

ASYMPTOTIC DISTRIBUTION OF PARAMETERS IN TRIVALENT MAPS AND LINEAR LAMBDA TERMS

Olivier Bodini¹, Alexandros Singh², and Noam Zeilberger³

¹*EREN, Université Sorbonne Paris Nord, 93430 Villetaneuse, France
olivier.bodini@sorbonne-paris-nord.fr*

²*LIASD, Université Paris 8, 93200 Saint-Denis, France
alexandros.singh07@univ-paris8.fr*

³*LIX, École Polytechnique, 91120 Palaiseau, France
noam.zeilberger@polytechnique.edu*

Submitted: Oct 6, 2021; Accepted: Feb 28, 2025; Published: Jul 15, 2025

© The authors. Released under the CC BY license (International 4.0).

Abstract. In this work, we study the limit distributions of various combinatorial parameters in trivalent maps, linear λ -terms, and other related families of objects. We focus on parameters in maps which naturally correspond to parameters in λ -terms and vice versa, allowing us to employ techniques from map theory and the λ -calculus in a combinatorial interplay. Some examples of the parameters we study are: the number of bridges in rooted trivalent maps and of subterms in closed linear λ -terms as well as the number of vertices of degree 1 in $(1, 3)$ -valent maps and of free variables in open linear λ -terms. To analyse their distributions, we introduce appropriate tools: a moment-pumping schema for differential equations and a composition schema inspired by Bender's theorem.

Keywords. Random maps on surfaces, lambda calculus, analytic combinatorics, limit laws

Mathematics Subject Classifications. 05A16, 05A19, 03B40, 05C30

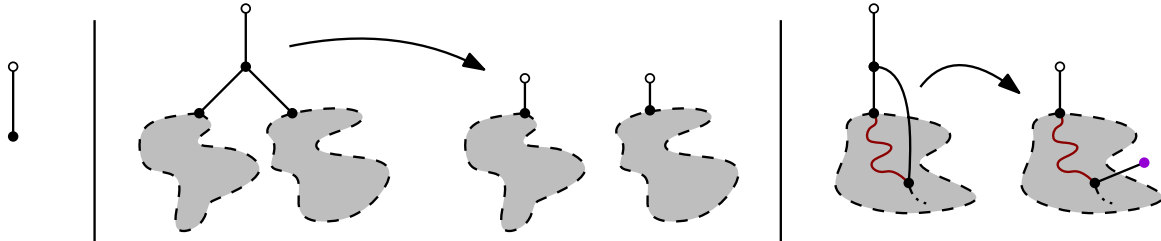
1. Introduction

In [BGJ13] a bijection between linear λ -terms and trivalent maps was presented, establishing a framework for studying the combinatorics of the linear λ -calculus and its relation to maps. The goal of this work is to further explore and extend this relation between the λ -calculus and maps, presenting a study of the asymptotic behaviour of certain combinatorial parameters defined on related families of λ -terms and maps. The parameters analysed have natural interpretations as observables of λ -terms and of maps, which allows us to employ a mix of techniques drawn from the study of both.

1.1. Motivation and main results

Maps, or graphs embedded on surfaces, are an important object of study in modern combinatorics and their presence in various areas, ranging from algebra to physics, forms bridges between seemingly disparate subjects. In recent years it has become apparent that such bridges extend to logic as well, stemming from bijections between various natural classes of rooted maps and certain subsystems of λ -calculus. This includes a natural bijection between rooted trivalent maps and linear lambda terms [BGJ13], as well as a somewhat more involved bijection between rooted planar maps of arbitrary vertex degrees and β -normal ordered linear lambda terms [ZG15], both of which have led to further study of the combinatorial interactions between lambda terms and maps.

To make the above correspondence concrete, let us briefly recall here the bijection of [BGJ13] following the analysis of [Zei16], which is itself inspired by Tutte’s classical approach to map enumeration via repeated root edge decomposition [Tut68]. Informally, a rooted trivalent map may be defined as a graph equipped with an embedding into an oriented surface of arbitrary genus, all of whose vertices have degree 3, and one of whose edges has been distinguished and oriented (see below for a more formal definition). For reasons that will be quickly apparent, it is pertinent to slightly extend the class of rooted trivalent maps by embedding it into the class of (1,3)-valent maps, that is, maps whose vertices all have degree 3 or 1. We will view 1-valent vertices as labelled “external” vertices, and the root itself as a distinguished external vertex. By considering what happens around the root, it is clear that such a map falls into one of three categories:



namely, it is (from left to right) either: the trivial one-edge map with no trivalent vertices and a single 1-valent vertex besides the root; or a map in which the deletion of the root and its unique trivalent neighbour yields a pair of disconnected maps which may be canonically rooted; or finally a map in which the same operation yields a connected map which may be again rooted canonically and which in addition has a distinguished degree-1 vertex.

Quite remarkably, this decomposition à la Tutte exactly mirrors the standard inductive definition of linear lambda terms. Informally, an arbitrary lambda term is either a *variable* x , an *application* $(t u)$ of a term t to another term u , or an *abstraction* $\lambda x.t$ of a term t in a variable x , with linearity imposing the condition that in an abstraction $\lambda x.t$, the variable x has to occur exactly once in t . Concretely, the differential equation resulting from this analysis

$$T(z, u) = zu + zT(z, u)^2 + z \frac{\partial}{\partial u} T(z, u) \quad (1.1)$$

can be seen as counting either (1,3)-valent maps or linear λ -terms, with the size variable z tracking edges or subterms and the “catalytic” variable u tracking non-root 1-valent vertices or free variables. Setting $u = 0$ then allows us to recover the ordinary generating function $T_{[0]}(z) = T(z, 0)$ counting rooted trivalent maps in the classical sense as well as closed linear lambda terms. As shown in [BGGJ13], this ordinary generating function also admits a more efficient specification making no use of the extra variable u :

$$T_{[0]}(z) = z^2 + zT_{[0]}(z)^2 + 2z^4 \frac{\partial}{\partial z} T_{[0]}(z).$$

The bijection between λ -terms and maps can be described graphically by representing the terms as certain decorated syntactic diagrams, in the manner of Figure 1.1. Such diagrams yield rooted trivalent maps with external 1-valent vertices simply by forgetting the labels of trivalent nodes, and conversely, these labels can be uniquely reconstructed from a given map by the recursive decomposition process shown above. A more comprehensive discussion of the correspondence between rooted trivalent maps and linear λ -terms is given in [BGJ13] and [Zei16], and we will review it further below.

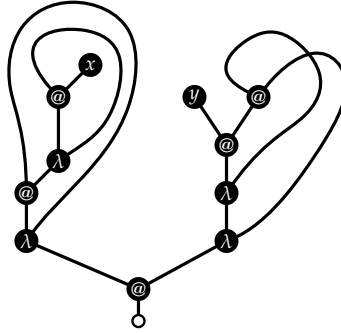


Figure 1.1: The syntactic diagram of the open linear λ -term $(\lambda a. a(\lambda b. b x))(\lambda c. \lambda d. y(cd))$.

By exploiting such bijective correspondences between families of maps and λ -terms, we identify and study pairs of corresponding parameters natural to both classes, focusing on their limit distributions. The parameters studied in this work, together with their limit distributions, are listed in Table 1.1.

Parameter on maps (number of)	Parameter on λ -terms (number of)	Limit distribution
Loops in rooted trivalent maps	Identity-subterms in closed linear λ -terms	$Poisson(1)$
Bridges in rooted trivalent maps	Closed subterms in closed linear λ -terms	$Poisson(1)$
Vertices of degree 1 in rooted (1, 3)-maps	Free variables in open linear λ -terms up to exchange	$\mathcal{N}((2n)^{1/3}, (2n)^{1/3})$
Vertices of degree 2 in rooted (2, 3)-maps	Unused abstractions in closed affine λ -terms	$\mathcal{N}\left(\frac{(2n)^{2/3}}{2}, \frac{(2n)^{2/3}}{2}\right)$

Table 1.1: Pairs of corresponding parameters in families of maps with n edges and λ -terms with n subterms together with their limit distributions, where $Poisson(\lambda)$ signifies the Poisson law of rate λ and $\mathcal{N}(\mu, \sigma^2)$ is shorthand for the corresponding random variables X_n converging in law to the standard normal distribution after being standardised as $\frac{X_n - \mu}{\sigma_n}$.

The first step of our approach is obtaining combinatorial specifications which allow us to capture the behaviour of our parameters of interest. This is done via a number of new decompositions valid for restricted families of maps and λ -terms. We are then faced with the task of asymptotically analysing these specifications, a task made difficult by the fact that the number of elements of a given size in these families exhibits rapid growth; this precludes a straightforward approach based on standard tools of analytic combinatorics as the corresponding generating functions are purely formal power series and do not represent functions analytic at 0. To facilitate our approach, we therefore develop two new schemas which serve to encompass the two general cases we have observed in our study: differential specifications giving rise to Poisson limit laws and composition-based specifications giving rise to Gaussian limit laws.

Our purpose in this present work is therefore twofold. On the one hand, we want to demonstrate how interesting insights on the typical structure of large random maps and λ -terms may be obtained by fruitfully making use of techniques drawn from the study of maps and λ -terms in tandem. On the other hand, we present two new tools which aid in the asymptotic analysis of parameters of fast-growing combinatorial classes; these tools are of independent interest, being applicable to the study of a wide class of combinatorial classes whose generating function is purely formal and obeys certain types of differential or functional equations.

Finally, we mention that in [Sin22] the present theme of the combinatorial interplay of λ -terms and maps is further developed and a number of parameters in maps and terms is studied using techniques related to the present work. A notable example comes in the form of a lower bound for the number of β -reduction steps required for a random closed linear λ -term to reach its normal form, which was obtained by studying the number of occurrences of certain patterns in (decorated) trivalent maps.

1.2. Related work

The structure and enumeration, both exact and asymptotic, of maps by their genus has been the subject of much study; see, for example, [CV81, BR86, BFSS01] for the planar case and [WL72, BIZ80, MS01] for the higher genus case. Our work focuses instead on the investigation of enumerative and statistical properties of rooted maps counted without regard to their genus. Such studies often involve purely formal power series – a typical example appearing in [AB00] where the number of maps with n edges is shown to be asymptotically $(2n - 1)!!$. This sometimes renders the application of analytic techniques difficult since, as Odlyzko notes in [Odl96]:

There are few methods for dealing with asymptotics of formal power series, at least when compared to the wealth of techniques available for studying analytic generating functions.

However the divergent nature of these power series can also lead to a simpler analysis, since it allows for the application of tools such as Bender’s theorem [Ben75]. Indeed, Odlyzko continues by noting that:

Fortunately, combinatorial enumeration problems that do require the use of formal power series often involve rapidly growing sequences of positive terms, for which some simple techniques apply.

A number of new such techniques will be developed in this work, as part of our approach.

The structure of large random such maps has only recently begun to be investigated. For example the distribution of genus in (half-edge-labelled) random maps was investigated in [Car19]. More recently, the authors of [BCDH18] investigated the asymptotic distributions for the number of vertices, root isthmus parts, root edges, root degree, leaves, and loops in random maps. In particular, comparing their results to ours, we note that for general maps the authors derived a *Poisson*(1) limit law for the number of leaves and a previously-unknown law for the number of loops. Both of these results stand in stark contrast to the case of leaves in $(1, 3)$ -valent maps, which we show is normally distributed when standardised using $\mu = \sigma^2 = n^{1/3}$, and to the case of loops in rooted trivalent maps which we show is *Poisson*(1). In terms of techniques employed, the authors of [BCDH18] show that for most of the statistics considered in their work the corresponding bivariate generating functions are formal solutions to Riccati equations, which may be linearised to yield recurrences on the coefficients of said generating functions which are amenable to study. We note here that an instance of a Riccati-type differential equation appears in our work too, but this time it is a differential equation with respect to the variable coupled to the statistic we are interested in, unlike the instances of [BCDH18] where the derivative was taken with regards to the size-coupled variable.

As for the λ -calculus, while of central importance to logic and theoretical computer science, it is a relatively new subject of study for combinatorialists. The combinatorial study of closed linear and affine λ -terms and their relaxations was introduced in [BGJ13, BGGJ13], and further analyzed and extended to open terms in [Zei16]. A bijective account of the interplay between some classes of planar linear normal λ -terms and planar maps is presented in [Fan23]. Finally, we note that there exists a number of combinatorial studies of λ -terms which use a size notion different than ours and that of [BGJ13, BGGJ13, Zei16]. For example, there exists a number of works focusing on a unary de Bruijn notation based model as in [BGLZ17]. This choice of size notion has the effect of altering the qualitative properties of our objects of study: in particular, the statistical results and the associated techniques of [BBD19] are not applicable to our model.

1.3. Structure of the article

Section 2 recalls the formal definitions of maps (Section 2.1) and lambda terms (Section 2.2), establishing notation that is assumed in the rest of the paper, and also presents some background material on analytic combinatorics (Section 2.3).

Sections 3 and 4 present the main results of our work in two rather independent sections. Section 3 focuses on closed linear λ -terms and closed rooted trivalent maps (as defined in Section 2.1). In this section we present a number of useful bijective results for linear and bridgeless linear terms, from which we then extract asymptotic results using a new moment-pumping framework presented in Section 3.2. The main results of this section are Theorems 3.10 and 3.22 where the asymptotic distributions of the numbers of loops and bridges in rooted trivalent maps, or equivalently those of identity and closed proper subterms in closed linear λ -terms is presented.

Section 4 focuses on open linear, closed affine λ -terms, open trivalent maps, and $(2, 3)$ -valent maps (once again, as defined in Section 2.1). The main tool employed in this section is a schema used to analyse composition-based specifications as presented in Section 4.1. The main results

for open linear terms and open trivalent maps are presented in Theorem 4.5 and Corollary 4.6 where the asymptotic distribution of the number of degree-1 vertices in rooted and unrooted open trivalent maps is given, which also yields the asymptotic distribution of the number of free variables in open linear terms. As for rooted and unrooted $(2, 3)$ -valent maps and closed affine terms, the main results are presented in Theorem 4.13 and Corollary 4.14, giving the asymptotic distributions for the number of degree-2 vertices in rooted and unrooted $(2, 3)$ -valent maps, as well as the number of unused abstractions in closed affine terms.

We conclude with Section 5, where we briefly discuss potential future directions dictated by this work.

2. Preliminaries

2.1. Graphs and maps

We begin by establishing some notation and definitions pertaining to graphs and maps. A comprehensive treatment of maps and various of their aspects is presented in [LZ04].

Graphs. We will consider finite undirected graphs, allowing for loops and multiple edges. Given a graph G , we will denote the set of its vertices by $V(G)$ and that of its edges by $E(G)$. For an edge $\{u, v\} \in E(G)$ we will write $e = uv = vu$ to denote the edge e between u and v and we will call u, v the *endpoints* of e . We will also say that e is *incident to* u, v .

Given a vertex $v \in V(G)$ we call the set $N_G(v) = \{u \mid u \in V(G), uv \in E(G)\}$ its *neighbourhood*. The *degree* or *valency* of v is $|N_G(v)|$.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph *induced by* $S \subseteq V(G)$, denoted by $G[S]$, is the subgraph of G consisting of all vertices in S and all edges in $E(G)$ that have both endpoints in S .

For a vertex $u \in V(G)$, we denote by $G \setminus u$ the subgraph induced by $V(G) \setminus u$. For an edge $e \in E(G)$, we denote by $G \setminus e$ the graph $(V(G), E(G) \setminus e)$. We will refer to the last two operations as *vertex deletion* and *edge deletion* respectively. We define G/v , the graph obtained from G by *dissolving* a degree 2 vertex $v \in V(G)$, to be the graph obtained by deleting v and adding an edge to its two adjacent neighbours.

An edge $e \in E(G)$ is a *bridge* if $G \setminus e$ has one more connected component than G .

We will denote by $G + H$ the disjoint union of two graphs G, H .

Maps as embedded graphs. A *map* is an embedding of a connected graph into a connected, closed, oriented surface such that all faces are homeomorphic to open disks. Maps are considered up to orientation preserving homeomorphisms of the underlying surface.

In Sections 4.2 and 4.3 we will also make use of the notion of a *disconnected map*. Such maps will be considered as embeddings of disconnected graphs not on a single surface but on the disjoint union of such surfaces, in a way such that each connected component of the graph is drawn on a different surface. When the need arises to consider both connected and disconnected maps as a single class, we will refer to them as *not-necessarily-connected maps*. In this work we focus on embeddings of degree-constrained graphs. We shall refer to maps whose underlying

graph's vertices are all of degree three as *trivalent*. More generally, if the set of allowed degrees is $k_1, \dots, k_n \in \mathbb{N}_{\geq 0}^n$ we will talk about a $(k_1, \dots, k_n)$ -valent map or just $(k_1, \dots, k_n)$ -map.

Finally, we will often make use of graph-theoretic notions when referring to a map, which are to be interpreted as identifying properties of its underlying graph. For example, a *bridge* of a map is a bridge of its underlying graph, i.e., an edge whose deletion results in a disconnected graph. Similarly, a *path* in a map is a path in its underlying graph. A *spanning tree* of a map is a spanning tree of its underlying graph.

A *submap* m' of a map m is an embedding of the subgraph corresponding to m' in m . If G is the graph of a map m and $v \subseteq V(G)$, $m[v]$ will denote the embedding of the induced subgraph $G[v]$. The embedding chosen for a subgraph m' is the restriction of the one of m , i.e., the embedding of m' for which the neighbours of any vertex in m' are oriented in exactly the same way as in m .

Maps and permutations. It is a classic result that embeddings of graphs may be represented, up to isomorphism, by certain systems of permutations [LZ04]. In particular, maps on connected closed oriented surfaces have the following equivalent purely algebraic definition: a finite set H of *half-edges* together with a pair (v, e) of permutations on H such that e is a fixed-point-free involution and the group $\langle v, e \rangle$ acts transitively on H . Such objects are sometimes referred to as *combinatorial maps*. More generally, if one drops the requirement that $\langle v, e \rangle$ acts transitively one obtains not-necessarily-connected maps.

Various properties of a map M may be read off from the tuple (v, e) . For example, vertices of M correspond to cycles of v , their degree being the length of said cycle. A cycle (h_1, h_2) of e similarly is to be interpreted as encoding an edge e formed by gluing two half-edges h_1, h_2 . Finally, the faces of the map may be read off as the cycles of the permutation $f = e^{-1}v^{-1}$ (thus satisfying the identity $vef = 1$).

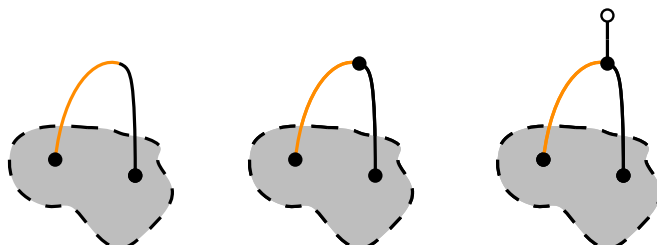


Figure 2.1: The bijection between half-edge rooted (1,3)-valent maps (left) and vertex-rooted open rooted trivalent maps (right) obtained by first subdividing the root-containing edge of the former (middle step) before finally adding a new vertex.

Open rooted trivalent maps with external vertices. Under the permutation-based definition, a popular way of rooting a map is by simply choosing an arbitrary half-edge, in which case the unique v -, e -, and f -cycles it forms a part of are marked as the *root vertex*, *root edge*, and *root face*, respectively. We shall adopt a slightly different convention for rooting trivalent maps, which is partly motivated by their correspondence with λ -terms, but may also be motivated by considering rooted maps as embeddings of graphs on surfaces with boundary. Indeed, consider an

embedding of a trivalent graph onto a compact oriented surface with a unique boundary component. The condition that the faces defined by the complement of the graph are all homeomorphic to open disks implies that if we *remove* the boundary of the surface, what is left is an embedding of a trivalent graph with some open edges, in the sense that they run into the boundary without including a vertex at the end. In turn, such open ends of edges may be closed by the addition of 1-valent vertices, which should then be interpreted as being “external” to the map, and moreover should carry extra labelling information specifying their order of attachment to the boundary. See Figure 2.2 for an illustration.

This leads to our definition of *open rooted trivalent maps* as combinatorial maps equipped with the following data and properties:

- a distinguished 1-valent vertex $r \in V$, called the *root*;
- an ordered list of 1-valent vertices $\Gamma = x_1, \dots, x_k \in V$ that are all mutually distinct and distinct from r ;
- such that the complement of $\Gamma \cup r$ in V consists of 3-valent vertices.

The number $k \in \mathbb{N}$ of non-root external vertices is called the *arity* of the open map, and in particular it is said to be *closed* if it has arity 0. In general, we refer to the $k + 1$ 1-valent vertices as *external*, and the remaining 3-valent vertices as *internal*. As a visual aid, we will draw internal vertices as solid black vertices while for external vertices we will use white vertices with a colored border. Specifically for the root, we will always represent it by a white vertex with black border. (We note that the unrooted versions of such uni-trivalent diagrams have been studied under different names in many contexts, particularly in knot theory [BN95] and physics [Pen71].)

From this definition, it is clear that there is a trivial bijection between closed rooted trivalent maps and standard half-edge rooted trivalent maps, as shown in Figure 2.1.¹ As far as enumeration is concerned, going from a half-edge-rooted trivalent map to the corresponding open rooted trivalent map increases the number of edges by 2 (equivalently, the number of half-edges by 4). At general arity k , we will be interested in enumerating open rooted trivalent maps modulo the $k!$ relabellings of the non-root external vertices, which is equivalent to enumerating unlabelled 1-valent-vertex-rooted (1,3)-valent maps or (by the same bijection of Figure 2.1) half-edge-rooted (1,3)-maps up to a size shift.

Certain graph-theoretic notions must be appropriately adapted to account for the internal vs. external distinction. In particular, we will say that bridge is an *internal bridge* if both of its ends are internal vertices. Indeed, as suggested above, one can think of external vertices as being implicitly connected via a path along what was formerly the boundary of the map.

¹Note that in the classical definition, the map with no half-edges is often treated as a special case and rooted by default, in order to ensure that the loop map  is in the image of the bijection. This shows one small advantage of our rooting convention, since we do not need to include any special cases.

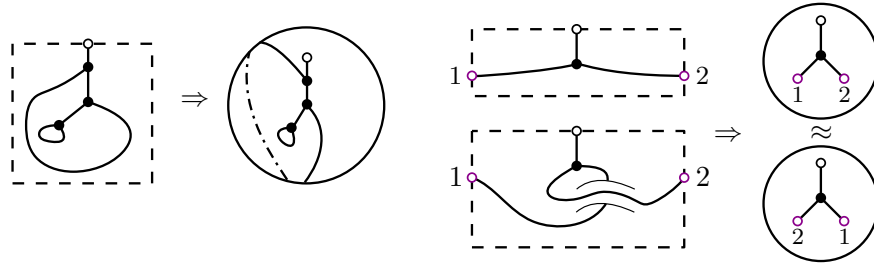


Figure 2.2: On the left, a trivalent graph embedded on a square together with the corresponding closed rooted trivalent map obtained by deleting the boundary and closing open edges. On the right, two trivalent graphs embedded in a rectangle and in a rectangle with a handle attached respectively, together with the corresponding open trivalent maps of arity 2, which are isomorphic up to relabelling of non-root external vertices.

2.2. λ -Calculus

Linear and affine λ -terms. The λ -calculus is, among other things, a computationally universal programming language. Traditionally, the set of λ -terms is defined inductively, as follows:

- A variable (taken from an infinite set $\{x, y, z, \dots\}$) is a valid term.
- If x a variable and t is a valid term, then so is $(\lambda x.t)$. Such a term is called an *abstraction*, the variable x in $(\lambda x.t)$ is considered *bound*, and we refer to t as the *body* of the abstraction.
- If t and u are valid terms, then so is $(t u)$. Such a term is called an *application*.

When it aids in readability, we shall do away with some of the parentheses when writing out a λ -term, following standard conventions. In particular, we omit outermost parentheses and associate applications to the left, while λ -abstraction is always assumed to take scope to the right by default. For example, $\lambda x.(\lambda y.(\lambda z.(x((yz)w))))$ is written more concisely as $\lambda x.\lambda y.\lambda z.x(yzw)$.

While the theory of the λ -calculus is a vast and important subject [BDS13], in this present work we will only deal with λ -terms as syntactic/combinatorial objects. As we shall see, even in this reduced capacity, considerations of λ -theoretic notions are quite fruitful and find natural counterparts in the realm of maps.

We recall some technical vocabulary. A term is called *closed* if all variables occurring in it are bound by some abstraction. Otherwise such a term is called *open* and the variables not bound by an abstraction are referred to as *free*. Two λ -terms are equivalent if, intuitively, they differ only in the names of variables. The precise notion of equivalence, α -equivalence necessitates the employment of *capture-avoiding substitutions* which we will not delve into.

