

ORTHOGONAL WEBS AND SEMISIMPLIFICATION

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Abstract. We define a diagrammatic category that is equivalent to tilting representations for the orthogonal group. Our construction works in characteristic not equal to two. We also describe the semisimplification of this category.

Keywords. Representations of linear algebraic groups, webs, Howe duality, diagrammatic presentation, positive characteristic, semisimplification

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1. Introduction

In this paper we define a diagrammatic category of *orthogonal webs* that is equivalent to tilting representations of the orthogonal group.

1.1. Motivation

Diagrammatic methods and diagrammatic presentations in representation theory have been studied for over a century.

An early example is Schur's celebrated *Schur–Weyl duality* [Sch01] relating representations of $SL_N(\mathbb{C})$ (or $GL_N(\mathbb{C})$) with the symmetric group. The diagrammatic description of a permutation gives rise to diagrammatic methods for $SL_N(\mathbb{C})$ -representations. In modern terms, we say that there is a functor from a generators and relations (i.e. diagrammatic) monoidal category to the monoidal category of representations of $SL_N(\mathbb{C})$, and Schur–Weyl duality says this functor is full.

The functor is not faithful, but there is an explicit diagrammatic formulation of the kernel which involves the antisymmetrizer on N strands, see e.g. [Här99]. Identification of this kernel yields a diagrammatic presentation of the monoidal category of representations of $SL_N(\mathbb{C})$.

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However, since the kernel is described by a complicated sum of diagrams, this presentation is not very aesthetically pleasing from a diagrammatic perspective. In the special case $N = 2$, the kernel has a particularly simple description, as shown by Rumer–Teller–Weyl [RTW32], which resulted in the Temperley–Lieb calculus, a presentation which does look nice diagrammatically.

Brauer extended Schur’s result to $O_N(\mathbb{C})$ (and $SP_{2N}(\mathbb{C})$) relating them via *Brauer duality* to Brauer’s diagrammatic algebra [Bra37]. Brauer’s duality, combined with an identification of the kernel in terms of Brauer’s diagrammatic algebra, yields a diagrammatic presentation for the monoidal category of representations of $O_N(\mathbb{C})$. Similarly as for $SL_N(\mathbb{C})$, the kernel is well-known, see for example [LZ15], but does not seem to admit a diagrammatic description without complicated sums when $N > 2$.

With the birth of quantum topology in the 1980s many more diagrammatic presentations were found and these often include the identification of the kernels. For example, Yamada’s presentation for $SO_3(\mathbb{C})$ -representations [Yam89], following ideas from [TL71], which employs the diagrammatics of *webs* (also known as birdtracks [Cvi08], spiders [Kup96] etc.). Yamada’s presentation is in the spirit of Rumer–Teller–Weyl and all relations are beautifully included in the diagrammatics. These webs were then extended to many other settings, and have been at the heart of diagrammatic representation theory every since.

The strategy that we run in this paper is inspired by the observation of Cautis–Kamnitzer–Morrison [CKM14] that the kernel under *Howe’s duality* for $GL_N(\mathbb{C})$ - $GL_m(\mathbb{C})$ [How95] has again a pleasing diagrammatic interpretation in terms of webs. They masterfully used Howe’s duality to define a presentation for $SL_N(\mathbb{C})$ -representations which generalizes Rumer–Teller–Weyl’s presentation. Since the kernel under Schur–Weyl duality for $N > 2$ is diagrammatically rather ugly, such a nice presentation did not seem possible from Schur–Weyl duality itself.

Another upshot of Cautis–Kamnitzer–Morrison’s description is that it works over any field when one slightly modifies the target category to be *tilting representations*. Taking the prime characteristic version of Howe’s $GL_N(\mathbb{C})$ - $GL_m(\mathbb{C})$ duality from [AR96] and running the Cautis–Kamnitzer–Morrison strategy, one gets a diagrammatic category equivalent to tilting representations of $SL_N(\overline{\mathbb{F}}_p)$ for any prime p . The technicalities however are much more involved, see [Eli15] for details.

Remark 1.1. In characteristic zero all representations are tilting, so the setting in the above paragraph generalizes Cautis–Kamnitzer–Morrison’s result. The same is true in the orthogonal world and we will state all of our results for tilting representations.

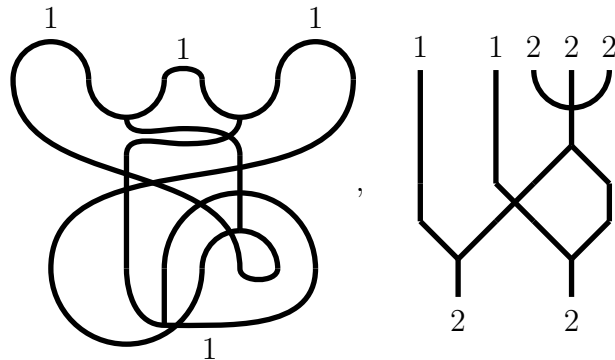
This is the starting point of our paper. We begin by fixing $p = 0$ or any prime $p \neq 2$. Let \mathbb{F}_ϑ be an infinite field of characteristic p containing $\sqrt{-1}$. Our main results are:

- (A) We give a diagrammatic presentation of tilting $O_N(\mathbb{F}_\vartheta)$ -representations using *orthogonal webs*. This extends the result of Sartori [ST19] to prime characteristic.
- (B) As an application, we give an orthogonal version of the main result of [BEAEO20], i.e. we describe the semisimplification of tilting $O_N(\mathbb{F}_\vartheta)$ -representations. Here p does not need to be bigger than N .

A key ingredient in our proofs is Howe’s orthogonal duality in prime characteristic [AR96]. Before coming to the main body of the paper we now explain our results in more detail.

1.2. What we prove

A *closed orthogonal pre-web* is a trivalent graph with edges labeled with integers $\{1, \dots, N\}$ such that we have k, l and $k+l$ around every trivalent vertex. A *closed orthogonal web* is a planar embedding of a closed orthogonal pre-web such that each point of intersection is a crossing in the usual sense. As usual in diagrammatic algebra, cutting these graphs open and putting them into a strip with bottom and top boundary points gives a way to define morphisms, called *orthogonal webs*, in a monoidal category. Here are two examples, the left one being closed:



Notation 1.2. *If in this or other illustrations an edge is not labeled, then its label is determined by other labels and we omitted it to avoid clutter.*

Orthogonal webs form a combinatorial and topological category akin to the Temperley–Lieb calculus and Yamada’s webs. Let $\mathbb{F}_\triangleright$ be as above. Enriching orthogonal webs $\mathbb{F}_\triangleright$ -linearly and imposing relations of the form

$$\begin{array}{c} >N \\ | \\ >N \end{array} = 0, \quad \begin{array}{c} k+l \\ \diamond \\ k+l \end{array} = \binom{k+l}{k} \cdot \begin{array}{c} k+l \\ | \\ k+l \end{array}, \quad k \bigcirc = \binom{N}{k}, \quad \begin{array}{c} l \\ \diamond \\ k \end{array} = 0 \text{ for } k > l,$$

together with associativity and coassociativity and some additional relations, gives a symmetric ribbon $\mathbb{F}_\triangleright$ -linear category $\mathbf{Web}_{\mathbb{F}_\triangleright}(\mathcal{O}(N))$, which is the main object under study in this paper. Precisely, let $\mathbf{Rep}_{\mathbb{F}_\triangleright}(\mathcal{O}(N))$ denote the symmetric ribbon $\mathbb{F}_\triangleright$ -linear category of finite dimensional $\mathcal{O}_N(\mathbb{F}_\triangleright)$ -representations. We show:

Theorem 1.3. *There is a fully faithful symmetric ribbon $\mathbb{F}_\triangleright$ -linear functor*

$$\mathbf{Web}_{\mathbb{F}_\triangleright}(\mathcal{O}(N)) \rightarrow \mathbf{Rep}_{\mathbb{F}_\triangleright}(\mathcal{O}(N)),$$

sending k to the k th exterior power of the vector $\mathcal{O}_N(\mathbb{F}_\triangleright)$ -representation. This functor induces an equivalence of symmetric ribbon (additive) $\mathbb{F}_\triangleright$ -linear categories between:

- (i) *The additive idempotent closure of $\mathbf{Web}_{\mathbb{F}_\triangleright}(\mathcal{O}(N))$.*
- (ii) *The category of all tilting $\mathcal{O}_N(\mathbb{F}_\triangleright)$ -representations.*

With respect to [Theorem 1.3](#) we note:

- ▶ The result generalizes [[ST19](#), Section 3] to positive characteristic and we get a nice diagrammatic calculus.
- ▶ The characteristic of our ground field is essentially arbitrary, except that we cannot work in characteristic two (a not surprising restriction for the orthogonal group).
- ▶ The points (i) and (ii) are the expected caveats in diagrammatic representation theory. Here we stress that an \mathbb{F}_ν -linear equivalence of additive \mathbb{F}_ν -linear categories is automatically additive as well, and we will omit the “additive” below.

Let $p = \infty$ in case the characteristic of \mathbb{F}_ν is zero. For a number $N \in \mathbb{Z}_{\geq 0}$ let N_i be the p -adic digits, i.e. numbers $N_i \in \{0, \dots, p-1\}$ such that $N = \sum_{i \in \mathbb{Z}_{\geq 0}} N_i p^i$. Finally, we write $\overline{\mathbf{Tilt}}_{\mathbb{F}_\nu}(O(N))$ for the category of $O_N(\mathbb{F}_\nu)$ -tilting representations, and we use overline to indicate semisimplifications. Having established [Theorem 1.3](#), we then prove:

Theorem 1.4. *There is an equivalence of symmetric ribbon \mathbb{F}_ν -linear categories*

$$\overline{\mathbf{Tilt}}_{\mathbb{F}_\nu}(O(N)) \cong \boxtimes_{i \in \mathbb{Z}_{\geq 0}} \overline{\mathbf{Tilt}}_{\mathbb{F}_\nu}(O(N_i)),$$

where \boxtimes is Deligne’s tensor product.

The categories $\overline{\mathbf{Tilt}}_{\mathbb{F}_\nu}(O(N_i))$ appearing in [Theorem 1.4](#) are *Verlinde categories*, so [Theorem 1.4](#) expresses $\overline{\mathbf{Tilt}}_{\mathbb{F}_\nu}(O(N))$ in terms of well-known categories.

Remark 1.5. We expect similar results for the symplectic group instead of the orthogonal group. However, the exterior powers are more complicated in this case, e.g. even in characteristic zero they are not simple. As a result of this complication, one needs new arguments to identify the semisimplification in terms of well-known categories.

1.3. How we prove this

Our proof of [Theorem 1.3](#) is, in broad outline, similar to the proof in Cautis–Kamnitzer–Morrison’s paper on type A webs [[CKM14](#)]. We will now sketch our approach and also indicate what the new steps are.

Remark 1.6. To keep things simple in this introduction, we are not precise with the ground rings and fields: there are no serious issues, but some care needs to be taken. We will give the precise statements and details in the main body of the paper.

1.3.1 Webs in type A in characteristic zero (known)

We first recall a simplified outline of the Cautis–Kamnitzer–Morrison proof when working over \mathbb{Q} (so $q = 1$, or as we say later, *dequantized*). (Technical side note: Their paper considers \mathfrak{sl}_N and not \mathfrak{gl}_N representations, but it is easy to adapt their arguments to work for \mathfrak{gl}_N .)

They consider the following objects connected with the exterior powers $\Lambda^k := \Lambda^k(\mathbb{Q}^N)$.

1. A generators and relations algebra, denoted $\dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m)$, which is finite dimensional, semi-simple, and is equipped with an isomorphism

$$\Phi_m : \dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m) \xrightarrow{\cong} \text{End}_{\text{GL}_N(\mathbb{Q})} \left(\bigoplus_{\substack{(k_1, \dots, k_m) \\ 0 \leq k_i \leq N}} \Lambda^{k_1} \otimes \dots \otimes \Lambda^{k_m} \right).$$

2. A full monoidal subcategory of $\mathbf{Rep}(\text{GL}_N(\mathbb{Q}))$, denoted $\mathbf{Fund}(\text{GL}_N(\mathbb{Q}))$, with objects $\Lambda^{k_1} \otimes \dots \otimes \Lambda^{k_m}$.
3. A generators and relations monoidal category, denoted $\mathbf{Web}(\text{GL}_N(\mathbb{Q}))$, which has objects (k_1, \dots, k_m) , and is equipped with a monoidal functor

$$\Gamma : \mathbf{Web}(\text{GL}_N(\mathbb{Q})) \rightarrow \mathbf{Fund}(\text{GL}_N(\mathbb{Q})).$$

Given the above, their proof that Γ is fully faithful proceeds as follows.

- (i) Using the generators and relations for the algebra $\dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m)$, they construct an algebra homomorphism

$$\Psi_m : \dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m) \longrightarrow \text{End}_{\mathbf{Web}(\text{GL}_N(\mathbb{Q}))} \left(\bigoplus_{\substack{(k_1, \dots, k_m) \\ 0 \leq k_i \leq N}} (k_1, \dots, k_m) \right).$$

Everything is set up so that $\Gamma \circ \Psi_m = \Phi_m$.

- (ii) One of the key features of Ψ_m is that the Chevalley generators $e_i, f_i, i = 1, \dots, m - 1$, and their divided powers $e_i^{(a)}$ and $f_i^{(a)}, i = 1, \dots, m - 1$, map to *ladder* shaped diagrams in the diagrammatic category $\mathbf{Web}(\text{GL}_N(\mathbb{Q}))$. This allows them to give a topological argument, using what they call *ladderization of webs*, to prove that Ψ_m is surjective.
- (iii) The proof that Γ is fully faithful uses nothing more than that for all $m \geq 1$, Φ_m is an isomorphism, Ψ_m is surjective, and $\Gamma \circ \Psi_m = \Phi_m$.

1.3.2 Webs in type A integrally (known, with a mild tweak)

Amazingly, this is not the only proof that Γ is fully faithful. Another proof, which avoids using $\dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m)$ entirely, was found by Elias [Eli15]. Elias’ result also generalized the result in [CKM14] from \mathbb{Q} to \mathbb{Z} in the following sense.

There are analogs of the source and target of Γ which can be defined over \mathbb{Z} , denoted by $\mathbf{Web}_{\mathbb{Z}}(\text{GL}(N))$ and $\mathbf{Fund}_{\mathbb{Z}}(\text{GL}(N))$. The former is easy to define, since any \mathbb{Q} -linear diagrammatic category such that the coefficients of the relations are in \mathbb{Z} can be defined over \mathbb{Z} . The definition of $\mathbf{Fund}_{\mathbb{Z}}(\text{GL}(N))$ is a bit more complicated, but it uses only standard technology from modular representation theory of reductive algebraic groups, like tilting modules. In [Eli15], Elias carefully studies these categories and actually proves that Γ is fully faithful over \mathbb{Z} .

It is a consequence of well-known results about tilting modules that $\mathbf{Fund}_{\mathbb{Z}}(\mathrm{GL}(N))$ has torsion free homomorphism spaces which are the same rank as the dimension of the analogous homomorphism space over \mathbb{Q} . In contrast, for $\mathbf{Web}_{\mathbb{Z}}(\mathrm{GL}(N))$ it is not at all clear from the definition that the homomorphism spaces are torsion free. However, it follows from Γ being an equivalence over \mathbb{Z} that the homomorphism spaces in $\mathbf{Web}_{\mathbb{Z}}(\mathrm{GL}(N))$ are torsion free and the same rank as the analogous homomorphism spaces over \mathbb{Q} .

Elias' proof is technically difficult, so let us sketch a potential alternative proof.

Remark 1.7. To the best of our knowledge, the arguments below are new, but were known to experts. Since we do not provide complete details, the present paper still relies on Elias' theorem that Γ is fully faithful over \mathbb{Z} .

Cautis–Kamnitzer–Morrison already knew that there was an integral version of the algebra $\dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m)$ [CKM14, Remark, Section 4.1], which we denote by $\dot{U}_{\mathbb{Z}}^N(\mathfrak{gl}_m)$. A key difference between $\dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m)$ and $\dot{U}_{\mathbb{Z}}^N(\mathfrak{gl}_m)$ is that the latter is defined using *higher Serre relations*, while the former only needs Serre relations and the higher Serre relations are consequences. It is natural to try to adapt Cautis–Kamnitzer–Morrison's proof to work over \mathbb{Z} .

Defining the \mathbb{Z} analog of Φ_m , and proving that it is an isomorphism is a classic result [Don93]. See [BEAEO20, Proposition 4.13] for a modern discussion of this result, which also explains the connection to diagrammatic algebra. Moreover, the ladderization of webs argument [CKM14, Theorem 5.3.1] also works over \mathbb{Z} , without any changes. Thus, the proof reduces to constructing a \mathbb{Z} analog of Ψ_m .

