and J_2 be distinct j-spaces with $J_1 \perp_\ell J_2$. Suppose $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$. Then $\dim(J \cap J_c) < k$ for $c = 1, 2$. Moreover, $\dim(J \cap J_c) < k - 1$ for $c = 1, 2$ in case $(*)$ of Lemma 3.13.*

Proof. As in the previous proof, we put $S_c := J \cap J_c$ for $c = 1, 2$. We firstly deal with the generic case of Lemma 3.13, where $J \subseteq \langle J_1, J_2\rangle$. So, assuming that $J \subseteq \langle J_1, J_2\rangle$, we introduce S_J as a subspace of J complementary to $\langle S_1, S_2\rangle$. Note that, by Lemma 3.14, S_J is non-empty. Without loss of generality, $\dim S_1 \leqslant \dim S_2$. We show that $\dim S_2 < k$. Suppose for a contradiction that $\dim S_2 \geqslant k$.

Suppose first that $\dim(S \cap J) \geqslant k$. We apply Lemma 3.12. We select $K_1 = K_2 \subseteq S \cap J$. Since $J_1 \neq J_2$, K_0' is non-empty and can be chosen such that it shares at least a point with S_J. Any $I \in \mathsf{N}(J_1, J_2)$ containing K_1, K_2, K_0' then meets J in a subspace of dimension at least $k+1$, and hence $J \nsim I$, a contradiction.

Next, suppose $\dim(S \cap J) < k$. Then we choose K_2 in S_2 such that $S \cap K_2 = S \cap S_2 = S \cap J$. If $S_1 \nsubseteq S$ then we choose K_1 such that it shares at least a point with $S_1 \setminus S$ (which is possible since $K_2 \cap S = S \cap J$ has dimension smaller than k). This again yields an element $I \in \mathsf{N}(J_1, J_2)$ with $\dim(I \cap J) \geqslant \dim(\langle K_1, K_2\rangle \cap J) \geqslant k+1$, a contradiction. So $S_1 \subseteq S$. Since $J \subseteq \langle J_1, J_2\rangle$ we have $\dim S \leqslant \dim S_1$. This means that $S = S_1 \subseteq S_2$. In particular, $S \subseteq J$. Note that this also means that $J = \langle S_2, S_J\rangle$.

We claim that we can select K_1, K_2 and K_0' (the part of I complementary to $\langle J_1, K_2\rangle$ and $\langle J_2, K_1\rangle$ in $\langle J_1, J_2\rangle$) in such a way that $\dim(\langle K_1, K_2\rangle \cap J) \geqslant k$ and $K_0' \cap S_J$ is non-empty. As before, let K_2 be any k-space in S_2 containing S. Then $S_J \cap \langle J_1, K_2\rangle \subseteq S_J \cap \langle J_1, S_2\rangle = \varnothing$. Note that, since now $S = S_1 \subseteq S_2$ and $\dim(S \cap J) < k$ by assumption, we can always assume to have chosen $K_2 \subseteq S_2$ in such a way that $K_2 \supset S$. With this choice of K_2, necessarily $K_1 \supset S$. We claim that K_1 can be chosen in such a way that $S_J \nsubseteq \langle K_1, J_2\rangle$. Indeed, suppose the contrary. Then $S_J \subseteq \cap(\langle X, J_2\rangle \mid X \in \mathcal{K}_1)$, where $\mathcal{K}_1$ is the family of k-subspaces of J_1 which contain S. However, if X an Y are two subspases of J_1 both containing S then $\langle X, J_2\rangle \cap \langle Y, J_2\rangle = \langle X \cap Y, J_2\rangle$, since $S = J_1 \cap J_2$. It follows that $\cap(\langle X, J_2\rangle \mid X \in \mathcal{K}_1) =$

$\langle\cap(X \mid X \in \mathcal{K}_1), J_2\rangle = J_2$. So, if $S_J \subseteq \langle K_1, J_2\rangle$ for every choice of $K_1 \in \mathcal{K}_1$ then $S_J \subseteq J_2$, hence $S_J = \emptyset$, a contradiction to Lemma 3.14. Therefore $S_J \not\subseteq \langle K_1, J_2\rangle$ for at least one choice of $K_1 \in \mathcal{K}_1$. With K_1 chosen in this way, we can choose K_0' in such a way that K_0' meets S_J' in at least one point, say p, as claimed.

We are now ready for a final contradiction, thus finishing the proof of the inequality $\dim S_2 < k$. Let $I \in \mathsf{N}(J_1, J_2)$ be such that $I \cap \langle J_1, J_2\rangle = \langle K_1, K_2, K_0'\rangle$ whith K_1, K_2, K_0' chosen as above. Then $I \cap J' \supseteq \langle K_2, p\rangle$ and $\dim\langle K_2, p\rangle = k+1$.

Let now J_1, J_2, J be as in case $(*)$ of Lemma 3.13. We put $J' = J \cap \langle J_1, J_2\rangle$ and now define S_J' as a complement of $\langle S_1, S_2\rangle$ in J'. Recall that $\dim(J' \cap I) = k-1$ for every $I \in \mathsf{N}(J_1, J_2)$. Assuming that $\dim S_1 \leqslant \dim S_2$, suppose for a contradiction that $\dim S_2 \geqslant k-1$. On the other hand, $\dim S_2 < k$, otherwise we can choose a k-space K_2 in S_2 which can work as a brick to build an i-space $I \in \mathsf{N}(J_1, J_2)$ such that $\dim(I \cap J') \geqslant k$. Therefore $\dim S_2 = k-1$. Accordingly, $\dim(S \cap J') \leqslant k-1$. The same argument as in the generic case, with K_1 and K_2 chosen in such a way that $K_c \cap J = K_c \cap S_c$ is as large as possibile (in particular, $K_2 \supset S_2$) shows that $\dim(S \cap J') < k-1$ and $S_1 \subseteq S$. However now, since $\langle J', J_1\rangle \subseteq \langle J_1, J_2\rangle$ but $\dim J' = j-1$, we can only claim that $\dim S_1 \geqslant \dim S - 1$. Hence either $S_1 = S$ or S_1 is a hyperplane of S. If $S_1 = S$ then an argument quite similar to the one we have used in the generic case, with J and S_J replaced by J' and S_J' and again with $K_2 \supset S_2$ leads to a contradiction with the condition $\dim(J' \cap I) = k-1$. Similarly if S_1 is a hyperplane of S but $S_2 \supseteq S$.

So, let S_1 be a hyperplane of S and $S_2 \cap S = S_1$. Hence S_2 is a hyperplane in $\langle S_2, S\rangle$. However, $\dim S_2 = k-1$. Therefeore $\dim\langle S_2, S\rangle = k$. As $S_1 \subseteq S_2$, we still have $J' = \langle S_2, S_J\rangle$. As $\langle S, S_2\rangle$ is a k-space, we can put $K_2 = \langle S, S_2\rangle$. Now we can go on just as in the generic case: $\langle K_2, J_1\rangle = \langle S_2, J_1\rangle$ is disjoint from S_J' and, if we can choose $K_1 \supseteq S = J_1 \cap K_2$ in such a way that $S_J' \not\subseteq \langle K_1, J_2\rangle$, then we can choose K_0' in such a way that $\dim(J' \cap \langle K_1, K_2, K_0'\rangle > k-1$, a contradiction. On the other hand, if $S_J' \subseteq \langle K_1, J_2\rangle$ for every k-subspace of J_1 containing S, then $S_J' \subseteq S_2$, hence $S_J' = \emptyset$, which contradicts Lemma 3.14. The proof is complete. $\square$

Proof of Proposition 3.11. Suppose for a contradiction that there is a $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$. Assume firstly that J_1, J_2 and J are not as in case $(*)$ of Lemma 3.13. By Lemmas 3.13 and 3.14, $J \subseteq \langle J_1, J_2\rangle$ and $J = \langle S_1, S_2, S_J\rangle$, where $S_c = J \cap J_c$, $c = 1, 2$, and S_J is a non-empty subspace of J complementary to $\langle S_1, S_2\rangle$. Moreover, by Lemma 3.15 we have $\dim S_c \leqslant k$ for $c = 1, 2$. Analogously as in the proof of Lemmas 3.14 and 3.15, we may assume that $\dim S_1 \leqslant \dim S_2$. So, $\dim S \leqslant \dim S_1 \leqslant \dim S_2$, as noticed in the proof of Lemma 3.15.