An occurrence of a free or bound variable is called a *use* of the variable. A term is said to be *linear* if every (free or bound) variable is used exactly once. For example the terms $\lambda x.x$ and $\lambda x.\lambda y.\lambda z.(xz)y$ are both linear while the terms $\lambda x.xx$ and $\lambda x.\lambda y.x$ are not. The latter is an example of an *affine* term, that is, a term in which every variable is used at most once. We will refer to abstractions whose bound variable is never used as *unused abstractions*.

To make the above notions more precise, we can consider λ -terms as indexed explicitly by lists of free variables, defining the relation $\Gamma \vdash t$ between an ordered list of free variables $\Gamma = (x_1, \dots, x_k)$ and a linear λ -term t by the following inductive rules:

$$\frac{}{x \vdash x} \quad \frac{\Gamma \vdash t \quad \Delta \vdash u}{\Gamma, \Delta \vdash (t \ u)} \quad \frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.t} \quad \frac{\Gamma, x, y, \Delta \vdash t}{\Gamma, y, x, \Delta \vdash t}$$

where we write (Γ, Δ) for the concatenation of two lists Γ and Δ . From left to right, the first three rules express formation of variables, applications, and abstraction terms, respectively, while the fourth rule (called the *exchange* rule) reflects the property that variables may be used in an arbitrary order in a linear term. To define affine terms, we add one more rule (called *weakening*):

$$\frac{\Gamma \vdash t}{\Gamma, x \vdash t}$$

which, reading from bottom to top, reflects the property that extra variables may be ignored.

Subterms and one-hole contexts. The *subterms* of a term t are defined as follows:

- t is a subterm of itself;
- if u is a subterm of t_1 or t_2 then u is a subterm of $(t_1 \ t_2)$;
- if u is a subterm of t then u is a subterm of $\lambda x.t$.

The *proper* subterms of t are all of its subterms except for t itself. We write $u \preceq t$ to indicate that u is a subterm of t , and $u \prec t$ that it is a proper subterm.

For many purposes, including ones of enumeration, it is important to distinguish between different occurrences of the same subterm. For example, up to α -equivalence, the *identity term* $\lambda x.x$ occurs twice as a subterm of $x(\lambda y.y)(\lambda z.z)$. A convenient way of distinguishing occurrences is through the notion of *one-hole context*, which is the analogue of differentiation in the symbolic method [McB01]. In our setting, one-hole contexts may be defined inductively as follows:

- the *identity context*, written $\square$, is a one-hole context;
- if c is a one-hole context and t is a term, then so are $(c \ t)$ and $(t \ c)$;
- if x is a variable and c is a one-hole context then so is $\lambda x.c$.

The result of “plugging” the hole of a context c with a term u is a term $c[u]$ defined by:

$$\begin{aligned} \square[u] &= u \\ (c \ t)[u] &= (c[u] \ t) \\ (t \ c)[u] &= (t \ c[u]) \\ (\lambda x.c)[u] &= \lambda x.(c[u]) \end{aligned}$$

It is easy to check that $u \preceq t$ (respectively $u \prec t$) iff there exists a one-hole context c (resp. $c \neq \square$) such that $t = c[u]$. Moreover, by distinguishing different contexts c_1, c_2 one can distinguish between different occurrences of the same subterm u within a term $t = c_1[u] = c_2[u]$. Finally, there

is an evident notion of composition of contexts, written $c_1 \circ c_2$, satisfying $(c_1 \circ c_2)[u] = c_1[c_2[u]]$ for all u . Given two one-hole contexts c_1, c_2 , we say c_1 is a *right subcontext* of c_2 if there exists a context c_3 such that $c_2 = c_3 \circ c_1$.

Now we can define the size $|t|$ of a λ -term to be the number of distinct factorizations $t = c[u]$ of t into a subterm u and surrounding one-hole context c . It is immediate that:

$$|x| = 1 \quad |(t\ u)| = 1 + |t| + |u| \quad |\lambda x.t| = 1 + |t|$$

For example, $\lambda x.x$ has size two and $\lambda x.x(\lambda y.y)(\lambda z.z)$ has size 8 under this metric. We define the size $|c|$ of a one-hole context similarly but assigning the identity context size zero:

$$|\square| = 0 \quad |(c\ t)| = 1 + |c| + |t| \quad |(t\ c)| = 1 + |t| + |c| \quad |\lambda x.c| = 1 + |c|$$

so that we have the identity $|c[u]| = |c| + |u|$ for all c and u .

Finally, observe that for any term with at least one free variable $\Gamma, x \vdash t$ there is a unique one-hole context c such that $c[u] = t[u/x]$, by replacing the variable x with a box $\square$. In this case, we say c is a *simple* context and write $\Gamma \vdash c$. By extension, we say c is closed if $\Gamma = \cdot$.

Lambda terms as invariants of rooted maps. As recalled in Section 1.1, there is a natural bijection τ from rooted trivalent maps to linear lambda terms, which may be understood either via repeated root edge decomposition à la Tutte (as advocated in [Zei16]), or alternatively (as in the original construction [BGJ13]) as building a canonical depth-first search spanning tree of a map. In either case, we adopt the viewpoint that the term $t = \tau(m)$ may be seen as an “invariant” of the map m , in other words that it extracts some important topological information. In particular, t describes a canonical spanning tree on m obtained by deleting in the map the edges corresponding to the bound variables of the term. We call this the *t-tree* of m . Moreover, following [Ber07], we will call the unique path in the *t-tree* between two vertices of m a *t-path*, and fixing some vertex x , we define the *parent* of x to be its neighbour along the *t-path* between the root vertex and itself. An example of two maps and their corresponding terms, with canonical spanning trees highlighted, is presented in Figure 2.3.

The bijection τ allows us to establish a dictionary of correspondences between structural properties of linear lambda terms and rooted trivalent maps. For example, it is not hard to see that loops in maps correspond to *identity-subterms* of lambda terms, that is, subterms α -equivalent to $\lambda x.x$, and that dually, (internal) bridges correspond to *closed proper subterms* [Zei16]. In fact, more generally any decomposition of a linear term $t = c[u]$ into a subterm u and surrounding one-hole context c may be interpreted as a $(k + 1)$ -cut of the corresponding map, where k is the number of free variables of u (see Figure 2.4 for an illustration). We represent the one-hole context itself as a map with a distinguished vertex, which we draw as a box, marking the hole. In particular, a closed simple one-hole context $\cdot \vdash c$ may be considered as a $(1,3)$ -valent map with *two* marked 1-valent vertices, one representing the hole in addition to the one representing the root. We write $\tau^{-1}(c)$ for this map, by extension of the original bijection.

Given these correspondences, we note that for many of the results in this work, the proofs may be given either purely in the language of lambda calculus or in the language of maps, and then automatically transported to the other side along a bijection. Nevertheless, we will oftentimes include in our proofs both complementary arguments, even if not strictly required, to illuminate how the arguments translate from one class to the other.

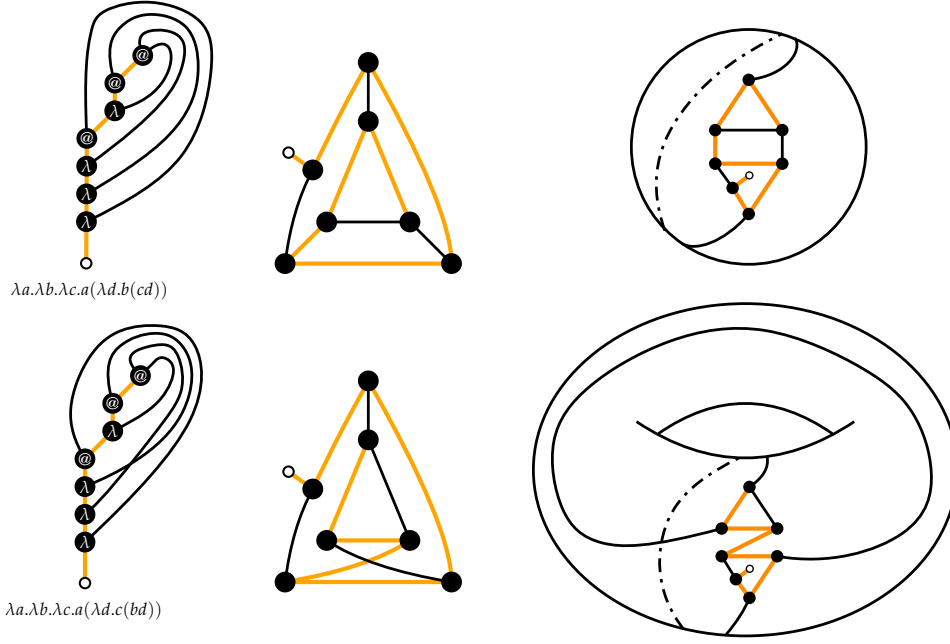


Figure 2.3: Two closed linear λ -terms and their equivalent rooted trivalent maps drawn as graphs with crossings and as embeddings. The spanning trees induced by the terms on each map are highlighted in orange.

2.3. Analytic combinatorics

Here we recall some notions from analytic combinatorics and establish some basic lemmas, whose proofs are found in Appendix A.

Combinatorial structures and the symbolic method. Let $\mathcal{A}$ be a set equipped with a *size function* $|\cdot| : \mathcal{A} \rightarrow \mathbb{N}$, such that the set of elements $\mathcal{A}_n = \{a \in \mathcal{A} \mid |a| = n\}$ of any given size n has a finite cardinality $a_n = |\mathcal{A}_n|$. Such a set is called a *combinatorial class*. To a combinatorial class $\mathcal{A}$ one can assign a power series $A(z)$, either a so-called *ordinary generating function* or an *exponential* one, defined as $\sum_n a_n \frac{z^n}{\omega_n}$ where the weight ω_n is given by $\omega_n = 1$ in the ordinary case and by $\omega_n = n!$ in the exponential case. We make use of the coefficient extraction operator $[z^n]A(z)$ to denote the coefficients of z^n in $A(z)$. We refer the reader to [FS09] for a description of the algebra of combinatorial classes and the corresponding algebra of generating functions. In particular, we make use of the operations of composition, disjoint union, cartesian product, and pointing which correspond to the composition, Cauchy product, and application of the operator $z\partial_z$ for power series respectively. We also make use of the *exponential Hadamard product* for exponential generating functions, defined as $A(z) \odot B(z) = \sum_n \frac{a_n b_n}{n!} z^n$, where $a_n = n![z^n]A(z)$, $b_n = n![z^n]B(z)$.

For a combinatorial class $\mathcal{A}$, we define a *combinatorial parameter*, or just parameter, to be a function $\chi : \mathcal{A} \rightarrow \mathbb{N}$. Again, these can be of ordinary or of exponential type. Then, if $a_{n,k} = |\{a \in \mathcal{A}_n \mid \chi(a) = k\}|$ is the number of objects of size n with parameter value k , the *bivariate generating function* $A(z, u)$ of $\mathcal{A}$ with respect to χ is $\sum_{n,k} \frac{a_{n,k}}{\omega_n \rho_k} z^n u^k$, where the

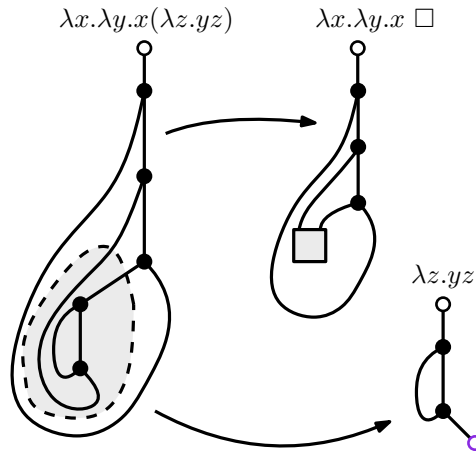


Figure 2.4: A decomposition of $t = \lambda x.\lambda y.x(\lambda z.yz)$, as $c[u]$ for $c = \lambda x.\lambda y.x \square$ and $u = \lambda z.yz$, corresponding to a 2-cut in the map $\tau^{-1}(t)$.

weight ρ_k is given by $\rho_k = 1$ if χ is of ordinary type and by $\rho_k = k!$ if it is of exponential type. We say that the variable u marks χ . For any $n \in \mathbb{N}$, the parameter χ determines a discrete random variable X_n over $\mathcal{A}_n$: $\mathbb{P}(X_n = k) = \frac{a_{n,k}/\rho_k}{\sum_k a_{n,k}/\rho_k}$. In such a case we say that X_n corresponds to χ taken over $\mathcal{A}_n$.

Finally, we can also form new combinatorial classes $\mathcal{A}|_{\chi=k}$ by restricting $\mathcal{A}$ to a particular value for the parameter χ , and keeping the same notion of size. An important recurring case is when χ corresponds to a natural “arity grading” for $\mathcal{A}$ distinct from its size grading, and we introduce a special notation for this, writing $\mathcal{A}_{[k]}$ for the set of elements of arity k . The use of iterated subscripts following these conventions should be clear from context. For example, we write $\dot{\mathcal{T}}$ to denote the combinatorial class of all linear λ -terms, $\dot{\mathcal{T}}_{[0]}$ for its restriction to the combinatorial class of closed terms (i.e., terms of arity 0), and $\dot{\mathcal{T}}_{[0]n}$ for the finite set of closed linear terms with n subterms.

A list of some of the main combinatorial classes considered in this work is given in Table 2.1.

Divergent generating functions. When enumerating various classes of non-planar maps and λ -terms, one quickly realises that the numbers involved grow rapidly and that the corresponding generating functions are everywhere divergent. In particular, they do not represent some function analytic at 0, and are to be interpreted as purely formal power series.

Combinatorial Class	Symbol	Size Notion
Open rooted trivalent maps and linear λ -terms	$\dot{\mathcal{T}}$	Num. of edges in map / subterms in term
Closed rooted trivalent maps and linear λ -terms	$\dot{\mathcal{T}}_{[0]}$	—//—
Affine linear λ -terms	$\mathcal{A}$	Num. of subterms
Unrooted (1,3)-valent maps	$\mathcal{T}$	Num. of edges
Unrooted (2,3)-valent maps	$\mathcal{D}$	—//—

Table 2.1: The main combinatorial classes considered in this work.

Such generating functions are not always amenable to straightforward analysis using standard tools of analytic combinatorics but instead require their own technical tools. One of our aims in this work is to develop such tools for analysing structural properties of combinatorial classes whose objects grow so rapidly so as to render their generating functions divergent.

We conclude these preliminaries with some basic results useful to the asymptotic and probabilistic study of such classes. The first shows that rooting a combinatorial structure does not affect the distribution of parameters over it.

Proposition 2.1. *Let $\mathcal{F}$ be a combinatorial class and χ some parameter defined on it. Then the limit distribution of χ taken over $\mathcal{F}_n^\bullet$ is the same as that of χ taken over $\mathcal{F}_n$.*

The above lemma can be iterated to show that applications of any operator of the form $z^n \partial_z^n$ result in limit distributions which converge in law to the limit distribution of χ taken over $\mathcal{F}_n$.

Next, we present our tools for the analysis of Cauchy products of the various divergent power-series that appear in the sequel, based on the notion of logarithmic convexity.

Definition 2.2 (Logarithmic convexity). A sequence $s_\ell, s_{\ell+1}, \dots, s_u$, with $s_i \in \mathbb{R}_{>0}$ for all $\ell \leq i \leq u$, is *log-convex on the interval $I = [\ell, \dots, u]$* if $s_i^2 \leq s_{i-1}s_{i+1}$ for all $\ell + 1 \leq i \leq u - 1$ or, equivalently, if the sequence $(s_i/s_{i-1})_{\ell < i < u}$ is increasing. A sequence $(s_n)_{n \in \mathbb{N}}$ is *asymptotically log-convex* if there exists $N \in \mathbb{N}$ such that the subsequence $(s_n)_{N \leq n \leq N'}$ is log-convex for any $N' > N$.

Lemma 2.3. *Let $f(z) = \sum_{n \geq 0} f_n z^n, g(z) = \sum_{n \geq 0} g_n z^n$ be two power series with coefficients in $\mathbb{R}_{\geq 0}$, such that the following conditions hold:*

- (i) $\text{Supp}(f) = a + d\mathbb{N}$ and $\text{Supp}(g) = b + d\mathbb{N}$.
- (ii) *The sequences $(f'_n)_{n \in \mathbb{N}}, (g'_n)_{n \in \mathbb{N}}$ are asymptotically log-convex, where $f'_n = f_{nd+a}$ and $g'_n = g_{nd+b}$.*
- (iii) *There exist $\sigma_1, \sigma_2 > 0$ such that for $n \geq a \wedge n \equiv a \pmod{d}$, $\frac{f_{n-d}}{f_n} = O(n^{-\sigma_1})$ and for $n \geq b \wedge n \equiv b \pmod{d}$, $\frac{g_{n-d}}{g_n} = O(n^{-\sigma_2})$.*
- (iv) *For $n \geq a + b \wedge n \equiv (a + b) \pmod{d}$, $g_{n-a} = O(f_{n-b})$.*

Then, we have

$$[z^n]f(z)g(z) = f_{n-b}g_b + f_a g_{n-a} + O(f_{n-b-d}), \quad \text{for } n \geq a+b \text{ and } n \equiv a+b \pmod{d}. \quad (2.1)$$

Applying the above lemma iteratively we obtain.

Corollary 2.4. *Let $f(z) = \sum_{n \geq m} f_n z^n$ be a power series such that*

- $\text{Supp}(f) = a + d\mathbb{N}$.
- *The sequence $(f'_n)_{n \in \mathbb{N}}$ is asymptotically log-convex, where $f'_n = f_{nd+a}$.*
- *There exists $\sigma > 0$ such that for $n \geq a \wedge n \equiv a \pmod{d}$, $\frac{f_{n-d}}{f_n} = O(n^{-\sigma})$.*

Then for $n \in a + d\mathbb{N}, k \in \mathbb{N}, l \in \mathbb{N}$

$$[z^n] \frac{k f(z)^l}{z^{a(l-1)}} = kl f_a f_n (1 + O(n^{-\sigma})). \quad (2.2)$$

3. Poisson distributions from differential equations

In this section our aim is to explore the limit distribution of the number of bridges in trivalent maps and of closed subterms in closed linear λ -terms. Our approach will be based on combinatorial specifications of maps and terms in $\dot{\mathcal{T}}_{[0]}$ respectively. As it turns out, these specifications yield differential equations governing the behaviour of our parameters of interest. To analyse these differential equations we will introduce, in Section 3.2, a schema providing sufficient conditions for the limit distribution of some combinatorial parameter of a divergent combinatorial class to weakly converge to a Poisson distribution of rate 1, or a shifted version of such a distribution. Armed with this schema we will then proceed to first prove a special case of our desired result: the limit distribution of the number of loops in trivalent maps and of identity-subterms in closed linear λ -terms is *Poisson*(1). Finally, we prove that the same holds for the number of bridges and subterms too.

As a warmup, we begin with a discussion of bridgeless trivalent maps and linear λ -terms.

3.1. Bridgeless maps and linear λ -terms

Let the class $\dot{\mathcal{B}}_{[0]}$ of *bridgeless* rooted trivalent maps and closed linear λ -terms be the subclass of $\dot{\mathcal{T}}_{[0]}$ consisting of rooted trivalent maps with no internal bridges, or equivalently to closed linear λ -terms which have no closed proper subterms.

We begin by stating the following trivial isomorphism between $\dot{\mathcal{B}}_{[0]}$ and the class $\dot{\mathcal{B}}_{[1]}$ of *one-variable-open* bridgeless linear terms, that is, linear terms $x \vdash t$ such that t has no closed subterm. Considered as maps, elements of $\dot{\mathcal{B}}_{[1]}$ contain no internal bridges and exactly two external vertices (one corresponding to the root and the other to the free variable).

Proposition 3.1.

$$\dot{\mathcal{B}}_{[0]} = \mathcal{Z}\dot{\mathcal{B}}_{[1]} \quad (3.1)$$

Proof. Let $l = \lambda x.u \in \dot{\mathcal{B}}_{[0]}$. Then by deleting the outermost abstraction of l we obtain $x \vdash u \in \dot{\mathcal{B}}_{[1]}$. For the opposite direction, we have that any one-variable-open bridgeless term $x \vdash u \in \dot{\mathcal{B}}_{[1]}$ uniquely yields an abstraction term $\lambda x.u \in \dot{\mathcal{B}}_{[0]}$.

In terms of maps, let $m \in \dot{\mathcal{B}}_{[0]}$ with root vertex r and a its unique neighbour. Then one direction of Equation (3.1) corresponds to the observation that such a map is either the one-edge map or is such that by deleting r and the first edge ax encountered after ra in a counterclockwise tour of a , one obtains, after rooting at a , a map $m \setminus r \setminus ax \in \dot{\mathcal{B}}_{[1]}$ which has two external bridges: one incident to a and the other to x . For the other direction, we note that for a map $m \in \dot{\mathcal{B}}_{[0]}$, m is either the one-edge map or we can use it to uniquely recreate a map $m' \in \dot{\mathcal{B}}_{[1]}$ by adding a new edge between the root and the unique degree-1 vertex of m before introducing a new root and an edge between it and the old one. $\square$

To construct the bijections in the rest of this subsection, we will rely on the following lemma.

Lemma 3.2. *Let $\Gamma, y \vdash t$ be a linear λ -term whose free variables include y . Then the set of subterms $S = \{s \preceq t \mid y \vdash s\}$ whose only free variable is y is linearly ordered by the subterm relationship. Since it is moreover non-empty (with $y \in S$), it contains a unique maximal element.*

Proof. We proceed by induction on t :

Case 0: $t = y$ is a variable. Then $S = \{y \preceq y\}$ is the trivial linear order on one element.

Case 1: $t = \lambda z.u$ is an abstraction. We have that $\Gamma, y, z \vdash u$, and by induction, the set $S' = \{s \preceq u \mid y \vdash s\}$ is linearly ordered. But either $S = S'$ (if Γ is non-empty) or else $S = S' \cup t$ (if Γ is empty), in which case we can uniquely extend the linear order on S' to S noting that every element of S' is a proper subterm of t .

Case 2: $t = (t_1 t_2)$ is an application. By linearity, we have that $\Gamma_1 \vdash t_1$ and $\Gamma_2 \vdash t_2$ for some Γ_1 and Γ_2 such that Γ, y is some shuffle of Γ_1 and Γ_2 . In particular, y must appear free in one of t_1 or t_2 , and without loss of generality suppose it is t_1 and that $\Gamma_1 = (\Gamma'_1, y)$ for some Γ'_1 . Then by induction the set $S' = \{s \preceq t_1 \mid y \vdash s\}$ is linearly ordered, and again, either $S = S'$ (if Γ is non-empty) or else $S = S' \cup t$ (if Γ is empty), in which case we can uniquely extend the linear order on S' to S . $\square$

We now proceed with an equation for the class $\dot{\mathcal{B}}_{[1]}$.

Lemma 3.3.

$$\dot{\mathcal{B}}_{[1]} = \mathcal{Z} + 2 \times \mathcal{Z}^2 \times \dot{\mathcal{B}}_{[1]} \times \dot{\mathcal{B}}_{[1]}^\bullet \quad (3.2)$$

where $2 = \mathcal{E} + \mathcal{E}$ stands for the class with two neutral objects ϵ_1, ϵ_2 and $\dot{\mathcal{B}}_{[1]}^\bullet$ denotes the pointing of $\dot{\mathcal{B}}_{[1]}$, that is, the class of one-variable-open linear terms with no closed proper subterms and a marked subterm, or equivalently rooted trivalent maps with two external vertices, no internal bridges and a marked edge.

Proof. Let $x \vdash t \in \dot{\mathcal{B}}_{[1]}$. Then t is either a variable, which is accounted for by the $\mathcal{Z}$ summand, or else it must be an abstraction term. Indeed t cannot be an application term $t = (t_1 t_2)$ since, by linearity, either t_1 or t_2 would have to be closed, contradicting the assumption that t has no closed subterms. Assume then that $t = \lambda y.t'$ for some $x, y \vdash t'$ with two free variables. Now, by Lemma 3.2, let $y \vdash t_m$ be the subterm of t' that is maximal among terms with free variable y , and let $x \vdash c_m$ be the corresponding context $t' = c_m[t_m]$. By assumption that t has no closed subterms, t_m must occur in an application of the form $(t_m u)$ or $(u t_m)$ for some u , that is, the context c_m must decompose as $c_m = c'_m \circ (\square u)$ or $c_m = c'_m \circ (u \square)$. In either case, by plugging u for the hole of c'_m we are left with a one-variable-open term $x \vdash c'_m[u]$ with no closed proper subterms and a marked subterm. But then the triple $(\epsilon_i, t_m, c'_m[u])$ forms an element of $2 \times \dot{\mathcal{B}}_{[1]} \times \dot{\mathcal{B}}_{[1]}^\bullet$, where the choice of ϵ_i records which of the two cases ($c_m = c'_m \circ (\square u)$ or $c_m = c'_m \circ (u \square)$) we are in, and conversely any such triple uniquely determines a term $t = \lambda y.c'_m[t_m u]$ or $t = \lambda y.c'_m[u t_m]$. This establishes the right summand on the right-hand side of (3.2), with the factor $\mathcal{Z}^2$ accounting for the fact that we removed one application and one abstraction in passing from t to $(\epsilon_i, t_m, c'_m[u])$. For a graphical example of Equation (3.2) see Figure 3.1. $\square$

Combining Equations (3.1) and (3.2) also yields an equation for $\dot{\mathcal{B}}_{[0]}$.