We know from working over \mathbb{Q} that, in order to have $\Gamma \circ \Psi_m = \Phi_m$, the generators $e_i^{(a)}$ and $f_i^{(a)}$ have to go to specific ladder shaped diagrams. So we just need to check these diagrams satisfy the defining relations of $\dot{U}_{\mathbb{Z}}(\mathfrak{gl}_m)$. However, it is highly nontrivial to directly verify the higher Serre relations for ladder webs. To the best of our knowledge, this calculation does not appear in the literature. Filling in this gap, would give an alternative proof of Elias' result.

On the other hand, Elias's result that Γ is an equivalence over \mathbb{Z} can be used to prove that the higher Serre relations hold for the ladder web diagrams in $\mathbf{Web}_{\mathbb{Z}}(\mathrm{GL}(N))$, see [Lemma 2.19](#).

1.3.3 Webs for orthogonal groups in characteristic zero (known before)

Following the breakthrough work of Cautis–Kamnitzer–Morrison on webs for $\mathrm{GL}(N)$, people immediately began hunting for the generalization to orthogonal groups. The (arguably) first result in this direction is the paper [ST19], which although quite subtle in the quantum case gives the following results over \mathbb{Q} .

[ST19] continues to consider exterior powers $\Lambda^k := \Lambda^k(\mathbb{Q}^N)$, but now as representations of $O_N(\mathbb{Q}) \subset \mathrm{GL}_N(\mathbb{Q})$, and studies the following objects related to these exterior powers.

1. A generators and relations algebra, denoted $\dot{U}_{\mathbb{Q}}^N(\mathfrak{so}_{2m})$, which is finite dimensional, semi-simple, and is equipped with an isomorphism

$$\Phi_m : \dot{U}_{\mathbb{Q}}^N(\mathfrak{so}_{2m}) \xrightarrow{\cong} \mathrm{End}_{O_N(\mathbb{Q})} \left(\bigoplus_{\substack{(k_1, \dots, k_m) \\ 0 \leq k_i \leq N}} \Lambda^{k_1} \otimes \dots \otimes \Lambda^{k_m} \right).$$

The Chevalley generators for this algebra are $e_i, f_i, i = 1, \dots, m-1$, and e_m, f_m . The algebra $\dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m)$ maps to $\dot{U}_{\mathbb{Q}}^N(\mathfrak{so}_{2m})$, sending the generators to Chevalley generators for $i = 1, \dots, m-1$.

2. A full monoidal subcategory of $\mathbf{Rep}(O_N(\mathbb{Q}))$, denoted $\mathbf{Fund}(O_N(\mathbb{Q}))$, with objects $\Lambda^{k_1} \otimes \dots \otimes \Lambda^{k_m}$.
3. A generators and relations monoidal category, denoted $\mathbf{Web}(O_N(\mathbb{Q}))$, which has objects (k_1, \dots, k_m) , and is equipped with a monoidal functor

$$\Gamma : \mathbf{Web}(O_N(\mathbb{Q})) \rightarrow \mathbf{Fund}(O_N(\mathbb{Q})).$$

Restriction induces a monoidal functor $\mathbf{Fund}(GL_N(\mathbb{Q})) \rightarrow \mathbf{Fund}(O_N(\mathbb{Q}))$. This is paralleled in the definition of orthogonal webs [ST19, Section 3], which are webs for $GL_N(\mathbb{Q})$ along with cups and caps realizing the symmetric form on \mathbb{Q}^N preserved by $O_N(\mathbb{Q})$. In particular, there is a monoidal functor $\mathbf{Web}(GL_N(\mathbb{Q})) \rightarrow \mathbf{Web}(O_N(\mathbb{Q}))$, and we refer to webs in the image as *type A webs*. (It is not clear from the definition that this functor is faithful, but one can show that it is.)

The proof that Γ is fully faithful follows the same outline.

- (i) Using the generators and relations for the algebra $\dot{U}_{\mathbb{Q}}^N(\mathfrak{so}_{2m})$, [ST19] constructs an algebra homomorphism (denoted Υ in [ST19, Section 6B])

$$\Psi_m : \dot{U}_{\mathbb{Q}}^N(\mathfrak{so}_{2m}) \longrightarrow \text{End}_{\mathbf{Web}(O_N(\mathbb{Q}))} \left(\bigoplus_{\substack{(k_1, \dots, k_m) \\ 0 \leq k_i \leq N}} (k_1, \dots, k_m) \right),$$

such that $\Gamma \circ \Psi_m = \Phi_m$.

- (ii) This Ψ_m still maps Chevalley generators e_i, f_i , and their divided powers $e_i^{(a)}$ and $f_i^{(a)}$, to the same ladder shaped type A web diagrams. Moreover, the Chevalley generators e_m, f_m , and their divided powers, also map to ladder shaped diagrams. A *ladderization of webs* argument then proves Ψ_m is surjective.
- (iii) The proof that Γ is fully faithful uses nothing more than that, for all $m \geq 1$, Φ_m is an isomorphism, Ψ_m is surjective, and $\Gamma \circ \Psi_m = \Phi_m$.

To construct Ψ_m , one needs to verify that the defining relations for $\dot{U}_{\mathbb{Q}}^N(\mathfrak{so}_{2m})$ are satisfied by the ladder webs corresponding to Chevalley generators. Of course, since the type A webs in $\mathbf{Web}(O_N(\mathbb{Q}))$ satisfy type A web relations, and the Chevalley generators $e_i, f_i \in \dot{U}_{\mathbb{Q}}^N(\mathfrak{so}_{2m})$, for $i = 1, \dots, m-1$, satisfy the same relations as in $\dot{U}_{\mathbb{Q}}^N(\mathfrak{gl}_m)$, it suffices to check relations involving e_m, f_m . The wonderful observation of Sartori, *Sartori's trick*, is that this can be done effortlessly [ST19, Section 3C], thanks to the topological intuition provided by the diagrammatic description of $\mathbf{Web}(O_N(\mathbb{Q}))$.

1.3.4 Webs for orthogonal groups integrally (new)

In [section 3](#), we define the orthogonal web category over \mathbb{Z} and establish its connection to the familiar orthogonal web category in [subsection 1.3.3](#) after base changing to \mathbb{Q} . We prove [Theorem 1.3](#) by combining various results to establish \mathbb{Z} analogs of everything above.

Remark 1.8. Up to this point we have just been discussing motivation for the present work (although the ideas discussed in [Remark 1.7](#) were new). For the rest of the introduction, the results are new (excluding the discussion about semisimplification in type A [subsection 1.3.5](#)). The new results are nontrivial, but can still be accomplished efficiently, and can be grouped into two classes.

On the one hand, the technical results like (a) and (c) below do not to our knowledge appear in the literature. For comparison, in type A, the analogous technical results were well-established prior to the study of the corresponding web categories.

On the other hand, many of the arguments in the present paper involve generators and relations checks, or ladderization arguments, the key steps of which have appeared in prior papers. Thus, our main task is to carefully “piece things together.” Indeed, a central and new observation of this paper is that this approach works.

1. In [section 2](#), we combine results of [\[Tak83\]](#) and [\[Lus10\]](#) to establish that $\dot{U}_{\mathbb{Z}}^N(\mathfrak{g})$ has a generators and relations presentation.
2. In [section 4](#), we construct a \mathbb{Z} analog of $\Phi_m : \dot{U}_{\mathbb{Z}}(\mathfrak{so}_{2m}) \rightarrow \mathbf{Fund}_{\mathbb{Z}}(O(N))$ and establish that it is an isomorphism. The existence of such an isomorphism was proven in [\[AR96\]](#), but in order to guarantee that Φ_m is compatible with our web diagrammatics, i.e. $\Gamma \circ \Psi_m = \Phi_m$, we provide more details.
3. In [subsection 7.2](#), we provide background on tilting modules for the disconnected group $O(N)$, over \mathbb{F}_{\heartsuit} . This is inspired by [\[AHR20\]](#). We use these results to define and understand $\mathbf{Fund}_{\mathbb{F}_{\heartsuit}}(O(N))$.
4. In [subsection 5.1](#), we construct $\Gamma : \mathbf{Web}_{\mathbb{Z}}(O(N)) \rightarrow \mathbf{Fund}_{\mathbb{Z}}(O(N))$. We then use Elias’ results on type A webs over \mathbb{Z} to prove that the relations for the divided power Chevalley generators in $\dot{U}_{\mathbb{Z}}^N(\mathfrak{gl}_m)$ are satisfied by the ladder webs in $\mathbf{Web}_{\mathbb{Z}}(GL(N))$.
5. In [subsection 3.2](#), we establish the existence of $\Psi_m : \dot{U}_{\mathbb{Z}}(\mathfrak{so}_{2m}) \rightarrow \mathbf{Web}_{\mathbb{Z}}(O(N))$, a \mathbb{Z} analog of Ψ_m from [subsection 1.3.3](#), by checking that the relations satisfied by divided power Chevalley generators in the presentation of $\dot{U}_{\mathbb{A}_{\heartsuit}}(\mathfrak{so}_{2m})$ are satisfied by the ladder webs in $\mathbf{Web}_{\mathbb{Z}}(O(N))$. This uses that we know the relations for $e_i^{(a)}, f_i^{(a)}$ are satisfied by type A ladder diagrams, and the observation that Sartori’s trick to establish relations between orthogonal ladder diagrams work in an identical manner over \mathbb{Z} .
6. The exact same ladderization argument for orthogonal webs over \mathbb{Q} still works over \mathbb{Z} , this establishing surjectivity of Ψ_m .

Remark 1.9. Everything above about type A webs has a quantum analog [CKM14, Eli15, LT21, Bru25]. However, even in characteristic zero, the quantum analog of [ST19] does not describe the monoidal category of representations of quantum $O_N(\mathbb{Q})$. This is because the orthogonal quantum skew Howe duality in [ST19] is for the pair $U'_q(O_N)$ - $U_q(\mathfrak{so}_{2m})$, where $U'_q(O_N)$ is not a Drinfeld–Jimbo quantum group. The nonstandard quantum group $U'_q(O_N)$ is not a Hopf algebra, but is a quantum symmetric pair, i.e. a coideal algebra inside the Drinfeld–Jimbo quantum group $U_q(\mathrm{GL}(N))$.

Later, the web category for the Drinfeld–Jimbo quantum group $U_q(\mathrm{O}(N))$ was constructed in [BW23], which proves an equivalence between quantum analogs of $\mathbf{Web}(O_N(\mathbb{Q}))$ and also $\mathbf{Fund}(O_N(\mathbb{Q}))$. This works only in characteristic zero when q is generic and does not use Howe duality, since the q -Howe duality for the pair $U_q(O_N)$ - $U'_q(\mathfrak{so}_{2m})$ has not been studied.

1.3.5 Semisimplification in type A (known)

Let p be a prime. The results above about type A webs over \mathbb{Z} can be base changed to give an equivalence

$$\mathbf{Web}_{\mathbb{F}_p}(\mathrm{GL}(N)) \rightarrow \mathbf{Fund}_{\mathbb{F}_p}(\mathrm{GL}(N)).$$

Taking the idempotent closure of $\mathbf{Fund}_{\mathbb{F}_p}(\mathrm{GL}(N))$, i.e. including all direct summands as objects, yields the category of tilting modules $\mathbf{Tilt}_{\mathbb{F}_p}(\mathrm{GL}(N))$.

As before, we count $p = 0$ as $p = \infty$ for the remainder of the introduction. The category of tilting modules has a unique semisimple monoidal quotient by the negligible ideal. The irreducible objects in the semisimple quotient correspond to the indecomposable tilting modules with nonzero dimension modulo p . When $p \geq N$, this category is well-studied under the name *Verlinde category*.

A consequence of main Theorem in [BEAEO20] is that for any p , this semisimplified tilting module category for $\mathrm{GL}(N)$ is equivalent to a tensor product of Verlinde categories for $\mathrm{GL}(N_i)$ where N_i is the p^i th term in the p -adic decomposition of N . Having established the connection between type A webs and tilting modules, their argument can be summarized as follows.

1. The tensor product of Verlinde categories for $\mathrm{GL}(N_i)$ is identified with a quotient of the colored oriented Brauer category [BEAEO20, Lemma 3.3]. The colored oriented Brauer category also maps to the semisimplification of $\mathbf{Web}_{\mathbb{F}_p}(\mathrm{GL}(N))$ [BEAEO20, Equation 3.4].
2. A general argument about semisimplification [BEAEO20, Lemma 2.6], along with some facts about exterior powers in characteristic p [BEAEO20, Lemma 3.4], reduces the claimed equivalence to showing that the functor from the colored oriented Brauer category to the semisimplification of the type A web category is full.
3. Motivated by a connection between endomorphism algebras of permutation modules for symmetric groups and type A webs, they find a spanning set of what they call chicken foot diagrams in $\mathbf{Web}_{\mathbb{F}_p}(\mathrm{GL}(N))$ [BEAEO20, Section 4], which we call *few-to-many-to-few (fmf)* diagrams.

4. They prove fullness by showing that the only fmf diagrams which can survive in the the semisimple quotient are in the image of the functor from the colored oriented Brauer category [BEAEO20, Lemma 4.16 and Theorem 4.17].

1.3.6 Semisimplification for orthogonal groups (new)

When $p > 2$, there is an analog of tilting modules for $O(N)$, see [section 7](#) (for the potentially first exposition of this). As before, when $p > N$, the semisimplification of this category is a well-understood Verlinde category for $O(N)$. For $p < N$, the semisimplification was not studied previously in the literature.

Upon establishing the connection between orthogonal webs and tilting modules in [Theorem 1.3](#), we spend [section 6](#) proving [Theorem 1.4](#), that the semisimplification of tilting modules of $\text{Tilt}_{\mathbb{F}_p}(O(N))$ is a tensor product of Verlinde categories for $O(N_i)$, where again N_i is the p^i th term in the p -adic expansion of N . Our arguments mirror the argument in [BEAEO20].

1. We recall the colored Brauer category (no orientation) in [subsection 6.2](#). We relate its semisimple quotient to the tensor product of $O(N_i)$ Verlinde categories in [Lemma 6.20](#), and to $\text{Web}_{\mathbb{F}_p}(O(N))$, see [Proposition 6.10](#) and [Lemma 6.25](#).
2. In [subsection 6.4](#), we use the exact same general arguments about semisimplification and exterior powers in characteristic p to reduce the desired equivalence to showing that the functor from the colored Brauer category to the semisimplification of the orthogonal web category is full.
3. Although there is no longer a connection between something like permutation modules of Brauer diagrams and orthogonal webs, we draw inspiration from the type A chicken foot diagrams and in [subsection 6.3](#), we find an analogous spanning set of fmf diagrams in $\text{Web}_{\mathbb{F}_p}(O(N))$ in [Proposition 6.17](#).
4. We deduce fullness by using [Lemma 6.24](#) to argue that the only fmf diagrams which can be nonzero in the quotient are in the image of the functor from the colored Brauer category.

All of this is new and requires some careful analysis of tilting modules for orthogonal groups which appears in [subsection 7.2](#).

2. Idempotent algebras and type A webs

In this section, we present a mix of well-known material and new, though expected (and well-known to experts), observations such as a presentation of $\dot{U}_{\mathbb{Z}}^N(\mathfrak{g})$ and [Lemma 2.18](#).

2.1. Idempotent version of divided powers form

Notation 2.1. *In this section (and in some sections below) we let $\mathbb{A} = \mathbb{Z}$ and not $\mathbb{A}_{\heartsuit} = \mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ as in e.g. [section 3](#). Note that the field of fractions of \mathbb{A} is \mathbb{Q} . We can go from \mathbb{A} to \mathbb{A}_{\heartsuit} by scalar extension, and this is how we use the results in this section in the rest of the paper.*

Let \mathfrak{g} be a semisimple Lie algebra. Write $\mathbb{Z}\Phi$ for the associated root lattice and X for the integral weight lattice. Fix a choice of simple roots Δ . We consider the *universal enveloping algebra* $U_{\mathbb{Q}} = U_{\mathbb{Q}}(\mathfrak{g})$, viewed as a \mathbb{Q} -algebra, which has a generators and relations presentation called the *Serre presentation*. In this section we recall a presentation of the idempotent \mathbb{A} -form $\dot{U}_{\mathbb{A}}$ of $U_{\mathbb{Q}}$ which is similar in spirit.

Remark 2.2. The main result of this section, the presentation in [Lemma 2.9](#), is well-known but difficult to find explicitly spelled out. So we decided to give details how to get it.