Let J_1' be a complement to S in J_1. Then $\langle J_1, J_2\rangle = \langle J_2, J_1'\rangle$ with $J_2 \cap J_1' = \emptyset$ and the projection of $\langle J_1, J_2\rangle$ onto J_1' along J_2 bijectively maps S_J onto a subspace S_J^1 of J_1', i.e., $S_J^1 = \langle S_J, J_2\rangle \cap J_1'$. Then S_J^1 is disjoint from S (since $S_J^1 \subseteq J_1'$) and $\dim S_J^1 = \dim S_J$. Note that $\langle S_J, J_1'\rangle$ is also disjoint from S (because $S_J \cap J_1 = \emptyset$). Therefore the projection of $\langle J_1, J_2\rangle = \langle J_2, J_1'\rangle$ onto J_2 along J_1' bijectively maps S_J onto a subspace S_J^2 of J_2 disjoint from S. Hence $S_J^2 \subseteq J_2'$ for a suitable complement J_2' to S in J_2 and we have $S_J^2 = \langle S_J, J_1\rangle \cap J_2'$. Moreover, if we project $\langle S_J, J_2\rangle = \langle S_J^1, J_2\rangle$ onto J_2 along S_J or S_J^1, we bijectively map S_J^1 or respectively S_J onto S_J^2. Accordingly, S_J^2 can also be described as $S_J^2 = \langle S_J, S_J^1\rangle \cap J_2$. So, we have constructed two disjoint subspaces S_J^1 and S_J^2 of dimension $\dim S_J$, contained in J_1 and J_2 respectively and both disjoint from S. Moreover, $S_J \subseteq \langle S_J^1, S_J^2\rangle$. In view of the way S_J^1 and S_J^2

have been defined, each point p of S_J is on a unique line p_1p_2 with $p_1 \in S_J^1$ and $p_2 \in S_J^2$ and the induced map ϕ between S_J^1 and S_J^2 with $p_1 \mapsto p_2$ is an isomorphism.

We claim that S_J^1 is disjoint from S_1. Indeed, suppose for a contradiction that $p_1 \in S_1^J \cap S_1$. Put $p_2 = \phi(p_1) \in S_J^2$ and $p = p_1p_2 \cap S_J$. Then $p_2 \in pp_1 \subseteq J$ and hence $p_2 \in J \cap J_2 = S_2$. But then $p \in p_1p_2 \in \langle S_1, S_2\rangle$, whereas $p \in S_J$ and S_J is disjoint from $\langle S_1, S_2\rangle$. The claim follows. Likewise, S_J^2 is disjoint from S_2. As a consequence, we obtain $\langle S_J^1, S_J^2\rangle \cap J = S_J$: by definition, $S_J \subseteq \langle S_J^1, S_J^2\rangle$. If S_J would be strictly contained in $\langle S_J^1, S_J^2\rangle \cap J$, then the latter is a subspace of dimension strictly bigger than $\dim S_J = \dim S_J^1 = \dim S_J^2$ in $\langle S_J^1, S_J^2\rangle$ and hence it would meet S_J^1 non-trivially, contradicting the previous claim. We now consider two cases.

Suppose first that $\dim\langle S_2, S_J^2\rangle \leqslant k$**.** In this case, we select the k-space K_2 in J_2 such that it contains $\langle S_2, S_J^2\rangle$. Since $\dim S_1 \leqslant \dim S_2$ by assumption, we also have $\dim\langle S_1, S_J^1\rangle \leqslant \dim\langle S_2, S_J^2\rangle \leqslant k$ and hence we can choose K_1 in J_1 such that it contains $\langle S_1, S_J^1\rangle$ and meets S in $K_2 \cap S = S_2 \cap S = J \cap S$ (recall that $S_J^2 \cap S = \varnothing$). Then $\langle K_1, K_2\rangle \cap J$ contains $\langle S_J^1, S_J^2\rangle \cap J = S_J$ and also $\langle S_1, S_2\rangle$, so $J \subseteq \langle K_1, K_2\rangle$.

This yields an i-space I in $\mathsf{N}(J_1, J_2)$ with $I \cap J \supseteq \langle K_1, K_2\rangle \cap J \supseteq J$, and hence $J \not\sim I$ (recall $k < j$).

Next, suppose that $\dim\langle S_2, S_J^2\rangle > k$**.** Since $\dim S_2 < k$ by Lemma 3.15, the assumption allows us to choose a subspace Y_2 in S_J^2 of the appropriate dimension such that $\langle S_2, Y_2\rangle$ is a k-space, which we select as K_2. Let Y_1 be the subspace of S_J^1 with $\phi(Y_1) = Y_2$. Then $\dim Y_1 = \dim Y_2$ and hence $\dim\langle S_1, Y_1\rangle \leqslant \dim\langle S_2, Y_2\rangle = k$, so we can choose a k-space K_1 in J_1 containing $\langle S_1, Y_1\rangle$ and meeting S in $K_2 \cap S$. Let Y be the subspace $\langle Y_1, Y_2\rangle \cap J$ and note that, analogously to $\langle S_J^1, S_J^2\rangle \cap J = S_J$ above, we have $\dim Y = \dim Y_2$. Then $\langle K_1, K_2\rangle \cap J \supseteq \langle S_1, S_2, Y\rangle$. Noting that $\dim\langle S_2, Y\rangle = \dim\langle S_2, Y_2\rangle = k$, we obtain that $\dim\langle S_1, S_2, Y\rangle > k$ provided that $S_1 \nsubseteq \langle S_2, Y\rangle$, i.e., provided that $S_1 \nsubseteq S$. In that case, this again yields an i-space I in $\mathsf{N}(J_1, J_2)$ with $\dim(I \cap J) > k$, and hence $J \not\sim I$. So, finally, suppose $S_1 \subseteq S$. Then, as $\dim S \leqslant \dim S_1$, we obtain that $S = S_1 \subseteq S_2$. Let q be a point of $S_J \setminus Y$ (which exists as $J = \langle S_1, S_2, S_J\rangle = \langle S_2, S_J\rangle$ and $\dim\langle S_2, Y\rangle = k < j$). Then q is not contained in $\langle J_1, K_2\rangle$, as $\langle J_1, K_2\rangle \cap J \subseteq \langle S_2, Y\rangle$. Likewise, q is not contained in $\langle J_2, K_1\rangle$. Hence we may select the subspace K_0' (recall Lemma 3.12) such that it contains q, leading to an i-space $I \in \mathsf{N}(J_1, J_2)$ with $\dim(I \cap J) > k$. This contradiction completes the proof in the generic case of Lemma 3.13.