Corollary 3.4.

$$\dot{\mathcal{B}}_{[0]} + 2\mathcal{Z}\dot{\mathcal{B}}_{[0]}^2 = \mathcal{Z}^2 + 2\mathcal{Z} \times \dot{\mathcal{B}}_{[0]} \times \dot{\mathcal{B}}_{[0]}^\bullet \quad (3.3)$$

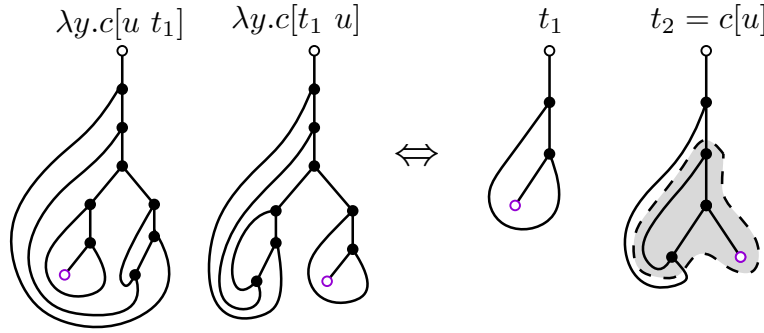


Figure 3.1: Two maps in $\dot{\mathcal{B}}_{[1]}$ together with the two corresponding maps $t_1 \in \dot{\mathcal{B}}_{[1]}$, $t_2 \in \dot{\mathcal{B}}_{[1]}^*$ used in their decomposition according to Equation (3.2).

The following lemma provides a bijection between non-identity/non-loop elements of $\dot{\mathcal{B}}_{[0]}$ and elements of $\dot{\mathcal{T}}_{[0]}$ having exactly one internal bridge/closed proper subterm.

Lemma 3.5. *The class $\dot{\mathcal{B}}_{[0]} \setminus \{ \circ \}$ is in bijection with the subclass of $\dot{\mathcal{O}}_{[0]} \subset \dot{\mathcal{T}}_{[0]}$ consisting of rooted trivalent maps having exactly one internal bridge and closed linear λ -terms having exactly one proper closed subterm.*

Proof. The bijection may be summarized schematically by the transformation

$$\lambda x.\lambda y.c[u] \leftrightarrow \lambda x.c[\lambda y.u]$$

where the left-to-right direction is *a priori* underspecified since there may be many ways to pick a subterm u with free variable y , but is determined canonically by picking the maximal one using Lemma 3.2. Visually, the bijection may also be summarized as a certain “sliding” operation on maps, see Figure 3.2.

In more detail, let us write $\phi : \dot{\mathcal{B}}_{[0]} \setminus \{ \lambda x.x \} \rightarrow \dot{\mathcal{O}}_{[0]}$ and $\phi^{-1} : \dot{\mathcal{O}}_{[0]} \rightarrow \dot{\mathcal{B}}_{[0]} \setminus \{ \lambda x.x \}$ for the two directions of the correspondence. We begin by defining these functions on lambda terms, and then give the equivalent definition on maps.

ϕ and ϕ^{-1} on lambda terms: Let $t \in \dot{\mathcal{B}}_{[0]} \setminus \{ \lambda x.x \}$ and note that t must be of the form $t = \lambda x.\lambda y.t_0$. Indeed, t is necessarily an abstraction $t = \lambda x.t_1$ by Proposition 3.1, and if t_1 were an application $t_1 = (t_2 \ t_3)$ then, by linearity, one of t_2 or t_3 would be closed, a contradiction. Therefore t_1 is also an abstraction $t_1 = \lambda y.t_0$ by the assumption that $t \neq \lambda x.x$. Consider now all possible ways of decomposing $t_0 = c[u]$ into a subterm $y \vdash u$ with free variable y and its surrounding context $x \vdash c$, and define

$$\phi(t) = \lambda x.c_m[\lambda y.u_m]$$

by taking the decomposition $t_0 = c_m[u_m]$ such that u_m is *maximal*, which exists by Lemma 3.2. Since u_m and c_m do not contain any closed proper subterms by assumption, the term $\phi(t)$ has exactly one closed proper subterm $\lambda y.u_m$.

Conversely, if $t' \in \dot{\mathcal{O}}_{[0]}$ is a term with exactly one closed proper subterm, then it necessarily decomposes as $t' = \lambda x.c[\lambda y.u]$ for some closed subterm $\lambda y.u$ with surrounding context $\lambda x.c$,

and we take $\phi^{-1}(t') = \lambda x.\lambda y.c[u]$. Observe that u is maximal among subterms of $\phi^{-1}(t')$ with free variable y , which ensures that ϕ^{-1} really is an inverse to ϕ .

This already completes the proof of the bijection, but we now describe it again on maps.

Direction: $\phi : \dot{\mathcal{B}}_{[0]} \setminus \{ \text{loop} \} \rightarrow \dot{\mathcal{O}}_{[0]}$ **on maps.** Let $m \in \dot{\mathcal{B}}_{[0]} \setminus \{ \text{loop} \}$ be a bridgeless rooted trivalent map that is not a loop, let $\tau(m) = t$ be its corresponding linear term, and let r be its root. Let, also, x, y be the child and grandchild (in the t -tree of m) of the root r . Then, by bridgelessness of m , we have that neither of x, y can be cut vertices and therefore there exists an edge $e = yz$ incident to y which does not belong to the t -tree of m . We then construct a new map $m' = (m \setminus e)/y$ and distinguish two cases based on whether m' is bridgeless or not. In the case where m' is bridgeless, we create the map $\phi(m)$ by introducing a new vertex q and two new edges qq and qz making q a loop and a neighbour of z . In the second case in which m' has bridges we note that they must all belong to unique path between r and z in the spanning tree $t' = t/y$ of m' ; indeed if there was another another bridge which was not in the t' -path between r and z it would necessarily also be present in the initial map m , contradicting its bridgelessness. Therefore we can choose uniquely the bridge $b = vw$ whose endpoints lie closest to the root r along the r - z path and delete it to form the map $m' \setminus b'$ to which we then introduce a new vertex q and three edges qv, qw, qz , making it adjacent to the former endpoints of b' and also z . In all of the above cases the maps have a unique bridge incident to q , yielding an element of $\dot{\mathcal{O}}_{[0]}$ as desired.

Direction: $\phi^{-1} : \dot{\mathcal{O}}_{[0]} \rightarrow \dot{\mathcal{B}}_{[0]} \setminus \{ \text{loop} \}$ **on maps.** Conversely, let $m \in \dot{\mathcal{O}}_{[0]}$ with r, x, v the root and its child and grandchild (in the $\tau(m)$ -tree). We denote the unique bridge of m by $b = vq$ and the two connected components of $m \setminus b$ by C_1, C_2 , with the convention that C_1 contains r and v while C_2 contains q . Note that since no other edge incident to q can be a bridge, there exists some edge $e = qz$ which does not belong to the $\tau(m)$ -tree of m . If q is a loop, then we form a new map $m' \setminus q$ and introduce to it a new vertex y along with three new edges xy, vy, zy . Otherwise we form the map $m' = (m \setminus e \setminus xv)/q$ and introduce to it a new vertex y and three new edges xy, vy, zy making it adjacent to x, v , and z . In either cases the new map m' , considered rooted at r , is trivalent and moreover is bridgeless since for any vertex formerly belonging to C_2 there now exists a path (via the newly added edge yz) connecting it to any of the vertices formerly belonging to C_1 .

Finally, to establish that the map operations ϕ, ϕ^{-1} are inverses of each other we let $m_1 \in \dot{\mathcal{B}}_{[0]} \setminus \{ \text{loop} \}$ be a non-loop bridgeless map, $m_2 \in \dot{\mathcal{O}}_{[0]}$ be a one-bridge map, and we label their vertices x, y, z, w, v, q as above. If $m_1 \setminus yz$ is bridgeless, then the map $\phi(m_1) \setminus q$ is by construction isomorphic to $(m_1 \setminus yz)/y$ which guarantees that $\phi^{-1}(\phi(m_1)) = m_1$ since ϕ^{-1} operates on $\phi(m_1)$ exactly by deleting the q and introducing a vertex y making it incident to x, v, z . Conversely, if $m_2 \in \dot{\mathcal{O}}_{[0]}$ is a map with a unique bridge incident to a loop, then $(\phi^{-1}(m_2) \setminus yz)/y$ is by construction isomorphic to $m_2 \setminus q$ and so we have $\phi(\phi^{-1}(m_2)) = m_2$ since ϕ operates on $\phi^{-1}(m_2)$ by deleting yz , dissolving y , and introducing a new loop vertex q making it a neighbour of z . Now, if $m_1 \setminus yz$ is not bridgeless, then the map $(\phi(m_1) \setminus qz)/q$ is by construction isomorphic to $(m_1 \setminus yz)/y$ and so once again by following the operation ϕ^{-1} on $\phi(m_1)$ we obtain $\phi^{-1}(\phi(m_1)) = m_1$. Finally, if $m_2 \in \dot{\mathcal{O}}_{[0]}$ has no loop, then $(\phi^{-1} \setminus yz)/y$ is isomorphic to $(m_2 \setminus qz)/q$ once again giving $\phi(\phi^{-1}(m_2)) = m_2$ as desired.

For graphical examples of the bijection, see Figure 3.2. □

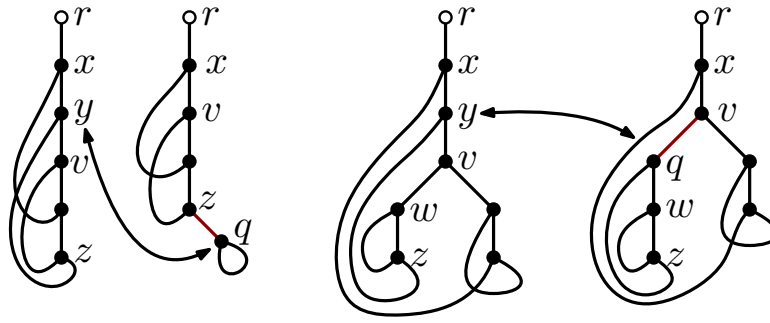


Figure 3.2: Two pairs of maps related by the bijection “ $\lambda x.\lambda y.c[u] \leftrightarrow \lambda x.c[\lambda y.u]$ ” explained in Lemma 3.5, with vertices labeled as in the proof.

Returning to the specification given in Equation (3.2), we see that it yields the following differential equation satisfied by the generating function $b(z)$ of $\dot{\mathcal{B}}_{[1]}$.

$$b(z) = z + 2z^3b(z)\frac{\partial}{\partial z}b(z) \quad (3.4)$$

We note that, by Equation (3.1), the generating function $B(z)$ of $\dot{\mathcal{B}}_{[0]}$ satisfies $B(z) = zb(z)$ and so we can focus on the easier-to-analyse $b(z)$ to obtain estimates for the asymptotic growth of $[z^n]B(z)$.

From Equation (3.4) one may extract the following recurrence for $b_n = [z^n]b(z)$, $n \geq 4$:

$$b_n = 2 \sum_{k=4}^n b_{k-3} (n-k+1) b_{n-k+1}.$$

We can obtain a lower bound for the sequence b_n (which generates **A267827**) by first isolating the summands corresponding to $k = 4$ and $k = n$ in the above recurrence, and then translating it to a differential equation taking into account the initial values $b_1 = 1, b_2 = b_3 = 0$ to obtain

$$\underline{b}(z) = z - 2z^4 + 2z^3\underline{b}(z) + 2z^4\frac{\partial}{\partial z}\underline{b}(z)$$

where now $\underline{b}(z)$ is such that $[z^n]\underline{b}(z) \leq b_n$ for all n . Let $\hat{\underline{b}}(z) = \sum_n \frac{b_n}{n!} z^n$ be the *Borel transform* of $\underline{b}$. Then $\hat{\underline{b}}$ satisfies

$$\hat{\underline{b}}(z) = z^2 - 2z\hat{\underline{b}}(z) + \frac{\partial^2}{\partial z^2}\hat{\underline{b}}(z)$$

which, for initial conditions $\hat{\underline{b}}(0), \hat{\underline{b}}'(0) = 0$, has a unique solution expressible in terms of the Airy functions

$$\hat{\underline{b}}(z) = \frac{z}{2} + \frac{\text{AiryBi}(2^{1/3}z)2^{2/3}3^{1/3}\pi}{12\Gamma(\frac{2}{3})} - \frac{\text{AiryAi}(2^{1/3}z)2^{2/3}3^{5/6}\pi}{12\Gamma(\frac{2}{3})}$$

Using the following closed form of Taylor series for *AiryAi*, *AiryBi* at $z = 0$

$$\frac{1}{3^c\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{(n+1)}{3}\right)}{n!} (3^{1/3}z)^n \left| \sin\left(\frac{2\pi(n+1)}{3}\right) \right|$$

where $c = 2/3$ for *AiryAi* and $c = 1/6$ for *AiryBi*, one obtains the following lower bound to b_n for $n = 3k + 1$:

$$b_n \geq n! [z^n] \hat{b}(z) = \frac{6^{k+1} \Gamma(k + \frac{2}{3})}{12 \Gamma(\frac{2}{3})} = \omega(5^k k!) \quad (3.5)$$

From this rough lower bound one is led to conjecture that bridgeless terms might make up a considerable percentage of all closed linear λ -terms (of which there are $O(6^k k!)$). This, coupled with Lemma 3.5, gives us a first clue of what the limit distribution looks like: it must obey $\mathbb{P}[X = 0] = \mathbb{P}[X = 1]$ and for $k \geq 2$, $\mathbb{P}[X = k]$ seems to decay fast. These observations suggest that we are looking at a *Poisson*(1) limit distribution for the number of bridges in $\hat{\mathcal{T}}_{[0]}$. Indeed, the following subsection provides the tool which will help us prove this conjecture.

Remark 3.6. As pointed out by one of the reviewers, a lower bound can also be derived directly by studying the sequence $\beta_n = b_{3n+2}$, which satisfies the recurrence

$$\begin{aligned} \beta_0 &= 1, \\ \beta_n &= 2 \sum_{0 \leq j < n} (3j + 1) \beta_j \beta_{n-1-j}. \end{aligned}$$

From the above one can then show

$$\beta_n \geq 2(3n - 1) \beta_{n-1} \geq \dots \geq 2^n \prod_{1 \leq j \leq n} (3j - 1) = \Theta(n^{-1/3} 6^n n!).$$

3.2. Poisson distributions from differential equations

Lemma 3.7 (Poisson Schema). *Let $F(z, u)$ be some bivariate power series. Furthermore, suppose that*

- *The power series $F(z, u)$ satisfies a first order differential equation with respect to u , which may be rearranged as*

$$\frac{\partial}{\partial u} F(z, u) = W_1(z, u, F(z, u)) \quad (3.6)$$

where W_1 is a rational function of $z, u, F(z, u)$.

- *There exist a constant $\sigma \in \mathbb{R}^+$ and constants $a \in \mathbb{N}^+, b \in \mathbb{N}$ such that²*

$$[z^n] F(z, 1) = 0 \quad n \not\equiv b \pmod{a} \quad (3.7)$$

$$\frac{[z^{n-b}] F(z, 1)}{[z^n] F(z, 1)} = O(n^{-\sigma}) \quad n \equiv b \pmod{a} \quad (3.8)$$

²The periodicity condition here is not essential to the lemma at all, the same holds for power series with non-zero coefficients for all $n \geq 0$. However given the fact that in this section we shall deal with power series which have non-zero coefficients only for $n \equiv 2 \pmod{3}$, we include this condition for ease of use.

Define $W_N(z, u, f)$, for $N \in \mathbb{N}_{\geq 2}$, such that $W_N(z, u, F(z, u)) = \frac{\partial^N}{\partial u^N} F(z, u)$. Then if for all $N \geq 1$:

$$[z^n] (\partial_f W_{N+1} \cdot W_1)|_{f=F(z,1), u=1} \sim [z^n] W_N|_{f=F(z,1), u=1}, \quad (3.9)$$

$$[z^n] (\partial_u W_{N+1})|_{f=F(z,1), u=1} = O([z^{n-b}] F(z, 1)), \quad (3.10)$$

$$[z^n] W_1|_{f=F(z,1), u=1} \sim [z^n] F(z, 1), \quad (3.11)$$

the random variables X_n whose probability generating function is given by

$$p_n(u) = \frac{[z^n] F(z, u)}{[z^n] F(z, 1)} \quad (3.12)$$

converge in distribution to a $Poisson(1)$ -distributed random variable X .

Proof. We have, by the chain rule, that

$$W_N = \frac{\partial}{\partial f} W_{N-1} W_1 + \frac{\partial}{\partial u} W_{N-1}. \quad (3.13)$$

Evaluating the above at $u = 1, f = F(z, 1)$ and extracting coefficients we have, by Equations (3.9) and (3.10),

$$[z^n] W_N(z, 1, F(z, 1)) \sim [z^n] W_{N-1}(z, 1, F(z, 1)) + O([z^{n-b}] F(z, 1)) \quad n \equiv b \pmod{a} \quad (3.14)$$

By Equation (3.11) we have that $[z^n] W_N(z, 1, F(z, 1))$ grows asymptotically as

$$[z^n] W_1(z, 1, F(z, 1)) \sim [z^n] F(z, 1)$$

and by Equation (3.8) we have $[z^{n-b}] F(z, 1) = o([z^n] F(z, 1))$. Therefore we have the following chain of asymptotic equivalences valid for $n \equiv b \pmod{a}$:

$$[z^n] W_N(z, 1, F(z, 1)) \sim [z^n] W_{N-1}(z, 1, F(z, 1)) \sim \dots \sim [z^n] W_1(z, 1, F(z, 1)) \sim [z^n] F(z, 1)$$

which translates to the following chain of asymptotic equalities between the factorial moments of X_n , valid once again for $n \equiv b \pmod{a}$:

$$\frac{[z^n] W_N(z, 1, F(z, 1))}{[z^n] F(z, 1)} \sim \frac{[z^n] W_{N-1}(z, 1, F(z, 1))}{[z^n] F(z, 1)} \sim \dots \sim \frac{[z^n] W_1(z, 1, F(z, 1))}{[z^n] F(z, 1)} \sim \frac{[z^n] F(z, 1)}{[z^n] F(z, 1)} = 1 \quad (3.15)$$

Using the following relation between r -th factorial and power moments of a random variable:

$$\mathbb{E}(X_n^r) = \sum_{k=0}^r \left\{ \begin{matrix} r \\ k \end{matrix} \right\} \mathbb{E}(X_n^k) \quad (3.16)$$

where $X_n^k = X_n(X_n - 1) \dots (X_n - k + 1)$ and $\left\{ \begin{matrix} r \\ k \end{matrix} \right\}$ denotes the Stirling numbers of the second kind, we have that $\lim_{n \rightarrow \infty} \mathbb{E}(X_n^r) = \mathbb{E}(X^r)$. Since the moment generating function $\mathbb{E}(e^{sX})$ exists in a neighbourhood of 0, we have that the $Poisson(1)$ distribution is determined by its moments (see [Bil08, Theorem 30.1]) and so, by the Markov–Fréchet–Shohat moment convergence theorem (see [Bil08, Theorem 30.2], [FS09, Theorem C.2]), we obtain our desired result. $\square$

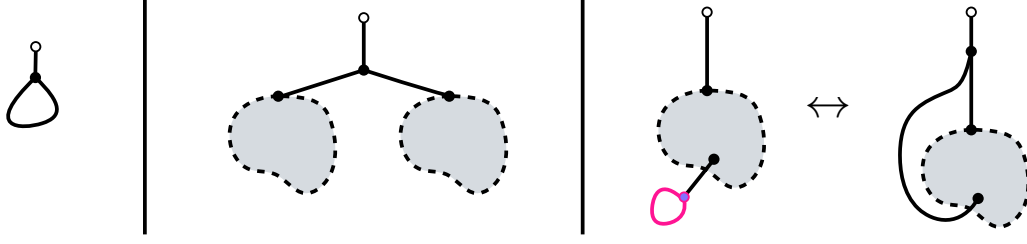


Figure 3.3: The three main cases of Equation (3.17): z^2 (left), $zT_{[0]}^{id}(z, u)^2$ (middle), and $\frac{\partial}{\partial u}T_{[0]}^{id}(z, u)$ (right), together with a transformation of the last into an abstraction term.

3.3. Identity subterms of closed linear terms and loops in trivalent maps

The goal of this subsection is to investigate the limit distribution of the number of identity-subterms (that is subterms equivalent to $\lambda x.x$) in closed linear λ -terms, or equivalently the number of loops in trivalent maps.

We begin by presenting a specification of the bivariate generating function $G(z, u)$ of closed linear λ -terms where u tags identity subterms.

Lemma 3.8. *Let $T_{[0]}^{id}(z, u)$ be the bivariate generating function enumerating closed linear λ -terms with respect to size and number of identity-subterms. Then,*

$$T_{[0]}^{id}(z, u) = (u - 1)z^2 + zT_{[0]}^{id}(z, u)^2 + \frac{\partial}{\partial u}T_{[0]}^{id}(z, u) \quad (3.17)$$

Proof. Consider the following rearrangement of the above equation:

$$T_{[0]}^{id}(z, u) + z^2 = uz^2 + zT_{[0]}^{id}(z, u)^2 + \frac{\partial}{\partial u}T_{[0]}^{id}(z, u) \quad (3.18)$$

which may be interpreted combinatorially in the following fashion: a closed linear λ -term is either a term of the form $\lambda x.x$, or an application of two closed terms, or is formed by taking some closed linear λ -term t , selecting some identity-subterm s , and replacing it with a free variable which is then bound by a newly-introduced lambda on top, to form the term $\lambda a.t[s := a]$, as on the right side of Figure 3.3.

Two subtler points of the abstraction case of the above construction are worth discussing. Firstly, note the use of the plain differential operator as opposed to the more usual pointing ($z\frac{\partial}{\partial u}$) operator. This is due to the fact that identity-terms are of size 2 which, after being pointed-at and replaced by a free variable, leads to a reduction of the size of the term by 1. This is balanced by the introduction of the new abstraction which, having size 1, causes the overall size to remain invariant. Secondly, we have the following “edge case”: applying the abstraction construction to the term $\lambda x.x$ leaves it invariant but removes its u mark. Such a term does not belong in $T_{[0]}^{id}(z, u)$, so we have to consider it separately, which is why the left-hand side of Equation (3.18) is $T_{[0]}^{id}(z, u) + z^2$ instead of just $T_{[0]}^{id}(z, u)$.

It is also of interest to note that this construction is highly reminiscent of the construction of the generating function enumerating open linear λ -terms with u tagging free variables presented

in Equation (1.1), with the term uz^2 enumerating the identity-term $\lambda x..x$ in Equation (3.17) playing a role analogous to the term uz enumerating the term x in Equation (1.1). $\square$

Before we proceed with the main result of this subsection, let us first address a technicality regarding the coefficient asymptotics of a power series which exhibits periodicities of the form

$$F(z) = \sum_{n=0}^{\infty} f_{an+b} z^{an+b},$$

with $f_n = 0$ for $n \not\equiv b \pmod{a}$, $a \in \mathbb{N}^+$, $b \in \mathbb{N}$. For such a power series, expressions of the form $[z^n]F(z) \sim f(n)$, where $f(n)$ is some function of n , are not well defined since they represent divergent limits. To correct for this periodic appearance of zeros in $[z^n]F(z)$, one can manipulate the power series so as to “skip” the problematic values of n :

$$[z^n] \frac{F(z^{1/a})}{z^{b/a}} \sim f(n). \quad (3.19)$$

To avoid having to use cumbersome case-based notation and/or manipulation of power series, we will instead follow [FS09] and write

$$[z^n]F(z) \sim f(n), \quad n \equiv b \pmod{a} \quad (3.20)$$

to mean that the limit is taken for the subsequence with $n = ak + b$, $k \in \mathbb{N}$, and is zero otherwise.

In the specific case of the generating function of closed linear λ -terms, $T_{[0]}(z)$, we have that:

Theorem 3.9 ([BGJ13, Theorem 3.3], [GIZ13, Lemma 37], [Jan03, Equation 3.8]). *The number of closed linear λ -terms of size $p \equiv 2 \pmod{3}$ is*

$$[z^p]T_{[0]}(z) \sim \frac{3}{\pi} 6^n n!, \quad p = 3n + 2. \quad (3.21)$$

We now proceed with the main result of this subsection.