Definition 2.3. The *divided power algebra* $U_{\mathbb{A}} = U_{\mathbb{A}}(\mathfrak{g})$ (for $U_{\mathbb{Q}}$) is the \mathbb{A} -subalgebra of $U_{\mathbb{Q}}$ generated by $e_{\alpha}^{(a)}$, $f_{\alpha}^{(b)}$, and $\binom{h_{\alpha}}{c}$ for all $\alpha \in \Delta$, and $a, b, c \in \mathbb{Z}_{\geq 0}$.

Fix formal variables t and u and define elements in $U_{\mathbb{A}}[[t, u]]$ by

$$e_{\alpha}(u) = \sum_{i \geq 0} e_{\alpha}^{(i)} u^i, \quad f_{\alpha}(u) = \sum_{i \geq 0} f_{\alpha}^{(i)} u^i, \quad h_{\alpha}(u) = \sum_{i \geq 0} \binom{h_{\alpha}}{i} u^i,$$

and similarly with t in place of u .

Lemma 2.4. As an \mathbb{A} -algebra $U_{\mathbb{A}}$ has a presentation with generators $e_{\alpha}^{(a)}$, $f_{\alpha}^{(b)}$, and $\binom{h_{\alpha}}{c}$ for all $\alpha \in \Delta$, and $a, b, c \in \mathbb{Z}_{\geq 0}$, and relations

$$e_{\alpha}^{(0)} = f_{\alpha}^{(0)} = \binom{h_{\alpha}}{0} = 1 \quad \text{for all } \alpha \in \Delta, \tag{2.1}$$

$$\begin{cases} h_{\alpha}(t)h_{\alpha}(u) = h_{\alpha}(t + u + tu) & \text{for all } \alpha \in \Delta, \\ h_{\alpha}(t)h_{\beta}(u) = h_{\beta}(u)h_{\alpha}(t) & \text{for all } \alpha, \beta \in \Delta, \\ h_{\alpha}(t)e_{\beta}(u) = e_{\beta}((1+t)^{\alpha^{\vee}(\beta)}u)h_{\alpha}(t) & \text{for all } \alpha, \beta \in \Delta, \\ h_{\alpha}(t)f_{\beta}(u) = f_{\beta}((1+t)^{-\alpha^{\vee}(\beta)}u)h_{\alpha}(t) & \text{for all } \alpha, \beta \in \Delta, \end{cases} \tag{2.2}$$

$$\{e_{\alpha}(t)e_{\alpha}(u) = e_{\alpha}(t + u), \quad f_{\alpha}(t)f_{\alpha}(u) = f_{\alpha}(t + u) \quad \text{for all } \alpha \in \Delta, \tag{2.3}$$

$$\{e_{\alpha}(t)f_{\alpha}(u) = f_{\alpha}\left(\frac{u}{1+tu}\right)h_{\alpha}(tu)e_{\alpha}\left(\frac{t}{1+tu}\right) \quad \text{for all } \alpha \in \Delta, \tag{2.4}$$

$$\{e_{\alpha}(t)f_{\beta}(u) = f_{\beta}(u)e_{\alpha}(t) \quad \text{for all } \alpha \neq \beta \in \Delta, \tag{2.5}$$

$$\begin{cases} \sum_{p+r=n} (-1)^r e_{\alpha}^{(p)} e_{\beta}^{(m)} e_{\alpha}^{(r)} = 0 & \text{whenever } n > -m \cdot \alpha_i^{\vee}(\beta) \text{ and } \alpha \neq \beta \in \Delta, \\ \sum_{p+r=n} (-1)^r f_{\alpha}^{(p)} f_{\beta}^{(m)} f_{\alpha}^{(r)} = 0 & \text{whenever } n > -m \cdot \alpha_i^{\vee}(\beta) \text{ and } \alpha \neq \beta \in \Delta. \end{cases} \tag{2.6}$$

The relations are interpreted as equalities in $U_{\mathbb{A}}[[t, u]]$.

Proof. This is [[Tak83](#), Corollary 5.2]. □

The rather trivial relation [Equation 2.1](#) is listed for completeness only.

We define the idempotent (universal enveloping) algebra as in [[Lus10](#), Section 23.1]. Note that this idempotent algebra is not unital.

Definition 2.5. Given $K, L \in X$, set

$${}^L U_{\mathbb{Q}}^K = U_{\mathbb{Q}} / \left(\sum_{\alpha \in \Delta} (h_{\alpha} - \alpha^{\vee}(L)) U_{\mathbb{Q}} + \sum_{\alpha \in \Delta} U_{\mathbb{Q}} (h_{\alpha} - \alpha^{\vee}(K)) \right),$$

and let $\mathbf{1}_K$ be the image of 1 under the canonical projection $U_{\mathbb{Q}} \twoheadrightarrow {}^K U_{\mathbb{Q}}^K$. Define the *idempotent enveloping algebra* as

$$\dot{U}_{\mathbb{Q}} = \bigoplus_{L, K \in X} {}^L U_{\mathbb{Q}}^K.$$

We consider $\dot{U}_{\mathbb{Q}}$ as an idempotent \mathbb{Q} -algebra with multiplication inherited from $U_{\mathbb{Q}}$.

Analogous to [Definition 2.3](#) we define:

Definition 2.6. The *divided power idempotent algebra* $\dot{U}_{\mathbb{A}}$ (for $\dot{U}_{\mathbb{Q}}$) is the \mathbb{A} -subalgebra of $\dot{U}_{\mathbb{Q}}$ generated by $e_{\alpha}^{(a)} \mathbf{1}_K, f_{\alpha}^{(a)} \mathbf{1}_K$ for all $\alpha \in \Delta, a \in \mathbb{Z}_{\geq 0}$, and $K \in X$.

Let $\mathfrak{n} = \mathfrak{n}_+$ and \mathfrak{n}_- be the spans of the positive and negative root spaces. Let B denote the *canonical basis* for $U_{\mathbb{Q}}^+ = U_{\mathbb{Q}}(\mathfrak{n})$, see e.g. [\[Lus10, Part IV\]](#). There is an isomorphism $U_{\mathbb{Q}}^+ \cong U_{\mathbb{Q}}^- = U_{\mathbb{Q}}(\mathfrak{n}_-)$, denoted $u^+ = u \mapsto u^-$, which is determined by $e_{\alpha} \mapsto f_{\alpha}$ for all $\alpha \in \Delta$.

Lemma 2.7. *We have*

$$\dot{U}_{\mathbb{A}} = \bigoplus_{b_1, b_2 \in B, K \in X} \mathbb{A} \cdot b_1^- \mathbf{1}_K b_2^+,$$

and the structure constants for the basis B are in \mathbb{A} , i.e. $x \cdot y \in \mathbb{A} \cdot B$ for $x, y \in B$.

Proof. This is [\[Lus10, Section 23.2.2\]](#). □

We have seen two procedures for extending enveloping algebras: adding idempotents and adding divided powers. The divided power idempotent algebra is defined by adding divided powers to the idempotent algebra. Next, we study adding divided powers first, and then adding idempotents.

Definition 2.8. Given $K, L \in X$, set

$${}^L V_{\mathbb{A}}^K = U_{\mathbb{A}} / \left(\sum_{\alpha \in \Delta, i \in \mathbb{Z}_{\geq 0}} \left(\binom{h_{\alpha}}{i} - \binom{\alpha^{\vee}(L)}{i} \right) U_{\mathbb{A}} + \sum_{\alpha \in \Delta, i \in \mathbb{Z}_{\geq 0}} U_{\mathbb{A}} \left(\binom{h_{\alpha}}{i} - \binom{\alpha^{\vee}(K)}{i} \right) \right)$$

and let $\mathbf{1}_K$ be the image of 1 under the canonical projection $V_{\mathbb{A}} \twoheadrightarrow {}^K V_{\mathbb{A}}^K$. Define the *idempotent divided power algebra* to be

$$\dot{V}_{\mathbb{A}} = \bigoplus_{K, L \in X} {}^L V_{\mathbb{A}}^K.$$

Similarly as before, we consider $\dot{V}_{\mathbb{A}}$ as an idempotent \mathbb{A} -algebra with multiplication inherited from $V_{\mathbb{A}}$.

Lemma 2.9. *As an \mathbb{A} -algebra $\dot{V}_{\mathbb{A}}$ has a presentation as an idempotented algebra with generators $e_{\alpha}^{(a)} \mathbf{1}_K$ and $f_{\alpha}^{(a)} \mathbf{1}_K$, for all $K \in X$, $\alpha \in \Delta$, and $a, b \in \mathbb{Z}_{\geq 0}$, and relations*

$$\{ \mathbf{1}_K \mathbf{1}_L = \delta_{K,L} \mathbf{1}_K, \quad e_{\alpha}^{(0)} \mathbf{1}_K = \mathbf{1}_K, \quad f_{\alpha}^{(0)} \mathbf{1}_K = \mathbf{1}_K, \tag{2.7}$$

$$\{ e_{\alpha}^{(a)} e_{\alpha}^{(b)} \mathbf{1}_K = \binom{a+b}{a} e_{\alpha}^{(a+b)} \mathbf{1}_K, \quad f_{\alpha}^{(a)} f_{\alpha}^{(b)} \mathbf{1}_K = \binom{a+b}{a} f_{\alpha}^{(a+b)} \mathbf{1}_K \quad \text{for all } \alpha \in \Delta, \tag{2.8}$$

$$\left\{ e_{\alpha}^{(a)} f_{\alpha}^{(b)} \mathbf{1}_K = \sum_{x \in \mathbb{Z}_{\geq 0}} \binom{\alpha^{\vee(K)} + a - b}{x} f_{\alpha}^{(b-x)} e_{\alpha}^{(a-x)} \mathbf{1}_K \quad \text{for all } \alpha \in \Delta, \tag{2.9}$$

$$\{ e_{\alpha}^{(a)} f_{\beta}^{(b)} \mathbf{1}_K = f_{\beta}^{(b)} e_{\alpha}^{(a)} \mathbf{1}_K \quad \text{for all } \alpha \neq \beta \in \Delta, \tag{2.10}$$

$$\left\{ \begin{aligned} \sum_{p+r=n} (-1)^r e_{\alpha}^{(p)} e_{\beta}^{(m)} e_{\alpha}^{(r)} \mathbf{1}_K &= 0 \quad \text{whenever } n > -m \cdot \alpha_i^{\vee}(\beta) \text{ and } \alpha \neq \beta \in \Delta, \\ \sum_{p+r=n} (-1)^r f_{\alpha}^{(p)} f_{\beta}^{(m)} f_{\alpha}^{(r)} \mathbf{1}_K &= 0 \quad \text{whenever } n > -m \cdot \alpha_i^{\vee}(\beta) \text{ and } \alpha \neq \beta \in \Delta. \end{aligned} \tag{2.11}$$

Proof. This follows from Lemma 2.4. The match between the relations is Equation 2.2 \iff Equation 2.7, Equation 2.3 \iff Equation 2.8, Equation 2.4 \iff Equation 2.9, Equation 2.5 \iff Equation 2.10, Equation 2.6 \iff Equation 2.11. \square

The following gives the promised presentation:

Proposition 2.10. *There is an isomorphism of \mathbb{A} -algebras $\eta: \dot{V}_{\mathbb{A}} \rightarrow \dot{U}_{\mathbb{A}}$ such that $e_{\alpha}^{(a)} \mathbf{1}_K \mapsto e_{\alpha}^{(a)} \mathbf{1}_K$ and $f_{\alpha}^{(a)} \mathbf{1}_K \mapsto f_{\alpha}^{(a)} \mathbf{1}_K$.*

Proof. Thanks to Lemma 2.9, $\dot{V}_{\mathbb{A}}$ has a generators and relations presentation, so the existence of η is a (omitted) relations check. Surjectivity is then immediate. We show η is injective. Let PBW be the \mathbb{A} -basis of $\dot{U}_{\mathbb{A}}$ from [Lus10, Section 23.2.1]. By the same arguments as in [Lus10], the preimage $\eta^{-1}(\text{PBW})$ is an \mathbb{A} -spanning set of $\dot{V}_{\mathbb{A}}$. Since PBW is an \mathbb{A} -basis, it follows that $\eta^{-1}(\text{PBW})$ is \mathbb{A} -linearly independent. Thus, η sends an \mathbb{A} -basis to an \mathbb{A} -basis and is therefore an \mathbb{A} -isomorphism. \square

Notation 2.11. *We use η from Proposition 2.10 to identify $\dot{V}_{\mathbb{A}}$ with $\dot{U}_{\mathbb{A}}$. Thus, we are justified to only refer to $\dot{U}_{\mathbb{A}}$.*

For certain results about Howe duality below, we need to pass from a semisimple Lie algebra to a reductive Lie algebra, namely the general linear group. For $\dot{U}_{\mathbb{A}}(\mathfrak{gl}_N)$, this combines the definition of $\dot{U}_q(\mathfrak{gl}_N)$ in [CKM14, Section 4.1] with the higher Serre relations (2.11). A more sophisticated approach could be developed using [Tak83, Theorem 5.1], but we do not pursue it in this paper.

2.2. A quick recap on type A webs

The following is a condensed reformulation of the webs from [CKM14].

$$\begin{array}{c} k+l \\ \diagup \quad \diagdown \\ k \quad l \\ \diagdown \quad \diagup \\ k+l \end{array} = \binom{k+l}{k} \cdot \begin{array}{c} k+l \\ | \\ k+l \end{array}, \quad \begin{array}{c} r \quad s \\ \diagdown \quad \diagup \\ k \quad l \end{array} = \sum_{\substack{a,b \geq 0 \\ k-a+b=r}} (-1)^{ab} \begin{array}{c} r \quad s \\ | \quad | \\ a \quad b \\ | \quad | \\ k \quad l \end{array}.$$

These are called, in order, *exterior relation*, *associativity*, *coassociativity*, *digon removal*, and (signed) *Schur relation*. The morphisms of $\mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$ are called (*integral type A exterior*) *webs*, while *diagrams* are webs obtained from the generating webs without taking \mathbb{A} -linear combinations.

Lemma 2.14. *The following relations hold in $\mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$.*

1. *The inverse Schur relations:*

$$\begin{array}{c} l \quad k \\ \diagdown \quad \diagup \\ k \quad l \end{array} = (-1)^{kl} \sum_{b-a=k-l} (-1)^{k-b} \begin{array}{c} l \quad k \\ | \quad | \\ a \quad b \\ | \quad | \\ k \quad l \end{array}.$$

2. *Square switch relations, also called the E-F-relation of type A:*

$$\begin{array}{c} k' \quad l' \\ | \quad | \\ a \quad b \\ | \quad | \\ k \quad l \end{array} = \sum_{x \geq 0} \binom{k'-l}{x} \begin{array}{c} k' \quad l' \\ | \quad | \\ b' \quad a' \\ | \quad | \\ k \quad l \end{array}$$

where $k' = k - b + a$, $l' = k - a + b$, $a' = a - x$ and $b' = b - x$.

Proof. The (non-signed) version of these relations are [BEAEO20, 4.26-4.28]. The proof that these non-signed relations are given in [BEAEO20, Appendix] and can easily be adapted to our signed case. \square

Lemma 2.15. *The category $\mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$ together with the crossings is symmetric. We additionally have the following compatibility between crossings and trivalent vertices*

$$\begin{array}{c} k+l \\ \diagup \quad \diagdown \\ k \quad l \\ \diagdown \quad \diagup \\ k+l \end{array} = (-1)^{kl} \begin{array}{c} k+l \\ \diagdown \quad \diagup \\ l \quad k \end{array}.$$

Proof. This can again be proven by adapting proofs from [BEAEO20, Appendix] to our signed case. To show the category is symmetric one must check the braid relations [BEAEO20, 4.32-4.33] and naturality of the braiding, e.g.

[BEAEO20, 4.31]. The claimed additional compatibility is [BEAEO20, 4.30]. \square

We use the following *ladder diagrams*, indicated as a composition of generators for the first one, and their shorthand notations $E_i^{(a)}$ and $F_i^{(a)}$:

$$\begin{aligned}
 E_i^{(a)} 1_{\mathbb{K}} &:= \begin{array}{c} k_i+a \quad k_{i+1}-a \\ | \quad | \\ \text{---} \text{---} \text{---} \\ | \quad | \\ k_i \quad k_{i+1} \end{array} = \left(\begin{array}{c} k_i+a \quad k_{i+1}-a \\ | \quad | \\ \text{---} \text{---} \\ | \quad | \\ k_i \quad a \end{array} \otimes \begin{array}{c} | \\ k_{i+1}-a \end{array} \right) \circ \left(\begin{array}{c} k_i \quad a \quad k_{i+1}-a \\ | \quad | \\ \text{---} \text{---} \\ | \quad | \\ k_i \quad k_{i+1} \end{array} \right), \\
 F_i^{(a)} 1_{\mathbb{K}} &:= \begin{array}{c} k_i-a \quad k_{i+1}+a \\ | \quad | \\ \text{---} \text{---} \\ | \quad | \\ k_i \quad k_{i+1} \end{array},
 \end{aligned} \tag{2.12}$$

where one acts on the i th and $(i+1)$ th entry, and the rest is the identity.