Suppose now that J_1, J_2 and J are as in case $(*)$ of Lemma 3.13. So, $J' := J \cap \langle J_1, J_2\rangle$ is a hyperplane of J and $\dim(I \cap J') = k - 1$ for every $I \in \mathsf{N}(J_1, J_2)$. Chosen a complement S_J' of $\langle S_1, S_2\rangle$ in J', we define S_J^1 and S_J^2 just as in the generic case, but for replacing S_J with S_J'. As in that case, there is an isomorphism $\phi : S_J^1 \to S_J^2$ such that, for every $p_1 \in S_J^1$, the line $p_1\phi(p_1)$ meets S_J' in a point and the mapping $p_1 \in S_J^1 \to p_1\phi(p_1) \cap S_J'$ is an isomorphism from S_J^1 to S_J'. Moreover, $S_J^1 \cap S1 = S_J^2 \cap S_2 = \varnothing$. Still assuming that $\dim S_1 \leqslant \dim S_2$, now $\dim S - 1 \leqslant \dim S_1$, as remarked in the proof of Lemma 3.15 and $\dim S_2 < k - 1$, by Lemma 3.15. So, $\dim S - 1 \leqslant \dim S_1 \leqslant \dim S_2 < k - 1$.

Suppose that $\dim\langle S_2, S_J^2\rangle \leqslant k - 1$. We select the k-space K_2 in J_2 such that it contains $\langle S_2, S_J^2\rangle$. As $\dim\langle S_1, S_J^1\rangle \leqslant \dim\langle S_2, S_J^2\rangle \leqslant k - 1$, we can choose K_1 in J_1 such that it contains $\langle S_1, S_J^1\rangle$ and meets S in $K_2 \cap S$. Then $\langle K_1, K_2\rangle \cap J'$ contains both $\langle S_J^1, S_J^2\rangle \cap J' = S_J'$ and $\langle S_1, S_2\rangle$. Therefore $J' \subseteq \langle K_1, K_2\rangle$. This yields an i-space I

in $\mathsf{N}(J_1, J_2)$ with $I \cap J' \supseteq \langle K_1, K_2\rangle \cap J' \supseteq J'$, and hence $J' \subseteq I$. However $\dim(J' \cap I) = k-1$. Consequently $k-1 = \dim J' = j-1$, namely $k = j$, a contradiction.

Therefore $\dim\langle S_2, S_J^2\rangle > k-1$. As in the generic case, let Y_2 be a subspace in S_J^2 such that $\langle S_2, Y_2\rangle$ is a k-space, to be selected as K_2. Put $Y_1 = \phi^{-1}(Y_2)$. So, $Y_1 \subseteq S_J^1$ and $\dim Y_1 = \dim Y_2$, hence $\dim\langle S_1, Y_1\rangle \leqslant \dim\langle S_2, Y_2\rangle = k$ and we can choose a k-space K_1 in J_1 containing $\langle S_1, Y_1\rangle$ and meeting S in $K_2 \cap S$. Let $Y := \langle Y_1, Y_2\rangle \cap J'$. Then $\dim Y = \dim Y_2$ and $\langle K_1, K_2\rangle \cap J' \supseteq \langle S_1, S_2, Y\rangle$. Also $\dim\langle S_1, S_2, Y\rangle \geqslant \dim\langle S_2, Y\rangle = \dim\langle S_2, Y_2\rangle = k$. Therefore $\dim(J' \cap \langle K_1, K_2\rangle) \geqslant k$. However $\dim(J' \cap \langle K_1, K_2\rangle) \leqslant k-1$, since $\dim(J' \cap I) = k-1$ for every $I \in \mathsf{N}(J_1, J_2)$. We have reached a final contradiction. □

This finishes the proof of Proposition 3.11.

We can now prove Proposition 3.2.

Recall the values α and β and the subspace $D = J_1^{J_2} \cap J_2^{J_1}$ as introduced in the beginning of this section.

Proof of Proposition 3.2. By Proposition 3.11, $J_1 \perp_\ell J_2$ implies that $\mathsf{N}(\mathsf{N}((J_1, J_2)) = \{J_1, J_2\}$. We now show that this does not occur for pairs $\{J_1, J_2\}$ with a different mutual position, unless $\dim(J_1 \cap J_2) = j-1$ but only in case that $h(\Delta) = 2$ and, if Δ is hyperbolic then $\ell = n-1$. So we assume that $\mathsf{N}(\mathsf{N}((J_1, J_2)) = \{J_1, J_2\}$ and look at the imposed conditions on J_1, J_2 and Δ. In particular, this means $\mathsf{N}(J_1, J_2) \neq \varnothing$. Hence, by Proposition 3.10, $k \leqslant s+\alpha+1$ and in case Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, we also have that β odd, and $J_1^{J_2}$ does not have type ℓ'. We now study the mutual position of J_1 and J_2 (in terms of the parameters α and β).

Case 1: Suppose $\alpha \geqslant 0$**.** Suppose first that $\beta = -1$. Then J_1 and J_2 are collinear subspaces and hence $J_1 \perp_\ell J_2$ unless if Δ is hyperbolic, $\ell \in \{(n-1)', (n-1)''\}$ and $J_1^{J_2} = \langle J_1, J_2\rangle$ has type ℓ'. In the latter case however we have $\mathsf{N}(J_1, J_2) = \varnothing$ by Proposition 3.10. So assume $\beta \geqslant 0$. Then $D = J_1^{J_2} \cap J_2^{J_1}$ is strictly contained in $J_1^{J_2}$ and hence there is a j-space J distinct from J_1 which is contained in $J_1^{J_2}$ and with $J \cap D = J_1 \cap D$ (recall that $J_1 \cap D = \mathsf{proj}_{J_1}(J_2)$). By Proposition 3.10, each $I \in \mathsf{N}(J_1, J_2)$ is collinear with $J_1^{J_2}$ since step 3 says it is contained in a singular subspace N_1 containing $J_1^{J_2}$, and $I \cap J_1^{J_2}\langle K_1, K_2\rangle \subseteq D$. Therefore, $J \perp I$ and $I \cap J = I \cap J \cap D = I \cap J_1 \cap D = K_1$, i.e., $I \sim J$. We conclude that $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$, a contradiction to our hypothesis.

Case 2: Suppose $\alpha = -1$**.** First note that this implies that $\beta \geqslant 0$ (since $\alpha + \beta + 1 = j - s - 1 \geqslant 0$). Moreover, the condition $k \leqslant s + \alpha + 1$ becomes $k \leqslant s$. Lemma 3.5(b) implies that, for each $I \in \mathsf{N}(J_1, J_2)$, $K_1 = K_2 \subseteq S$ since $K_c \subseteq \mathsf{proj}_{J_c}(J_d) = S$ in this case. Let J be a j-space containing S, and let B_1, B_2 and B denote the β-spaces in $\mathrm{Res}_\Delta(S)$ corresponding to J_1, J_2, J. Note that B_1 and B_2 are opposite and hence $B_1^\perp \cap B_2^\perp$ is a (non-degenerate) polar space of rank $n-j-1 \geqslant 1$ of the same hyperbolic order as Δ. Note that $B_1^\perp \cap B_2^\perp$ is disjoint from $(B_1^\perp \cap B_2^\perp)^\perp$ as $B_1^\perp \cap B_2^\perp$ is non-degenerate.

Suppose that we can choose B (this amounts to choosing J) such that it is contained in $(B_1^\perp, B_2^\perp)^\perp$, but $B \neq B_1, B_2$. We show that $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$. To that end, take any $I \in \mathsf{N}(J_1, J_2)$. As $I \perp J_1$, in particular $I \perp S$ and hence we can consider the subspace I' in $\mathrm{Res}_\Delta(S)$ corresponding to I (i.e., to $\langle I, S\rangle$). Now $I \perp J_c$ (and hence $\langle I, S\rangle \perp J_c$) implies that I' is contained in $B_1^\perp \cap B_2^\perp$. Therefore, $I' \perp B$ and hence $\langle I, S\rangle \perp \langle J, S\rangle = J$, so $I \perp J$. Finally, $I' \cap B = \varnothing$, from which we conclude that $J \cap I \subseteq S$, meaning that $J \cap I = S \cap I = K_1$, so $I \sim J$. Now, $(B_1^\perp \cap B_2^\perp)^\perp$ contains β-spaces other than B_1 and B_2 if and only if either $\beta > 0$, or, if $\beta = 0$ and B_1 and B_2 determine a proper hyperbolic line, i.e., $h(\Delta) > 2$. If Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, then β needs to be odd in order for $\mathsf{N}(J_1, J_2) \neq \varnothing$, so the second option does not occur in this case.