Theorem 3.10. *Let χ_{id} be the parameter corresponding to the number of identity-subterms in a closed linear λ -term or, equivalently, to loops in rooted trivalent maps. Then for the random variables X_n^{id} corresponding to χ_{id} taken over $\dot{\mathcal{T}}_{[0]n}$ respectively we have:*

$$X_n^{id} \xrightarrow{D} \text{Poisson}(1) \quad (3.22)$$

Proof. Let $W_N(z, u, f)$ be such that $W_N(z, u, T_{[0]}^{id}) = \frac{\partial^N}{\partial u^N} T_{[0]}^{id}(z, u)$. For $N = 1$ we have, by Equation (3.17),

$$W_1 = f - zf^2 - (u - 1)z^2.$$

By induction, we will show that $W_N = \partial_f W_{N-1} W_1 + \partial_u W_{N-1}$ consists of terms of the form $f + R$ where R is a finite sum of monomials of the form $c_i f^j u^k z^l$ where c_i is a constant, $j, k \geq 0$, and $l \geq 1$.

Indeed, W_1 is of this form and for every $N \geq 2$, if W_{N-1} is of this form, we have

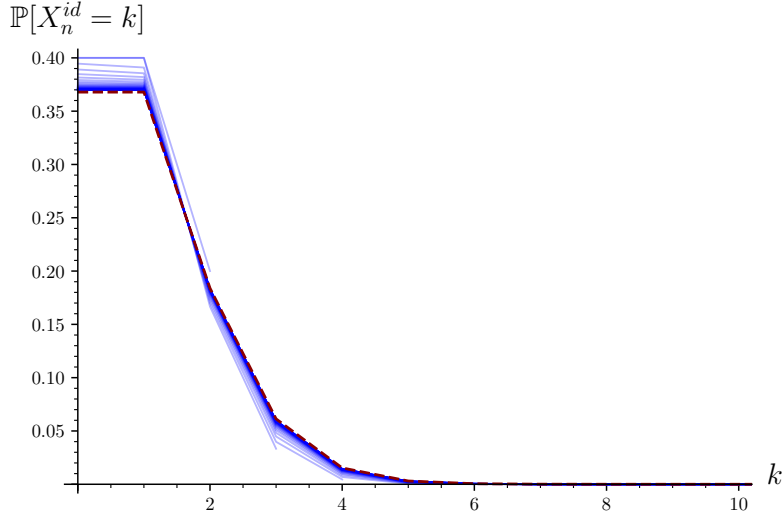


Figure 3.4: Density plots of X_n^{id} for $n = 2 \dots 100$ along with $Poisson(1)$ in red.

$$\frac{\partial}{\partial f} W_{N-1} W_1 + \frac{\partial}{\partial u} W_{N-1} = \left(1 + \frac{\partial}{\partial f} R\right) \cdot W_1 + \frac{\partial}{\partial u} R.$$

But a term-by-term differentiation of R with respect to either f or u maintains all desired properties of R and so finally, by grouping together the monomials of $\partial_f R$ and $\partial_u R$ as R' and noticing that W_1 contributes the sole f summand of W_N we see that $W_N = f + R'$ with R' as desired.

An application of Corollary 2.4 shows that $[z^n] z^k T_{[0]}^{id}(z, 1)^l$ for $k \geq 1, l \geq 0$ is asymptotically negligible compared to $[z^n] T_{[0]}^{id}(z, 1)$, for $n \equiv 2 \pmod{3}$. Since the R summand of $W_{n-1}|_{u=1}$ consists precisely of a finite number of such terms we have

$$\partial_f R|_{f=T_{[0]}^{id}(z,1), u=1} = O([z^{n-3}] T_{[0]}^{id}(z, 1)) \quad \text{and} \quad \partial_u R|_{f=T_{[0]}^{id}(z,1), u=1} = O([z^{n-3}] T_{[0]}^{id}(z, 1)),$$

again for $n \equiv 2 \pmod{3}$.

Finally, we have that

$$\begin{aligned} [z^n] \frac{\partial}{\partial f} W_N \cdot W_1 \Big|_{f=T_{[0]}^{id}(z,1), u=1} &\sim [z^n] W_N|_{f=T_{[0]}^{id}(z,1), u=1} \\ [z^n] \frac{\partial}{\partial u} W_N \Big|_{f=T_{[0]}^{id}(z,1), u=1} &= O([z^{n-3}] T_{[0]}^{id}(z, 1)), \quad n \equiv 2 \pmod{3} \\ [z^n] W_1|_{f=T_{[0]}^{id}(z,1), u=1} &\sim [z^n] T_{[0]}^{id}(z, 1), \quad n \equiv 2 \pmod{3} \end{aligned}$$

so by Lemma 3.7 we obtain the desired result. $\square$

3.4. Closed proper subterms of closed linear λ -terms and internal bridges in trivalent maps

Identity-subterms are the simplest case of a more general notion, that of closed proper subterms. Equivalently, loops are the smallest possible rooted trivalent map which can appear at one side of a bridge. In this subsection we are going to generalise the result of the previous subsection by investigating the limit distribution of closed proper subterms of linear λ -terms and of internal bridges in rooted trivalent maps. As in the previous subsection, we will rely on a specification for the bivariate generating function $T_{[0]}^{sub}(z, v)$ of closed linear lambda terms where v tags closed proper subterms.

Before presenting said specification, we begin by defining the following classes which will provide the building blocks for our decomposition of $\dot{\mathcal{T}}_{[0]}^\lambda$.

Definition 3.11 (The class $\mathcal{K}$ of simple closed non-trivial one-hole contexts). Let $\mathcal{K}$ be the combinatorial class consisting of simple closed one-hole contexts other than $\square$. In terms of maps, elements of $\mathcal{K}$ correspond to *doubly-rooted trivalent maps*, which have the following structure:

- There is a distinguished vertex r of degree 1, called the *root vertex* (which as usual we will draw as a white vertex with black border).
- There is a distinguished vertex v of degree 1, called the *box vertex* (which as at the end of Section 2.2 we will draw as a grey vertex with black border).
- All other vertices have degree 3, and there is at least one such vertex.

Elements of $\mathcal{K}$ will be enumerated by their size as contexts, which equals the number of edges in the corresponding map **minus** 1.

Definition 3.12 (The class $\mathcal{Q}$). Let $\mathcal{Q} \subseteq \mathcal{K}$ be the combinatorial class formed by restricting $\mathcal{K}$ to one-hole contexts of which every proper right subcontext is either $\square$ or has a free variable. Viewed as maps, the elements $q \in \mathcal{Q}$ have the additional property that:

- No edge belonging to the $\tau(q)$ -path from r to v is an internal bridge, where r is the root vertex, v is the distinguished 1-valent vertex, and $\tau(q)$ is the one-hole context corresponding to q .

Let $\dot{\mathcal{T}}_{[0]}^\lambda$ be the class consisting of *closed non-identity linear abstraction terms*; note that the map corresponding to an element of $\dot{\mathcal{T}}_{[0]}^\lambda$ is exactly a map of size bigger than 2 with the property that deleting the root and its unique neighbour leaves a connected map.

Lemma 3.13. For the combinatorial classes $\dot{\mathcal{T}}_{[0]}^\lambda, \mathcal{Q}$ we have

$$\dot{\mathcal{T}}_{[0]}^\lambda = \mathcal{Z}^2 \mathcal{Q} + \mathcal{Q} \dot{\mathcal{T}}_{[0]}^\lambda \quad (3.23)$$

At the level of generating functions we have

$$T_{[0]}^{\lambda, sub}(z, v) = z^2 Q(z, v) + Q(z, v) T_{[0]}^{\lambda, sub}(z, v) \quad (3.24)$$

where $T_{[0]}^{\lambda, sub}(z, v)$ is the generating function enumerating elements of $\dot{\mathcal{T}}_{[0]}^\lambda$ and $Q(z, v)$ the one enumerating elements of $\mathcal{Q}$ with v in both cases tagging proper subterms of terms, or equivalently internal bridges in rooted trivalent maps.

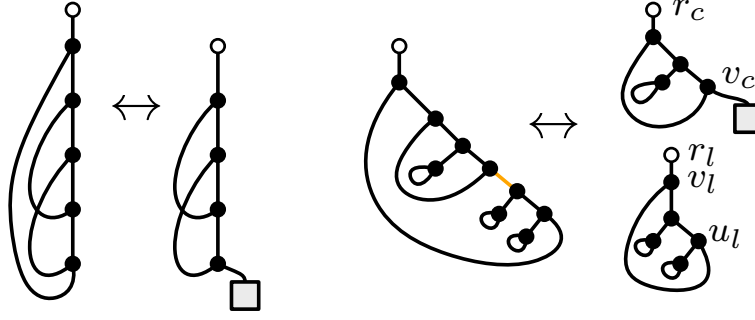


Figure 3.5: Maps corresponding to elements of $\dot{\mathcal{T}}_{[0]}^\lambda$ and their decompositions. For the subfigure on the left, we have a decomposition of $\lambda x.\lambda y.\lambda z.xzy$ into $\lambda y.\lambda z.\Box zy$, which is of the form $\mathcal{Z}^2\mathcal{Q}$. For the map on the right we have a decomposition of $\lambda x.\lambda y.x(\lambda a.a)(\lambda b.b)y(\lambda c.c)$ into $\lambda y.\Box y(\lambda c.c)$ and $\lambda x.x(\lambda a.b)(\lambda b.b)$ of the form $\mathcal{Q}\dot{\mathcal{T}}_{[0]}^\lambda$. Vertices in the right subfigure are labelled as in the proof of Lemma 3.13 and the internal bridge b is highlighted.

Proof. We establish the desired bijection of Equation (3.23) by providing two mappings ψ, ψ^{-1} between the left-hand side and the right-hand side and vice-versa. We then verify that this is indeed a bijection and that it leads to the equality of generating functions presented in Equation (3.24).

Direction $\psi : \dot{\mathcal{T}}_{[0]}^\lambda \rightarrow \mathcal{Z}^2\mathcal{Q} + \mathcal{Q}\dot{\mathcal{T}}_{[0]}^\lambda$. Let l be some element of $\dot{\mathcal{T}}_{[0]}^\lambda$, viewed as a closed abstraction term and write $l = \lambda x.c[x]$ for some one-hole context c . We now distinguish cases based on the nature of right subcontexts of c :

Case 1.1: All proper right subcontexts of c are either $\Box$ or have a free variable. In this case c is an element of $\mathcal{Q}$ by definition. To account for the change of size resulting from the deletion of the outermost abstraction and its bound variable, a $\mathcal{Z}^2$ factor is introduced, yielding the $\mathcal{Z}^2\mathcal{Q}$ summand of Equation (3.23). Therefore we let $\psi(l) = (\bullet^2, c)$, for $\bullet^2 \in \mathcal{Z}^2$.

In terms of maps, let m be the map corresponding to l and r be the root, a be the vertex representing the outermost abstraction, and v be the vertex representing its bound variable. The map m_c corresponding to the context c is obtained from the map $(m \setminus av)/a$ by adding a new box vertex which is made adjacent to v . The current case then corresponds to the non-existence of an internal bridge along the c -path from the root of m_c to its unique box vertex. As such, m_c yields a unique member of $\mathcal{Q}$.

Case 1.2: There exists a proper closed right subcontext of c other than $\Box$. In this case let c' be the biggest such proper right subcontext, with c decomposing as $c = c_1 \circ c'$ for some $c_1 \neq \Box$. Let c'' be an arbitrary proper right subcontext of c_1 . Then $c_1 = c_0 \circ c''$ for some context c_0 and $c = c_0 \circ c'' \circ c'$. Notice then that any such c'' must either be $\Box$ or have a free variable, for otherwise $c'' \circ c'$ would be a closed proper right subcontext of c bigger than c' , violating maximality of c' . Since c'' was an arbitrary proper right subcontext of c_1 , we have that c_1 belongs to $\mathcal{Q}$ by definition. As for c' , since by linearity x does not appear in c' , we have that $\lambda x.c'[x]$ belongs to $\dot{\mathcal{T}}_{[0]}^\lambda$. Together, these yield the $\mathcal{Q}\dot{\mathcal{T}}_{[0]}^\lambda$ summand of Equation (3.23) and so

we let $\psi(l) = (c_1, \lambda x.c'[x])$.

In terms of maps, we have in this case that the map m_c corresponding to c , constructed as detailed above, has at least one internal bridge in the c -path between the root and the box vertex. Let b be the unique such bridge closest to the root (in terms of distances along the c -tree of m_c). Then the deletion of b results in two connected components which yield $m_q = \tau^{-1}(q)$ and (after some manipulation) $m_{\lambda x.c'[x]} = \tau^{-1}(\lambda x.c'[x])$.

Direction $\psi^{-1} : \mathcal{Z}^2\mathcal{Q} + \mathcal{Q}\dot{\mathcal{T}}_{[0]}^\lambda \rightarrow \dot{\mathcal{T}}_{[0]}^\lambda$. For the other direction, let $t \in \mathcal{Z}^2\mathcal{Q} + \mathcal{Q}\dot{\mathcal{T}}_{[0]}^\lambda$. We distinguish the following two possibilities based on the nature of t .

Case 2.1: $t \in \mathcal{Z}^2\mathcal{Q}$. In this case, t consists of an element of $\mathcal{Z}^2$ together with a context $c \in \mathcal{Q}$. Then, assuming without loss of generality that x does not appear in c , $t' = \lambda x.c[x]$ belongs in $\dot{\mathcal{T}}_{[0]}^\lambda$, with $\mathcal{Z}^2$ accounting for the increase in size $|t'| = |c| + 2$, so that the preimage of t is $\psi^{-1}(t) = (\bullet^2, \lambda x.c[x])$.

In terms of maps, let $r, v_\square$ be the root and box vertices of $m_c = \tau^{-1}(c)$ respectively. Then this case amounts to identifying r with $v_\square$ and introducing a new root vertex r' which we make adjacent to r .

Case 2.2: $t \in \mathcal{Q}\dot{\mathcal{T}}_{[0]}^\lambda$. In this case, t consists of a pair of $c \in \mathcal{Q}$ and $l \in \dot{\mathcal{T}}_{[0]}^\lambda$. Suppose, without loss of generality, that $l = \lambda x.d[x]$. Then $\lambda x.c[d[x]]$ belongs in $\dot{\mathcal{T}}_{[0]}^\lambda$ and $\psi^{-1}(t) = \lambda x.c[d[x]]$.

In terms of maps, let r_l be the root of $m_l = \tau^{-1}(l)$, let v_l be the unique neighbour of r_l and let u_l be the neighbour of v_l which is furthest from v_l in the l -tree of m . Let also $r_c, v_\square, v_c$ be the root, box vertex, and the unique neighbour of the box vertex in $m_c = \tau^{-1}(c)$, respectively. Then by taking the disjoint union $(m_l \setminus r_l \setminus v_l u_l) + (m_c \setminus v_\square) + r_{new}$, where r_{new} is a new vertex, introducing two new edges $r_{new}r_c, r_c u_l$, and identifying v_c with v_l , we form a rooted trivalent map as desired. See the right subfigure of Figure 3.5 for an example.

We now proceed to verify that ψ^{-1} is a two-sided inverse of ψ . Let $l \in \dot{\mathcal{T}}_{[0]}^\lambda$ and write $l = \lambda x.c[x]$. We then have

$$\psi^{-1}(\psi(t)) = \psi^{-1}((\bullet^2, c)) = \lambda x.c[x]$$

or

$$\psi^{-1}(\psi(t)) = \psi^{-1}((q, \lambda x.c'[x])) = \lambda x.q[c'[x]] = \lambda x.c[x]$$

depending on whether we fall under Case 1.1 or 1.2 respectively. Conversely, if $(\bullet^2, c) \in \mathcal{Z}^2\mathcal{Q}$, we have $\psi(\psi^{-1}((\bullet^2, c))) = \psi(\lambda x.c[x]) = (\bullet^2, c)$ since $c \in \mathcal{Q}$ satisfies the properties of Case 1.1. Lastly, if $(c, \lambda x.d[x]) \in \mathcal{Q}\dot{\mathcal{T}}_{[0]}^\lambda$, $\psi(\psi^{-1}((c, \lambda x.d[x]))) = \psi(\lambda x.c[d[x]]) = (c, \lambda x.d[x])$ since d by construction is closed and so we fall under Case 1.2.

Finally, we note that in both directions of the above bijection, no new bridges/closed subterms are created, so that the number of bridges/closed subterms in the left-hand-side of Equation (3.23) equals that on the right, yielding Equation (3.24). $\square$

Now, in order to obtain the bivariate generating function for closed linear lambda terms by size and number of closed proper subterms, let $\Theta_{sub}\dot{\mathcal{T}}_{[0]}^\lambda$ be the class of *closed linear* λ -terms

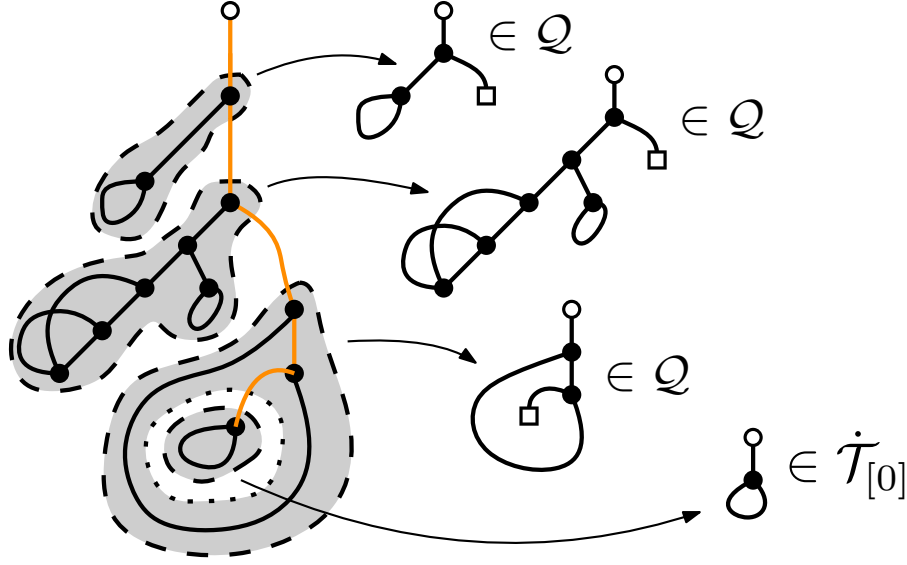


Figure 3.6: A map m along with a highlighted path between the root and a bridge in m . The decomposition of m into components yielding maps in $\mathcal{Q}$ and $\dot{\mathcal{T}}_{[0]}$ is depicted by grey borders.

with a distinguished closed proper subterm or, equivalently, rooted trivalent maps with a distinguished internal bridge. The following proposition just recapitulates the fact that such pointed objects may be decomposed in terms of the class of one-hole contexts/doubly-rooted maps $\mathcal{K}$.

Proposition 3.14. *For the combinatorial classes $\Theta_{sub}\dot{\mathcal{T}}_{[0]}$, $\dot{\mathcal{T}}_{[0]}$, and $\mathcal{K}$ we have*

$$\Theta_{sub}\dot{\mathcal{T}}_{[0]} = \mathcal{K} \dot{\mathcal{T}}_{[0]}. \quad (3.25)$$

At the level of generating functions, if $K(z, v)$ is the generating function enumerating objects of $\mathcal{K}$, with v tagging internal bridges/closed proper subterms, then

$$v \frac{\partial}{\partial v} T_{[0]}^{sub}(z, v) = K(z, v) T_{[0]}(z, v). \quad (3.26)$$

Proof. Let $t \in \Theta_{sub}\dot{\mathcal{T}}_{[0]}$ be a closed linear λ -term with a distinguished closed proper subterm u . By definition, this means that $t = c[u]$ for some non-trivial one-hole context $c \in \mathcal{K}$ and $u \in \dot{\mathcal{T}}_{[0]}$, establishing Equation (3.25).

In terms of maps, if $m = \tau(t)$ is the map corresponding to t with $b = vw$ a distinguished internal bridge, where without loss of generality we assume v is an ancestor of w in the t -tree of m , then C is the component of $m \setminus b$ containing the root, with a new box vertex added and made adjacent to v , while u is the remaining component with a new root vertex added and made adjacent to w . $\square$

We now show proceed to show that elements of $\mathcal{K}$ factor into non-empty sequences of elements in $\mathcal{Q}$.

Lemma 3.15. *For the combinatorial classes $\mathcal{K}$ and $\mathcal{Q}$ we have*

$$\mathcal{K} = SEQ_{\geq 1}(\mathcal{Q}) \quad (3.27)$$

At the level of generating functions, if $Q(z, v)$ is the generating function enumerating objects of $\mathcal{Q}$, with v tagging internal bridges/closed proper subterms, then

$$K(z, v) = \frac{vQ(z, v)}{1 - vQ(z, v)} \quad (3.28)$$

Proof. Let $m \in \mathcal{K}$ be a doubly-rooted trivalent map corresponding to a one-hole context c , with $b_s = uv$ the unique external bridge of m adjacent to its box vertex v . Let, also, r be the root of m and p be the c -path between the unique neighbour of r and the box vertex v .

Let i be the number of bridges belonging to p and let us label these bridges as $b_1, b_2, \dots, b_i$, ordered by their proximity to the root (so that $b_i = b_s$). We note that $i \geq 1$ by definition of p . Let $C_1, C_2, \dots, C_{i+1}$ be the $i + 1$ connected components of $m' = m \setminus b_1 \setminus \dots \setminus b_i \setminus r$, ordered in a way such that C_i is the unique component of m' containing the endpoint of b_i closest to the root.

Consider now a connected component $C_k, k \in \{1, \dots, i\}$, and let p_k be the restriction of p in C_k , i.e., $p_k = p[V(p) \cap V(C_k)]$. By construction, there exist exactly two degree-2 vertices in C_k , say s_k and t_k , with s_k being the one closer to the root in m , in terms of distances along the c -tree of m . Modifying each C_k by adding a new root vertex r_k which we make adjacent to s_k as well as a new box vertex v_k which we make adjacent to t_k we produce a map C'_k satisfying the restriction of maps in $\mathcal{Q}$: the unique path between r_k and v_k along the $\tau(C'_k)$ -spanning tree of C'_k is exactly p_k together with the edges $r_k s_k$ and $t_k v_k$, which by construction contains no internal bridges of C_k . Finally, we note that $C_{i+1} = v$ is just the box vertex of m which we are free to discard.

In terms of contexts, the above decomposition translates uniquely to a factorisation $c = c_1 \circ \dots \circ c_i$ where $c_k = \tau(C'_k)$ for $1 \leq k \leq i$.

The above arguments provide a mapping from structures in $\mathcal{K}$ to a unique non-empty sequence of elements $C'_k \in \mathcal{Q}$. With respect to enumeration we note again that each $C'_k, 1 \leq k \leq i$, can be paired with a unique bridge of m , the one incident to the vertex of C_k closest to the root, and so we introduce v -factors in the generating function $\frac{vQ(z, v)}{1 - vQ(z, v)}$ to keep track of that data. Furthermore, the decomposition of m described above is invertible: given the sequence of maps $(C'_1, \dots, C'_i) \in \mathcal{Q}^i$ we may uniquely reconstruct m by deleting the box vertex of each $C'_k \in \{C'_1, \dots, C'_{i-1}\}$ and identifying the resulting unique degree-2 vertex of each C'_k , for $k < i$, with the root of C_{k+1} . This new map, rooted at the root of C_1 , is an element of $\mathcal{K}$. Furthermore, all bridges/closed subterms are accounted for and so we have the desired equality between the corresponding generating functions.

Finally, in terms of contexts this corresponds to the fact that from the sequence of contexts $(c_1, \dots, c_i) \in \mathcal{Q}^i$, we can uniquely reconstruct the element $c \in \mathcal{K}$ as $c = c_1 \circ \dots \circ c_i$. $\square$

From the above bijections we finally obtain

Lemma 3.16. *Let $T_{[0]}^{sub}$ be the generating function enumerating closed linear lambda terms where v tags closed proper subterms. We have that*

$$\frac{\partial}{\partial v} T_{[0]}^{sub}(z, v) = -\frac{v^2 z T_{[0]}^{sub}(z, v)^3 + z^2 T_{[0]}^{sub}(z, v) - T_{[0]}^{sub}(z, v)^2}{(v^3 - v^2) z T_{[0]}^{sub}(z, v)^2 + v z^2 - (v - 1) T_{[0]}^{sub}(z, v)}. \quad (3.29)$$

Proof. By Equations (3.26) and (3.28) we have

$$v \frac{\partial}{\partial v} T_{[0]}^{sub}(z, v) = K(z, v) T_{[0]}^{sub}(z, v) = \frac{vQ(z, v)}{1 - vQ(z, v)} T_{[0]}^{sub}(z, v) \quad (3.30)$$

The result then follows by substituting $Q = \frac{T_{[0]}^{\lambda, sub}}{z^2 + T_{[0]}^{\lambda, sub}}$ (which follows from Equation (3.24)) and $T_{[0]}^{\lambda, sub} = T_{[0]}^{sub} - zv^2(T_{[0]}^{sub})^2 - z^2$ (which follows from the definition of $\mathcal{T}_{[0]}^{\lambda}$) into Equation (3.30) and finally dividing both sides by v . $\square$

Before we proceed with the main result of this section, we present a number of definitions and lemmas useful for the proof.