Lemma 2.16. *The E - F relations and the type A Serre relations hold in $\mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$, that is, for all $a \in \mathbb{Z}_{\geq 0}$:*

$$\begin{aligned}
 E_i F_i 1_{\mathbb{K}} &= F_i E_i 1_{\mathbb{K}} + (K_{i-1} + K_i - N) \cdot 1_{\mathbb{K}}, \\
 E_i F_j 1_{\mathbb{K}} &= F_j E_i 1_{\mathbb{K}} \text{ if } i \neq j, \\
 a \cdot E_i^{(a)} 1_{\mathbb{K}} &= E_i^a 1_{\mathbb{K}}, \quad E_i E_j 1_{\mathbb{K}} = E_j E_i 1_{\mathbb{K}} \text{ if } |i-j| \neq 1, \\
 2 \cdot E_i E_j E_i 1_{\mathbb{K}} &= E_i^2 E_j 1_{\mathbb{K}} + E_j E_i^2 1_{\mathbb{K}} \text{ if } |i-j| = 1.
 \end{aligned}$$

Similarly for F s.

Proof. See [CKM14, Lemma 2.2.1 and Proposition 5.2.1]. \square

Notation 2.17. *We consider $\dot{\mathbf{U}}_{\mathbb{A}}(\mathfrak{g})$ as an \mathbb{A} -linear category with the $1_{\mathbb{K}}$ being the objects. In this convention it suffices to indicate one $1_{\mathbb{K}}$ in a given expression and we will use this below. The reader unfamiliar with this is referred to [CKM14, Section 4.1].*

The following statement is known to experts. However, we are not aware of a proof in the literature so we give one.

Lemma 2.18. *There is a full \mathbb{A} -linear functor*

$$\mathcal{H}_{\mathbb{A}}^{gl}(N, m): \dot{U}_{\mathbb{A}}(\mathfrak{gl}_m) \rightarrow \mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$$

such that, in the notation of [Equation 2.12](#),

$$e_i^{(a)} \mathbf{1}_{\mathbb{K}} \mapsto E_i^{(a)} \mathbf{1}_{\mathbb{K}} = \begin{array}{c} k_i+a \quad k_{i+1}-a \\ | \quad | \\ \text{---} a \text{---} \\ | \quad | \\ k_i \quad k_{i+1} \end{array}, \quad f_i^{(a)} \mathbf{1}_{\mathbb{K}} \mapsto F_i^{(a)} \mathbf{1}_{\mathbb{K}} = \begin{array}{c} k_i-a \quad k_{i+1}+a \\ | \quad | \\ \text{---} a \text{---} \\ | \quad | \\ k_i \quad k_{i+1} \end{array}. \quad (2.13)$$

The kernel of this functor is the ideal generated by $\mathbf{1}_{\mathbb{K}}$ with at least one entry $> N$.

Proof. In this proof we also work with \mathbb{Q} , the fraction field of \mathbb{A} .

Consider an injective \mathbb{A} -module map $f_{\mathbb{A}}: \mathbb{A}^r \rightarrow \mathbb{A}^s$, which can be viewed as a matrix with s rows, r columns, and entries in \mathbb{A} . Write $\mathrm{coker}_{\mathbb{A}}$ and $\mathrm{coker}_{\mathbb{Q}}$ for the cokernel of $f_{\mathbb{A}}$ and $\mathbb{Q} \otimes f_{\mathbb{A}}: \mathbb{Q}^r \rightarrow \mathbb{Q}^s$, respectively. The torsion \mathbb{A} -submodule of $\mathrm{coker}_{\mathbb{A}}$ is the set

$$T = \{c \in \mathrm{coker}_{\mathbb{A}} \mid \text{there exists } a \in \mathbb{A} \text{ such that } a \cdot c = 0\}.$$

Claim in Lemma 2.18. We have $\mathrm{coker}_{\mathbb{A}}/T \cong \mathbb{A} \cdot \mathrm{coker}_{\mathbb{Q}}$.

Proof of Claim in Lemma 2.18. There is a homomorphism $g: \mathrm{coker}_{\mathbb{A}} \rightarrow \mathrm{res}_{\mathbb{A}}^{\mathbb{Q}} \mathrm{coker}_{\mathbb{Q}}$, defined by $v + \mathrm{im}(f_{\mathbb{A}}) \mapsto 1 \otimes v + \mathrm{im}(\mathbb{Q} \otimes f_{\mathbb{A}})$. We denote the image by $\mathbb{A} \cdot \mathrm{coker}_{\mathbb{Q}}$.

We claim that $T \subset \ker(g)$. To see this, let $t + \mathrm{im}(f_{\mathbb{A}}) \in T$, so there is non-zero $a \in \mathbb{A}$ such that $a \cdot t = f_{\mathbb{A}}(v)$. Then modulo $\mathrm{im}(\mathbb{Q} \otimes f_{\mathbb{A}})$ we have

$$g(t + \mathrm{im}(f_{\mathbb{A}})) \equiv 1 \otimes t \equiv a^{-1} \otimes a \cdot t \equiv a^{-1} \otimes f_{\mathbb{A}}(v) \equiv a^{-1} \cdot \mathbb{Q} \otimes f_{\mathbb{A}}(1 \otimes v) \equiv 0.$$

It follows that there is a \mathbb{A} -linear map $\bar{g}: \mathrm{coker}_{\mathbb{A}}/T \rightarrow \mathbb{A} \cdot \mathrm{coker}_{\mathbb{Q}}$.

Since \mathbb{A} is a PID, a finitely generated and torsion free \mathbb{A} -module is free. Therefore, one can easily argue that $\mathrm{coker}_{\mathbb{A}}/T$ and $\mathbb{A} \cdot \mathrm{coker}_{\mathbb{Q}}$ are free \mathbb{A} -modules. Finally, we observe that the \mathbb{Q} span of $\mathbb{A} \cdot \mathrm{coker}_{\mathbb{Q}}$ is equal to $\mathrm{coker}_{\mathbb{Q}}$, so $\mathrm{rank}_{\mathbb{A}} \mathbb{A} \cdot \mathrm{coker}_{\mathbb{Q}} = \dim_{\mathbb{Q}} \mathrm{coker}_{\mathbb{Q}} = s - r$. Also, $\mathrm{rank}_{\mathbb{A}} \mathrm{coker}_{\mathbb{A}}/T$ is less than or equal to $s - r$. Thus, surjectivity of g implies injectivity of \bar{g} and we conclude that \bar{g} is an isomorphism $\mathrm{coker}_{\mathbb{A}}/T \cong \mathbb{A} \cdot \mathrm{coker}_{\mathbb{Q}}$. \square (Claim)

In [[CKM14](#), Proposition 5.21], it is shown that the assignments in [Equation 2.13](#) determine a functor

$$\mathcal{H}_{\mathbb{Q}}^{gl}(N, m): \dot{U}_{\mathbb{Q}}(\mathfrak{gl}_m) \rightarrow \mathbf{Web}_{\mathbb{Q}}(\mathrm{GL}(N)).$$

Restricting this functor to $\dot{U}_{\mathbb{A}} \subset \dot{U}_{\mathbb{Q}}$ we obtain a functor

$$\mathcal{H}_{\mathbb{Q}}^{gl}(N, m)|_{\dot{U}_{\mathbb{A}}}: \dot{U}_{\mathbb{A}}(\mathfrak{gl}_m) \rightarrow \mathbf{Web}_{\mathbb{Q}}(\mathrm{GL}(N)).$$

Let $\mathbb{A} \cdot \mathbf{Web}_{\mathbb{Q}}(\mathrm{GL}(N))$ denote the category obtained by the \mathbb{A} -span of coefficient-free diagrams in $\mathbf{Web}_{\mathbb{Q}}(\mathrm{GL}(N))$. The functor $\mathcal{H}_{\mathbb{Q}}^{gl}(N, m)|_{\dot{U}_{\mathbb{A}}}$ has image in $\mathbb{A} \cdot \mathbf{Web}_{\mathbb{Q}}(\mathrm{GL}(N))$.

It follows from [Eli15, Theorem 2.58] that the homomorphism spaces in $\mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$ are free and finitely generated \mathbb{A} -modules. In particular, the homomorphism spaces are torsion free.

It is immediate from the generators and relations description of $\mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$ that there is a functor

$$\mathcal{I}: \mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N)) \rightarrow \mathbb{A} \cdot \mathbf{Web}_{\mathbb{Q}}(\mathrm{GL}(N))$$

that sends diagrams to their diagrammatic counterparts in $\mathbb{A} \cdot \mathbf{Web}_{\mathbb{Q}}(\mathrm{GL}(N))$, and therefore is full. ‘Claim in Lemma 2.18’ implies that \mathcal{I} is an equivalence.

The composition

$$\mathcal{I}^{-1} \circ \mathcal{H}_{\mathbb{Q}}^{gl}(N, m)|_{\dot{U}_{\mathbb{A}}} : \dot{U}_{\mathbb{A}}(\mathfrak{gl}_m) \rightarrow \mathbb{A} \cdot \mathbf{Web}_{\mathbb{Q}}(\mathrm{GL}(N)) \rightarrow \mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$$

is the desired functor $\mathcal{H}_{\mathbb{A}}^{gl}(N, m)$.

Finally, in (a \mathfrak{gl}_m version of) [CKM14, Theorem 5.3.1], the authors argue that $\mathcal{H}_{\mathbb{Q}}^{gl}(N, m)$ is full by showing any web can be rewritten as a composition of $E^{(a)}$ and $F^{(a)}$ webs. This claim is still true over \mathbb{A} , and therefore $\mathcal{H}_{\mathbb{A}}^{gl}(N, m)$ is full. \square

Lemma 2.18 implies that the higher type A Serre relations hold in $\mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$. In fact, as far as we are aware, the following is not entirely explicit in [CKM14] and related literature:

Lemma 2.19. Higher E - F relations and higher type A Serre relations hold in $\mathbf{Web}_{\mathbb{A}}(\mathrm{GL}(N))$, that is, for all $a, b \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} E_i^{(a)} F_i^{(b)} 1_{\mathbb{K}} &= \sum_{x \in \mathbb{Z}_{\geq 0}} \binom{k_i - k_{i+1} + a - b}{x} \cdot F_i^{(b-x)} E_i^{(a-x)} 1_{\mathbb{K}}, \\ E_i^{(a)} F_j^{(b)} 1_{\mathbb{K}} &= F_j^{(a)} E_i^{(b)} 1_{\mathbb{K}} \text{ if } i \neq j, \\ \binom{a+b}{a} \cdot E_i^{(a)} E_i^{(b)} 1_{\mathbb{K}} &= E_i^{(a+b)} 1_{\mathbb{K}}, \quad E_i^{(a)} E_j^{(b)} 1_{\mathbb{K}} = E_j^{(b)} E_i^{(a)} 1_{\mathbb{K}} \text{ if } |i - j| \neq 1, \\ \sum_{p+r=n} E_i^{(p)} E_j^{(m)} E_i^{(r)} 1_{\mathbb{K}} &= 0 \text{ if } |i - j| = 1 \text{ and } n > m. \end{aligned}$$

Similarly for F s.

Proof. We know that the relation in Equation 2.11 holds in $\dot{U}_{\mathbb{A}}(\mathfrak{gl}_m)$ by Proposition 2.10. The claim then follows from applying the functor $\mathcal{H}_{\mathbb{A}}^{gl}(N, m)$ defined in Lemma 2.18 to these relations. \square

Example 2.20. The relation $E_i E_{i+1} E_i = E_i^{(2)} E_{i+1} + E_{i+1} E_i^{(2)}$, in terms of diagrams, is

and $E_i^{(a)} E_j^{(b)} = E_j^{(b)} E_i^{(a)}$, for $|i - j| > 1$ follows from far commutativity illustrated in Notation 2.12.

3. Orthogonal diagrammatics

Throughout, our choices of the ground ring for orthogonal groups are:

- (i) The ring $\mathbb{A}_\vartheta = \mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ for integral results (the appearance of $\frac{1}{2}$ is probably not surprising, and see [Remark 4.26](#) for why we need $\sqrt{-1}$). Working *integrally* means working over \mathbb{A}_ϑ .
- (ii) The field \mathbb{F}_ϑ over \mathbb{A}_ϑ whenever we need to avoid torsion. We assume that \mathbb{F}_ϑ is infinite (to avoid using group schemes).
- (iii) The field $\mathbb{Q}_\vartheta = \mathbb{Q}(\sqrt{-1})$, the fraction field of \mathbb{A}_ϑ , whenever we need characteristic zero. Note that \mathbb{Q}_ϑ is a special case of a field \mathbb{F}_ϑ .

Note that we can always scalar extend \mathbb{A}_ϑ -linear categories and functors from \mathbb{A}_ϑ to \mathbb{F}_ϑ (and hence, to \mathbb{Q}_ϑ). Our notation to indicate this will be as in the following example: an \mathbb{A}_ϑ -linear functor $\mathcal{E}_{\mathbb{A}_\vartheta}^{\text{GL} \rightarrow \text{O}} : \mathbf{Web}_{\mathbb{A}_\vartheta}(\text{GL}(N)) \rightarrow \mathbf{Web}_{\mathbb{A}_\vartheta}(\text{O}(N))$ scalar extends to an \mathbb{F}_ϑ -linear functor $\mathcal{E}_{\mathbb{F}_\vartheta}^{\text{GL} \rightarrow \text{O}} : \mathbf{Web}_{\mathbb{F}_\vartheta}(\text{GL}(N)) \rightarrow \mathbf{Web}_{\mathbb{F}_\vartheta}(\text{O}(N))$.

3.1. Integral orthogonal webs

The category of *integral orthogonal (exterior) webs* $\mathbf{Web}_{\mathbb{A}_\vartheta}(\text{O}(N))$ is defined as:

Definition 3.1. Let $\mathbf{Web}_{\mathbb{A}_\vartheta}(\text{O}(N))$ denote the pivotal \mathbb{A}_ϑ -linear category \otimes -generated by objects $k \in \mathbb{Z}_{\geq 0}$ with $0 = \mathbb{1}$ and $k^* = k$, and \circ - \otimes -generated by morphisms

$$\begin{array}{ccc}
 \begin{array}{c} k+l \\ | \\ \text{Y} \\ | \\ k \quad l \end{array} : k \otimes l \rightarrow k+l, &
 \begin{array}{c} k \quad l \\ \text{Y} \\ | \\ k+l \end{array} : k+l \rightarrow k \otimes l, &
 \begin{array}{c} l \quad k \\ \text{X} \\ k \quad l \end{array} : k \otimes l \rightarrow l \otimes k,
 \end{array}$$

called *merge*, *split*, and *crossing*, subject to the relations \circ - \otimes -generated by the morphisms depicted below. To state the relations we display the pivotal structure using *caps and cups*:

$$\begin{array}{cc}
 \mathbb{1} & k \quad k \\
 \text{Cup} & \text{Cup} \\
 : k \otimes k \rightarrow \mathbb{1}, & : \mathbb{1} \rightarrow k \otimes k.
 \end{array}$$

The relations are:

1. The (exterior) type A web relations, *exterior relation*, *associativity*, *coassociativity*, *digon removal*, and (signed) *Schur relation* as depicted in [Definition 2.13](#).

To state the remaining relations, we recall that we can use the caps and cups to rotate merges and splits, e.g.:

$$\begin{array}{c} l \\ \diagdown \quad \diagup \\ k \quad k+l \end{array} = \begin{array}{c} \text{cap} \\ k \quad k \end{array} \quad \circ \quad \begin{array}{c} l \quad k \\ \diagup \quad \diagdown \\ k+l \end{array} .$$

2. Circle removal and lollipop relations depicted as

$$k \bigcirc = \binom{N}{k}, \quad k-a \begin{array}{c} l \\ \diagdown \quad \diagup \\ a \\ k \end{array} = 0 \text{ for } k > l.$$

3. Higher even orthogonal E-F relations depicted as

$$\begin{array}{c} k+a-b \quad l+a-b \\ b \\ \text{cap} \\ a \\ k \quad l \end{array} = \sum_{x \in \mathbb{Z}_{\geq 0}} \binom{k+l-N+a-b}{x} \cdot \begin{array}{c} k+a-b \quad l+a-b \\ a-x \\ \text{cup} \\ b-x \\ k \quad l \end{array} .$$

The morphisms of $\mathbf{Web}_{\mathbb{A}_\nabla}(O(N))$ are called (*integral orthogonal exterior*) webs, while *diagrams* are webs obtained from the generating webs without taking \mathbb{A}_∇ -linear combinations. We also use the notation for objects as in [Notation 2.12](#).