We conclude that, if $\mathsf{N}(\mathsf{N}(J_1, J_2) = \{J_1, J_2\}$ but not $J_1 \perp_\ell J_2$, then necessarily $\alpha = -1$ and $\beta = 0$ and therefore $\dim(J_1 \cap J_2) = j - 1$, and this only happens if $h(\Delta) = 2$, and additionally, if Δ is hyperbolic, then $\ell = n - 1$. The lemma follows. $\square$

4. Reduction to known graphs

We define two new graphs on Ω_j and Ω_i, respectively. Note that, so far, in all we did, the roles of i and j are interchangeable because we did not impose any order on i, j.

Definition 4.1. *Let $\Gamma'_j = (\Omega_j, \sim_j)$ be the graph with vertex set Ω_j, where two vertices $J_1, J_2 \in \Omega_j$ are adjacent precisely if $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$. Likewise, we define the graph $\Gamma'_i := (\Omega_i, \sim_i)$.*

According to Proposition 3.2, the adjacency relation $\sim_j$ in the graph Γ'_j goes as follows (likewise for $\sim_i$):

(i) If $h(\Delta) > 2$ or if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$), then $J_1 \sim_j J_2$ if and only if $J_1 \perp_\ell J_2$, i.e., $\sim_j = \perp_\ell$ on Ω_j

(ii) If $h(\Delta) = 2$ and, if Δ is hyperbolic, $\ell = n - 1$, then $J_1 \sim_j J_2$ if and only if $J_1 \perp_\ell J_2$ or $\dim(J_1 \cap J_2) = j - 1$. Note that, since $\ell = n - 1$ in this case, $\perp_\ell$ is the ordinary $\perp$.

In case (ii), the difficulty is that the two relations "being collinear" and "intersecting each other in a hyperplane" are somehow fused in Γ'_j. We will reduce this case to case (i) by distinguishing the two types of relations. In Case (i), our goal will be to recognise the maximal singular subspaces (in terms of the i-spaces or j-spaces they contain), as then we can construct a graph from which the automorphism group is known. A first step towards this is to determine whether two vertices generate a maximal singular subspace.

Recall that a clique in a graph is a set of pairwise adjacent vertices.

Lemma 4.2. *Suppose $J_1, J_2 \in \Omega_j$ are Γ'_j-adjacent. Then $\mathsf{N}(J_1, J_2)$ (with respect to the graph Γ) is a non-empty clique in Γ'_i if and only if one of the following applies:*

- *J_1 and J_2 are contained in a unique ℓ-space;*
- *J_1 and J_2 are not collinear and $\dim(J_1 \cap J_2) = j - 1$, and $j = n - 2$ and $i = k + 1 \in \{0, n-2\}$, furthermore $h(\Delta) = 2$ and, if Δ is hyperbolic then $\ell = n - 1$.*

Proof. Since $J_1 \sim_j J_2$, the set $\mathsf{N}(J_1, J_2)$ is non-empty. By Proposition 3.2, either $J_1 \perp_\ell J_2$ or $\dim(J_1 \cap J_2) = j - 1$, with the latter option only occurring if $h(\Delta) = 2$ and, if Δ is hyperbolic, $\ell = n - 1$. We distinguish between those two cases.

We first show the equivalence in case J_1 and J_2 are ℓ-collinear. Recall that $\mathcal{N}_1$ denotes the set of ℓ-spaces containing $J_1^{J_2} = \langle J_1, J_2\rangle$. Then $\mathcal{N}_1$ is non-empty since $J_1 \perp_\ell J_2$ and coincides with $\mathcal{N}_2$. We need to show that $\mathcal{N}_1$ contains a unique member if and only if $\mathsf{N}(J_1, J_2)$ is a clique in Γ_i'.

By steps 3 and 4 of Proposition 3.10, each $I \in \mathsf{N}(J_1, J_2)$ is contained in a member of $\mathcal{N}_1 = \mathcal{N}_2$. If $\mathcal{N}_1$ only has one element N, then all $I \in \mathsf{N}(J_1, J_2)$ are contained in the ℓ-space N, so they are pairwise ℓ-collinear and hence adjacent in Γ_i'. Conversely, suppose there are distinct members $N, N' \in \mathcal{N}_1$ containing $\langle J_1, J_2\rangle$. Then we claim that $\mathsf{N}(J_1, J_2)$ is not a clique in Γ_i'. Let $I, I' \in \mathsf{N}(J_1, J_2)$ be such that $I \subseteq N$ and $I' \subseteq N'$. Then $\langle I, J_1\rangle = N$ and $\langle I', J_1\rangle = N'$. If $I \perp I'$ then also $N \perp N'$, and by maximality we would obtain $N = N'$. So I and I' are not collinear. So, if $I \sim_i I'$ then this means that $\dim(I \cap I') = i - 1$ and that we are in case (ii) of the above case distinction. So suppose $I \cap I' \subseteq N \cap N'$ has dimension $i - 1$. If $\dim(N \cap N') < n-2$, then I has at least a line in $N \setminus N'$ (as $N = \langle J_1, I\rangle = \langle N \cap N', I\rangle$) and hence $\dim(I \cap I') < i-1$, a contradiction. So $\dim(N \cap N') = n-2$ for all choices of N and N', and hence $\dim\langle J_1, J_2\rangle = n - 2$. We now look inside $\langle J_1, J_2\rangle$. Then $\dim(I \cap I') = i - 1$ only if the $(i-1)$-spaces $I \cap \langle J_1, J_2\rangle$ and $I' \cap \langle J_1, J_2\rangle$ coincide. If $k \geqslant 0$ then we can choose the k-spaces $K_1 = I \cap J_1$ and $K_1' = I' \cap J_1$ distinct, so necessarily $k = -1$. In that case, $I \cap \langle J_1, J_2\rangle$ and $I' \cap \langle J_1, J_2\rangle$ can be any common complements of J_1 and J_2 in $\langle J_1, J_2\rangle$, and there is more than one such complement. The claim follows.

Next, we show the equivalence in case J_1 and J_2 are not collinear and $\dim(J_1 \cap J_2) = j - 1$, under the assumption that $h(\Delta) = 2$ and that, if Δ is hyperbolic, $\ell = n - 1$. Let $I, I' \in \mathsf{N}(J_1, J_2)$ be arbitrary. Recall from Lemma 3.5(b) that $K_c = I \cap J_c$ is contained in $\mathsf{proj}_{J_c}(J_d) = S$, so $K_1 = K_2$. Hence $I = \langle K_1, K_0\rangle$ where K_0 is a subpace collinear to both J_1 and J_2. Likewise for $I' = \langle K_1', K_0'\rangle$. In $\mathrm{Res}_\Delta(S)$, K_0 and K_0' correspond to $(i-k-1)$-spaces $\tilde{K}_0$ and $\tilde{K}_0'$ in $p_1^\perp \cap p_2^\perp$, where p_1, p_2 are the (opposite) points corresponding to J_1, J_2, respectively. We choose $\tilde{K}_0$ and $\tilde{K}_0'$ such that they correspond to opposite (and hence non-collinear) subspaces in $p_1^\perp \cap p_2^\perp$ (which is a polar space of rank $n - j - 1 \geqslant 1$). Then I and I' can only be adjacent in Γ_i' if they intersect each other in a hyperplane, so if $K_1 = K_1'$ and $\tilde{K}_0$ and $\tilde{K}_0'$ are just points. The first fact implies that either $k = -1$ or $k = \dim S = j - 1$; the second fact implies that $i - k - 1 = 0$, or equivalently (since $i + j - k = n - 1$) $j = n - 2$. So in case $k = -1$, we obtain $i = k+1 = 0$; and in case $k = j-1 = n-3$ we obtain $i = k+1 = n-2$. We verify that in these cases, $\mathsf{N}(J_1, J_2)$ is indeed a clique. Firstly, if $i = 0$, then Γ_i' is a complete graph, so any subset is a clique. Secondly, if $(k, i, j) = (n-3, n-2, n-2)$, then all members of $\mathsf{N}(J_1, J_2)$ contain the $(n-3)$-space S and hence they are adjacent in Γ_i'. The lemma follows. □