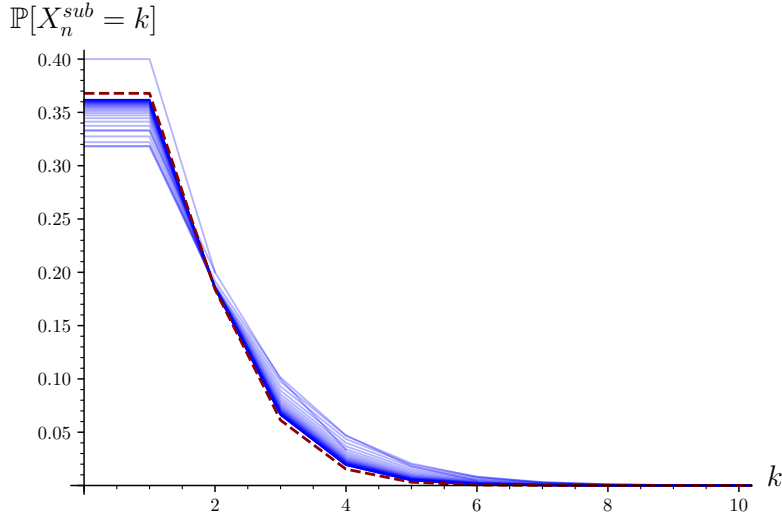


Figure 3.7: Density plots of X_n^{sub} for $n = 2 \dots 100$ along with $Poisson(1)$ in red.

Definition 3.17 (Operators B_k). We define a family of operators B_k , parameterised by $k \in \{-1\} \cup \mathbb{N}$, which extract the *balanced part* of a polynomial $\eta \in \mathbb{Z}[f, z, v]$, defined as follows

$$B_k(\eta) = \sum_{\substack{i,j \\ i \leq k+1}} \bar{\eta}_{j,i} v^j z^{2k-2(i-1)} f^i. \quad (3.31)$$

where $\bar{\eta}_{j,i} = [v^j f^i z^{2k-2(i-1)}] \eta$.

Definition 3.18 (k -admissible polynomial). A polynomial $\eta \in \mathbb{Z}[f, z, v]$ is k -admissible if it can be written as a sum of monomials

$$\sum_{l \geq \max\{0, 2k-2(i-1)\}} a_{j,i,l} f^i v^j z^l \quad (3.32)$$

Lemma 3.19. The operator B_k is linear, that is, if $\eta = \eta_1 + \eta_2$,

$$B_k(\eta) = B_k(\eta_1) + B_k(\eta_2) \quad (3.33)$$

Proof. Follows immediately from Equation (3.31). $\square$

Lemma 3.20. Let $\eta = \eta_1 \cdot \eta_2$, where η_1 is k_1 -admissible and η_2 is k_2 -admissible. Then η is $(k_1 + k_2 + 1)$ -admissible and

$$B_{k_1+k_2+1}(\eta) = B_{k_1}(\eta_1) \cdot B_{k_2}(\eta_2) \quad (3.34)$$

Proof. The product of two monomials $a_{j,i,l} z^l f^i v^j$ taken from η_1 and $b_{j',i',l'} z^{l'} f^{i'} v^{j'}$ taken from η_2 yields a monomial in $B_{k_1+k_2+1}(\eta)$ if and only if $l = 2k_1 - 2(i-1)$ and $l' = 2k_2 - 2(i'-1)$, i.e. only if both monomials belong to $B_{k_1}(\eta_1)$ and $B_{k_2}(\eta_2)$ respectively. Otherwise, if either $l > 2k_1 - 2(i-1)$ or $l' > 2k_2 - 2(i'-1)$, the product will result in a monomial of degree in z at least $2(k_1 + k_2) - 2((i + i') - 1) + 1$. Since η_1, η_2 are k_1 - and k_2 -admissible respectively, these are the only cases of monomials possible, therefore η is $(k_1 + k_2 + 1)$ admissible and its balanced part is given by Equation (3.34). $\square$

Lemma 3.21. Let η be k -admissible for $k \geq 0$. Then $\partial_f \eta$ is $(k-1)$ -admissible and

$$B_{k-1}(\partial_f \eta) = \sum_{i,j} i \bar{\eta}_{i,j} z^{2k-2(i-1)} f^i v^j \quad (3.35)$$

Proof. The k -admissibility of η implies that η can be written as a sum of monomials of the form

$$B_k(\eta) + R = \sum_{i,j} \bar{\eta}_{i,j} z^{2k-2(i-1)} f^i v^j + R \quad (3.36)$$

where R is a sum of monomials of the form $\underline{\eta}_{i,j} z^p f^i v^j$ for constants $\underline{\eta}_{i,j}$ and $p < 2k - 2(i-1)$. Then, any monomial $(i+1)\bar{\eta}_{i,j} z^l f^i v^j$ in $\partial_f B_k(\eta)$ satisfies the conditions of $(k-1)$ -admissibility, while any monomial $i\underline{\eta}_{i,j} z^p f^{i-1} v^j$ in $\partial_f R$ does not contribute to $B_{k-1}(\eta)$ and, since η is k -admissible, has a degree $p > 2k - 2(i-1) = 2(k+1) - 2i > 2(k-1) - 2(i-1)$ which also satisfies the conditions of $(k-1)$ -admissibility. $\square$

Theorem 3.22. Let χ_{sub} be the parameter corresponding to the number of closed proper sub-terms in a closed linear λ -term or, equivalently, of internal bridges in rooted trivalent maps. Then for the random variables X_n^{sub} corresponding to χ_{sub} taken over $\dot{\mathcal{T}}_{[0]n}$ respectively we have:

$$X_n^{sub} \xrightarrow{D} \text{Poisson}(1). \quad (3.37)$$

Proof. Let $W_N(z, v, f)$ be such that $W_N(z, v, T_{[0]}^{sub}(z, v)) = \frac{\partial^N}{\partial v^N} T_{[0]}^{sub}(z, v)$. By successively differentiating Equation (3.29) we have that W_N is a rational function

$$W_N = \frac{h_N}{g^k} \quad (3.38)$$

where $k = 2N - 1$, $h_N \in \mathbb{Z}[f, z, v]$, and

$$g = f^2 v^3 z - f^2 v^2 z + v z^2 - f v + f. \quad (3.39)$$

By definition, $W_N|_{f=T_{[0]}^{sub}(z,v)}$ equals the N -th derivative of $T_{[0]}^{sub}(z, v)$ with respect to v , which counts rooted trivalent maps in which N bridges are chosen to have their v -mark erased. As such, $W_N|_{f=T_{[0]}^{sub}(z,1),v=1}$ counts properly-sized objects: the coefficients $[z^n]W_N(z, v, T_{[0]}^{sub}(z, v))$ will be 0 for $n \not\equiv 2 \pmod{3}$. This implies that $[f^i]h_N$ has minimum degree in z at least $2k - 2(i - 1)$, otherwise

$$[z^p] \frac{[f^i]h_N}{g^k} \Big|_{f=T_{[0]}^{sub}(z,1),v=1} = [z^p] \frac{[f^i]h_N|_{f=T_{[0]}^{sub}(z,1),v=1}}{z^{2k}}$$

would be non-zero for some $p < 2$ which we know not to be the case. Therefore, using the B_k operator, we may expand h_N as the following sum of its *balanced* and *unbalanced* parts

$$h_N = B_k(h_N) + R = \sum_j v^j \sum_i \alpha_{N,j,i} z^{2k-2(i-1)} f^i + R_j \quad (3.40)$$

where $R = h_N - B_k(h_N)$, $R_j = [v^j]R$, and $\alpha_{N,j,i} = [v^j f^i z^{2k-2(i-1)}]B_k(h_N)$. The above argument then implies that R is a sum of monomials of the form $\beta_{N,j,i} v^j f^i z^l$ where $i \geq 3$, $l > 2k - 2(i - 1)$, and $\beta_{N,j,i} \in \mathbb{Z}$ and therefore h_N is k -admissible.

Let us now introduce the following two operators obtained by evaluating $\partial_f (B_k(\cdot))$ and $B_k(\cdot)$ at $f = z = v = 1$:

$$\begin{aligned} C_k(\eta) &= (\partial_f B_k(\eta))|_{f=1,z=1,v=1} = \sum_{i,j} i \bar{\eta}_{N,j,i}, \\ D_k(\eta) &= (B_k(\eta))|_{f=1,z=1,v=1} = \sum_{i,j} \bar{\eta}_{N,j,i}. \end{aligned}$$

Now, by Corollary 2.4, a term of W_N whose numerator is given by $\alpha_{N,j,i} z^{2k-2(i-1)} f^i$ is such that

$$[z^n] \frac{\alpha_{N,j,i} z^{2k-2(i-1)} f^i}{g^k} \Big|_{v=1, f=T_{[0]}(z)} = [z^n] \frac{\alpha_{N,j,i} T_{[0]}(z)^i}{z^{2(i-1)}} \sim i \alpha_{N,j,i} [z^n] T_{[0]}(z), \quad n \equiv 2 \pmod{3}$$

and so we have

$$[z^n] \left(\frac{B_k(h_N)}{g^k} \Big|_{v=1, f=T_{[0]}(z)} \right) \sim \sum_{i,j} i \alpha_{N,j,i} [z^n] T_{[0]}(z) = C_k(h_N) [z^n] T_{[0]}(z) \quad (3.41)$$

while the coefficients of z^n of monomials in the unbalanced part R evaluated at $v = 1$, $f = T_{[0]}(z)$ are asymptotically $O([z^{n-3}]T_{[0]}(z))$ for $n \equiv 2 \pmod{3}$ and since there are only finitely many of them, we have

$$[z^n] \left(\frac{R}{g^k} \Big|_{v=1, f=T_{[0]}(z)} \right) = O([z^{n-3}]T_{[0]}(z)) \quad (3.42)$$

We will now proceed via induction to show that for any $N \geq 1$ the following hold:

$$C_k([v^0]h_N) = \sum_i i\alpha_{N,0,i} = 1, \quad (3.43)$$

$$D_k([v^0]h_N) = \sum_i \alpha_{N,0,i} = 0, \quad (3.44)$$

and for $j \geq 1$

$$C_k([v^j]h_N) = \sum_i i\alpha_{N,j,i} = 0, \quad (3.45)$$

$$D_k([v^j]h_N) = \sum_i \alpha_{N,j,i} = 0. \quad (3.46)$$

For the inductive base, notice that all four equations hold for

$$W_1 = -\frac{f^3 v^2 z}{g} - \frac{f z^2}{g} + \frac{f^2}{g},$$

which has balanced part $B_1(h_1) = -f z^2 + f^2$. For our inductive step, supposing that the desired properties hold for W_N , we begin by noting that, by the chain rule, W_{N+1} may be written as

$$W_{N+1} = \partial_f W_N W_1 + \partial_v W_N = \frac{(\partial_f h_N \cdot g^k - h_N \cdot \partial_f g^k) h_1}{g^{2k+1}} + \frac{(\partial_v h_N \cdot g^k - h_N \cdot \partial_v g^k) g}{g^{2k+1}}.$$

By grouping together the summands and partially simplifying them, we obtain

$$W_{N+1} = \frac{\partial_f h_N \cdot g h_1 - k h_N \partial_f g \cdot h_1 + \partial_v h_N \cdot g^2 - k h_N \partial_v g \cdot g}{g^{k+2}}$$

which brings W_{N+1} into the form of Equation (3.38) with

$$h_{N+1} = \partial_f h_N \cdot g h_1 - k h_N \partial_f g \cdot h_1 + \partial_v h_N \cdot g^2 - k h_N \partial_v g \cdot g. \quad (3.47)$$

Then, Lemma 3.19 allows us to compute the balanced part of h_{N+1} as

$$\begin{aligned} B_{k+2}(h_{N+1}) &= B_{k+2}(\partial_f h_N \cdot g \cdot h_1) + B_{k+2}(-k \cdot h_N \cdot \partial_f g \cdot h_1) \\ &\quad + B_{k+2}(\partial_v h_N \cdot g^2) + B_{k+2}(-k \cdot h_N \cdot \partial_v g \cdot g) \end{aligned} \quad (3.48)$$

Therefore, to validate Equations (3.43) to (3.46), it suffices to sum the contributions of each summand of Equation (3.48) to Equations (3.43) to (3.46), which we now proceed to do.

1. For the first summand of Equation (3.48) we have

$$\begin{aligned} B_{k+2}(\partial_f h_N \cdot g \cdot h_1) &= B_{k+1}(\partial_f h_N \cdot h_1) \cdot B_0(g) && \text{by Lemma 3.20} \\ &= B_{k-1}(\partial_f h_N) \cdot B_1(h_1) \cdot B_0(g) && \text{by Lemma 3.20} \end{aligned}$$

By Lemma 3.21 we have

$$\begin{aligned} B_{k-1}(\partial_f h_N) \cdot B_1(h_1) &= \left(\sum_i i \alpha_{N,j,i} v^j z^{2k-2(i-1)} f^{i-1} \right) (-z^2 f + f^2) \\ &= \sum_{i,j} ((i-1) \alpha_{N,j,i-1} - i \alpha_{N,j,i}) v^j z^{2(k+1)-2(i-1)} f^i. \end{aligned} \quad (3.49)$$

and so, by letting $a'_{j,i} = ((i-1) \alpha_{N,j,i-1} - i \alpha_{N,j,i})$,

$$\begin{aligned} B_{k+2}(\partial_f h_N \cdot g \cdot h_1) &= B_{k-1}(\partial_f h_N) \cdot B_1(h_1) \cdot B_0(g) \\ &= \left(\sum_{i,j} a'_{j,i} v^j z^{2(k+1)-2(i-1)} f^i \right) (vz^2 - fv + f) \\ &= \sum_{i,j} (\alpha'_{j-1,i} - \alpha'_{j-1,i-1} + \alpha'_{j,i-1}) z^{2(k+2)-2(i-1)} v^j f^i \end{aligned}$$

Therefore, for any $j \geq 0$,

$$\begin{aligned} C_{k+2}([v^j](\partial_f h_N \cdot h_1 \cdot g)) &= \sum_i i \alpha'_{j-1,i} - \sum_i i \alpha'_{j-1,i-1} + \sum_i i \alpha'_{j,i-1} \\ &= \sum_i i \alpha'_{j-1,i} - \sum_i (i+1) \alpha'_{j-1,i} + \sum_i (i+1) \alpha'_{j,i} \\ &= C_{k+1}([v^{j-1}](\partial_f h_N \cdot h_1)) \\ &\quad - C_{k+1}([v^{j-1}](\partial_f h_N \cdot h_1)) + D_{k+1}([v^{j-1}](\partial_f h_N \cdot h_1)) \\ &\quad + C_{k+1}([v^j](\partial_f h_N \cdot h_1)) + D_{k+1}([v^j](\partial_f h_N \cdot h_1)) \end{aligned} \quad (3.50)$$

$$\begin{aligned} D_{k+2}([v^j](\partial_f h_N \cdot h_1 \cdot g)) &= \sum_i \alpha'_{j-1,i} - \sum_i \alpha'_{j-1,i-1} + \sum_i \alpha'_{j,i-1} \\ &= D_{k+1}([v^j](\partial_f h_N \cdot h_1)) \end{aligned} \quad (3.51)$$

By Equation (3.49) we have

$$\begin{aligned}
 C_{k+1}([v^j](\partial_f h_N \cdot h_1)) &= \sum_i i((i-1)\alpha_{N,j,i-1} - i\alpha_{N,j,i}) \\
 &= -\alpha_{N,j,1} + (2\alpha_{N,j,1} - 4\alpha_{N,j,2}) + (6\alpha_{N,j,2} - 9\alpha_{N,j,3}) + \dots \\
 &= \alpha_{N,j,1} + 2\alpha_{N,j,2} + 3\alpha_{N,j,3} + \dots = \sum_i i\alpha_{N,j,i}
 \end{aligned} \tag{3.52}$$

$$\begin{aligned}
 D_{k+1}([v^j](\partial_f h_N \cdot h_1)) &= \sum_i ((i-1)\alpha_{N,j,i-1} - i\alpha_{N,j,i}) \\
 &= -\alpha_{N,j,1} + (\alpha_{N,j,1} - 2\alpha_{N,j,2}) + (2\alpha_{N,j,2} - 3\alpha_{N,j,3}) + \dots \\
 &= 0
 \end{aligned} \tag{3.53}$$

Since $[v^j]h_N = 0$ for $j < 0$, evaluating Equation (3.50) at $j = 0$ and applying Equations (3.52) and (3.53) yields

$$C_{k+1}([v^0](\partial_f h_N \cdot h_1)) + D_{k+1}([v^0](\partial_f h_N \cdot h_1)) = \sum_i i\alpha_{N,0,i} = 1$$

where the last step follows by applying Equation (3.43) inductively. Similarly, evaluating Equation (3.50) at $j \geq 1$ yields

$$D_{k+1}([v^{j-1}](\partial_f h_N \cdot h_1)) + C_{k+1}([v^j](\partial_f h_N \cdot h_1)) + D_{k+1}([v^j](\partial_f h_N \cdot h_1)) = 0$$

by applying Equation (3.45) inductively. Finally, Equation (3.51) yields 0 for any $j \geq 0$, due to Equation (3.53). Therefore the contribution of the first summand to Equation (3.43) is 1, while for Equations (3.44) to (3.46) we have a contribution of 0.

2. Moving on to the next summand of Equation (3.48), we have

$$\begin{aligned}
 B_{k+2}(-k \cdot h_N \cdot \partial_f g \cdot h_1) &= B_k(-k \cdot h_N \cdot \partial_f g) \cdot B_1(h_1) && \text{by Lemma 3.20} \\
 &= -k \cdot B_k(h_N) \cdot B_{-1}(\partial_f g) \cdot B_1(h_1) && \text{by Lemma 3.20}
 \end{aligned}$$

From this we obtain

$$\begin{aligned}
 B_{k+2}(-k \cdot h_N \cdot \partial_f g \cdot h_1) &= -k \cdot B_k(h_N) \cdot B_{-1}(\partial_f g) \cdot B_1(h_1) \\
 &= -k \cdot \left(\sum_i \alpha_{N,j,i} v^j f^i z^{2k-2(i-1)} \right) \cdot (-v+1) \cdot (-fz^2 + f^2) \\
 &= -k \sum_i (-(\alpha_{N,j,i-1} - \alpha_{N,j-1,i-1}) + (\alpha_{N,j,i-2} - \alpha_{N,j-1,i-2})) v^j f^i z^{2(k+2)-2(i-1)}
 \end{aligned} \tag{3.54}$$

As argued before, $a_{N,j,i} = 0$ for $j \leq 0$, so from Equation (3.54) we then obtain

$$\begin{aligned}
 & C_{k+2}([v^j](-k \cdot h_N \cdot \partial_f g \cdot h_1)) \\
 &= -k \sum_i i(-(\alpha_{N,j,i-1} - \alpha_{N,j-1,i-1}) + (\alpha_{N,j,i-2} - \alpha_{N,j-1,i-2})) \\
 &= -k(2(-\alpha_{N,j,1} + \alpha_{N,j-1,1}) + 3(-\alpha_{N,j,2} + \alpha_{N,j-1,2} + \alpha_{N,j,1} - \alpha_{N,j-1,1}) + \dots) \\
 &= -k((a_{N,j,1} - \alpha_{N,j-1,1}) + (\alpha_{N,j,2} - \alpha_{N,j-1,2}) + \dots) \\
 &= -k\left(\sum_i a_{N,j,1} - \sum_i a_{N,j-1,1}\right)
 \end{aligned} \tag{3.55}$$

and

$$\begin{aligned}
 & D_{k+2}([v^j](-k \cdot h_N \cdot \partial_f g \cdot h_1)) \\
 &= \sum_i (-(\alpha_{N,j,i-1} - \alpha_{N,j-1,i-1}) + (\alpha_{N,j,i-2} - \alpha_{N,j-1,i-2})) \\
 &= (-\alpha_{N,j,1} + \alpha_{N,j-1,1}) + (-\alpha_{N,j,2} + \alpha_{N,j-1,2} + \alpha_{N,j,1} - \alpha_{N,j-1,1}) + \dots \\
 &= 0
 \end{aligned} \tag{3.56}$$

For $j = 0$, Equation (3.55) becomes $-k \sum_i a_{N,0,i}$ which yields 0 by induction and Equation (3.44), while for $j \geq 1$, Equation (3.55) again yields 0 due to Equations (3.44) and (3.46). Given that, the contribution of the second summand to Equations (3.43) and (3.45) is 0. Finally, its contribution to Equations (3.44) and (3.46) is obtained by evaluating Equation (3.56) at any $j \geq 0$, which is also 0.

3. For the third summand of Equation (3.48) we have

$$B_{k+2}(\partial_v h_N \cdot g^2) = B_k(\partial_v h_N) \cdot B_1(g^2)$$

by Lemma 3.20. It is straightforward to notice that differentiation by v preserves the balanced part of h_N and only affects by a change of coefficients from $\alpha_{N,j,i}$ to $i\alpha_{N,j-1,i}$ so that

$$\begin{aligned}
 & B_{k+2}(\partial_v h_N \cdot g^2) = B_k(\partial_v h_N) \cdot B_1(g^2) \\
 &= \left(\sum_{i,j} j \alpha_{N,j,i} v^{j-1} z^{2k-2(i-1)} f^i \right) (v^2 z^4 - 2fv^2 z^2 + f^2 v^2 + 2fvz^2 - 2f^2 v + f^2) \\
 &= \sum_i ((j-1)\alpha_{N,j-1,i} - 2(j-1)\alpha_{N,j-1,i-1} + (j-1)\alpha_{N,j-1,i-2} \\
 &\quad + 2j\alpha_{N,j,i-1} - 2j\alpha_{N,j,i-2} + (j+1)\alpha_{N,j+1,i-2}) v^j z^{2(k+2)-2(i-1)} f^i
 \end{aligned} \tag{3.57}$$

From the above we obtain

$$\begin{aligned} & C_{k+2}([v^j](\partial_v h_N \cdot g^2)) \\ &= \sum_i i(j-1)\alpha_{N,j-1,i} - \sum_i i2(j-1)\alpha_{N,j-1,i-1} + \sum_i i(j-1)\alpha_{N,j-1,i-2} \\ &+ \sum_i i2j\alpha_{N,j,i-1} - \sum_i i2j\alpha_{N,j,i-2} + \sum_i i(j+1)\alpha_{N,j+1,i-2} \end{aligned} \quad (3.58)$$

and

$$\begin{aligned} & D_{k+2}([v^j](\partial_v h_N \cdot g^2)) \\ &= \sum_i (j-1)\alpha_{N,j-1,i} - \sum_i 2(j-1)\alpha_{N,j-1,i-1} + \sum_i (j-1)\alpha_{N,j-1,i-2} \\ &+ \sum_i 2j\alpha_{N,j,i-1} - \sum_i 2j\alpha_{N,j,i-2} + \sum_i (j+1)\alpha_{N,j+1,i-2} \end{aligned} \quad (3.59)$$

For $j = 0$, Equation (3.58) reduces to $\sum_i i\alpha_{N,1,i-2}$ which is zero due to induction and Equation (3.45). Similarly, for $j \geq 1$, all summands of Equation (3.58) are zero again due to Equation (3.45). By similar arguments, Equation (3.59) is zero for any $j \geq 0$ due to induction and Equation (3.46). Therefore the contributions of the third summand to each of Equations (3.43) to (3.46) are 0.