Example 3.2. For $l = 0$ and k even the lollipop relation becomes

$$\begin{array}{c} k/2 \\ \text{lollipop} \\ k \end{array} = 0.$$

The name lollipop relation originates in this picture. The representation theoretical interpretation of this relation and of the digon removal is a nonsemisimple version of *Schur's lemma* which holds over \mathbb{A}_∇ . Indeed, the reader can compare [Lemma 7.52](#) with the usual well-known results in the theory, e.g. [\[APW91, Corollary 7.4\]](#) or [\[AT17, Remark 2.29\]](#).

Recall $\mathbf{Web}_{\mathbb{A}_\nabla}(GL(N))$ as in [Definition 2.13](#). The defining relations of $\mathbf{Web}_{\mathbb{A}_\nabla}(GL(N))$ are a subset of those in $\mathbf{Web}_{\mathbb{A}_\nabla}(O(N))$.

Definition 3.3. There is a monoidal \mathbb{A}_\heartsuit -linear functor

$$\mathcal{E}_{\mathbb{A}_\heartsuit}^{\text{GL} \rightarrow \text{O}}: \mathbf{Web}_{\mathbb{A}_\heartsuit}(\text{GL}(N)) \rightarrow \mathbf{Web}_{\mathbb{A}_\heartsuit}(\text{O}(N))$$

sending k to k and diagrams in $\mathbf{Web}_{\mathbb{A}_\heartsuit}(\text{GL}(N))$ to their counterparts in $\mathbf{Web}_{\mathbb{A}_\heartsuit}(\text{O}(N))$.

In particular, all relations listed in [subsection 2.2](#) hold in $\mathbf{Web}_{\mathbb{A}_\heartsuit}(\text{O}(N))$ as well. We will use this (mostly) silently throughout.

Proposition 3.4. *The functor $\mathcal{E}_{\mathbb{A}_\heartsuit}^{\text{GL} \rightarrow \text{O}}$ is faithful.*

Proof. Proven in [subsection 5.3](#). □

Remark 3.5. We will not use [Proposition 3.4](#) until we prove it. For now, we are establishing results needed to prove it.

Recall the notion of a *ribbon category* from, for example, [\[EGNO15, Section 8.10\]](#). By definition, any ribbon category is braided monoidal and pivotal.

Lemma 3.6. *The category $\mathbf{Web}_{\mathbb{A}_\heartsuit}(\text{O}(N))$ together with the crossings is symmetric. Moreover, $\mathbf{Web}_{\mathbb{A}_\heartsuit}(\text{O}(N))$ together with this symmetric and pivotal structure is ribbon. We additionally have the Reidemeister I relations:*

(Note that being symmetric also implies the Reidemeister II and III relations.)

Proof. Because of [Lemma 2.15](#) and the functor from [Definition 3.3](#), we just need to derive relations involving the compatibility of cups and caps with the braiding, e.g. the Reidemeister I relation which can be proven by using

and then by applying [Lemma 2.14](#). Note that the naturality of the braiding with respect to the cups and caps follows from being able to rotate diagrams and the Reidemeister II relation. □

Using the Reidemeister I relation from [Lemma 3.6](#), we can derive the following relation.

3.2. Diagrammatic orthogonal Howe duality

Recall $\dot{U}_{\mathbb{A}_\nabla}(\mathfrak{g})$ as in subsection 2.1. In this section we will prove the analog of Lemma 2.18, replacing $\dot{U}_{\mathbb{A}_\nabla}(\mathfrak{gl}_m) \rightarrow \mathbf{Web}_{\mathbb{A}_\nabla}(\mathrm{GL}(N))$ with $\dot{U}_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m}) \rightarrow \mathbf{Web}_{\mathbb{A}_\nabla}(\mathrm{O}(N))$.

To simplify notation, we continue to use $E_i^{(a)}$ and $F_i^{(a)}$ to denote diagrams as in Equation 2.12, viewed as diagrams in $\mathbf{Web}_{\mathbb{A}_\nabla}(\mathrm{O}(N))$. In the same spirit we write

$$\begin{aligned}
 e_i^{(a)} 1_K &:= \begin{array}{c} k_i+a \quad k_{i+1}+a \\ \text{---} \text{---} \\ \text{---} \text{---} \\ k_i \quad k_{i+1} \end{array} = \left(\begin{array}{c} k_i+a \quad k_{i+1}+a \\ \text{---} \text{---} \\ \text{---} \text{---} \\ k_i \quad a \quad a \quad k_{i+1} \end{array} \right) \circ \left(\begin{array}{c} k_i \quad a \quad a \quad k_{i+1} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ k_i \quad \mathbb{1} \quad k_{i+1} \end{array} \right), \\
 f_i^{(a)} 1_K &:= \begin{array}{c} k_i-a \quad k_{i+1}-a \\ \text{---} \text{---} \\ \text{---} \text{---} \\ k_i \quad k_{i+1} \end{array}.
 \end{aligned} \tag{3.1}$$

Here, for clarity, we illustrated how $e_i^{(a)}$ is obtained from the generating morphisms.

The relations in the following Lemma hold in $\mathbf{Web}_{\mathbb{A}_\nabla}(\mathrm{O}(N))$. Combined with the relations we proved among the E_i and F_i , these relations are sufficient to prove that there is a functor $\dot{U}_{\mathbb{Q}_\nabla}(\mathfrak{so}_{2m}) \rightarrow \mathbf{Web}_{\mathbb{Q}_\nabla}(\mathrm{O}(N))$.

Lemma 3.11. *We have the following in $\mathbf{Web}_{\mathbb{A}_\nabla}(\mathrm{O}(N))$.*

1. *The even orthogonal E-F relations, that is:*

$$\begin{aligned}
 e_i f_i 1_K &= f_i e_i 1_K + (K_{i-1} + K_i - N) \cdot 1_K, \\
 E_i f_m 1_K &= f_m E_i 1_K \quad \text{and} \quad F_i e_m 1_K = e_m F_i 1_K.
 \end{aligned}$$

2. *The even orthogonal Serre relations hold, that is:*

$$\begin{aligned}
 e_i E_j 1_K &= E_j e_i 1_K \text{ if } |i - j| \neq 1, \\
 2 \cdot e_i E_j e_i 1_K &= e_i^2 E_j 1_K + E_j e_i^2 1_K \text{ if } |i - j| = 1, \\
 2 \cdot E_i e_j E_i 1_K &= E_i^2 e_j 1_K + E_j e_i^2 1_K \text{ if } |i - j| = 1.
 \end{aligned}$$

Similarly for f s and F s.

Proof. 1. The first relation follows by definition. The others are easy to prove using e.g. associativity.

2. We leave it as an easy exercise. For hints, compare to [ST19, Lemmas 3.9 to 3.14]. □

Using the associativity and bigon relations, it is easy to establish the *orthogonal divided power relations* in $\mathbf{Web}_{\mathbb{A}_\nabla}(\mathcal{O}(N))$, that is for all $a, b \in \mathbb{Z}_{\geq 0}$:

$$a! \cdot e_i^{(a)} \mathbf{1}_K = e_i^a \mathbf{1}_K \quad \text{and} \quad e_i^{(a)} e_i^{(b)} \mathbf{1}_K = \binom{a+b}{a} \cdot e_i^{(a+b)} \mathbf{1}_K,$$

and similarly for f s. In order to prove there is a functor $\dot{U}_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m}) \rightarrow \mathbf{Web}_{\mathbb{A}_\nabla}(\mathcal{O}(N))$, it remains to show the following relations hold.

Lemma 3.12. *We have the following in $\mathbf{Web}_{\mathbb{A}_\nabla}(\mathcal{O}(N))$.*

1. *The higher even orthogonal E-F relations, that is, for all $a, b \in \mathbb{Z}_{\geq 0}$:*

$$e_i^{(a)} f_i^{(b)} \mathbf{1}_K = \sum_{x \in \mathbb{Z}_{\geq 0}} \binom{K_{i-1} + K_i - N + a - b}{x} \cdot f_i^{(b-x)} e_i^{(a-x)} \mathbf{1}_K.$$

2. *The higher even orthogonal Serre relations hold*

$$\begin{aligned} e_i^{(a)} E_j^{(b)} \mathbf{1}_K &= E_j^{(b)} e_i^{(a)} \mathbf{1}_K \text{ if } |i - j| \neq 1, \\ \sum_{p+r=n} e_i^{(p)} E_j^{(m)} e_i^{(r)} \mathbf{1}_K &= 0 \text{ if } |i - j| = 1 \text{ and } n > m, \\ \sum_{p+r=n} E_i^{(p)} e_j^{(m)} E_i^{(r)} \mathbf{1}_K &= 0 \text{ if } |i - j| = 1 \text{ and } n > m. \end{aligned}$$

Similarly for f s and F s.

Proof. 1. By definition.

2. We use Sartori's trick as in [ST19, Sections 3 and 4] to prove all relations inductively by using sequences of type A relations. The first relation follows easily from far commutativity in $\mathbf{Web}_{\mathbb{A}_\nabla}(\mathcal{O}(N))$. It is explained in the proofs of [ST19, Lemma 3.13, Lemma 3.14] that for ladder web diagrams, the remaining two even orthogonal Serre relations are a consequence of the type A Serre relations. Since the type A web relations hold in $\mathbf{Web}_{\mathbb{A}_\nabla}(\mathcal{O}(N))$, it follows from Lemma 2.19 that the higher type A Serre relations hold in $\mathbf{Web}_{\mathbb{A}_\nabla}(\mathcal{O}(N))$ as well. Thus, the higher even orthogonal Serre relations follows from a completely analogous calculation to the proofs of [ST19, Lemma 3.13, Lemma 3.14]. \square

Let $\mathbf{I}_{>N}$ be the \circ - \otimes -ideal in $\dot{U}_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$ generated by $\mathbf{1}_K$ with at least one entry $> N$.

Proposition 3.13. *(Diagrammatic Howe duality.) There is a full \mathbb{A}_∇ -linear functor*

$$\mathcal{H}_{\mathbb{A}_\nabla}^{\text{so}}(N, m): \dot{U}_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m}) \rightarrow \mathbf{Web}_{\mathbb{A}_\nabla}(\mathcal{O}(N))$$

such that Equation 2.13 holds for $i \neq m$ and also, in the notation of Equation 3.1,

$$e_m^{(a)} \mathbf{1}_K \mapsto e_m^{(a)} \mathbf{1}_K = \begin{array}{c} k_{m-1}+a \quad k_m+a \\ \text{---} \quad \text{---} \\ \quad \cup \\ \quad \text{---} \\ \quad a \\ \quad \text{---} \\ k_{m-1} \quad k_m \end{array}, \quad f_m^{(a)} \mathbf{1}_K \mapsto f_m^{(a)} \mathbf{1}_K = \begin{array}{c} k_{m-1}-a \quad k_m-a \\ \text{---} \quad \text{---} \\ \quad \cap \\ \quad \text{---} \\ \quad a \\ \quad \text{---} \\ k_{m-1} \quad k_m \end{array}.$$

The kernel of $\mathcal{H}_{\mathbb{A}_\nabla}^{\text{so}}(N, m)$ contains $\mathbf{I}_{>N}$, and the kernel of $\mathcal{H}_{\mathbb{F}_\nabla}^{\text{so}}(N, m)$ is spanned by $\mathbf{I}_{>N}$.

Proof of Proposition 3.13 excluding the identification of the kernel. To show existence, it follows from Lemma 2.9 that it suffices to check that the $\dot{U}_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$ relations are satisfied by $E_i^{(a)}$, $F_i^{(a)}$, $e_m^{(i)}$, and $f_m^{(i)}$ in $\mathbf{Web}_{\mathbb{A}_\nabla}(O(N))$. The most interesting case is the higher Serre relations, which hold thanks to Lemma 2.19 (interpreted in $\mathbf{Web}_{\mathbb{A}_\nabla}(O(N))$) and Lemma 3.12.

To see fullness we can copy [CKM14, Proof of Theorem 5.3.1] as follows. We rewrite

$$\begin{array}{c} v-a \quad w-a \\ | \quad | \\ \text{---} \text{---} \\ | \quad | \\ v \quad w \end{array} = \begin{array}{c} v-a \\ | \\ v-a \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ a \quad a \end{array} \begin{array}{c} w-a \\ | \\ w-a \end{array} \circ \left(\begin{array}{c} v-a \quad a \\ \diagdown \quad \diagup \\ v \end{array} \otimes \begin{array}{c} a \quad w-a \\ \diagdown \quad \diagup \\ w \end{array} \right),$$

and similarly for the $e_m^{(a)} \mathbf{1}_K$ s. Now apply the strategy in [CKM14, Proof of Theorem 5.3.1].

The statement about containing the \circ - \otimes -ideal generated by $\mathbf{1}_K$ with at least one entry $> N$ follows by construction. □

The remaining statement in Proposition 3.13 concerning identification of the kernel will be proven at the end of subsection 5.3 below.

3.3. Redundancy of some relations

The relations in Definition 3.1.(c) may all be redundant. We were able to prove the following two lemmas, showing that some of them are redundant.

Lemma 3.14. *For $a = b = 1$, the higher even orthogonal E-F relations become:*

$$\begin{array}{c} k \quad l \\ | \quad | \\ \text{---} \text{---} \\ | \quad | \\ k \quad l \end{array} = (k + l - N) \cdot \begin{array}{c} k \quad l \\ | \quad | \\ \text{---} \text{---} \\ | \quad | \\ k \quad l \end{array} + \begin{array}{c} k \quad l \\ | \quad | \\ | \quad | \\ k \quad l \end{array}. \tag{3.2}$$

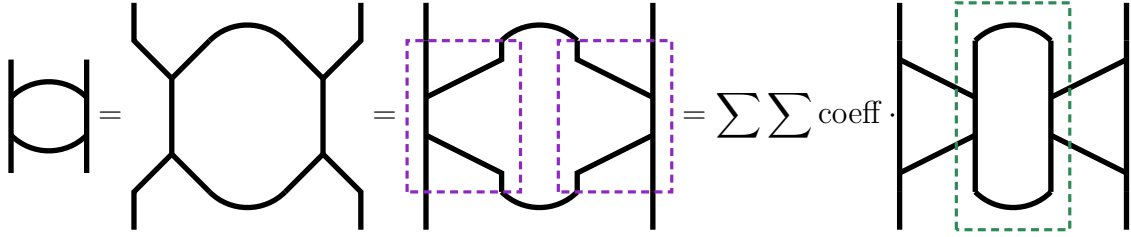
This relation is a consequence of the other (not higher even orthogonal E-F) relations.

Proof. To see why, observe that the diagram on the left-hand side of Equation 3.2 contains two merge-split subdiagrams. Applying the (signed) Schur relation to each merge-split results in a sum of four diagrams. Applying the Reidemeister II relation to one summand results in the ef diagram on the right hand side of Equation 3.2. Applying circle removal to another summand results in $-N$ times the identity diagram. Applying Reidemeister I and then sideways digon relations to the remaining two summands, results in $k + l$ times the identity diagram. □

Lemma 3.15. *If one only assumes the higher even orthogonal E-F relations for the values $k, l \leq \min\{a, b\}$, then they follow in general.*

Proof. Below we will suppress coefficients and labels to highlight the main steps. We again use a version of Sartori’s trick.

The case $a = b = 0$ is trivial. To see how this can be verified when $a = b = 1$, see [ST19, Lemma 3.9]. A similar argument works when $\min\{a, b\} = 1$. For $\min\{a, b\} > 1$ we write



where the third step uses Lemma 2.14.(b). Now, the marked piece in the right-hand diagram is a higher even orthogonal E - F relation diagram where $k, l \leq \min\{a, b\}$. \square

Remark 3.16. A similar argument as in the proof of Lemma 3.14 above should work when $\min\{a, b\} = 1$. That there is such an argument suggests the higher even orthogonal E - F relations may follow from the other orthogonal web relations. For example, we convinced ourselves that the case $a = b = 2$ also follows from the other relations, and the argument seems to work in general. However, the calculation was rather complicated, and we were unable to definitively prove that the relations in Definition 3.1.(c) are redundant.

4. Howe's action integrally

Let us denote by $\Lambda^*(-)$ the *exterior algebra*.

Notation 4.1. To avoid clutter we will omit the \wedge and write xy instead of $x \wedge y$.