As a corollary, we have:

Corollary 4.3. *The graph Γ_j' has cliques of the form $N(I_1, I_2)$ with $I_1 \sim_i I_2$ if and only if*

$$\begin{cases} i \geqslant \frac{n-2}{2} & \text{if } \Delta \text{ is not hyperbolic or if } \ell = n-1 \text{ when } \Delta \text{ is hyperbolic} \\ i \geqslant \frac{n-3}{2} & \text{if } \Delta \text{ is hyperbolic and } \ell \in \{(n-1)', (n-1)''\} \end{cases}$$

Moreover, at least one of i, j satisfies these inequalities.

Proof. Suppose that i is strictly smaller than the given values, but that there is a clique $\mathsf{N}(I_1, I_2)$ as described in the statement. Noting that $\frac{n-3}{2} < \frac{n-2}{2} \leqslant n-2$ because $n \geqslant 2$, Lemma 4.2 implies that I_1 and I_2 are contained in a unique ℓ-space. In case Δ is not hyperbolic, or if $\ell = n-1$ when Δ is hyperbolic, this means that $\dim\langle I_1, I_2\rangle = n-1$. However, $\dim\langle I_1, I_2\rangle < 2\frac{n-2}{2}+1 = n-1$, a contradiction. In case Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, it means that $\dim\langle I_1, I_2\rangle \geqslant n-2$, whereas this time $2i+1 < 2\frac{n-3}{2}+1 = n-2$. The claim follows. The converse statement is easily checked: if $i \geqslant \frac{n-2}{2}$, then each $n-1$-space contains two i-spaces I_1, I_2 generating it and then $N(I_1, I_2)$ gives such a clique; likewise if $i \geqslant \frac{n-3}{2}$ when Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, then each $(n-2)$-space contains two i-spaces generating it.

Finally, suppose $i < \frac{n-a}{2}$ for $a \in \{2, 3\}$, then $j = n-1+k-i \geqslant n-i-2 > n-2-\frac{n-a}{2} > \frac{n+a-4}{2} \geqslant \frac{n-2}{2}$. □

So if $\ell = n-1$ and $\min\{i, j\} < \frac{n-2}{2}$, or Δ is hyperbolic, $\ell \in \{(n-1)', (n-1)''\}$ and $\min\{i, j\} < \frac{n-3}{2}$, the above corollary gives us a way to detect which bipart of Γ contains the i-spaces and which contains the j-spaces.

4.1. Distinguishing the adjacency relations

Throughout this subsection we assume that we are in Case (ii): $h(\Delta) = 2$ with, if Δ is hyperbolic, also $\ell = n-1$, and hence adjacency in Γ'_i and Γ'_j is given by "being (ℓ-)collinear" or "intersecting each other in a hyperplane". Note that ℓ-collinearity coincides with ordinary collinearity here.

Recall that $\max\{i, j\} < n-1$ by Remark 1.5. If $\max\{i, j\} < n-2$, Lemma 4.2 allows us to determine whether two distinct j-spaces J_1 and J_2 are contained in a unique ℓ-space, this is namely the case when $J_1 \sim_j J_2$ and $\mathsf{N}(J_1, J_2)$ (with respect to Γ) is a non-empty clique in Γ'_i; likewise for the i-spaces. If, say, $j = n-2$ however, then in certain circumstances (such as $\ell = n-1$ if Δ is hyperbolic) there can also be j-spaces J_1, J_2 like that which are not contained in a unique ℓ-space (instead, they meet each other in a subspace of dimension $j-1 = n-3$). Note that two $(n-2)$-spaces which are contained in a unique ℓ-space also meet each other in a $(n-3)$-space. This observation will be a key to deal with the case $\max\{i, j\} = n-2$ in a different way.

Lemma 4.4. *Suppose $h(\Delta) = 2$ and, if Δ is hyperbolic, then $\ell = n-1$. Then in the graph Γ'_{n-2}, with Ω_{n-2} as vertex set, adjacency is given by intersecting each other in a subspace of dimension $n-3$. Moreover, each automorphism of Γ'_{n-2} is induced by an automorphism of Δ and vice versa.*

Proof. Suppose that $X_1 \sim_{n-2} X_2$. Then $X_1 \perp X_2$ or $\dim(X_1 \cap X_2) = n-3$, according to the definition of Γ'_{n-2}. In the first case, $X_1 \neq X_2$ implies that $\dim\langle X_1, X_2\rangle = n-1$ and hence $\dim(X_1 \cap X_2) = n-3$ too. By Lemma 5.2 of [DSVM18], we can construct the $(n-2)$-Grassmann graph $\mathsf{G}_{n-2}(\Delta)$ from $\Gamma'_{n-2}(\Delta)$, i.e., the graph on the $(n-2)$-spaces of Δ where two such vertices are adjacent if they correspond to *collinear* subspaces that meet each other in a subspace of dimension $(n-3)$. It is well known (we refer to Corollary 5.4 and Lemma 5.5

in [DSVM18] for further reading, but this is not the first occurrence by far) that the $(n-2)$-Grassmann graph $\mathsf{G}_{n-2}(\Delta)$ uniquely determines Δ if Δ is not hyperbolic, and possibly up to duality (interchanging the two families of maximal singular subspaces) if Δ is hyperbolic (no trialities occur if $n = 4$ since $j = 2$, so the planes are preserved). This means that each automorphism ρ of Γ'_{n-2} extends naturally to an automorphism of Δ and hence we say that ρ is induced by an automorphism of Δ. The result follows. $\square$

An important remark to be made here is that we can only use Lemma 4.4 when $\max\{|i|,|j|\} = n-3$ provided that we can recognise which bipart of Γ contains the $(n-2)$-spaces (i.e., whether it is Γ'_i or Γ'_j which contains the $(n-2)$-spaces). So we now summarize the situations under which we can do this. Recall that a polar space Δ of rank n admits order (s,t) if every line of Δ contains exactly $s+1$ points and every singular subspace of dimension $n-2$ is contained in just $t+1$ maximal singular subspace. Note that every polar space of rank $n \geqslant 3$ admits an order. Given $n \geqslant 3$, the classification of polar spaces gives us that $s < \infty$ implies $t < \infty$.

Lemma 4.5. *Suppose $h(\Delta) = 2$ and, if Δ is hyperbolic, then $\ell = n-1$. If $\max\{i,j\} = n-2$, then we can recognise which one of Γ'_i, Γ'_j is Γ'_{n-2}, in case one of the following occurs:*

(A1) $\min\{i,j\} < \frac{n-2}{2}$

(A2) $i = j = n-2$

(A3) $\frac{n-2}{2} \leqslant \min\{i,j\} < n-2$ *and Δ has order $(2,t)$ with $t < \infty$.*

Proof. First, if $\min\{i,j\} < \frac{n-2}{2}$ then this follows from Corollary 4.3. This allows us to detect which bipart contains the $(n-2)$-spaces. Second, if $i = j = n-2$ then both biparts contain $(n-2)$-spaces so either choice is fine. Third, if the order (s,t) of Δ is such that $s = 2$ then we want to tell the biparts apart by counting. The required calculations have been done by the second author in a note [Pas21] standing in support of this paper (as alluded to in the introduction). For ease of notation, we put $i = \min\{i,j\}$ and $j = \max\{i,j\} = n-2$. Let χ_i be the degree of a vertex in Γ'_i, likewise we define χ_{n-2}. It follows from Proposition 3.6 of [Pas21] that $\chi_i > \chi_{n-2}$, enabling us to distinguish the biparts. Worthwhile mentioning is that $|\Omega_i| \neq |\Omega_{n-2}|$, and in most cases even $|\Omega_i| > |\Omega_{n-2}|$ unless if $n = 5$, $i = 2$ and $t = 2$, as then $|\Omega_i| = |\Omega_{n-2}|$ and possibly if i is very close to $\frac{n}{2}$, it could happen that $|\Omega_i| < |\Omega_{n-2}|$. For more details we refer to [Pas21] (in particular, Proposition 2.9).