4. Finally, for the fourth summand of Equation (3.48) we have

$$\begin{aligned} B_{k+2}(-k \cdot h_N \cdot \partial_v g \cdot g) &= -k \cdot B_{k+1}(h_N \cdot \partial_v g) \cdot B_0(g) && \text{by Lemma 3.20} \\ &= -k \cdot B_k(h_N) \cdot B_0(\partial_v g) \cdot B_0(g) && \text{by Lemma 3.20} \end{aligned}$$

From this we obtain

$$\begin{aligned} B_{k+2}(-k \cdot h_N \cdot \partial_v g \cdot g) &= -k \cdot B_k(h_N) \cdot B_0(\partial_v g) \cdot B_0(g) \\ &= -k \left(\sum_i \alpha_{N,j,i} v^i z^{2k-2(i-1)} f^i \right) (z^2 - f) \\ &= k \sum_i (\alpha_{N,j,i-1} - \alpha_{N,j,i}) v^i z^{2(k+1)-2(i-1)} f^i. \end{aligned} \quad (3.60)$$

and so, by letting $a''_{j,i} = k(\alpha_{N,j,i-1} - \alpha_{N,j,i})$, we have

$$\begin{aligned} B_{k+2}(-k \cdot h_N \cdot \partial_v g \cdot g) &= (vz^2 - fv + f) \left(\sum_{i,j} a''_{j,i} v^j z^{2(k+1)-2(i-1)} f^i \right) \\ &= \sum_{i,j} (\alpha''_{j-1,i} - \alpha''_{j-1,i-1} + \alpha''_{j,i-1}) z^{2(k+2)-2(i-1)} v^j f^i. \end{aligned} \quad (3.61)$$

Summing the coefficients of Equation (3.60) we obtain

$$\begin{aligned}
 C_{k+1}([v^j](-k \cdot h_N \cdot \partial_v g)) &= k \sum_i i(\alpha_{N,j,i-1} - \alpha_{N,j,i}) \\
 &= k(-\alpha_{N,j,1} + 2(\alpha_{N,j,1} - \alpha_{N,j,2}) + 3(\alpha_{N,j,2} - \alpha_{N,j,3}) + \dots) \\
 &= k(\alpha_{N,j,1} + \alpha_{N,j,2} + \alpha_{N,j,3} + \dots),
 \end{aligned} \tag{3.62}$$

$$\begin{aligned}
 D_{k+1}([v^j](-k \cdot h_N \cdot \partial_v g)) &= \sum_i (\alpha_{N,j,i-1} - \alpha_{N,j,i}) \\
 &= -\alpha_{N,j,1} + (\alpha_{N,j,1} - \alpha_{N,j,2}) + (\alpha_{N,j,2} - \alpha_{N,j,3}) + \dots \\
 &= 0.
 \end{aligned} \tag{3.63}$$

By induction, Equation (3.62) gives 0 for any $j \geq 0$ due to Equations (3.44) and (3.46). Adapting the arguments used for the first summand of Equation (3.48) by swapping a'' for a' , we obtain, for all $j \geq 0$

$$\begin{aligned}
 C_{k+2}([v^j](-k \cdot h_N \cdot \partial_v g \cdot g)) &= 0, \\
 D_{k+2}([v^j](-k \cdot h_N \cdot \partial_v g \cdot g)) &= 0.
 \end{aligned}$$

Therefore the contributions of the fourth summand to each of Equations (3.43) to (3.46) are 0.

Finally, we have, by Equations (3.41) and (3.42), that $[z^n]W_N \sim [z^n] \frac{B_{k+2}(h_N)}{g^{k+2}}$. Since

$$B_{k+2}(\partial_f W_N \cdot W_1) = B_{k+2}(\partial_f h_N \cdot g \cdot h_1) - B_{k+2}(-k \cdot h_N \cdot \partial_f g \cdot h_1) \tag{3.64}$$

and

$$B_{k+2}(\partial_v W_N) = B_{k+2}(\partial_v h_N \cdot g^2) - B_{k+2}(k \cdot h_N \cdot \partial_v g \cdot g), \tag{3.65}$$

the computations of steps 1 to 4 above together with Equations (3.43) to (3.46) show that

$$\begin{aligned}
 [z^n](\partial_f W_{N-1} W_1)|_{f=T_{[0]}(z), v=1} &\sim [z^n]W_{N-1}|_{f=T_{[0]}(z), v=1} \sim [z^n]T_{[0]}(z), \\
 [z^n](\partial_v W_{N-1})|_{f=T_{[0]}(z), v=1} &= O([z^{n-3}]T_{[0]}(z)).
 \end{aligned}$$

Therefore Lemma 3.7 applies for $n \equiv 2 \pmod{3}$, yielding our desired result. $\square$

An application of the bijection shown in Figure 2.1 and of Proposition 2.1 shows that Theorem 3.22 holds for unrooted trivalent maps too:

Corollary 3.23. *Let χ_b be the parameter corresponding to the number of internal bridges in unrooted trivalent maps. Then we have $X_n^b \xrightarrow{D} \text{Poisson}(1)$ for the random variables X_n^b counting χ_b over maps of size n .*

Finally, by Theorem 3.22 we have that the probability of an object in $\dot{\mathcal{T}}_{[0]}$ to be bridgeless is $1/e$ which together with Theorem 3.9 yields the following asymptotic form for the number of bridgeless rooted trivalent maps and closed linear λ -terms without closed proper subterms:

Corollary 3.24. *The number of bridgeless rooted trivalent maps and closed linear λ -terms without closed proper subterms of size $p \equiv 2 \pmod{3}$ is*

$$[z^p] \sim \frac{3}{e\pi} 6^n n!, \quad p = 3n + 2. \quad (3.66)$$

Note 3.25. In this note, inspired by the suggestion of one of the reviewers, we sketch an alternative approach to proving Theorem 3.22 by studying directly the quantities $[z^n v^k] T_{[0]}^{sub}(z, v)$.

Our starting point is Equation (3.30):

$$[z^n v^k] v \frac{\partial}{\partial v} T_{[0]}^{sub} = \sum_{\substack{n_1+n_2=n \\ k_1+k_2=k}} [z^{n_1} v^{k_1}] \frac{vQ}{1-vQ} [z^{n_2} v^{k_2}] T_{[0]}^{sub},$$

in which we isolate the term corresponding to $n_2 = 2$, rewriting it as:

$$[z^n v^k] v \frac{\partial}{\partial v} T_{[0]}^{sub} = z^2 [z^{n-2} v^{k-1}] \frac{Q}{1-vQ} + R \quad (3.67)$$

where R stands for the rest of the summands (those corresponding to $n_2 > 2$). The idea then is to derive a recurrence for $[z^n v^k] T_{[0]}^{sub}$ using Equation (3.67), which one then solves to obtain the desired Poisson law as follows:

Step 1: Prove that the sequence of random variables X_n^{sub} is tight, for example by showing that the mean and variance are bounded. This can be done by studying Equation (3.29) and its derivative using Corollary 2.4. This step guarantees the existence of a discrete limit law.

Step 2: Show that $[z^n v^k] z^2 Q \sim [z^n v^k] T_{[0]}^{sub}$. From Equation (3.24) we have:

$$z^2 Q = \frac{z^2 T_{[0]}^{\lambda, sub}}{z^2 + T_{[0]}^{\lambda, sub}}.$$

Unfortunately this composition does not fall under the schema of Theorem 4.4 that we propose in the next section, as $F(z, u, y) = \frac{z^2 y}{z^2 + y}$ is not analytic at $(0, 0)$. However, the same type of approach works here too. We begin by expanding the above as:

$$\frac{z^2 T_{[0]}^{\lambda, sub}}{z^2 + T_{[0]}^{\lambda, sub}} = T_{[0]}^{\lambda, sub} - \sum_{k=2}^{\infty} (-1)^k \frac{\left(T_{[0]}^{\lambda, sub}\right)^k}{z^{2(k-1)}}. \quad (3.68)$$

Next, using Theorem 4.4 we have that $[z^n] T_{[0]}^{\lambda, sub}(z, 1) \sim [z^n] T_{[0]}^{sub}$ and using Corollary 2.4 that:

$$[z^n] \frac{\left(T_{[0]}^{\lambda, sub}\right)^k}{z^{2(k-1)}} = O([z^{n-3(k-1)}] T_{[0]}^{sub}). \quad (3.69)$$

so that, for $v = 1$, extracting the coefficients in Equation (3.68) we get:

$$\frac{[z^n]z^2Q}{[z^n]T_{[0]}^{\lambda,sub}} = 1 - \sum_{k=2}^{\infty} O(n^{-k}) = 1 + O(n^{-1}). \quad (3.70)$$

Given the above, arguments very similar to those used in the proof of Theorem 4.4 prove that the probability generating function induced by z^2Q converges to that of $T_{[0]}^{\lambda,sub}$. Finally, an application of Theorem 4.4 shows that, in turn, the probability generating function of $T_{[0]}^{\lambda,sub}$ converges to that of $T_{[0]}^{sub}$ which together with Step 1 give us the desired result as convergence in distribution and in probability are equivalent for discrete limit laws.

Step 3: Show that $[z^n v^k] \frac{Q}{1-vQ} \sim [z^n v^k]Q$ and $\frac{[z^n v^k]R}{[z^n]T_{[0]}^{sub}(z,1)} = o(1)$. The first statement follows from Theorem 4.4, while the second one follows from the same theorem together with the fact that $[z^n v^k]R \leq [z^n]R|_{v=1}$.

Step 4: From the above, derive a recurrence of the form:

$$\frac{[z^n v^k]v \frac{\partial}{\partial v} T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)} = \frac{[z^n v^k]T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)} + o(1),$$

which yields

$$\frac{[z^n v^k]T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)} \sim \frac{1}{k!} \frac{[z^n v^0]T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)}. \quad (3.71)$$

Therefore we are left with the determination of the constant $\frac{[z^n v^0]T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)}$. As an upper bound, since identity subterms are a special case of closed subterms, we have:

$$\frac{[z^n v^0]T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)} \geq \frac{[z^n v^0]T_{[0]}^{id}}{T_{[0]}^{sub}(z,1)} \sim e^{-1},$$

the asymptotic equivalence from Theorem 3.10.

Using Lemma 3.5 and Equation (3.68) evaluated at $k = 1$ we get:

$$\frac{[z^n v^0]T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)} = \frac{[z^n v^1]T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)} = \frac{[z^{n-2}v^0]Q}{[z^n]T_{[0]}^{sub}(z,1)} + o(1). \quad (3.72)$$

Interpreting the above combinatorially, we see asymptotically almost all terms with one closed proper subterm are constructed by taking a *bridgeless* term in Q and replacing its box vertex by a loop. Since the collection of such maps forms a subset of those maps with one loop, we have an upper bound of the form:

$$\frac{[z^{n-2}v^0]Q}{[z^n]T_{[0]}^{sub}(z,1)} \leq e^{-1},$$

which finally yields: $\frac{[z^n v^0]T_{[0]}^{sub}}{[z^n]T_{[0]}^{sub}(z,1)} \sim e^{-1}$. Equation (3.71) and the method of moments then implies that the limit distribution is indeed *Poisson*(1), as desired.

4. Compositions of divergent powerseries and parameter distributions

A large not-necessarily-connected trivalent map is almost surely connected, as we will see later on in this section. Such phenomena are abundant in the study of maps and graphs. As another example of this phenomenon, let us take $G(z)$ to be the exponential generating function for not-necessarily-connected labeled graphs and $C(z)$ the one enumerating connected labeled graphs. Then,

$$\begin{aligned} 1 + G(z) &= e^{C(z)} \\ C(z) &= \ln(G(z) + 1) \end{aligned} \tag{4.1}$$

and a famous theorem of Bender, [Ben75, Theorem 1], may then be used to show that

$$[z^n] \ln(G(z) + 1) \sim [z^n] G(z)$$

proving that asymptotically almost all labelled graphs of size n are connected. A crucial element of this proof is the fact that the number of labelled graphs, $n![z^n]G(z) = 2^{\binom{n}{2}}$, grows much faster than $n!$.

More generally, the aforementioned theorem by Bender shows that for appropriate formal power series $F(z, y)$ analytic at 0 and $G(z)$ satisfying (among other requirements) $[z^{n-1}]G(z) = o([z^n]G(z))$, one can deduce the coefficient asymptotics of the composition $F(z, G(z))$ from those of $G(z)$. The example given above corresponds to $F(z, y) = \ln(y + 1)$. Other examples of this can be given by taking instead F and G to be enumerating objects of some classes $\mathcal{F}$ and $\mathcal{G}$, for which the number of structures of size n grows moderately for $\mathcal{F}$ and rapidly for $\mathcal{G}$. In such cases, the combinatorial intuition behind Bender's result is that, asymptotically, most of the structures in $\mathcal{F}(\mathcal{G})$ are constructed by taking a small $\mathcal{F}$ -structure and replacing one of its atoms with the biggest appropriate $\mathcal{G}$ -structure. The rapid growth conditions on the coefficients of $G(z)$ then precisely reflect the fact that there are many more ways to pick an element of $\mathcal{G}_n$ and compose it into a small $\mathcal{F}$ -structure than there are ways to pick one in $\mathcal{G}_{n-1}$ and do so.

Given the above, one then expects that if χ is some parameter of a rapidly-growing class $\mathcal{C}$ of connected structures, then the parameter χ^* defined by summing χ over the connected components of not-necessarily-connected $\mathcal{G}$ -structures behaves similarly. In the same vein, one would expect that for combinatorial classes $\mathcal{F}, \mathcal{G}$ where $\mathcal{G}$ is rapidly-growing while $\mathcal{F}$ only grows moderately, parameters χ defined over $\mathcal{G}$ and its natural extension over $\mathcal{F}(\mathcal{G})$ formed by summing χ over $\mathcal{G}$ -substructures both behave in a similar way. Indeed, the result of the following subsection serves to formalise this intuition.

4.1. Composition schema

In this subsection we present a theorem inspired by Bender's theorem ([Ben75, Theorem 1]), as well as its extensions presented in [Bor18, Theorem 32] and [Pan19, Lemma B.8], which formalises the above discussion: under sufficient conditions, the limit law of a parameter marked by u in a power series $G(z, u)$ remains unchanged when composing with some $F(z, u, y)$ provided that $F(z, 1, y)$ is analytic at the origin and the coefficients of $G(z, 1)$ grow rapidly enough.

Before proceeding with the main result of this subsection, we present a series of useful lemmas first.

Lemma 4.1. *Let $(a_n)_{n \in \mathbb{N}}$ be a positive asymptotically log-convex sequence such that $\frac{a_{n-1}}{a_n} = O(n^{-\sigma})$ for $\sigma > 0$. Then for $n_0 \in \mathbb{N}$*

$$\sum_{k=n_0}^{n-n_0} a_k a_{n-k} = O(a_{n-n_0}). \quad (4.2)$$

Proof. Applying Corollary 2.4 to $f = \sum_{n \in \mathbb{N}} a_n$ for $k = 1, l = 2$, we obtain

$$\sum_{k=n_0}^{n-n_0} a_k a_{n-k} = 2a_{n_0} a_{n-n_0} (1 + O(n^{-\sigma})) = O(a_{n-n_0}), \quad (4.3)$$

as desired. $\square$

Lemma 4.2. *Let $(a_n)_{n \in \mathbb{N}}$ be a positive asymptotically log-convex sequence such that $\frac{a_{n-1}}{a_n} = O(n^{-\sigma})$ for $\sigma > 0$. Then there exists a constant $C > 0$ such that for all $q \geq 2$*

$$\sum_{\substack{k_1 + \dots + k_q = n \\ \forall j, k_j \geq 1}} \prod_{j=1}^q a_{k_j} \leq C^{q-1} a_{n-q+1} \quad (4.4)$$

Proof. We proceed by induction. For $q = 2$ the result holds by Lemma 4.1. Let $q > 2$ and rewrite the sum as

$$\sum_{k_q=1}^{n-q+1} a_{k_q} \sum_{\substack{k_1 + \dots + k_{q-1} = n-k_q \\ \forall j, k_j \geq 1}} \prod_{j=1}^{q-1} a_{k_j} \quad (4.5)$$

we then have by our inductive hypothesis

$$\sum_{k_q=1}^{n-q+1} a_{k_q} \sum_{\substack{k_1 + \dots + k_{q-1} = n-k_q \\ \forall j, k_j \geq 1}} \prod_{j=1}^{q-1} a_{k_j} \leq C^{q-2} \sum_{k_q=1}^{n-q+1} a_{k_q} a_{n-k_q-q+2} \quad (4.6)$$

from which, by applying once more our inductive hypothesis for $q = 2$, we obtain the desired result. $\square$

Lemma 4.3. *Let $(a_n)_{n \in \mathbb{N}}$ be a positive asymptotically log-convex sequence such that $\frac{a_{n-1}}{a_n} = O(n^{-\sigma})$ for $\sigma > 0$. Then for $n_0 \in \mathbb{N}$ and sufficiently large n*

$$\sum_{k=n_0}^n C^k P(k) a_{n-k} = O(a_{n-n_0}) \quad (4.7)$$

for any constant $C \in \mathbb{R}_{\geq 1}$ and $P \in \mathbb{R}[k]$ a polynomial in k .

Proof. The rapid growth of (a_n) implies that there exists a constant $D > 0$ such that for every $k \geq 1$, $C^k P(k) \leq D a_k$ so that

$$\sum_{k=n_0}^n C^k P(k) a_{n-k} \leq D \sum_{k=n_0}^n a_k a_{n-k} \quad (4.8)$$

For $k \leq n - n_0$, Lemma 4.1 implies

$$\sum_{k=n_0}^{n-n_0} C^k P(k) a_{n-k} = O(a_{n-n_0}) \quad (4.9)$$

while for $n - n_0 < k \leq n$ we have

$$\sum_{k=n-n_0+1}^n C^k P(k) a_{n-k} = \sum_{k'=0}^{n_0-1} C^{n-k'} P(n-k') a_{k'} = O(a_{n-n_0}) \quad (4.10)$$

by the rapid growth of $(a_n)_{n \in \mathbb{N}}$. $\square$

Theorem 4.4. Let $G(z, u)$ be a bivariate powerseries $\sum_{n \geq 1} g_n(u) z^n$ such that $(g_n(1))_{n \in \mathbb{N}}$ is an asymptotically log-convex sequence and

$$\frac{g_{n-1}(1)}{g_n(1)} = O(n^{-\sigma}) \quad (4.11)$$

for $\sigma > 0$.

Let also $F(z, u, y)$ be a power series $\sum_{i \geq 0} \sum_{j \geq 0} f_{i,j}(u) z^i y^j$ with $f_{0,1}(u) = 1$, such that $F(z, 1, y)$ represents a function analytic at the origin. Then the random variables X_n whose probability generating functions are given by

$$p_n(u) = \frac{[z^n] F(z, u, G(z, u))}{[z^n] F(z, 1, G(z, 1))} \quad (4.12)$$

admit the same limit distribution as the random variables Y_n whose probability generating functions are given by

$$q_n(u) = \frac{[z^n] G(z, u)}{[z^n] G(z, 1)}. \quad (4.13)$$

That is, if $Y_n \xrightarrow{D} Y$, then $X_n \xrightarrow{D} Y$ too.

Proof. We begin by formally expanding $F(z, u, G(z, u))$ as a power series and extracting the coefficient of z^n :

$$[z^n] F(z, u, G(z, u)) = f_{n,0} + \sum_{t=0}^{n-1} \sum_{q=1}^{n-t} f_{t,q}(u) \sum_{k_1+\dots+k_q=n-t} \prod_{j=1}^q g_{k_j}(u). \quad (4.14)$$

Our goal then, is to show that after making the change of variables $u = e^{i\tau}$, the summand of Equation (4.14) corresponding to $t = 0$, $q = 1$ (which is exactly $g_n(e^{i\tau})$ by our assumption that $f_{0,1} = 1$), yields asymptotically the dominant contribution for any $\tau \in \mathbb{R}$. This, after normalising by $[z^n]F(z, 1, G(z, 1))$, is then enough to prove that the characteristic functions $p_n(e^{i\tau})$ converge to the characteristic function $q_n(e^{i\tau})$ corresponding to Y_n . To this end, we will need to provide upper bounds for the summands of Equation (4.14) evaluated at $u = e^{i\tau}$. We will do so by exploiting the fact that $|g_{k_j}(e^{i\tau})|$ is bounded above for any real τ by $\bar{g}_{k_j} := g_{k_j}(1)$, which allows us to make use of Lemmas 4.1 to 4.3. We provide bounds for the summands of Equation (4.14) as follows:

- Firstly, we deal with the summands corresponding to $([z^{t \geq 1}]F(z, u, y))|_{y=G(z,u)}$ (i.e., the summands of Equation (4.14) in which $f_{t,q}(u)$ is such that $t \geq 1$), showing they are asymptotically negligible.
- Secondly, we deal with the summands corresponding to $([z^0]F(z, u, y))|_{y=G(z,u)}$ (i.e., the summands of Equation (4.14) which contain only factors of the form $f_{0,q}(u)$ for $q \geq 1$). Here, we distinguish two sub-cases:
 - One in which $k_1 = n$ is the sole summand appearing in the innermost sum of Equation (4.14). This case provides the main asymptotic contributions.
 - The other corresponding to summands with $k_j < n - 1$ for all j , which we show are asymptotically negligible.

Summands corresponding to $([z^{t \geq 1}]F(z, u, y))|_{y=G(z,u)}$. By the growth of the coefficients of G and F , we have $f_{n,0} = o(g_{n-1})$, therefore we can focus on the case of $f_{t,q}$ with $q \geq 1$. Since f is analytic, there exists some D such that $|f_{t,q}(1)| \leq D^{t+q}$ so that by Lemmas 4.1 to 4.3 we have that the restriction of Equation (4.14) to $t \geq 1$ is

$$\begin{aligned}
 & \left| \sum_{t=1}^{n-1} \sum_{q=1}^{n-t} f_{t,q}(e^{i\tau}) \sum_{k_1+\dots+k_q=n-t} \prod_{j=1}^q g_{k_j}(e^{i\tau}) \right| \leq \sum_{t=1}^{n-1} \sum_{q=1}^{n-t} |f_{t,q}(1)| \sum_{k_1+\dots+k_q=n-t} \prod_{j=1}^q \bar{g}_{k_j} \\
 & \leq \sum_{t=1}^{n-1} \sum_{q=1}^{n-t} D^{t+q} C^{q-1} \bar{g}_{n-t-q+1} \quad \text{by Lemma 4.2} \\
 & = \sum_{t=1}^{n-1} D^t C^{-1} \sum_{q=1}^{n-t} D^q C^q \bar{g}_{n-t-q+1} \\
 & \leq \sum_{t=1}^{n-1} D^t C^{-1} \sum_{q=1}^{n-t} K \bar{g}_q \bar{g}_{n-t-q+1} \quad \text{by Equation (4.11)} \rightarrow \exists K. \forall k. D^k C^k \leq K \bar{g}_k \\
 & \leq \sum_{t=1}^{n-1} D^t C^{-1} K \cdot O(\bar{g}_{n-t}) \quad \text{by Lemma 4.1} \\
 & = O(\bar{g}_{n-1}) \quad \text{by Lemma 4.3}
 \end{aligned} \tag{4.15}$$

Summands corresponding to $([z^0]F(z, u, y))|_{y=G(z, u)}$. We will now focus on the summand of Equation (4.14) corresponding to $t = 0$

$$\sum_{q=1}^n f_{0,q}(u) \sum_{k_1+\dots+k_q=n} \prod_{j=1}^q g_{k_j}(u).$$

We may rewrite this sum as follows, depending on whether $q = 1$ or $q = 2$

$$f_{0,1}(u)g_n(u) + \sum_{q=2}^n f_{0,q}(u) \sum_{\substack{k_1+\dots+k_q=n \\ \forall j, k_j > 1}} \prod_{j=1}^q g_{k_j}(u) \quad (4.16)$$

We proceed by providing bounds for the second term of Equation (4.16) evaluated at $u = e^{i\tau}$. Once again, we note that since f is analytic at 0, $|f_{0,q}(1)| \leq D^q$ for some constant D . As such we have,

$$\begin{aligned} & \left| \sum_{q=2}^n f_{0,q}(e^{it}) \sum_{\substack{k_1+\dots+k_q=n \\ \forall j, k_j > 1}} \prod_{j=1}^q g_{k_j}(e^{it}) \right| \leq \sum_{q=2}^n D^q \sum_{\substack{k_1+\dots+k_q=n \\ \forall j, k_j > 1}} \prod_{j=1}^q \bar{g}_{k_j} \\ & \leq \sum_{q=2}^n D^q C^{q-1} \bar{g}_{n-q+1} \quad \text{by Lemma 4.2} \\ & \leq C^{-1} K \sum_{q=2}^{n-1} \bar{g}_q \bar{g}_{n-q+1} + D^n C^{n-1} \bar{g}_1 \quad \text{by Equation (4.11)} \rightarrow \exists K, \forall k, D^k C^k \leq K \bar{g}_k \\ & = O(\bar{g}_{n-1}) \quad \text{by Lemma 4.1} \end{aligned} \quad (4.17)$$

Now, by [Ben75, Theorem 1] together with the fact that $f_{0,1} = 1$, we have that the coefficients of $F(z, u, G(z, 1))$ grow asymptotically as $\bar{g}_n$. Therefore the second term of Equation (4.16), divided by $\bar{g}_n$, tends to 0 as n tends to infinity, due to the bound demonstrated in Equation (4.17). Similarly, the terms corresponding to $t \geq 1$ in Equation (4.14), when divided by $\bar{g}_n$, also tend to 0, due to Equation (4.15).

Finally, since $f_{0,1}(u) = 1$, we obtain that,

$$p_n(e^{it}) = g_n(e^{it}) + O\left(\frac{1}{n^\sigma}\right) \quad (4.18)$$

for any real t as $n \rightarrow \infty$, with the error being uniform in t , which by Lévy's convergence theorem leads to our desired result. $\square$

4.2. Distribution of degree 1 vertices in $\mathcal{T}$ and of free variables in $\dot{\mathcal{T}}$

In this section, we will apply Theorem 4.4 in order to determine the distribution of 1-valent vertices in (1,3)-valent maps, from which we derive the distribution of free variables in linear lambda terms considered up to variable exchange.