We study the free $\mathbb{A}_\triangleright$ -module $\Lambda^*(V \otimes \mathbb{A}_\triangleright^m)$ for $m \geq 0$, where V is a free $\mathbb{A}_\triangleright$ -module of rank $\text{rank}_{\mathbb{A}_\triangleright} V = N$. Thus, we have two crucial (fixed) numbers $N, m \in \mathbb{Z}_{\geq 1}$:

- (i) N is the rank of the left space in $V \otimes \mathbb{A}_\triangleright^m$, or equivalently as we will see later, the *maximal thickness* (of horizontal cuts) of strands in $\mathbf{Web}_{\mathbb{A}_\triangleright}(O(N))$;
- (ii) m is the rank of the right space in $V \otimes \mathbb{A}_\triangleright^m$, or equivalently as we will see later, the *total thickness* of strands in $\mathbf{Web}_{\mathbb{A}_\triangleright}(O(N))$.

Here, and throughout, $O(N) = O(V)$.

Notation 4.2. The notation that we will use for $O(N)$ is adapted to the diagram $\bullet \dashrightarrow \bullet \dashrightarrow \bullet$, if N is odd, or $\bullet \dashrightarrow \bullet \curvearrowright$, if N is even, where we have N nodes.

4.1. An $U_{\mathbb{A}_\triangleright}(\mathfrak{so}_{2m})$ -action on $\Lambda^*(\mathbb{A}_\triangleright^m)$

We need some notation:

Notation 4.3. We fix the following.

1. Write $[i, j] = \{i, i + 1, \dots, j\}$ for $i \leq j \in \mathbb{Z}$, and $\square_{N \times m} = [1, N] \times [1, m]$. In this notation, N indexes rows and m indexes columns in illustrations.
2. Actions are always left actions.

As before, we specify more notation as we go.

Any basis of $\Lambda^*(V \otimes \mathbb{A}_\nabla^m)$ is indexed by subsets of $[1, N] \times [1, m]$.

Suppose $N = 1$, and consider the \mathbb{A}_∇ -module $\Lambda^*(V \otimes \mathbb{A}_\nabla^m) \cong \Lambda^*(\mathbb{A}_\nabla^m)$. Write $\{x_1, \dots, x_m\}$ for the standard basis of \mathbb{A}_∇^m . Given a subset $S \subset [1, m]$, such that $S = \{s_1, \dots, s_k\}$ and $s_1 < \dots < s_k$, we write $x_S = x_{s_1} \dots x_{s_k}$, and get the well-known lemma:

Lemma 4.4. The set $\{x_S | S \subset [1, m]\}$ is an \mathbb{A}_∇ -basis of $\Lambda^*(\mathbb{A}_\nabla^m)$. □

As usual, there are differential operators

$$\begin{aligned} x_i : \Lambda^*(\mathbb{A}_\nabla^m) &\rightarrow \Lambda^*(\mathbb{A}_\nabla^m), & x_S &\mapsto x_i \cdot x_S = \begin{cases} (-1)^{|S \cap [1, i-1]|} \cdot x_{S \cup \{i\}} & \text{if } i \notin S, \\ 0 & \text{if } i \in S, \end{cases} \\ \partial_i : \Lambda^*(\mathbb{A}_\nabla^m) &\rightarrow \Lambda^*(\mathbb{A}_\nabla^m), & x_S &\mapsto \partial_i \cdot x_S = \begin{cases} (-1)^{|S \cap [1, i-1]|} \cdot x_{S \setminus \{i\}} & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases} \end{aligned} \tag{4.1}$$

We have the Leibniz rule $x_i \circ \partial_j + \partial_j \circ x_i = \delta_{i,j}$, as one directly verifies.

Remark 4.5. The operators in Equation 4.1 generate an action of the Clifford algebra

$$\text{Cl}(\mathbb{A}_\nabla^m \oplus (\mathbb{A}_\nabla^m)^*, (-, -)), \quad \text{where } (x_i, x_j^*) = \delta_{ij},$$

on the spin representation.

Let $a, b \in \{1, \dots, m\}$. Consider the operators on $\Lambda^*(\mathbb{A}_\nabla^m)$ given by

$$e_{a,b}^- = \partial_a \circ \partial_b, \quad e_{a,b}^o = x_a \circ \partial_b - \delta_{a,b} \frac{1}{2}, \quad e_{a,b}^+ = x_a \circ x_b.$$

Note that $e_{a,b}^\pm = -e_{b,a}^\pm$ and $e_{a,a}^\pm = 0$.

Definition 4.6. We write

$$\begin{aligned} e_i &= e_{i,i+1}^o, & f_i &= e_{i+1,i}^o, & h_i &= e_{i,i}^o - e_{i+1,i+1}^o & \text{for } i \in [1, m-1], \\ e_m &= e_{m-1,m}^+, & f_m &= e_{m,m-1}^-, & h_m &= e_{m-1,m-1}^o + e_{m,m}^o. \end{aligned}$$

These are called *Howe operators* for \mathfrak{so}_{2m} (with $\mathfrak{so}_{2m} = \mathfrak{so}_{2m}(\mathbb{Q}_\nabla)$).

Remark 4.7. The name in Definition 4.6 comes from Howe’s discussion on the skew-symmetric FFT of invariant theory in [How95, Section 4].

Notation 4.8. Our notation for \mathfrak{so}_{2m} is adapted to the diagram $\bullet \cdots \bullet \leftarrow \bullet$ with m nodes.

In particular, for later use we specify some root conventions. Let $\mathbb{Z}\Phi \subset \bigoplus_{i=1}^m \mathbb{Z} \frac{\epsilon_i}{2}$ be the root lattice for \mathfrak{so}_{2m} . We choose the simple roots by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i \in [1, m-1]$, and $\alpha_m = \epsilon_{m-1} + \epsilon_m$. Let $X(\mathfrak{so}_{2m}) \subset \bigoplus_{i=1}^m \mathbb{Z} \frac{\epsilon_i}{2}$ be the set of integral weights for \mathfrak{so}_{2m} . Then $X(\mathfrak{so}_{2m})_+ = \bigoplus_{i=1}^m \mathbb{Z} \varpi_i$, where

$$\begin{aligned} \varpi_1 &= \epsilon_1, \\ \varpi_2 &= \epsilon_1 + \epsilon_2, \\ &\vdots \\ \varpi_{m-2} &= \epsilon_1 + \cdots + \epsilon_{m-2}, \\ \varpi_{m-1} &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{m-1} - \epsilon_m), \\ \varpi_m &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{m-1} + \epsilon_m), \end{aligned}$$

with $i \in [1, m-2]$. Let moreover $W = \langle s_1, \dots, s_m \rangle$ be the Weyl group of \mathfrak{so}_{2m} generated by the simple reflections corresponding to our choice of simple roots.

Lemma 4.9. The operators e_i , f_i and h_i act on $\Lambda^*(\mathbb{A}_{\diamond}^m)$ by:

$$\begin{aligned} e_i \cdot x_S &= \begin{cases} x_{(S \setminus \{i+1\}) \cup \{i\}} & \text{if } i+1 \in S, i \notin S, \\ 0 & \text{else,} \end{cases} & f_i \cdot x_S &= \begin{cases} x_{(S \setminus \{i\}) \cup \{i+1\}} & \text{if } i \in S, i+1 \notin S, \\ 0 & \text{else,} \end{cases} \\ h_i \cdot x_S &= \begin{cases} x_S & \text{if } S \cap \{i, i+1\} = \{i\}, \\ -x_S & \text{if } S \cap \{i, i+1\} = \{i+1\}, \\ 0 & \text{if } S \cap \{i, i+1\} = \emptyset \text{ or } \{i, i+1\}, \end{cases} \\ e_m \cdot x_S &= \begin{cases} x_{S \cup \{m-1, m\}} & \text{if } m-1, m \notin S, \\ 0 & \text{else,} \end{cases} & f_m \cdot x_S &= \begin{cases} x_{S \setminus \{m-1, m\}} & \text{if } m-1, m \in S, \\ 0 & \text{else,} \end{cases} \\ h_m \cdot x_S &= \begin{cases} x_S & \text{if } S \cap \{m, m+1\} = \{m, m+1\}, \\ -x_S & \text{if } S \cap \{m, m+1\} = \emptyset, \\ 0 & \text{if } S \cap \{m, m+1\} = \{m\} \text{ or } \{m+1\}. \end{cases} \end{aligned}$$

In the first two displays we have $i \in [1, m-1]$.

Proof. Following for example [BT25], a nice way to think about the action is as follows. First, we imagine a row with m nodes, where each node is empty or filled with one dot. These correspond to $S \subset [1, m]$ by, for example,

$$S = \{1, 5, 6 = m\} \leftrightarrow \begin{array}{|c|c|c|c|c|c|} \hline \bullet & & & & \bullet & \bullet \\ \hline \end{array}.$$

In this notation, the operator x_i adds a dot in the i th box, or acts as zero if there is a dot in the box, up to a factor of -1 for each dot to the left of the i th box. The operator ∂_i removes a dot from the i th box, or acts by zero if the box is empty, up to a factor of -1 for each dot to the left

of the i th box. Using this notation the lemma is easy to verify, and we only give an example. For $m = 6$ and $S = \{1, 3, 4\}$ we have

$$e_m = (x_{m-1} \circ x_m) \cdot x_1 x_3 x_4 = x_1 x_3 x_4 x_5 x_6$$

$$\rightsquigarrow e_m \cdot \begin{array}{|c|c|c|c|c|c|} \hline \bullet & & \bullet & \bullet & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline \bullet & & \bullet & \bullet & \bullet & \bullet \\ \hline \end{array}.$$

All other necessary calculations are similar. □

The graphical notation in the above proof is called a *dot diagram*. We will use these throughout and identify S with such diagrams.

Example 4.10. Here is an explicit example:

$$f_1 \cdot x_{\{1,5,6\}} = x_{\{2,5,6\}} \rightsquigarrow f_1 \cdot \begin{array}{|c|c|c|c|c|c|} \hline \bullet & & & & \bullet & \bullet \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline & \bullet & & & \bullet & \bullet \\ \hline \end{array}, \tag{4.2}$$

which is a graphical version of the formulas in [Lemma 4.9](#).

The point is that the action in this notation is visually easier to remember. Explicitly, the action from [Lemma 4.9](#) in this dot diagram notation is as follows.

- (i) For $i \in [1, m - 1]$ the operators e_i and f_i move dots rightwards or leftwards, if possible, and annihilate the diagram otherwise. See [Equation 4.2](#).
- (ii) The operators e_m and f_m add or remove dots in the final two columns if possible and annihilate the diagram otherwise. For example:

$$f_{6=m} \cdot x_{\{1,5,6\}} = x_{\{1\}} \rightsquigarrow f_6 \cdot \begin{array}{|c|c|c|c|c|c|} \hline \bullet & & & & \bullet & \bullet \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline \bullet & & & & & \\ \hline \end{array}.$$

- (iii) The h_k operators essentially only add signs in case they find dots in certain spots.

More generally, from [subsection 4.2](#) below we will use dot diagrams for $S \subset \square_{N \times m}$, where we have N rows. For example,

$$\begin{array}{c}
 \begin{array}{|c|c|c|c|c|c|} \hline \bullet & \bullet & & & & \\ \hline \bullet & & & & & \bullet \\ \hline \bullet & & & & \bullet & \\ \hline \end{array}
 \end{array}
 \rightsquigarrow
 S = \{(1, 1), (2, 1), (3, 1), (1, 2), (3, 5), (2, 6)\}.$$

One can check that W acts by even-signed permutations on $\oplus_{i=1}^m \mathbb{Z} \frac{\epsilon_i}{2}$, preserving $X(\mathfrak{so}_{2m})$ set-wise, where the simple reflection generators s_1, \dots, s_{m-1} permute the ϵ_i and s_m scales the ϵ_{m-1} and ϵ_m coordinates by -1 . The longest element $w_0 \in W$ acts by

$$w_0(a_1, \dots, a_{m-1}, a_m) = \begin{cases} (-a_1, \dots, -a_{m-1}, -a_m) & \text{if } m \text{ is even} \\ (-a_1, \dots, -a_{m-1}, a_m) & \text{if } m \text{ is odd.} \end{cases}$$

The action of W by even-signed permutations on $\{\frac{1}{2}(\pm\epsilon_1, \pm\epsilon_2, \dots, \pm\epsilon_m)\}$ has two orbits: the vectors with an odd number of signs and the vectors with an even number of signs. These orbits are the weight spaces of the $U_{\mathbb{A}_\vee}(\mathfrak{so}_{2m})$ -representations $\nabla_{\mathbb{A}_\vee}(\varpi_{m-1})$ and $\nabla_{\mathbb{A}_\vee}(\varpi_m)$, respectively. The (dual) Weyl representations $\Delta_{\mathbb{A}_\vee}(-), \nabla_{\mathbb{A}_\vee}(-)$ are recalled in [section 7](#) below.

Remark 4.11. The $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ -representations $\nabla_{\mathbb{A}_\heartsuit}(\varpi_{m-1})$ and $\nabla_{\mathbb{A}_\heartsuit}(\varpi_m)$ are *minuscule*. It follows that $\nabla_{\mathbb{A}_\heartsuit}(\varpi_{m-1}) \cong \Delta_{\mathbb{A}_\heartsuit}(\varpi_{m-1})$ and $\nabla_{\mathbb{A}_\heartsuit}(\varpi_m) \cong \Delta_{\mathbb{A}_\heartsuit}(\varpi_m)$.

Using the convention that $\varpi_0 = 0$, we see that we have $\text{wt}_{\mathfrak{so}_{2m}}(x_i) = -\varpi_{i-1} + \varpi_i - \varpi_m$, $\text{wt}_{\mathfrak{so}_{2m}}(x_{m-1}) = -\varpi_{m-3} + \varpi_{m-2}$ and $\text{wt}_{\mathfrak{so}_{2m}}(x_m) = -\varpi_{m-1}$. In terms of the ϵ_i basis we have $\text{wt}_{\mathfrak{so}_{2m}}(x_i) = \frac{1}{2}(-1, -1, \dots, -1, 1, -1, \dots, -1, -1)$ with the 1 in the i th entry. More generally, if $S \subset [1, m]$, then $\text{wt}_{\mathfrak{so}_{2m}}(x_S) = \sum_{i \in S} \text{wt}_{\mathfrak{so}_{2m}}(x_i)$, and in the ϵ_i basis we have 1 in the i th entry for all $i \in S$. In terms of dot diagrams this reads:

$$\text{wt}_{\mathfrak{so}_{2m}}(x_S) = \frac{1}{2}(d_1(S), d_2(S), \dots, d_m(S)),$$

where $d_i(S)$ is 1, if the i th node for the dot diagram for S contains a dot, and -1 otherwise.

Example 4.12. We get

$$x_S = \begin{array}{|c|c|c|c|c|c|} \hline \bullet & & \bullet & \bullet & & \\ \hline \end{array} \rightsquigarrow \text{wt}_{\mathfrak{so}_{2m}}(x_S) = \frac{1}{2}(1, -1, 1, 1, -1, -1),$$

where $m = 6$ and $S = \{1, 3, 4\}$.

Recall the $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ -action on $\Lambda^*(\mathbb{A}_\heartsuit^m)$ from [Lemma 4.9](#). Over \mathbb{C} the following is classical, see for example [[How95](#), Section 4.3] and the discussion leading to that section.

Proposition 4.13. *The operators e_i , f_i and h_i , for $i \in [1, m]$, give rise to an $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ -action on $\Lambda^*(\mathbb{A}_\heartsuit^m)$ such that the i th \mathfrak{sl}_2 -triple (e_i, f_i, h_i) corresponds to the simple root α_i and $h_i = \alpha_i^\vee$. Moreover,*

$$\Lambda^*(\mathbb{A}_\heartsuit^m) \cong \nabla_{\mathbb{A}_\heartsuit}(\varpi_{m-1}) \oplus \nabla_{\mathbb{A}_\heartsuit}(\varpi_m)$$

as $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ -representations.

Proof. As recalled in [section 7](#) below, the $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ -representation $\nabla_{\mathbb{A}_\heartsuit}(\lambda)$ specializes to the simple $U_{\mathbb{Q}_\heartsuit}(\mathfrak{so}_{2m})$ -representation $L_{\mathbb{Q}_\heartsuit}(\lambda)$.