We conclude that in all three cases we are indeed able to select a bipart of Γ containing $(n-2)$-spaces and hence we know which of the two graphs introduced in Definition 4.1 is Γ'_{n-2}. $\square$

In case $\max\{i,j\} = n-2$ but the situations described in (A1), (A2) and (A3) do not apply (meaning that $\frac{n-2}{2} \leqslant \min\{i,j\} < n-2$ and $s > 2$); or in case $\max\{i,j\} < n-2$, we want to be able to determine when two vertices of either Γ'_i or Γ'_j correspond to collinear subspaces. In case $\min\{i,j\} < \frac{n-2}{2}$, Corollary 4.3 again enables us to recognise which of Γ'_i and Γ'_j contains the larger subspaces and in this one we will be able to determine whether two adjacent vertices are collinear subspaces or not; In case $\min\{i,j\} \geqslant \frac{n-2}{2}$ we do not have a way to distinguish Γ'_i and Γ'_j but for both i and j we will be able to detect whether adjacent vertices correspond to collinear subspaces or not.

Lemma 4.6. *Suppose $h(\Delta) = 2$ and, if Δ is hyperbolic, then $\ell = n-1$. We assume that either $\frac{n-2}{2} \leqslant \min\{i,j\} < \max\{i,j\} = n-2$ and $s > 2$ or that $\max\{i,j\} < n-2$. Suppose $x \in \{i,j\}$ has $x \geqslant \frac{n-2}{2}$ and let $y := \{i,j\} \setminus \{x\}$. Then $X_1, X_2 \in \Omega_x$ are collinear if and only if there are Y_1, Y_2 in Ω_y such that $\mathsf{N}(Y_1,Y_2)$ is a clique in Γ'_x containing X_1, X_2.*

Proof. For ease of notation we write $i = \min\{i,j\}$ and $j = \max\{i,j\}$. We focus on the case where $x = i$ and hence $y = j$. If $i \geqslant \frac{n-2}{2}$, we also incorporate the case where $x = j$ and $y = i$ by changing the roles of the i- and j-spaces and putting additional arguments, if any are needed, in italics, not to break with the flow of the argument too much.

Suppose that I_1 and I_2 are collinear. We apply the construction mentioned in Proposition 3.10 to I_1, I_2. Let $N \in \mathcal{N}_1$ be an ℓ-space containing $\langle I_1, I_2\rangle$. We show that we can select $J_1, J_2 \in \mathsf{N}(I_1,I_2)$ such that $\langle J_1, J_2\rangle = N$. If J_1 and J_2 are just any j-spaces generating N, then $2j - \dim(J_1 \cap J_2) = n-1 = j+i-k$ and hence $\dim(J_1 \cap J_2) = j-i+k$. So if we can choose J_1 and J_2 in $\mathsf{N}(J_1,J_2)$ in N such that $J_1 \setminus J_2$ contains a subspace of dimension $j-(j-i+k)-1 = i-k-1$, we have $\langle J_1, J_2\rangle = N$. *Observe that this will also work with i-spaces and j-spaces interchanged, provided that $i \geqslant \frac{n-2}{2}$, because we only use that two j-spaces can generate a maximal singular subspace.* We write $J_1 = \langle K_1, K_2, K'_0, K''_0\rangle$, with notation as before: $K_c = J_1 \cap I_c$, $c = \{1,2\}$, K'_0 is a common complementary subspace of $\langle I_1, K_2\rangle$ and $\langle I_2, K_1\rangle$ inside $\langle I_1, I_2\rangle$ and finally K''_0 is a subspace of N complementary to $\langle I_1, I_2\rangle$; likewise, $J_2 = \langle \overline{K}_1, \overline{K}_2, \overline{K}'_0, \overline{K}''_0\rangle$.

Suppose first that $\dim(I_1 \cap I_2) \leqslant k$ and let $K_1 = \overline{K}_1$, $K_2 = \overline{K}_2$ be such that they contain $I_1 \cap I_2$. Then $\dim K'_0 = \dim \overline{K}'_0 = i-k-1$ because $\langle K'_0, I_2, K_1\rangle = \langle I_1, I_2\rangle = \langle \overline{K}'_0, I_2, K_1\rangle$. Put $K := \langle K_1, K_2\rangle$ and $I := \langle I_1, I_2\rangle$. We claim that we can choose K'_0 and $\overline{K}'_0$ such that in $\mathrm{Res}_I(K)$ they correspond to $(i-k-1)$-spaces which are disjoint from each other and from the two $(i-k-1)$-spaces corresponding to $\langle I_1, K_2\rangle$ and $\langle I_2, K_1\rangle$. Indeed, if $i-k-1 > 0$ this is clearly possible; If $i-k-1 = 0$, then $\mathrm{Res}_I(K)$ is a line containing $s+1$ points. We need to choose the points corresponding to K'_0 and $\overline{K}'_0$ as distinct points on this line, different from the points corresponding to $\langle I_1, K_2\rangle$ and $\langle I_2, K_1\rangle$. Since $i = k+1$ implies $j = n-2$ (because $i+j-k = n-1$), our assumptions imply that $s > 2$ and hence the points can be chosen as required. *Note that, when interchanging the i- and j-spaces, then $i = n-2$ also implies $j = n-2$ and therefore the rest of the argument remains the same.* The claim follows. Therefore $J_1 \setminus J_2$ contains K'_0, and hence $\langle J_1, J_2\rangle = N$ and $J_1, J_2 \in \mathsf{N}(I_1,I_2)$.

Next, suppose that $d := \dim(I_1 \cap I_2) > k$. We choose $K_1 = K_2$ and $\overline{K}_1 = \overline{K}_2$ such that $K_1 \cap \overline{K}_1$ is minimal, so if $d \leqslant 2k+1$, then $\langle K_1, \overline{K}_1\rangle = I_1 \cap I_2$ and if $d > 2k+1$, then K_1 and $\overline{K}_1$ are disjoint. Let M be a subspace of K_1 complementary to $K_1 \cap \overline{K}_1$ in K_1 and let $\overline{M}$ be defined likewise. Put $m := \dim M = \dim \overline{M}$ and note that

$$m = \begin{cases} d-k-1 & \text{if } d \leqslant 2k+1 \\ k & \text{if } d > 2k+1 \end{cases}.$$

Now $\dim K'_0 = \dim \overline{K}'_0 = i-d-1$ since K'_0 is complementary to $\langle I_1, K_2\rangle = I_1$ in $\langle I_1, I_2\rangle$. Therefore

$$\dim\langle M, K'_0\rangle = \dim\langle \overline{M}, \overline{K}'_0\rangle = \begin{cases} (d-k-1)+(i-d-1)+1 = i-k-1 & \text{if } d \leqslant 2k+1 \\ k+(i-d-1)+1 = i-d+k & \text{if } d > 2k+1 \end{cases}.$$

This time we want to choose K_0' and $\overline{K}_0'$ such that they correspond to disjoint $(i-d-1)$-spaces in $\mathrm{Res}_I(I_1 \cap I_2)$ (again $I = \langle I_1, I_2\rangle$), and also disjoint from the $(i-d-1)$-spaces corresponding to I_1 and I_2. As above, this can only fail if $i-d-1=0$ and $s=2$. We now treat this problematic case.