To this end, we will first need to derive the generating function of vertex-rooted $(1, 3)$ -valent maps. To do so, it is more convenient to use an approach based on exponential generating functions, unlike our previous derivations which were done using the ordinary counterparts. The validity of this approach is guaranteed by the fact that open and closed rooted trivalent maps have no automorphisms: every such unlabelled map of size n corresponds to $n!$ labelled maps, a property known as *rigidity*. This leads to an equality between the ordinary generating functions enumerating unlabelled open rooted trivalent maps and the corresponding exponential generating function enumerating labelled such maps.

Let $\mathcal{T}^d$ be the class of *not-necessarily-connected* $(1, 3)$ -valent maps. Viewed as combinatorial maps, elements of $\mathcal{T}^d$ consist of a permutation v having cycles of length 3 or 1 and a fixed-point free involution e . Using the symbolic method, we obtain the exponential generating function of maps in $\mathcal{T}^d$ counted by number of half-edges (tracked by the variable h), which we can moreover refine to a bivariate generating function also keeping track of the number of 1-valent vertices (tracked by u):

$$(\exp(h^2/2) \odot \exp(h^3/3 + uh)) . \quad (4.19)$$

Taking the logarithm of this expression, in full analogy with Equation (4.1), yields the generating function of *connected* $(1, 3)$ -maps, while an application of the $h\partial_h$ operator yields *half-edge rooted* connected $(1, 3)$ -maps counted by number of half-edges and 1-valent vertices. Now, to switch from half-edge rooted $(1, 3)$ -maps to vertex-rooted $(1, 3)$ -valent maps, we apply the bijection explained in Section 2.1 and seen in Figure 2.1, which corresponds to multiplying by h^4 and adding the initial conditions³ $uh^2 + h^4$ to yield

$$uh^2 + h^4 + h^5 \frac{\partial}{\partial h} (\ln (\exp(h^2/2) \odot \exp(h^3/3 + uh))) , \quad (4.20)$$

and finally, to switch from counting half-edges to counting edges, we apply the change of variables $h^2 \mapsto z$:

$$T(z, u) = uz + z^2 + 2z^3 \frac{\partial}{\partial z} (\ln ((\exp(h^2/2) \odot \exp(h^3/3 + uh))|_{h=z^{1/2}})) . \quad (4.21)$$

Although this equation was derived in a completely different manner from Equation (1.1), they both speak about the same bivariate generating function $T(z, u)$, which as explained in Section 1.1 has an interpretation as counting linear λ -terms by number of subterms and free variables. To be completely precise, the coefficient $[z^n u^k]T(z, u)$ counts the number of linear lambda terms with n subterms and k free variables, *considered up to exchange of free variables*. Equivalently, it counts the number of open rooted trivalent maps with n edges and $k + 1$ external vertices, considered up to relabelling of the non-root external vertices. Finally, since each of the $k!$ possible relabellings of the free variables/external vertices yields a distinct labelled object, the variable u in $T(z, u)$ may be interpreted as either of exponential type (when tracking variables in linear lambda terms or non-root external vertices in open rooted trivalent maps) or of ordinary type (when counting 1-valent vertices in vertex-rooted $(1, 3)$ -maps). See Figure 4.1 for an example making this correspondence more concrete.

³See Section 2.1 for an explanation of why  must be considered as a special case; a similar argument applies to .

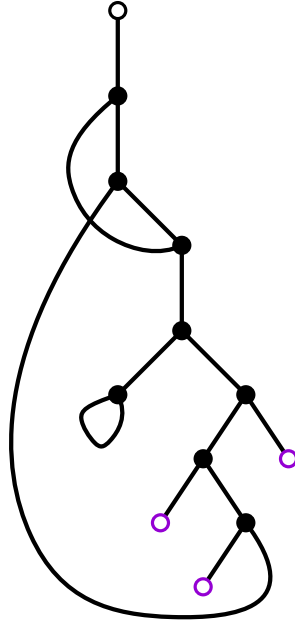


Figure 4.1: Example of a vertex-rooted (1,3)-valent map corresponding to the underlying map of the linear term $\lambda x.\lambda y.(((a((yc)b))(\lambda z.z))x)$. Since the 1-valent vertices are unlabelled (compare with the open rooted trivalent maps in Figure 2.2), the map captures the structure of the λ -term up to permutation of the free variables (a, b, c) .

Our aim in this subsection is to prove the following.

Theorem 4.5. *Let χ_{uni} be the combinatorial parameter corresponding to the number of 1-valent vertices in unrooted (1,3)-maps taken from $\mathcal{T}$. Let, also, X_n^{uni} be the random variable corresponding to χ_{uni} taken over $\mathcal{T}_n$. Then the mean and variance of X_n^{uni} are asymptotically $\mu_n \sim \sigma_n^2 \sim (2n)^{1/3}$ and the standardised random variables converge to a Gaussian law:*

$$\frac{X_n^{uni} - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0, 1) \quad (4.22)$$

An application of Proposition 2.1 and a change of variables $n \mapsto n - 2$, which does not affect the asymptotic mean and variance, yields the following corollary.

Corollary 4.6. *Let χ_{free} be the combinatorial parameter of exponential type corresponding to the number of non-root external vertices in open rooted trivalent maps and the number of free variables in open linear λ -terms. Then for $\mu_n \sim \sigma_n^2 \sim (2n)^{1/3}$, the random variables X_n^{free} corresponding to χ_{free} taken over $\dot{\mathcal{T}}_n$, properly standardised, converge to a Gaussian law:*

$$\frac{X_n^{free} - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0, 1) \quad (4.23)$$

Our plan is to first determine the probability generating function for the number of degree 1 vertices in large random maps in $\mathcal{T}^d$. We will then make use of Theorem 4.4 and Proposition 2.1

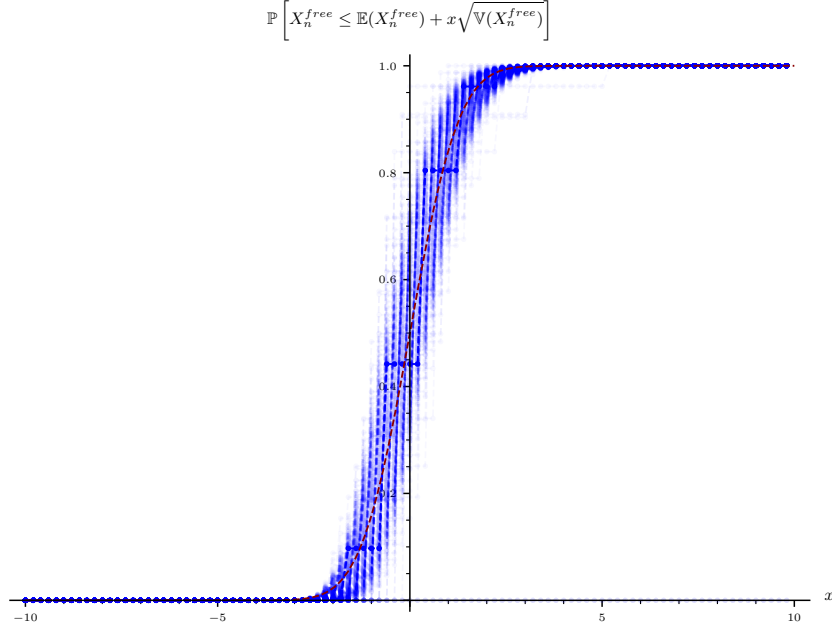


Figure 4.2: In hues of blue: the overlaid plots of $\mathbb{P}\left[X_n^{free} \leq \mathbb{E}(X_n^{free}) + x\sqrt{\mathbb{V}(X_n^{free})}\right]$, for $n = 0 \dots 300$, computed by discretising the x -axis using a step of size 0.2. In red: the cumulative distribution function of $\mathcal{N}(0, 1)$.

to obtain the analogous result for $\mathcal{T}$. Exploiting the structure of exponential Hadamard products, we obtain the result for $\mathcal{T}^d$ by studying the function $\exp(z^3/3 + uz)$ using the framework of *e-admissibility* as introduced in [DGK05]. In this work, the authors introduce the class $\mathcal{E}_R$, an extension of Hayman's original class of admissible functions (see [Hay56]) to bivariate generating functions of the form $f(z, u)$, for which Hayman's original result applies uniformly for a neighbourhood of $u = 1$. The following theorem then provides us with a general tool for obtaining limit distributions for random variables induced by functions in $\mathcal{E}_R$.

Theorem 4.7 (Adapted from [DGK05, Theorem 2]). *Let $f(z, u) \in \mathcal{E}_R$ be such that for sufficiently large n all coefficients $[z^n u^k]f(z, u)$ are nonnegative. Let, also, X_n be a sequence of random variables defined by*

$$\mathbb{P}(X_n = k) = \frac{[z^n u^k]f(z, u)}{[z^n]f(z, 1)}.$$

Then the following hold

$$\begin{aligned}\mathbb{E}(X_n) &= \bar{a}(r_n, 1) + o\left(\frac{|B(r_n, 1)|}{b(r_n, 1)}\right), \\ \mathbb{V}(X_n) &\sim \frac{|B(r_n, 1)|}{b(r_n, 1)}, \\ \frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} &\xrightarrow{D} \mathcal{N}(0, 1),\end{aligned}$$

where

$$\begin{aligned}a(z, u) &= z \frac{\partial_z f(z, u)}{f(z, u)}, \quad b(z, u) = z \partial_z a(z, u) \\ \bar{a}(z, u) &= u \frac{\partial_u f(z, u)}{f(z, u)}, \quad \bar{b}(z, u) = u \partial_u \bar{a}(z, u), \quad c(z, u) = u \partial_u a(z, u), \\ B(r, u) &= \begin{pmatrix} b(r, u) & c(r, u) \\ c(r, u) & \bar{b}(r, u) \end{pmatrix},\end{aligned}$$

r_n is the positive solution of $a(r_n, 1) = n$, and $|B|$ denotes the determinant.

To apply Theorem 4.7, we must first verify that our functions of interest are members of $\mathcal{E}_R$. To do this, we will make use of [GM06, Theorem 3], for which we would like to note that, as pointed out by Bernhard Gittenberger (one of the authors of [GM06]), there is a certain typo in its condition labelled (c). We present here a corrected version of a part of said theorem.

Theorem 4.8 (Adapted from [GM06, Theorem 3]). *Let $P(z, u)$ be a polynomial in z and u with real coefficients written in the form $P(z, u) = \sum_n p_n z^{k_n} u^{l_n}$ by choosing an arbitrary order of the monomials. Furthermore, let $P(z, 1) = \sum_m b_m z^m$, i.e., $b_m = \sum_{\{n|k_n=m\}} p_n$. Finally, set*

$$K = \max E \text{ with } E = \left\{ k_i + k_j \mid \det \begin{pmatrix} k_i & l_i \\ k_j & l_j \end{pmatrix} \neq 0 \right\}$$

and $I = \{(i, j) \mid k_i + k_j = K\}$. Then $e^{P(z, u)} \in \mathcal{E}_\infty$ if and only if the following conditions are satisfied:

(a) For every $d > 1$, there exists an $m \not\equiv 0 \pmod d$ such that $b_m \neq 0$. Moreover, for $m_d = \max\{m \not\equiv 0 \pmod d \mid b_m \neq 0\}$ we have $b_{m_d} > 0$.

(b) E is not empty and

$$\sum_{(\mu, \nu) \in I} p_\mu p_\nu \det \begin{pmatrix} k_\mu & l_\mu \\ k_\nu & l_\nu \end{pmatrix}^2 > 0.$$

(c) $\max\{k_j \mid p_j \neq 0\} + 2/3 \max\{k_j \mid p_j \neq 0 \wedge l_j \neq 0\} < 3K/5$.

Given the above tools of e-admissibility, we now proceed with the analysis of our parameters of interest, starting with the the number of 1-valent vertices in $\mathcal{T}^d$. Let $\chi_{uni,d}$ be the following extension of χ_{uni} from $\mathcal{T}$ to $\mathcal{T}^d$: if a map $m \in \mathcal{T}^d$ is connected, then $\chi_{uni,d}(m) = \chi_{uni}$, otherwise

$$\chi_{uni,d} = \sum_{C \text{ is a connected component of } m} \chi_{uni}(C).$$

Lemma 4.9. *Let $X_n^{uni,d}$ be the random variable corresponding to $\chi_{uni,d}$ taken over the set $\mathcal{T}_n^d$ of not-necessarily-connected $(1, 3)$ -valent maps with n half-edges. Then,*

$$\begin{aligned} \mathbb{E}(X_n^{uni,d}) &\sim \mathbb{V}(X_n^{uni,d}) \sim n^{1/3}, \\ X_n^{uni,d} &\xrightarrow{D} \mathcal{N}(1, 0). \end{aligned}$$

Proof. Firstly, by definition of $X_n^{uni,d}$ and the exponential Hadamard product, we have that (for $n \in 2\mathbb{N}$),

$$\begin{aligned} X_n^{uni,d} &= \frac{[z^n u^k] (\exp(h^2/2) \odot \exp(h^3/3 + uh))}{[z^n] (\exp(h^2/2) \odot \exp(h^3/3 + h))} \\ &= \frac{n! ([z^n] \exp(h^2/2) \cdot [z^n u^k] \exp(h^3/3 + uh))}{n! ([z^n] (\exp(h^2/2) \cdot [z^n] \exp(h^3/3 + h)))} \\ &= \frac{[z^n u^k] \exp(h^3/3 + uh)}{[z^n] \exp(h^3/3 + h)}, \end{aligned}$$

so that the limit law of $X_n^{uni,d}$ can be determined by studying $\exp(h^3/3 + uh)$ alone. An application of Theorem 4.8 then shows that $\exp(h^3/3 + uh)$ is e-admissible and so we may carry out a saddle-point analysis of said function, as dictated by Theorem 4.7, from which the desired result follows. $\square$

We also have the following asymptotics for the number of not-necessarily-connected $(1, 3)$ -valent maps, relying on a theorem of Moser and Wyman.

Theorem 4.10 (Adapted from [MW56]). *Let $P(x) = \sum_{i=1}^m a_i x^i$, where $a_i \in \mathbb{R}_{>0}$ for all $i \geq 0$. Then for*

$$e^{P(x)} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n,$$

we have

$$B_n \sim \frac{n! e^{P(R)}}{2\pi R^n} \left(\frac{2}{C_2(R)} \right)^{1/2} \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} \Psi_k(\phi) e^{-\phi^2} \frac{d\phi}{R^k} \right),$$

where

$$F(r, \theta) = P(re^{i\theta}) - P(r) - in\theta = \sum_{j=1}^{\infty} C_j(r) \frac{(i\theta)^j}{j!},$$

R is given by

$$C_1(R) = \sum_{k=1}^m k a_k R^k - n = 0,$$

and

$$\begin{aligned}\overline{C}_j(z) &= z^{2m} C_j(z^{-2}) = \sum_{k=1}^m k^j a_k z^{2m-2k}, \\ f_j(z) &= z^{m(j-2)} \overline{C}_j(z) \left(\frac{2}{\overline{C}_2(z)} \right)^{j/2}, \\ \psi(z, \phi) &= \sum_{j=3}^{\infty} \frac{f_j(z)}{j!} (i\phi)^j,\end{aligned}$$

so that, finally,

$$e^{\psi(z, \phi)} = \sum_{r=0}^{\infty} \Psi_r(\phi) z^r, \quad \Psi_0(\phi) = 1.$$

Lemma 4.11. For $n \in 2\mathbb{Z}^+$,

$$[z^n] (\exp(z^2/2) \odot \exp(z^3/3 + z)) = \frac{n!}{2^{n/2}(n/2)!} \frac{R^{-n} \exp(R^3/3 + R)}{\sqrt{2\pi(3R^3 + R)}} (1 - 4R^{-6} + 4R^{-10} + O(R^{-12})) \quad (4.24)$$

where R is the unique real solution to $R^3 + R - n = 0$.

Proof. Using Theorem 4.10 we have

$$[z^n] \exp\left(\frac{z^3}{3} + z\right) = \frac{R^{-n} \exp\left(\frac{R^3}{3} + R\right)}{\sqrt{2\pi(3R^3 + R)}} (1 - 4R^{-6} + 4R^{-10} + O(R^{-12})) \quad (4.25)$$

where R is the unique real solution of the saddle-point equation $R^3 + R - n = 0$. The result then follows by extracting the coefficients of $\exp(z^2/2)$ and applying the definition of the exponential Hadamard product. $\square$

Using [Ben75, Theorem 1] one can then derive the following asymptotic estimate for the number of rooted connected $(1, 3)$ -valent maps.

Corollary 4.12. Let $a_{(1,3)}(n)$ be given by the right-hand side of Equation (4.24). Then,

$$\dot{\mathcal{T}}_n \sim 2(n-2)a_{(1,3)}(n-2).$$

We can now proceed with a proof of Theorem 4.5.

Proof of Theorem 4.5. Note that $f(z, u, g(z, u))$ with $f(z, u, y) = \ln(1 + y)$ and

$$g(z, u) = (\exp(h^2/2) \odot \exp(h^3/3 + uh)) \Big|_{h=z^{1/2}} - 1, \quad (4.26)$$

is aperiodic and furthermore, by Lemma 4.11,

$$\frac{\bar{g}_n}{\bar{g}_{n-1}} = (2n)^{1/3} + \frac{2}{3}(2n)^{-1/3} + O(n^{-2/3}) \quad (4.27)$$

where $\bar{g}_n = [z^n]g(z, 1)$, which in particular implies the fast growth and asymptotic log-convexity of $\bar{g}_n$. Therefore the composition $f(z, u, g(z, u))$ falls under the schema presented in Theorem 4.4 and so we conclude that the limit distribution of X_n^{uni} is the same as that of $X_{2n}^{uni,d}$ (with the shift $n \mapsto 2n$ in size corresponding to the substitution $h \mapsto z^{1/2}$ above). The last is given in Lemma 4.9, yielding the desired result. $\square$

4.3. Distribution of degree 2 vertices in $\mathcal{D}$ and unused abstractions in $\mathcal{A}$

So far we have studied families of λ -terms which satisfy a condition of linearity: a bound variable must appear exactly once inside the body of its corresponding abstraction. By relaxing this condition, we obtain the notion of *affine λ -terms*, in which bound variables may occur *at most once* inside the body of their abstractions. We will call abstractions whose variable does not occur inside their body *unused abstractions*. Affine λ -terms are in bijection with $(2, 3)$ -valent maps, with the vertices of degree 2 corresponding to unused abstractions.

In this subsection we show that the limit distribution of vertices of degree 2 in $(2,3)$ -valent maps, as well as the limit distribution of unused abstractions in affine λ -terms, admit a Gaussian limit law.

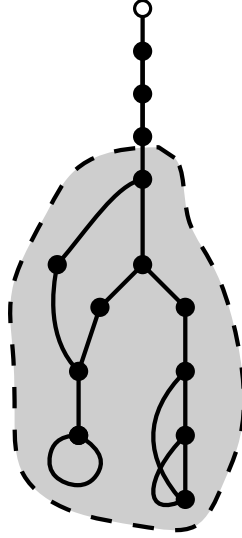


Figure 4.3: An element of $\dot{\mathcal{D}} \times \mathcal{P}$ corresponding to $\lambda x. \lambda y. (\lambda z. \lambda a. \lambda b. ba)(\lambda w. ((\lambda d. d)(\lambda e. y)))$. The rooted $(2, 3)$ -valent submap corresponding to the element of $\dot{\mathcal{D}}$ is highlighted in grey.

A specification for the generating function $A(z, u)$ of affine terms with unused abstractions marked by u can be obtained by arguments similar to those used to derive Equation (4.21):

$$D(h, u) = h \frac{\partial}{\partial h} \left(\ln \left(\exp \left(\frac{h^2}{2} \right) \odot \exp \left(\frac{h^3}{3} + \frac{uh^2}{2} \right) \right) \right) \quad (4.28)$$

$$A(z, u) = \frac{z^2 D(z^{1/2}, u)}{1 - uz} \quad (4.29)$$

Starting with Equation (4.28) we note that the presence of the term $uz^2/2$ inside the Hadamard product denotes that we allow for vertices of degree 2, marked by u , in our maps. This yields the ordinary generating function $D(h, u)$ of the class $\dot{\mathcal{D}}$ of *rooted* $(2, 3)$ -maps with vertices of degree 2 tagged by u . Next, we notice that an affine λ -term can start with an arbitrary number of abstractions whose bound variable is discarded. Letting $\mathcal{P}$ denote the class of paths, we see that this construction corresponds to an element of $\dot{\mathcal{D}} \times \mathcal{P}$: a rooted $(2, 3)$ -valent map m together with a path $p \in \mathcal{P}$ which we can visualise as grafted on the root of m as in Figure 4.3. To count such maps we multiply the generating function of rooted $(2, 3)$ -valent maps by the generating function $\frac{1}{1-uz}$ of $\mathcal{P}$, with u marking the vertices of the path, which gives Equation (4.29).

Using Equations (4.28) and (4.29) we will now proceed to show the following.

Theorem 4.13. *Let χ_{bi} be the combinatorial parameter of $\mathcal{D}$ corresponding to the number of degree 2 vertices. Let, also, X_n^{bi} be the random variable corresponding to χ_{bi} taken over $\mathcal{D}_n$. Then the mean and variance of X_n are asymptotically $\mu_n \sim \sigma_n^2 \sim \frac{(2n)^{2/3}}{2}$, while the standardised random variables converge to a Gaussian law:*

$$\frac{X_n^{bi} - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0, 1) \quad (4.30)$$

Once again, we obtain our desired result for λ -terms as a corollary of the above.

Corollary 4.14. *Let χ_λ be the combinatorial parameter of $\mathcal{A}$ corresponding to the number of abstractions discarding their variable in closed affine λ -terms. Then for $\mu_n \sim \sigma_n^2 \sim \frac{(2n)^{2/3}}{2}$, the standardised random variable X_n^λ corresponding to χ_λ taken over $\mathcal{A}_n$ converges to a Gaussian law:*

$$\frac{X_n^\lambda - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0, 1). \quad (4.31)$$

Our analysis closely mirrors that of the last section. We begin by determining the limit distribution for the random variable corresponding to the extension $\chi_{bi,d}$ of χ_{bi} obtained by summing the number of vertices of degree 2 for every connected component of a not-necessarily-connected $(2, 3)$ -valent map.

Lemma 4.15. *Let $X_n^{bi,d}$ be the random variable corresponding to $\chi_{bi,d}$ taken over the set $\mathcal{D}_n^d$ of not-necessarily-connected $(1, 3)$ -valent maps with n half-edges. Then,*

$$\begin{aligned} \mathbb{E}(X_n^{bi,d}) &\sim \mathbb{V}(X_n^{bi,d}) \sim \frac{n^{2/3}}{2}, \\ X_n^{bi,d} &\xrightarrow{D} \mathcal{N}(1, 0). \end{aligned}$$

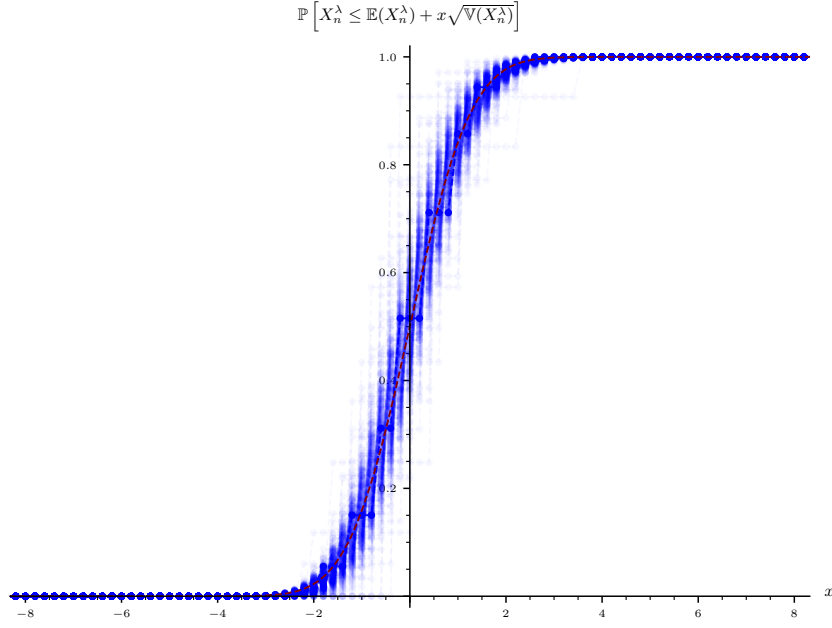


Figure 4.4: In hues of blue: the overlaid plots of $\mathbb{P}\left[X_n^\lambda \leq \mathbb{E}(X_n^\lambda) + x\sqrt{\mathbb{V}(X_n^\lambda)}\right]$, for $n = 0 \dots 300$, computed by discretising the x -axis using a step of size 0.2. In red: the cumulative distribution function of $\mathcal{N}(0, 1)$.

Proof. As in Lemma 4.9 we have that, by definition of $X_n^{bi,d}$ and the exponential Hadamard product, the limit law of $X_n^{bi,d}$ can be determined by studying $\exp(h^3/3 + uh^2/2)$ alone. An application of Theorem 4.8 then shows that $\exp(h^3/3 + uh^2/2)$ is e-admissible and the result follows once again from Theorem 4.7. $\square$

For the asymptotic number of not-necessarily-connected $(2, 3)$ -valent maps, we have the following lemma.