Using [Lemma 4.9](#), one can easily verify that $\mathbb{Q}_\heartsuit \otimes_{\mathbb{A}_\heartsuit} \Lambda^*(\mathbb{A}_\heartsuit^m) \cong L_{\mathbb{Q}_\heartsuit}(\varpi_{m-1}) \oplus L_{\mathbb{Q}_\heartsuit}(\varpi_m)$, as a $U_{\mathbb{Q}_\heartsuit}(\mathfrak{so}_{2m})$ -representation, such that the operators e_i, f_i, h_i correspond to α_i . Moreover, the operators e_i, f_i, h_i preserve the lattice $\Lambda^*(\mathbb{A}_\heartsuit^m)$ and the higher divided powers, $e_i^{(d)}$ and $f_i^{(d)}$, act as zero. Thus, $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ acts on $\Lambda^*(\mathbb{A}_\heartsuit^m)$. Analyzing [Lemma 4.9](#), it is easy to see that $\Lambda^*(\mathbb{A}_\heartsuit^m)$ is generated over $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ by the highest weight vectors $x_1 x_2 \dots x_{m-1}$ and $x_1 x_2 \dots x_{m-1} x_m$. Indeed, recall that f_i acts by moving the i th dot to the $i + 1$ st box (if the $i + 1$ st box is empty), e.g.

$$f_3 \cdot \begin{array}{|c|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & & \bullet & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline \bullet & \bullet & & \bullet & \bullet & \\ \hline \end{array},$$

and f_m acts by removing the last two dots (if the last two boxes contain dots), e.g.

$$f_6 \cdot \begin{array}{|c|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & & \bullet & \bullet \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & & & \\ \hline \end{array}.$$

Using this one can see that all dot configurations can be generated from the vectors $x_1 x_2 \dots x_{m-1}$ and $x_1 x_2 \dots x_{m-1} x_m$. \square

4.2. Two bases for $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$

We now consider the general case with arbitrary (but fixed) $N \in \mathbb{Z}_{\geq 1}$, and a free \mathbb{A}_ϑ -module V of dimension N .

Notation 4.14. *Throughout, we let the group $O(N)$ act diagonally on tensor products, as usual. Moreover, we consider $U_{\mathbb{A}_\vartheta}(\mathfrak{so}_{2m})$ as a Hopf algebra by extending $\Delta(x) = 1 \otimes x + x \otimes 1$ for all $x \in \mathfrak{so}_{2m}$.*

With the structure from [Notation 4.14](#) it follows from [Proposition 4.13](#) that $U_{\mathbb{A}_\vartheta}(\mathfrak{so}_{2m})$ also acts on $\Lambda^*(\mathbb{A}_\vartheta^m)^{\otimes N} \cong \Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$. Similarly, $O(N)$ acts on $\Lambda^*(V)^{\otimes m} \cong \Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$. We will use these two actions below.

Lemma 4.15. *The $U_{\mathbb{A}_\vartheta}(\mathfrak{so}_{2m})$ -representation $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$ has a filtration by Weyl representations, and a filtration by dual Weyl representations.*

Proof. Directly from [Proposition 4.13](#), [Remark 4.11](#) and [Remark 7.1](#). □

The following classical fact, which can be compared with [[How95](#), Section 4.3] again, is fundamental to what follows. We prove this result in [subsection 5.1](#) by identifying the action of e_i and f_i as compositions of tensor products of $O(N)$ intertwiners.

Proposition 4.16. *(Howe’s actions integrally.) The actions of $O(N)$ and $U_{\mathbb{A}_\vartheta}(\mathfrak{so}_{2m})$ on $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$ commute. Thus, $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$ is an $O(N)$ - $U_{\mathbb{A}_\vartheta}(\mathfrak{so}_{2m})^{op}$ -birepresentation.*

Remark 4.17. We have two left actions, and thus we need an op in [Proposition 4.16](#).

We now specify explicit isomorphisms

$$\Lambda^*(V)^{\otimes m} \cong \Lambda^*(V \otimes \mathbb{A}_\vartheta^m) \quad \text{and} \quad \Lambda^*(\mathbb{A}_\vartheta^m)^{\otimes N} \cong \Lambda^*(V \otimes \mathbb{A}_\vartheta^m),$$

see [Lemma 4.22](#) below.

Write $V = \bigoplus_{i=1}^N \mathbb{A}_\vartheta \cdot v_i$. Given, $(i, j) \in \square_{N \times m}$, write

$$w_{i,j} = v_i \otimes x_j \in \Lambda^*(V \otimes \mathbb{A}_\vartheta^m).$$

Definition 4.18. Let $S \subset \square_{N \times m}$. We will consider two orderings of $\square_{N \times m}$, *horizontal or row reading* and *vertical or column reading*:

$$\begin{aligned} (1, 1) <_h (1, 2) \dots <_h (1, m) <_h (2, 1) <_h (2, 2) <_h \dots <_h (N, m - 1) <_h (N, m), \\ (1, 1) <_v (2, 1) \dots <_v (N, 1) <_v (1, 2) <_v (2, 2) <_v \dots <_v (N - 1, m) <_v (N, m). \end{aligned}$$

Suppose that $S = \{(i_1, j_1) <_h \dots <_h (i_k, j_k)\}$ and, at the same time, $S = \{(i'_1, j'_1) <_v \dots <_v (i'_k, j'_k)\}$. Define

$$w_S^h = w_{i_1, j_1} \dots w_{i_k, j_k}, \quad w_S^v = w_{i'_1, j'_1} \dots w_{i'_k, j'_k}, \tag{4.3}$$

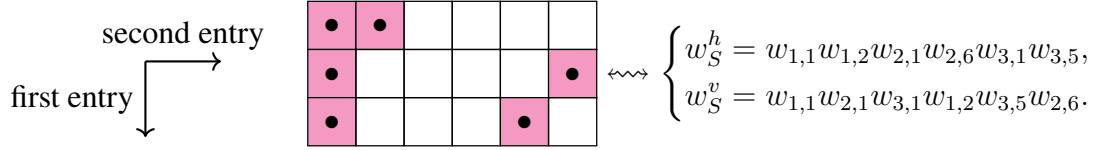
as the given elements of $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$.

Essentially by definition we get:

Lemma 4.19. *The sets $\{w_S^h | S \subset \square_{N \times m}\}$, $\{w_S^v | S \subset \square_{N \times m}\}$ are \mathbb{A}_∇ -bases of $\Lambda^*(V \otimes \mathbb{A}_\nabla^m)$. \square*

There are projections $\pi_h: \square_{N \times m} \rightarrow [1, m]$ and $\pi_v: \square_{N \times m} \rightarrow [1, N]$. Write $S_i = \pi_h^{-1}(\{i\})$ to denote the subset of S which projects to i under π_h and ${}_j S = \pi_v^{-1}(\{j\})$ to denote the subset of S which projects to j under π_v . Note that $\coprod_{i=1}^N {}_i S = S = \coprod_{j=1}^m S_j$.

Example 4.20. In dot diagram notation, now with an N -by- m rectangle, the elements in Equation 4.3 are simply given by reading along columns or rows, e.g.:



Here $(N, m) = (3, 6)$. Moreover, we have

$$\begin{aligned} S_1 &= \{(1, 1), (2, 1), (3, 1)\}, & S_2 &= \{(1, 2)\}, & S_5 &= \{(3, 5)\}, & S_6 &= \{(2, 6)\}, \\ {}_1 S &= \{(1, 1), (1, 2)\}, & {}_2 S &= \{(2, 1), (2, 6)\}, & {}_3 S &= \{(3, 1), (3, 5)\}, \end{aligned}$$

and all other of these sets are empty. In other words, the sets ${}_i S$ and S_j are row and column reading projections.

The following lemma is easy and omitted:

Lemma 4.21. *The set $\{v_T | T \subset [1, N]\}$ is an \mathbb{A}_∇ -basis of $\Lambda^*(V)$. \square*

Lemma 4.22. *Horizontal and vertical reading give isomorphisms of free \mathbb{A}_∇ -modules by*

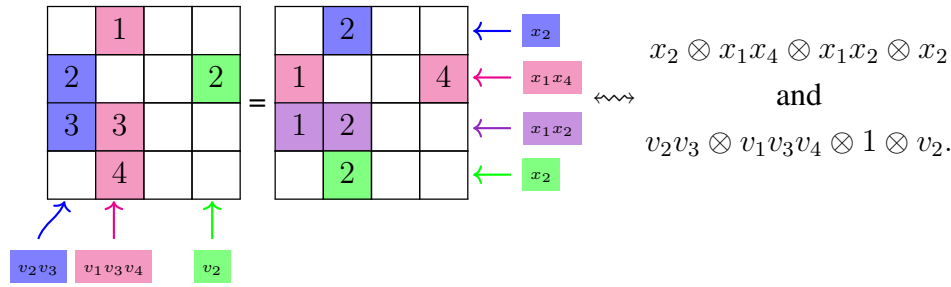
$$\begin{aligned} \phi_h: \Lambda^*(V \otimes \mathbb{A}_\nabla^m) &\rightarrow \Lambda^*(\mathbb{A}_\nabla^m)^{\otimes N}, & w_S^h &\mapsto x_{\pi_h(1S)} \otimes \dots \otimes x_{\pi_h(NS)}, \\ \phi_v: \Lambda^*(V \otimes \mathbb{A}_\nabla^m) &\rightarrow \Lambda^*(V)^{\otimes m}, & w_S^v &\mapsto v_{\pi_v(S_1)} \otimes \dots \otimes v_{\pi_v(S_m)}. \end{aligned}$$

Proof. We combine Lemma 4.4 and Lemma 4.21. \square

In identifications we fix the isomorphisms in Lemma 4.22.

Example 4.23. Let $N = 4$, $m = 4$, and $S = \{(2, 1), (3, 1), (1, 2), (3, 2), (4, 2), (2, 4)\}$. Then:

$$S = \begin{matrix} \begin{matrix} \square & \bullet & \square & \square \\ \bullet & \square & \square & \bullet \\ \bullet & \bullet & \square & \square \\ \square & \bullet & \square & \square \end{matrix}, & \begin{cases} w_S^h = w_{(1,2)}w_{(2,1)}w_{(2,4)}w_{(3,1)}w_{(3,2)}w_{(4,2)} \xrightarrow{\phi_h} x_2 \otimes x_1x_4 \otimes x_1x_2 \otimes x_2, \\ w_S^v = w_{(2,1)}w_{(3,1)}w_{(1,2)}w_{(3,2)}w_{(4,2)}w_{(2,4)} \xrightarrow{\phi_v} v_2v_3 \otimes v_1v_3v_4 \otimes 1 \otimes v_2. \end{cases} \end{matrix}$$



We have also illustrated how to get the expression for w_S^v and w_S^h on the bottom and right, respectively. Note that $w_S^h = -w_S^v$.

Let $\sigma_h^v(S)$ be the permutation that sends the ordered set $\{(i_1, j_1), \dots, (i_k, j_k)\}$ to the ordered set $\{(i'_1, j'_1), \dots, (i'_k, j'_k)\}$. Let ℓ denote its length.

Lemma 4.24. We have $w_S^h = (-1)^{\ell(\sigma_h^v(S))} \cdot w_S^v$.

Proof. Directly from the signed commutation rules of the exterior algebra. □

4.3. Two explicit actions on $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$

Each of the two bases $\{w_S^v\}$ and $\{w_S^h\}$ for $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$ are adapted to the action by $O(N)$ and $U_{\mathbb{A}_\vartheta}(\mathfrak{so}_{2m})$, respectively. We describe these actions now. The two actions commute, see [Proposition 4.16](#).

4.3.1 The $O(N)$ -action

The group $O(N)$ acts naturally on $\Lambda^*(V)^{\otimes m}$ and on the exterior algebra $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$ of $V \otimes \mathbb{A}_\vartheta^m \cong V \oplus \dots \oplus V$. Fixing this $O(N)$ -action we have:

Lemma 4.25. The map ϕ_ϑ intertwines the $O(N)$ -action on $\Lambda^*(V \otimes \mathbb{A}_\vartheta^m)$.

Proof. A direct check. □

Remark 4.26. There are two common choices of symmetric bilinear form when defining $O(N)$. These have *Gram matrices* either the diagonal or the antidiagonal. Since $\sqrt{-1} \in \mathbb{A}_\vartheta$ one can prove that these two forms are equivalent.

We define another basis for V , such that $(-, -)$ will have antidiagonal Gram matrix with respect to the new basis.

Definition 4.27. For $i \in \{1, \dots, n\}$, define

$$a_i = v_i - \sqrt{-1} \cdot v_{N-i+1}, \quad b_i = \frac{v_i + \sqrt{-1} \cdot v_{N-i+1}}{2}, \quad \text{and}$$

$$u = v_{n+1} \text{ if } N = 2n + 1.$$

(Note that we define u only if N is odd.)

For $i \in \{1, \dots, n\}$ we can express the v_i basis in terms of this new basis as

$$v_i = \frac{a_i + 2 \cdot b_i}{2}, \quad v_{N-i+1} = \sqrt{-1} \cdot \frac{a_i - 2 \cdot b_i}{2}, \quad \text{and} \\ v_{n+1} = u, \quad \text{if } N = 2n + 1.$$

Lemma 4.28. *The pairing $(v_i, v_j) = \delta_{ij}$ gives*

$$(a_i, b_i) = 1 = (b_i, a_i) = (u, u),$$

while all other pairings of basis vectors a_i , b_j and u vanish.

Proof. A routine calculation. □

Definition 4.29. Let the *split torus* $T \subset \text{SO}(N)$ be the diagonal matrices in $\text{SO}(N) \subset \text{O}(N)$ with respect to the basis in [Definition 4.27](#).

The group T is the subgroup generated by operators $\alpha_{\epsilon_i}^\vee(t) \in \text{SO}(N)$, for $i \in \{1, \dots, n\}$, determined by the action $\alpha_{\epsilon_i}^\vee(t) \cdot a_j = t^{(\epsilon_i, \epsilon_j)} \cdot a_j$, $\alpha_{\epsilon_i}^\vee(t) \cdot b_j = t^{(\epsilon_i, -\epsilon_j)} \cdot b_j$ and $\alpha_{\epsilon_i}^\vee(t) \cdot u = u$.

Remark 4.30. The action of T in the v_i basis is now easily computed, for example:

$$\alpha_{\epsilon_1}^\vee(t) \cdot v_1 = \frac{t + t^{-1}}{2} \cdot v_1 - \sqrt{-1} \cdot \frac{t - t^{-1}}{2} \cdot v_N.$$

This action of T in the v_i basis is not as easy to work with for our purposes, which is the reason we introduce the new basis.

Definition 4.31. Let $\sigma \in \text{O}(N)$ be the following element determined by its action.

1. If $N = 2n + 1$, then $\sigma \in \text{O}(N)$ acts on V by

$$\sigma \cdot a_i = a_i, \quad \sigma \cdot b_i = b_i, \quad \sigma \cdot u = -u.$$

2. If $N = 2n$, then $\sigma \in \text{O}(N)$ acts on V by

$$\sigma \cdot a_i = a_i, \quad \sigma \cdot b_i = b_i, \quad \sigma \cdot a_N = \frac{1}{2} \cdot b_N, \quad \sigma \cdot b_N = 2 \cdot a_N.$$

Here $i \in \{1, \dots, n-1\}$.

Note that conjugation by σ preserves $\text{SO}(N)$ and induces an automorphism of \mathfrak{so}_N , which agrees with the automorphism, also denoted by σ , defined in [Definition 7.15](#)

4.3.2 The $U_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$ -action

Definition 4.32. Using the isomorphism ϕ_h and the action of $U_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$ on $\Lambda^*(\mathbb{A}_\nabla^m)^{\otimes N}$ from [Notation 4.14](#), we can define an action of $U_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$ on $\Lambda^*(V \otimes \mathbb{A}_\nabla^m)$ by

$$u \cdot w = \phi_h^{-1}(u \cdot \phi_h(w)),$$

for all $u \in U_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$ and $w \in \Lambda^*(V \otimes \mathbb{A}_\nabla^m)$.

Notation 4.33. For $S \subset \square_{N \times m}$, write $d_j(S) = \sum_{i=1}^N d_j(iS)$ for $j \in [1, m]$. That is $d_j(S)$ is equal to the number of boxes in the j th column with a dot minus the number of boxes in the j th column without a dot.

Example 4.34. For the dot diagram

$$S = \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline \bullet & & & \bullet \\ \hline \bullet & \bullet & & \\ \hline & \bullet & & \\ \hline \end{array} \rightsquigarrow d_1 = 2 - 2 = 0, d_2 = 3 - 1 = 2, d_3 = 0 - 4 = -4, d_4 = 1 - 3 = -2,$$

where $d_i = d_i(S)$.

Lemma 4.35. The following defines a $U_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$ -action on $\Lambda^*(V \otimes \mathbb{A}_\nabla^m)$:

$$\begin{aligned} e_j \cdot w_S^h &= \sum_{\substack{1 \leq i \leq N \\ (i,j+1) \in S \\ (i,j) \notin S}} w_{S \setminus \{(i,j+1)\} \cup \{(i,j)\}}^h, & f_j \cdot w_S^h &= \sum_{\substack{1 \leq i \leq N \\ (i,j) \in S \\ (i,j+1) \notin S}} w_{S \setminus \{(i,j)\} \cup \{(i,j+1)\}}^h, \\ e_m \cdot w_S^h &= \sum_{\substack{1 \leq i \leq N \\ (i,m-1) \notin S \\ (i,m) \notin S}} w_{S \cup \{(i,m-1), (i,m)\}}^h, & f_m \cdot w_S^h &= \sum_{\substack{1 \leq i \leq N \\ (i,m-1) \in S \\ (i,m) \in S}} w_{S \setminus \{(i,m-1), (i,m)\}}^h, \\ h_k \cdot w_S^h &= \alpha_k^\vee \left(\frac{1}{2} (d_1(S), \dots, d_m(S)) \right) \cdot w_S^h, \end{aligned}$$

for $j \in [1, m - 1]$ and $k \in [1, m]$.