So suppose $i = d+1$ and $s = 2$. This means that there is a unique i-space I^* distinct from I_1 and I_2 containing $I_1 \cap I_2$; and I^* will contain both $\langle M, K_0'\rangle$ and $\langle \overline{M}, \overline{K}_0'\rangle$; which have dimension $m+1$. We can choose them disjoint provided that $2(m+1)+1 \leqslant i = d+1$, or equivalently, $2(m+1) \leqslant d$. Geometrically this means that $\langle M, \overline{M}\rangle$ is a strict subspace of $I_1 \cap I_2$. This is the case unless $d = 2k+1$ and $k = m$. So if $d \neq 2k+1$, we can make sure that $\langle M, K_0'\rangle$ and $\langle \overline{M}, \overline{K}_0'\rangle$ are disjoint. If $d = 2k+1$ and $k = m$ however, i.e., if K_1 and $\overline{K_1}$ are disjoint subspaces generating $I_1 \cap I_2$, then each choice of K_0' and $\overline{K}_0'$ is such that $\langle M, K_0'\rangle$ and $\langle \overline{M}, \overline{K}_0'\rangle$ share a point. So, $J_1 \setminus J_2$ contains the k-space $M = K_1$, and noting that $i = 2k+2$ in this case and hence $i-k-1 = k+1$, this is just one dimension short of what we need in order for J_1 and J_2 to generate N. By our assumption, $s = 2$ implies that $\max\{i,j\} < n-2$, and hence $\langle I_1, I_2\rangle$ (which has dimension $i+1 < n-1$) is strictly contained in the maximal singular subspace N. In this case $\langle K_1, K_0', \overline{K}_1, \overline{K}_0'\rangle$ is an i-space in the $(i+1)$-space I, and in N there are at least two $(i+1)$-spaces through $\langle K_1, K_0', \overline{K}_1, \overline{K}_0'\rangle$ distinct from I, so we can take points p_0 and $\overline{p}_0$ in these respective $(i+1)$-spaces and select $K_0'' \ni p_0$ and $\overline{K}_0'' \ni \overline{p}_0$ and the remaining part (if any) of K_0'' and $\overline{K}_0''$ can be chosen as coinciding subspaces in N outside the $(i+2)$-space $\langle I, p_0, \overline{p}_0\rangle$.

So, except for the case we just dealt with in the last few lines, we have constructed (parts of the) j-spaces J_1, J_2 such that $J_1 \setminus J_2$ contains $\langle M, K_0'\rangle$. If $d \leqslant 2k+1$ then $\dim\langle M, K_0''\rangle = i-k-1$ (see above), which as explained before suffices to obtain $J_1, J_2 \in \mathsf{N}(I_1, I_2)$ with $\langle J_1, J_2\rangle = N$. Therefore, if $d \leqslant 2k+1$ we are done. If $d > 2k+1$, then we have chosen $M = K_1$, $\overline{M} = \overline{K}_1$, K_0' and $\overline{K}_0'$ such that $\langle M, K_0'\rangle$ and $\langle \overline{M}, \overline{K}_0'\rangle$ are disjoint. The subspaces K_0'' and $\overline{K}_0''$ can be choosen freely in $N \setminus \langle I_1, I_2\rangle$ and hence $J_1 = \langle M, K_0', K_0''\rangle$ and $J_2 = \langle \overline{M}, \overline{K}_0', \overline{K}_0''\rangle$ generate N.

Proposition 3.10 (step 3) readily gives that each element of $\mathsf{N}(J_1, J_2)$ is contained in $\langle J_1, J_2\rangle = N$ and hence $\mathsf{N}(J_1, J_2)$ is a clique in Γ_i', containing I_1, I_2 by construction.

Conversely, suppose $\mathsf{N}(J_1, J_2)$ is a clique in Γ_i' containing I_1, I_2. It then follows from Lemma 4.2 that either there is a unique ℓ-space N containing J_1 and J_2, or a second option in which $j = n-2$ and $i \in \{0, n-2\}$. However, our assumptions imply that if $j = n-2$, then $0 < \frac{n-2}{2} \leqslant i < n-2$, excluding the second option. Again, by step 3 of Proposition 3.10, $I_1, I_2 \subseteq N$ and hence $I_1 \perp I_2$. *If $i \geqslant \frac{n-2}{2}$ and we interchange the role of the i- and j-spaces in this paragraph, then $i = n-2$ is excluded by the hypothesis of this lemma. So also here, Lemma 4.2 implies that I_1 and I_2 are contained in a unique ℓ-space, which contains J_1 and J_2.* □

Hence we can, from at least one of Γ_i', Γ_j' deduce the graph on the i-spaces or j-spaces of Δ with collinearity (which is the same as ℓ-collinearity, since $\ell = n-1$) as adjacency relation. We treat the non-bipartite graphs with ℓ-collinearity as adjacency relation in the next section.

5. Conclusion

Finally, we show the main theorem.

Proposition 5.1. *Let ρ be any automorphism of $\Gamma^n_{i,j;k,\ell}(\Delta)$, where $i, j < n-1$ and $i+j-k = n-1$. Then there is an automorphism $\tilde{\rho}$ of Δ (possibly a duality if Δ is hyperbolic) such that $\rho(X) = \tilde{\rho}(X)$ for each $X \in \Omega_i \cup \Omega_j$. Only if $i = j$, ρ possibly switches the biparts. Conversely, each automorphism of Δ induces a unique automorphism of $\Gamma^n_{i,j;k,\ell}(\Delta)$ that does not interchange the biparts.*

Proof. Clearly, each automorphism α of Δ induces a unique automorphism of $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$ that does not change biparts, by letting α act on both biparts. If $i = j$, then we can compose this automorphism of Γ with an automorphism of Γ that interchanges biparts.

Conversely, let ρ be an automorphism of Γ that does not switch the biparts. Let $x \in \{i, j\}$. We introduce the graph $\Gamma''_x = (\Omega_x, \perp_\ell)$, where adjacent vertices are ℓ-collinear. There are three scenarios:

1. If $h(\Delta) > 2$ or if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, then Γ''_x coincides with Γ'_x for $x \in \{i, j\}$ by Lemma 3.2(i) and Definition 4.1.

2. If $h(\Delta) = 2$ and $\ell = n-1$ if Δ is hyperbolic, then two vertices of Γ'_x can also be adjacent if they meet in an $(x-1)$-space. So here we use the material of the previous section:

   (a) If $\max\{i, j\} = n-2$ and at least one of (A1), (A2) or (A3) of Lemma 4.5 holds, then it follows from Lemma 4.4 that ρ is induced by an automorphism of Δ.

   (b) If $\max\{i, j\} = n-2$ and none of (A1), (A2) and (A3) holds (which means that $\frac{n-2}{2} \leqslant \min\{i, j\} < n-2$ and $s > 2$) or if $\max\{i, j\} < n-2$, then we use Lemma 4.6 to construct Γ''_x for $x \in \{i, j\}$ with $x \geqslant \frac{n-2}{2}$.

So suppose we are in case 1 or 2(b) and we obtained Γ''_x (with $x \geqslant \frac{n-2}{2}$ in case 2(b)). In both cases, we can additionally determine whether or not two ℓ-collinear x-spaces are contained in a unique ℓ-space with the help of Lemma 4.2. We suppose $x \in \{i, j\}$ is such that there are two x-spaces that are contained in a unique ℓ-space. Note that, according to Corollary 4.3, this is the case precisely if $x \geqslant \frac{n-2}{2}$ (if Δ is not hyperbolic or $\ell = n-1$) or if $x \geqslant \frac{n-3}{2}$ (if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$); and this is the case for at least one $x \in \{i, j\}$.