Lemma 4.16. *For $n \in 2\mathbb{Z}^+$*

$$[z^n](\exp(z^2/2) \odot \exp(z^3/3 + z^2/2)) = \frac{n!}{2^{n/2}(n/2)!} \frac{R^{-n} \exp(R^3/3 + R/2)}{\sqrt{2\pi(3R^3 + R)}} (1 - 4R^{-6} + 36R^{-8} + O(R^{-10})) \quad (4.32)$$

where R is the unique real solution of the equation $R^3 + R^2 - n = 0$.

Proof. Follows in a manner analogous to that used to derive Lemma 4.11, by using Theorem 4.10. $\square$

Using [Ben75, Theorem 1] we have the following estimate for the number of rooted connected $(2, 3)$ -maps.

Corollary 4.17. *Let $a_{(2,3)}(n)$ be equal to the right-hand side of Equation (4.32). Then,*

$$\mathcal{A}_n \sim 2(n-2)a_{(2,3)}(n-2)$$

Finally, we have:

Proof of Theorem 4.13. Let $f(z, u, y) = \ln(y + 1)$ and

$$g(z, y) = (\exp(z^2/2) \odot \exp(z^3/3 + uz^2/2))|_{z=z^{1/2}} - 1.$$

Then, by Lemma 4.16 we have for $\bar{g}_n = [z^n]g(z, 1)$:

$$\frac{\bar{g}_n}{\bar{g}_{n-1}} = (2n)^{1/3} + \frac{2}{3} + \frac{1}{9}(2n)^{-1/3} + O(n^{-2/3})$$

so we can apply Theorem 4.4 to conclude that the limit distribution of X_n^{bi} is the same as that of $X_{2n}^{bi,d}$, the last of which we obtain using Lemma 4.15. $\square$

Proof of Corollary 4.14. Firstly, by Proposition 2.1 we have that the limit distribution of χ_b taken over $\dot{\mathcal{D}}$ is the same as that of χ_b taken over $\mathcal{D}$. Then, an application of Theorem 4.4 for $F(z, u, y) = \frac{z^2 y}{1-uz}$ and $G(z, u) = D(z^{1/2}, u)$ shows that limit distribution of X_n^λ is the same as that of X_n^{bi} . As such, the results of Theorem 4.13 remain the same when passing from $\mathcal{D}$ to $\mathcal{A}$ (apart from a shift of size $n \mapsto n - 2$). $\square$

We conclude this subsection by sketching an approach to deriving a local limit law for X_n^{bi} . The basis for our approach is the following observation: affine terms may be obtained from linear λ -terms by replacing every vertex in their syntactic diagrams by a non-empty path. We therefore have the following equality between classes $\mathcal{A}$ of closed affine terms and $T_{[0]}$ of closed linear terms:

$$\mathcal{A} = \dot{T}_{[0]}(SEQ_{\geq 1} \mathcal{Z}) \quad (4.33)$$

Translating Equation (4.33) to an equality between generating functions and marking unused abstractions/vertices of degree 2 with u we have

$$A(z, u) = T_{[0]} \left(\frac{z}{1-uz} \right), \quad (4.34)$$

from which, by extracting coefficients, we obtain

$$a_n = \sum_{k=0}^n u^k \binom{n-1}{k} \ell_{n-k} = \sum_{k=0}^{\lceil \frac{n}{3} \rceil} u^{n-1-(3k+1)} \binom{n-1}{3k+1} \ell_k^* \quad (4.35)$$

where $a_n = [z^n]A$, $\ell_n = [z^n]T_{[0]}(z)$, $\ell_n^* = \ell_{3n+2}$. The above recurrence can be used to derive a local limit law for the number of unused abstractions and vertices of degree 2, as we demonstrate below briefly and informally.

Let $S(n, k) = \binom{n-1}{3k+1} \ell_k^*$ be the summand of Equation (4.35), evaluated at $u = 1$. We note that $S(n, k)$ is unimodal and so we may estimate, asymptotically as $n \rightarrow \infty$, the index k for which it is maximum by solving the following equation for k

$$\frac{S(n, k+1)}{S(n, k)} = 1. \quad (4.36)$$

Plugging $\ell_k^* \sim \frac{3}{\pi} 6^k k!$ into Equation (4.36) we obtain, asymptotically, a solution of the form

$$k_m \sim \frac{n}{3} - \frac{(2n)^{2/3}}{6} + \frac{(2n)^{1/3}}{9} - \frac{19}{18} + O(n^{-1/3}).$$

Doing the same for $S'(n, k) = (n - 1 - (3k + 1)) \binom{n-1}{3k+1} \ell_k^*$, the summand of the derivative with respect to u of Equation (4.35) evaluated at $u = 1$, we obtain an optimal index of the form

$$k'_m = \frac{n}{3} - \frac{(2n)^{2/3}}{6} + \frac{(2n)^{1/3}}{9} - \frac{25}{18} + O(n^{-1/3}).$$

Therefore an estimate for the mean μ_n is:

$$\mu_n = \frac{\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} S'(n, k)}{\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} S(n, k)} \geq \frac{S'(n, k'_m)}{S(n, k_m)} = \frac{(2n)^{2/3}}{2} - \frac{(2n)^{1/3}}{3} - \frac{1}{6} + O(n^{1/3}).$$

In the same manner, we may derive an asymptotic estimate for the variance σ_n^2 of the form:

$$\sigma_n^2 \geq s_n^2 = \frac{(2n)^{2/3}}{2} - 2 \frac{(2n)^{1/3}}{3} + \frac{1}{2} + O(n^{-1/3}).$$

The above estimates suggest that we study the asymptotics of $S(n, k)$ as we perturb k by $x\sqrt{s_n^2}$ around its peak at $k = k_m$:

$$\frac{\binom{n-1}{3k_m+1} \ell_{k_m}^*}{\binom{n-1}{3(k_m+x\sqrt{s_n^2})+1} \ell_{k_m+x\sqrt{s_n^2}}^*} = e^{-\frac{9x^2}{2}} \left(1 + O\left(\frac{x^3}{n^{1/3}}\right) \right),$$

which yields a local Gaussian approximation valid for $x = o(n^{1/9})$. Finally, to obtain the desired local limit law, one can show that indices outside this range have negligible contributions.

5. Conclusion and future directions

We have explored how tools drawn from the theory of maps and λ -terms may be used in tandem to study various classes of random maps and λ -terms which are bijectively related: trivalent maps and linear λ -terms, $(1, 3)$ -valent maps and open linear λ -terms, $(2, 3)$ -valent maps and closed affine terms. Making use of the interplay between maps and terms we have derived a number of new bijections and decompositions for our families of interest, which we then analysed using two new schemas developed here: one based on differential equations and another based on functional compositions. Using this toolkit we have studied combinatorial parameters which find natural interpretations in both maps and λ -terms, as presented in Table 1.1.

In this final section, we outline our results, some combinatorial implications of them, and suggest future topics for research.

The results derived in Section 3 show that the number of bridges in rooted trivalent maps has a *Poisson*(1) limit law, and the same is true for the number of loops. Since each loop induces exactly one bridge, we deduce that asymptotically almost surely all bridges of a rooted trivalent map in $\tilde{\mathcal{T}}_{[0]}$ are incident to loops. The corresponding result for closed linear λ -terms is that the number of subterms has a *Poisson*(1) limit law and these subterms are asymptotically almost surely all α -equivalent to the identity term $\lambda x.x$. The ideas sketched in Note 3.25 reveal more information on the structure of large random trivalent maps with n edges and k bridges: the dominant contribution to their number comes from taking maps in $\mathcal{Z}^2\mathcal{Q}$ with $n - 2$ edges and $k - 1$ bridges and replacing the box vertex with a loop. It would therefore be of interest to further investigate the class $\mathcal{Q}$ and its interplay with $\tilde{\mathcal{T}}_{[0]}$.

In Section 4 we relaxed the requirement for 3-regularity of the maps investigated, studying the number of unary and binary vertices in $(1, 3)$ - and $(2, 3)$ -valent maps respectively. We showed that these numbers have Gaussian limit laws with mean and variance asymptotic to $(2n)^{1/3}$ and $(2n)^{2/3}$ respectively. From a λ -calculus perspective, we studied two superclasses of closed linear terms: open linear terms and closed affine terms, deriving limit laws for the number of free variables and unused abstractions which are as mentioned above. As a potential extension of these results, we suggest an investigation of the corresponding local limit laws, from which more combinatorial information can be extracted. As an initial step, we sketched in Section 4.3 the derivation of such a law for affine terms/ $(2, 3)$ -valent maps. Since any affine term t can be constructed by taking some linear term $t' \in \tilde{\mathcal{T}}_{[0]}$ and inserting unused abstractions in it, according to Equation (4.34), the local limit law yields the distribution of the size of the term t , giving us further details on the relation of the two classes $\mathcal{A}$ and $\tilde{\mathcal{T}}_{[0]}$. Equivalent results for $(2, 3)$ -valent maps and their relation to trivalent maps can also be extracted from the aforementioned.

As for directions for future work, we mention the following:

- The results of Section 4 can also be obtained via direct analysis of the corresponding probability generating functions. This analysis reveals underlying Poisson laws with parameters which are functions of the size parameter n . This is also hinted to by the fact that the mean and the variance for the limit laws we derive are asymptotically equal. These occurrences of Poisson limit laws with unbounded parameters complement those with bounded parameter appearing in Section 3. Is there a simple explanation for this ubiquity of Poisson laws?
- In Section 4.3 we sketched an approach to deriving a local limit law for number of unused abstractions/binary vertices in affine terms and $(2, 3)$ -valent maps. Is there an equivalent result for free variables/unary vertices in open linear terms and $(1, 3)$ -valent maps?

Finally, let us mention that tools and techniques presented in this work have been applied to the analysis of normalisation of random closed linear λ -terms via β -reduction [Sin22], resulting in an asymptotic lower bound of $11n/240$ for the mean number of β -reduction steps required to normalise a random linear λ -term. It remains to strengthen this lower bound to an asymptotic estimate for the mean number of steps, and to understand the meaning of the β -reduction rule in the context of map theory.

A. Appendix

We begin with a proof of the fact that rooting does not affect the distribution of parameters in a combinatorial class.

Proof of Proposition 2.1. We have the following probability generating function for χ taken over $\mathcal{F}_n^\bullet$

$$p_n(u) = \frac{[z^n]\mathcal{F}^\bullet(z, u)}{[z^n]\mathcal{F}^\bullet(z, 1)} = \frac{n[z^n]\mathcal{F}(z, u)}{n[z^n]\mathcal{F}(z, 1)} = \frac{[z^n]\mathcal{F}(z, u)}{[z^n]\mathcal{F}(z, 1)}. \quad (\text{A.1})$$

□

Next we present the basic tools that lead to Lemma 2.3. First, we note that log-convexity is preserved under reflections: if $(s_i)_{i \in I}$ is log-convex on some interval $I = [\ell, \dots, u]$ then so is $(s_{u-i})_{i \in I}$. It is, furthermore, closed under multiplication: if $(s_i)_{i \in I}$ and $(s'_i)_{i \in I}$ are both log-convex on I , then so is $(s_i s'_i)_{i \in I}$.

The following lemmas show that log-convex sequences attain their maximum at a boundary point of their interval of definition, and how this can be leveraged to provide bounds on the coefficients of the Cauchy product of two such sequences.

Lemma A.1. *Let $(s_i)_{i \in I}$ be a log-convex sequence on some interval $I = [\ell, \dots, u]$. Then $\max((s_i)_{i \in I}) \in \{s_u, s_\ell\}$.*

Proof. Let $\ell \leq m \leq u$ be such that $\min((s_i)_{i \in I}) = s_m$. Then since $(s_i)_{i \in I}$ is log-convex,

$$\frac{s_{\ell+1}}{s_\ell} \leq \dots \leq \frac{s_m}{s_{m-1}} \leq \frac{s_{m+1}}{s_m} \leq \dots \leq \frac{s_u}{s_{u-1}}. \quad (\text{A.2})$$

As s_m is a minimum of $(s_i)_{i \in I}$, we have $\frac{s_m}{s_{m-1}} \leq 1$ and so for $\ell < i \leq m$, $\frac{s_i}{s_{i-1}} \leq 1$, therefore the subsequence $(s_i)_{\ell < i \leq m}$ is decreasing and $\max((s_i)_{\ell < i \leq m}) = s_\ell$. Similarly, since $\frac{s_{m+1}}{s_m} \geq 1$, the subsequence $(s_i)_{m < i < u}$ is increasing and $\max((s_i)_{m < i < u}) = s_u$. The lemma follows by combining these two results. □

Lemma A.2. *Let $(a_i)_{i \in I}$, $(b_i)_{i \in I}$ be two log-convex sequences on some interval $I = [\ell, \dots, u]$. Then for every $j < \frac{u-\ell}{2}$,*

$$\sum_{i=\ell}^u a_{n-i} b_i \leq \left(\sum_{i=\ell}^{\ell+j} a_{n-i} b_i \right) + \left(\sum_{i=u-j}^u a_{n-i} b_i \right) + (u - \ell - 2j - 1) (a_{\ell+j+1} b_{u-j-1} + a_{u-j-1} b_{\ell+j+1}). \quad (\text{A.3})$$

Proof. As log-convexity is preserved under multiplication and reflection, $c_i = (a_i \cdot b_{u-i})_{i \in I}$ is log-convex. By Lemma A.1, we have $c_j \leq c_u + c_\ell$ for all $u < j < \ell$. The lemma then follows by stripping the j extremal terms of the sum in the left-hand-side of Equation (A.3) and bounding the rest of the terms, of which there are $(u - \ell - 2j - 1)$, by the new maximal terms $c_{\ell+j+1}$, c_{u-j-1} . □

We are now ready to present the proof of the main lemma.

Proof of Lemma 2.3. For $n \geq a + b$ and $n \equiv a + b \pmod{d}$, let $m = \frac{n-(a+b)}{d}$. We then have

$$[z^n]f(z)g(z) = \sum_{k=a}^{n-b} f_k g_{n-k} = \sum_{k=0}^m f'_k g'_{n-k}. \quad (\text{A.4})$$

Let $N \in \mathbb{N}$ be the smallest index for which $(f'_n)_{n \geq N}$ and $(g'_n)_{n \geq N}$ are both log-convex, which exists due to Item (ii). Then, for $m > 2N$, using the shorthand notation $S_l^u = \sum_l^u f'_k g'_{n-k}$ we rewrite the sum in Equation (A.4) as follows into three parts:

$$\begin{aligned} \sum_{k=0}^m f'_k g'_{m-k} &= \sum_{k=0}^{N-1} f'_k g'_{m-k} + \sum_{k=N}^{m-N} f'_k g'_{m-k} + \sum_{k=m-N+1}^m f'_k g'_{m-k} \\ &= S_0^{N-1} + S_N^{m-N} + S_{m-N+1}^m. \end{aligned}$$

We now proceed to analyse each of S_0^{N-1} , S_N^{m-N} , S_{m-N+1}^m .

Analysis of S_0^{N-1} : Extracting the first term of the sum and using Item (iii) to bound the remaining $(N-1)$ terms, we have:

$$S_0^{N-1} = \sum_{k=0}^{N-1} f'_k g'_{m-k} = f'_0 g'_m + O(g'_{m-1}). \quad (\text{A.5})$$

Analysis of S_N^{m-N} : In the range $k \in \{N, \dots, m-N\}$, (f'_k) and (g'_{m-k}) are both log-convex and so, for m large enough, we may apply Lemma A.2 with $j = \max(\sigma_1^{-1}, \sigma_2^{-1})$ to obtain

$$\begin{aligned} S_N^{m-N} = \sum_{k=N}^{m-N} f'_k g'_{m-k} &\leq \sum_{k=N}^{N+j} f'_k g'_{m-k} + \sum_{k=m-N-j}^{m-N} f'_k g'_{m-k} \\ &\quad + (m-2N-2j-1) (f'_{N+j+1} g'_{m-N-j-1} + f'_{m-N-j-1} g'_{N+j+1}) \end{aligned} \quad (\text{A.6})$$

Using Item (iii) we then have:

$$\sum_{k=N}^{N+j} f'_k g'_{m-k} = f'_N g'_{m-N} + O(g'_{m-N-1}) \quad (\text{A.7})$$

$$\sum_{k=m-N-j}^{m-N} f'_k g'_{m-k} = f'_{m-N} g'_N + O(f'_{m-N-1}) \quad (\text{A.8})$$

Since $j = \max(\sigma_1^{-1}, \sigma_2^{-1})$, $\frac{f'_{m-N-j-1}}{f'_{m-N-1}} = O(m^{-1})$ and $\frac{g'_{m-N-j-1}}{g'_{m-N-1}} = O(m^{-1})$ and so

$$(m-2N-2j+1) (f'_{N+j+1} g'_{m-N-j-1} + f'_{m-N-j-1} g'_{N+j+1}) = O(f'_{m-N-1}) + O(g'_{m-N-1}) \quad (\text{A.9})$$

Analysis of S_{m-N-1}^m : Similarly to the analysis of S_0^N , we have $S_{m-N}^m = f'_m g'_0 + O(f'_{m-1})$.

Finally, combining the above and using Items (iii) and (iv), we have $S_0^m = f'_0 g'_m + f'_m g'_0 + O(f'_{m-1}) = f_{m-b} g_b + f_a g_{m-a} + O(f_{m-b-d})$ as desired. $\square$

Acknowledgements

We would like to thank the reviewers for their comments and suggestions, which we believe have greatly improved the content and presentation of this work. The authors are partially supported by ANR project LambdaComb (ANR-21-CE48-0017).

References

- [AB00] Didier Arques and Jean-François Béraud. Rooted maps on orientable surfaces, Riccati's equation and continued fractions. *Discrete Mathematics*, 215(1-3):1–12, 2000. doi:10.1016/S0012-365X(99)00197-1.
- [BBD19] Maciej Bendkowski, Olivier Bodini, and Sergey Dovgal. Statistical properties of lambda terms. *The Electronic Journal of Combinatorics*, 26(4):P4–1, 2019. doi:10.37236/8491.
- [BCDH18] Olivier Bodini, Julien Courtiel, Sergey Dovgal, and Hsien-Kuei Hwang. Asymptotic distribution of parameters in random maps. In *29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2018)*, pages 13–1. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2018.
- [BDS13] Hendrik Pieter Barendregt, Wil Dekkers, and Richard Statman. *Lambda calculus with types*. Cambridge University Press, 2013. doi:10.1017/CB09781139032636.
- [Ben75] Edward A Bender. An asymptotic expansion for the coefficients of some formal power series. *Journal of the London Mathematical Society*, s2-9(3):451–458, 1975. doi:10.1112/jlms/s2-9.3.451.
- [Ber07] Olivier Bernardi. Bijective counting of Kreweras walks and loopless triangulations. *Journal of Combinatorial Theory, Series A*, 114(5):931–956, 2007. doi:10.1016/j.jcta.2006.09.009.
- [BFSS01] Cyril Banderier, Philippe Flajolet, Gilles Schaeffer, and Michele Soria. Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures & Algorithms*, 19(3-4):194–246, 2001. doi:10.1002/rsa.10021.
- [BGGJ13] Olivier Bodini, D Gardy, Bernhard Gittenberger, and Alice Jacquot. Enumeration of generalized BCI lambda-terms. *The Electronic Journal of Combinatorics*, 20(4):23, 2013. doi:10.37236/3051.
- [BGJ13] Olivier Bodini, Danièle Gardy, and Alice Jacquot. Asymptotics and random sampling for BCI and BCK lambda terms. *Theoretical Computer Science*, 502:227–238, 2013. doi:10.1016/j.tcs.2013.01.008.

- [BGLZ17] Maciej Bendkowski, Katarzyna Grygiel, Pierre Lescanne, and Marek Zaionc. Combinatorics of λ -terms: a natural approach. *Journal of Logic and Computation*, 27(8):2611–2630, 2017. doi:10.1093/logcom/exx018.
- [Bil08] Patrick Billingsley. *Probability and measure*. John Wiley & Sons, 2008.
- [BIZ80] Daniel Bessis, Claude Itzykson, and Jean-Bernard Zuber. Quantum field theory techniques in graphical enumeration. *Advances in Applied Mathematics*, 1(2):109–157, 1980. doi:10.1016/0196-8858(80)90008-1.
- [BN95] Dror Bar-Natan. On the Vassiliev knot invariants. *Topology*, 34(2):423–472, 1995. doi:10.1016/0040-9383(95)93237-2.
- [Bor18] Michael Borinsky. Generating asymptotics for factorially divergent sequences. *The Electronic Journal of Combinatorics*, 25(4):4–1, 2018. doi:10.37236/5999.
- [BR86] Edward A Bender and L Bruce Richmond. A survey of the asymptotic behaviour of maps. *Journal of Combinatorial Theory, Series B*, 40(3):297–329, 1986. doi:10.1016/0095-8956(86)90086-9.
- [Car19] Ariane Carrance. Uniform random colored complexes. *Random Structures & Algorithms*, 55(3):615–648, 2019. doi:10.1002/rsa.20845.
- [CV81] Robert Cori and Bernard Vauquelin. Planar maps are well labeled trees. *Canadian Journal of Mathematics*, 33(5):1023–1042, 1981. doi:10.4153/CJM-1981-078-2.
- [DGK05] Michael Drmota, Bernhard Gittenberger, and Thomas Klausner. Extended admissible functions and Gaussian limiting distributions. *Mathematics of Computation*, 74(252):1953–1966, 2005. doi:10.1090/S0025-5718-05-01744-8.
- [Fan23] Wenjie Fang. Bijections between planar maps and planar linear normal λ -terms with connectivity condition. *Advances in Applied Mathematics*, 148:102532, 2023. doi:10.1016/j.aam.2023.102532.
- [FS09] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University press, 2009. doi:10.1017/cbo9780511801655.
- [GIZ13] Katarzyna Grygiel, Paweł M Idziak, and Marek Zaionc. How big is BCI fragment of BCK logic. *Journal of Logic and Computation*, 23(3):673–691, 2013. doi:10.1093/logcom/exs017.
- [GM06] Bernhard Gittenberger and Johannes Mandlburger. Hayman admissible functions in several variables. *The Electronic Journal of Combinatorics*, 13:R106–R106, 2006. doi:10.37236/1132.
- [Hay56] WK Hayman. A generalisation of Stirling’s formula. *Journal für die reine und angewandte Mathematik*, 1956(196):67–95, 1956. doi:10.1515/crll.1956.196.67.
- [Jan03] Svante Janson. The Wiener index of simply generated random trees. *Random Structures & Algorithms*, 22(4):337–358, 2003. doi:10.1002/rsa.10074.
- [LZ04] Sergei K Lando and Alexander K Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer Berlin Heidelberg, 2004. doi:10.1007/978-3-540-38361-1.

- [McB01] Conor McBride. The derivative of a regular type is its type of one-hole contexts. Unpublished manuscript, 2001.
- [MS01] Michel Marcus and Gilles Schaeffer. Une bijection simple pour les cartes orientables. Intern report A01-R-366 || marcus_01a, 2001.
- [MW56] Leo Moser and Max Wyman. Asymptotic expansions. *Canadian Journal of Mathematics*, 8:225–233, 1956. doi:10.4153/cjm-1956-026-x.
- [Odl96] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of Combinatorics (Vol. 2)*, page 1063–1229. MIT Press, Cambridge, MA, USA, 1996.
- [Pan19] Élie Panafieu. Analytic combinatorics of connected graphs. *Random Structures & Algorithms*, 55(2):427–495, 2019. doi:10.1002/rsa.20836.
- [Pen71] Roger Penrose. Applications of negative dimensional tensors. In *Combinatorial mathematics and its applications*, pages 221–244. Academic Press, New York, USA, 1971.
- [Sin22] Alexandros Singh. *Une approche unifiée de la combinatoire du lambda-calcul et des cartes: bijections et propriétés limites*. PhD thesis, Université Paris-Nord-Paris XIII, 2022.
- [Tut68] WT Tutte. On the enumeration of planar maps. *Bull. Amer. Math. Soc.*, 74:64–74, 1968. doi:10.1090/s0002-9904-1968-11877-4.
- [WL72] Timothy RS Walsh and Alfred B Lehman. Counting rooted maps by genus. i. *Journal of Combinatorial Theory, Series B*, 13(3):192–218, 1972. doi:10.1016/0095-8956(72)90056-1.
- [Zei16] Noam Zeilberger. Linear lambda terms as invariants of rooted trivalent maps. *Journal of Functional programming*, 26, 2016. doi:10.1017/s095679681600023x.
- [ZG15] Noam Zeilberger and Alain Giorgetti. A correspondence between rooted planar maps and normal planar lambda terms. *Logical Methods in Computer Science*, 11(3):1–39, 2015. doi:10.2168/lmcs-11(3:22)2015.