Let X_1 and X_2 be two x-spaces which are contained in a unique ℓ-space. We then look at the set S of vertices of Γ''_x which are collinear to both X_1 and X_2. If $\dim\langle X_1, X_2\rangle = n-1$ (so $\langle X_1, X_2\rangle$ has type ℓ), then S is the set of all x-spaces in $\langle X_1, X_2\rangle$ and all vertices in S are pairwise collinear and hence adjacent in Γ''_x (note that S is a maximal clique in Γ''_x even). Now suppose $\dim\langle X_1, X_2\rangle = n-2$, which only occurs if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$. Then there are two maximal singular subspaces N_1 and N_2 containing $\langle X_1, X_2\rangle$. In this case, S is the union of two maximal cliques S_1 and S_2 of Γ''_x, one containing all x-spaces in N_1 and the other containing all x-spaces in N_2, respectively (note that $S_1 \cap S_2$ corresponds to the set of all x-spaces in $\langle X_1, X_2\rangle$). We want to recognise which maximal clique corresponds to the ℓ-space containing $\langle X_1, X_2\rangle$. If $x \geqslant \frac{n-2}{2}$, we can find two x-spaces X'_1 and X'_2 in S_1 which generate N_1, likewise for N_2. Exactly one pair of them is ℓ-collinear, so with the help of Lemma 4.2 we can decide which one of N_1, N_2 is the ℓ-space. If $x = \frac{n-3}{2}$, then all pairs of x-spaces in S_1 and S_2 are adjacent in Γ''_x and hence we can recognise this situation. In this case, we change x to $\{i,j\} \setminus \{x\}$ (and we also denote this by x), because at least one of $\{i,j\}$ is greater or equal to $\frac{n-2}{2}$ (since $i+j-k = n-1$).

In all cases we can hence obtain the ℓ-spaces of Δ in terms of the x-spaces they contain. So we can construct a bipartite graph isomorphic to the *containment graph* $\mathsf{C}^n_{x,\ell}(\Delta)$ containing Ω_x and Ω_ℓ as bipartition classes, where two vertices are adjacent when one is contained in the other. Result [DSVM18, Proposition 7.4] now implies that each automorphism of this graph is induced by an automorphism $\tilde{\rho}$ of Δ (possibly a duality in case Δ is hyperbolic and $\ell = n-1$) and vice versa.

We thus have that $\tilde{\rho}$ and ρ coincide on the bipart Ω_x of Γ. We claim that $\tilde{\rho}$ and ρ also coincide on the other bipart of Γ. Indeed, consider the composition τ of ρ^{-1} and the automorphism of Γ induced by $\tilde{\rho}$. Then τ is an automorphism of Γ which is the identity on Ω_x. If τ is not the identity on Ω_y, the other bipart, then there are Y and $\tau(Y)$ in Ω_y with $Y \neq \tau(Y)$, and for such Y we have $\mathsf{N}(Y) = \mathsf{N}(\tau(Y))$. One then deduces that $Y^\perp = \tau(Y)^\perp$, which is only possible if $Y = \tau(Y)$, a contradiction. So τ is the identity on Γ and hence $\tilde{\rho}$ and ρ coincide on $\Omega_i \cup \Omega_j$ indeed.

Finally, suppose that ρ is an automorphism of Γ which switches the biparts. We claim that this is only possible if $i = j$. Suppose for ease of notation that $i \leqslant j$. If the above only works for one value $x \in \{i,j\}$ (meaning that meaning that $i < \frac{n-3}{2}$ if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$ and $i < \frac{n-2}{2}$ otherwise) then the biparts of Γ behave differently and hence cannot be interchanged. If the above works for both $x = i$ and $x = j$ (meaning that $i \geqslant \frac{n-3}{2}$ if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$ and $i \geqslant \frac{n-2}{2}$ otherwise, we need to do some more work. Let M be an ℓ-space and let C^j_M be the set of all j-spaces contained in M, C^i_M is defined likewise. Put $C^j = \{C^j_M \mid \mathsf{t}(M) = \ell\}$ and likewise C^i is defined. Given C^j_M, we want to recognise which member of C^i corresponds to C^j_M. So suppose J_1 and J_2 are j-spaces in C^j_M with $M = \langle J_1, J_2 \rangle$ (since $j \geqslant \frac{n-3}{2}$ when Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$ and $j \geqslant \frac{n-2}{2}$ otherwise, this is possible). By Proposition 3.10 step 3), $I \in \mathsf{N}(J_1, J_2)$ is contained in M. Conversely, analogously as before one deduces from the construction of Proposition 3.10 that each point of $M \setminus (J_1 \cup J_2)$ is contained in a member of $I \in \mathsf{N}(J_1, J_2)$. The only subspace containing the points $M \setminus (J_1 \cup J_2)$ is M. We obtain that the members of $\mathsf{N}(J_1, J_2)$ generate M and hence C^i_M is the unique member of C^i containing $\mathsf{N}(J_1, J_2)$. We conclude that each maximal singular subspace of Δ of type ℓ can be expressed both in terms of the i-spaces and in terms of the j-spaces it contains. Now, for any $I \in \Omega_i$, let $\mathcal{M}_I$ be the set of ℓ-spaces containing I, and likewise we define $\mathcal{M}_J$ for any $J \in \Omega_j$. Then it is clear that $I \subsetneq J$ if and only if $\mathcal{M}_J \subsetneq \mathcal{M}_I$. We shall now prove that, if ρ switches the two biparts of Γ, then $i = j$. We have shown that the containement graphs $C^n_{i,\ell}(\Delta)$ and $C^n_{j,\ell}(\Delta)$ can be recovered from Γ. Accordingly, ρ also induces an isomorphism $\rho_{i,j}$ from $C^n_{i,\ell}(\Delta)$ onto $C^n_{j,\ell}(\Delta)$ as well as an isomorphism $\rho_{j,i}$ from $C^n_{j,\ell}(\Delta)$ onto $C^n_{i,\ell}(\Delta)$. In particular, it defines two permutations of Ω_ℓ, which we can denote by $\rho_{i,j}$ and $\rho_{j,i}$. Clearly, $\rho_{i,j}(\mathcal{M}_I) = \mathcal{M}_{\rho(I)}$ and $\rho_{j,i}(\mathcal{M}_J) = \mathcal{M}_{\rho(J)}$. Suppose now $i < j$ and let $I \subsetneq J$. Then $\mathcal{M}_J \subsetneq \mathcal{M}_I$. Accordingly, $\mathcal{M}_{\rho(J)} = \rho_{j,i}(\mathcal{M}_J) \subsetneq \rho_{j,i}(\mathcal{M}_I)$. However, $I = \rho^{-1}(J')$ for a (unique) $J' \in \Omega_j$. Hence $\mathcal{M}_I = \mathcal{M}_{\rho^{-1}(J')} = \rho^{-1}_{j,i}(\mathcal{M}_{J'})$. Hence $\rho_{j,i}(\mathcal{M}_I) = \rho_{j,i}\rho^{-1}_{j,i}(\mathcal{M}_{J'}) = \mathcal{M}_{J'}$.

Thus, with $I' := \rho(J)$, we have $\mathcal{M}_{I'} \subsetneq \mathcal{M}_{J'}$, which forces $I' \supsetneq J'$. Contradiction. So, $i = j$.

Now, if $i = j$, then consider the bipart-switching automorphism s of Γ induced by the identity on Δ. Then $\rho \circ s$ is an automorphism of Γ which does not switch the biparts, and hence by the above, $\rho \circ s$ is induced by an automorphism of Δ. It follows that ρ itself is also induced by an automorphism of Δ. □

Proposition 5.1 concludes the proofs of Theorem 1.2 and 1.3.

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