Proof. Using the definition of ϕ_h , this follows from the description of the $U_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$ -action on $\Lambda^*(\mathbb{A}_\nabla^m)^{\otimes N}$ in [Notation 4.14](#). In fact, ϕ_h is designed so that this lemma is true.

More explicitly, in dot diagrams the action is row-wise. For example,

$$f_1 \cdot \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & & & \\ \hline \bullet & & & & \bullet \\ \hline \bullet & & & \bullet & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & & & \\ \hline & \bullet & & & \bullet \\ \hline \bullet & & & \bullet & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & & & \\ \hline \bullet & & & & \bullet \\ \hline & \bullet & & \bullet & \\ \hline \end{array},$$

which is the rule for rows as exemplified in [Equation 4.2](#) plus the coproduct. □

4.4. Semisimple Howe duality

For the following *semisimple Howe duality* we recall that $\Lambda_{+, \leq m}^{O(N)}$ denotes the set of m -restricted dominant $O(N)$ -weights, cf. [Appendix A](#), which index the simple $O(N)$ -representations (as well as Weyl and indecomposable tilting representations). Moreover, below we will give a combinatorial map \dagger , again defined later in [Appendix A](#), that takes a dominant $O(N)$ -weight and produces a dominant \mathfrak{so}_{2m} -weight. The following is our version of [[How95](#), Section 4.3.5]:

Proposition 4.36. (*Semisimple Howe duality.*) *As an $O(N)$ - $U_{\mathbb{Q}_\nu}(\mathfrak{so}_{2m})^{op}$ -birepresentation we have:*

$$\mathbb{Q}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m) \cong \bigoplus_{\lambda \in \Lambda_{+, \leq m}^{O(N)}} L_{\mathbb{Q}_\nu}(\lambda) \boxtimes L_{\mathbb{Q}_\nu}(\lambda^\dagger).$$

Proof. The key ingredient is the result [Proposition 4.16](#), which makes the question about a $O(N)$ - $U_{\mathbb{Q}_\nu}(\mathfrak{so}_{2m})^{op}$ -birepresentation decomposition of $\Lambda^*(V \otimes \mathbb{A}_\nu^m)$ well-defined, and character calculations in the spirit of Howe's original construction.

In more details, let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$. By [Lemma A.28](#), [Remark A.29](#), and [Definition A.34](#), it follows from complete reducibility that $\bigoplus_{\lambda \in \Lambda_{+, \leq m}^{O(N)}} L_{\mathbb{Q}_\nu}(\lambda) \boxtimes L_{\mathbb{Q}_\nu}(\lambda^\dagger)$ is isomorphic to a direct summand $\mathbb{Q}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)$. The claim then follows from the character calculation in [[How95](#), Section 4.3.5] or [[AR96](#), Proposition 3.2]. \square

Let $\Pi_{m, +}^{\leq N}$ denote the set of N -restricted dominant \mathfrak{so}_{2m} -weights, specified in [Appendix A](#).

Proposition 4.37. *We have*

$$\dim_{\mathbb{F}_\nu} \text{End}_{O(V)}(\mathbb{F}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)) = \sum_{\lambda \in \Lambda_{+, \leq m}^{O(N)}} (\dim_{\mathbb{Q}_\nu} L_{\mathbb{Q}_\nu}(\lambda^\dagger))^2 = \sum_{\kappa \in \Pi_{m, +}^{\leq N}} (\dim_{\mathbb{Q}_\nu} L_{\mathbb{Q}_\nu}(\kappa))^2.$$

Proof. Using that $\mathbb{F}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)$ is a tilting representations for both actions, as follows from [Lemma 4.15](#), and [Lemma 7.47](#), a standard argument, similar to [[AST17](#), Proposition 2.3], yields

$$\dim_{\mathbb{F}_\nu} \text{End}_{O(V)}(\mathbb{F}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)) = \dim_{\mathbb{Q}_\nu} \text{End}_{O(V)}(\mathbb{Q}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m))$$

Therefore, [Proposition 4.36](#) implies

$$\dim_{\mathbb{F}_\nu} \text{End}_{O(V)}(\mathbb{F}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)) = \sum_{\lambda \in \Lambda_{+, \leq m}^{O(N)}} (\dim_{\mathbb{Q}_\nu} L_{\mathbb{Q}_\nu}(\lambda^\dagger))^2,$$

and the final claim then follows from [Proposition A.35](#) proven later on. \square

[Proposition 4.37](#) gives an effective way to compute the dimension of the endomorphism space over \mathbb{F}_ν since $\dim_{\mathbb{Q}_\nu} L_{\mathbb{Q}_\nu}(\lambda^\dagger)$ can be computed using *Weyl's dimension formula*.

5. The diagrammatic presentation

We now prove our first main theorem: the equivalence of symmetric ribbon \mathbb{F}_φ -linear categories between $\mathbf{Web}_{\mathbb{F}_\varphi}(O(N))$ and $\mathbf{Fund}_{\mathbb{F}_\varphi}(O(N))$, see [Theorem 5.16](#). Upon additive idempotent completion, $\mathbf{Web}_{\mathbb{F}_\varphi}(O(N))$ is thus a *diagrammatic version* of $\mathbf{Tilt}_{\mathbb{F}_\varphi}(O(N))$ (recall tilting representations from [subsection 7.2](#)). Our main tool is a version of Howe’s $O(N)$ - \mathfrak{so}_{2m} duality in positive characteristic, cf. [subsection 5.2](#).

5.1. From webs to reps

We begin by defining some $O(N)$ -equivariant maps which correspond to our generating webs.

Notation 5.1. We let Λ^i denote $\Lambda^i(V)$ and write $\Lambda^{(i_1, \dots, i_m)} = \Lambda^{i_1}(V) \otimes \dots \otimes \Lambda^{i_m}(V)$.

Let $T = \{t_1 < \dots < t_i\}$ and $U = \{u_{i+1} < \dots < u_{i+j}\}$ be subsets of $[1, N]$ such that $U \cap T = \emptyset$ and write $T \cup U = \{s_1 < \dots < s_{i+j}\}$. Consider the permutation $\sigma_{T,U} \in S(i+j)$ in the symmetric group $S(i+j)$ of $[1, i+j]$ which sends $\{t_1 < \dots, t_i, u_{i+1} < \dots < u_{i+j}\}$ to $\{s_1 < \dots < s_{i+j}\}$. We let $\ell(T, U) = \ell(\sigma_{T,U})$ (here the length ℓ is in terms of number of simple transpositions).

Definition 5.2. We define *merge, split and crossing maps* to be

$$\begin{aligned} \mathbf{A}_{k,l}^{k+l} &: \Lambda^k \otimes \Lambda^l \rightarrow \Lambda^{k+l}, v_T \otimes v_U \mapsto (-1)^{\ell(T,U)} v_{T \cup U}, \\ \mathbf{Y}_{k+l}^{k,l} &: \Lambda^{k+l} \rightarrow \Lambda^k \otimes \Lambda^l, v_S \mapsto \sum_{\substack{S=T \amalg U \\ |T|=k, |U|=l}} (-1)^{\ell(T,U)} v_T \otimes v_U, \\ \mathbf{X}_{k,l}^{l,k} &: \Lambda^k \otimes \Lambda^l \rightarrow \Lambda^l \otimes \Lambda^k, v_T \otimes v_S \mapsto v_S \otimes v_T, \end{aligned}$$

and *cap and cup maps*

$$\begin{aligned} \mathbf{\Omega}_k &: \Lambda^k \otimes \Lambda^k \rightarrow \mathbb{1}, v_T \otimes v_U \mapsto (-1)^{\ell(T,U)} v_{T \cup U}, \\ \mathbf{U}_k &: \mathbb{1} \rightarrow \Lambda^k \otimes \Lambda^k, 1 \mapsto (-1)^{\binom{i}{2}} \sum_{\substack{S \subset [1,N] \\ |S|=k}} v_S \otimes v_S. \end{aligned}$$

(The notation is hopefully suggestive.)

Recall that $O(N)$ acts on the spaces in [Definition 5.2](#).

Lemma 5.3. *The maps in [Definition 5.2](#) are $O(N)$ -equivariant.*

Proof. A direct calculation. □

If $K \in \Pi_m^{\leq N}$, then $K = (k_1, \dots, k_m)$ where $k_i \in \{0, 1, \dots, N\}$. We write $\Lambda^K := \Lambda^{k_1} \otimes \dots \otimes \Lambda^{k_m}$ as in [Notation 5.1](#).

Definition 5.4. Define $\mathbf{Fund}_{\mathbb{F}_\varphi}(O(N))$ as the category with objects Λ^K for all $m \in \mathbb{Z}_{\geq 0}$ and all $K \in \Pi_m^{\leq N}$, and morphisms all \mathbb{F}_φ -linear maps which commute with $O(N)$.

In other words, $\mathbf{Fund}_{\mathbb{F}_\diamond}(\mathbf{O}(N))$ is the category of *fundamental* $\mathbf{O}(N)$ -representations, i.e. the full subcategory of all finite dimensional $\mathbf{O}(N)$ -representations monoidally generated by the Weyl representations for the fundamental $\mathbf{O}(N)$ -weights.

Lemma 5.5. *The category $\mathbf{Fund}_{\mathbb{F}_\diamond}(\mathbf{O}(N))$ is a symmetric ribbon \mathbb{F}_\diamond -linear category with:*

- (i) *with monoidal structure given by the usual tensor product of $\mathbf{O}(N)$ -representations;*
- (ii) *with symmetry given by the tensor flip;*
- (iii) *with pivotal structure given by $X^* = \mathrm{Hom}_{\mathbb{F}_\diamond}(X, \mathbb{F}_\diamond)$ and $X \rightarrow X^{**}$ defined by $x \mapsto (f \mapsto f(x))$.*

Proof. $\mathbf{Fund}_{\mathbb{F}_\diamond}(\mathbf{O}(N))$ is a full subcategory of all finite dimensional $\mathbf{O}(N)$ -representations, and inherits all structures from the parent category. \square

Proposition 5.6. *(The (diagrammatic) presentation functor.) There is a symmetric ribbon \mathbb{F}_\diamond -linear functor*

$$\mathcal{P}_{\mathbb{F}_\diamond}^{\mathbf{O}(N)}: \mathbf{Web}_{\mathbb{F}_\diamond}(\mathbf{O}(N)) \rightarrow \mathbf{Fund}_{\mathbb{F}_\diamond}(\mathbf{O}(N)),$$

$$k \mapsto \Lambda^k,$$

Proof. If $\mathcal{P}_{\mathbb{F}_\diamond}^{\mathbf{O}(N)}$ is well-defined, the other claims follow easily. Hence, it suffices to check that the defining relations of $\mathbf{Web}_{\mathbb{F}_\diamond}(\mathbf{O}(N))$, see [Definition 3.1](#), are satisfied in $\mathbf{Fund}_{\mathbb{F}_\diamond}(\mathbf{O}(N))$. That the exterior type A web relations hold follows from [\[CKM14, Proposition 5.2.1\]](#) via [Proposition 3.4](#). Circle removal relations hold since Λ^k has categorical dimension $\binom{N}{k}$ and lollipop relations hold by Schur's lemma and the fact that Λ^k are simple and pairwise nonisomorphic, cf. [Lemma 7.52](#).

It remains to prove that the higher even orthogonal E - F relations hold in $\mathbf{Fund}_{\mathbb{F}_\diamond}(\mathbf{O}(N))$. As explained in [Lemma 7.52](#), the objects in $\mathbf{Fund}_{\mathbb{F}_\diamond}(\mathbf{O}(N))$ are tensor products of the objects

$$\Lambda^i \mathbb{F}_\diamond^N \cong \mathbb{F}_\diamond \otimes_{\mathbb{A}_\diamond} \Lambda^i \cong \mathbb{F}_\diamond \otimes_{\mathbb{A}_\diamond} \Delta_{\mathbb{A}_\diamond}(1^i) \cong \mathbb{F}_\diamond \otimes_{\mathbb{A}_\diamond} \nabla_{\mathbb{A}_\diamond}(1^i), \text{ for } i = 1, \dots, N.$$

By [Lemma 7.50](#) and [Lemma 7.47](#) it follows that the dimension of homomorphism spaces in $\mathbf{Fund}_{\mathbb{F}_\diamond}(\mathbf{O}(N))$ are independent of \mathbb{F}_\diamond . Thus, the corresponding homomorphism spaces are free, finitely generated, and torsion free over \mathbb{A}_\diamond , and it follows that we can check relations over \mathbb{Q}_\diamond . Working over \mathbb{Q}_\diamond , we can use $X^{(a)} = a!X^a$ to replace divided powers with Chevalley generators, so it suffices to prove the orthogonal E - F relation in [Equation 3.2](#). But, as explained in [Lemma 3.14](#) this relation is known to be a consequence of the other orthogonal web relations and therefore holds in $\mathbf{Fund}_{\mathbb{Q}_\diamond}(\mathbf{O}(N))$. \square

Definition 5.7. We define *ladder operators* as follows. First:

$$\begin{aligned} E_{i,j}^{i+1,j-1} &= (\mathbf{A}_{i,1}^{i+1} \otimes \text{id}_{\Lambda^{j-1}}) \circ (\text{id}_{\Lambda^i} \otimes \mathbf{Y}_j^{1,j-1}): \Lambda^i \otimes \Lambda^j \rightarrow \Lambda^{i+1} \otimes \Lambda^{j-1}, \\ F_{i,j}^{i-1,j+1} &= (\text{id}_{\Lambda^{i-1}} \otimes \mathbf{A}_{1,j}^{j+1}) \circ (\mathbf{Y}_i^{i-1,1} \otimes \text{id}_{\Lambda^j}): \Lambda^i \otimes \Lambda^j \rightarrow \Lambda^{i-1} \otimes \Lambda^{j+1}, \\ e_{i,i}^{i+1,i+1} &= (\mathbf{A}_{i,1}^{i+1} \otimes \mathbf{A}_{1,i}^{i+1}) \circ (\text{id}_{\Lambda^i} \otimes \mathbf{U}_1 \otimes \text{id}_{\Lambda^i}): \Lambda^i \otimes \Lambda^i \rightarrow \Lambda^{i+1} \otimes \Lambda^{i+1}, \\ f_{i,i}^{i-1,i-1} &= (\text{id}_{\Lambda^{i-1}} \otimes \mathbf{O}_1 \otimes \text{id}_{\Lambda^{i-1}}) \circ (\mathbf{Y}_i^{i-1,1} \otimes \mathbf{Y}_i^{1,i-1}): \Lambda^i \otimes \Lambda^i \rightarrow \Lambda^{i-1} \otimes \Lambda^{i-1}. \end{aligned}$$

Moreover, let $\mathbf{K} \in \Pi_m$. We define operators on $\Lambda^*(V \otimes \mathbb{A}_v^m)$ by

$$\begin{aligned} \mathbf{E}_j \mathbf{1}_{\mathbf{K}} \cdot w_S^v &= \text{id}_{\Lambda^{(k_1, \dots, k_{j-1})}} \otimes E_{k_j, k_{j+1}}^{k_j+1, k_{j+1}-1} \otimes \text{id}_{\Lambda^{(k_{j+2}, \dots, k_m)}} (\phi_v(w_S^v)), \\ \mathbf{F}_j \mathbf{1}_{\mathbf{K}} \cdot w_S^v &= \text{id}_{\Lambda^{(k_1, \dots, k_{j-1})}} \otimes F_{k_j, k_{j+1}}^{k_j-1, k_{j+1}+1} \otimes \text{id}_{\Lambda^{(k_{j+2}, \dots, k_m)}} (\phi_v(w_S^v)), \\ \mathbf{e}_m \mathbf{1}_{\mathbf{K}} \cdot w_S^v &= \text{id}_{\Lambda^{(k_1, \dots, k_{m-2})}} \otimes e_{k_{m-1}, k_m}^{k_{m-1}+1, k_m+1} (\phi_v(w_S^v)), \\ \mathbf{f}_m \mathbf{1}_{\mathbf{K}} \cdot w_S^v &= \text{id}_{\Lambda^{(k_1, \dots, k_{m-2})}} \otimes e_{k_{m-1}, k_m}^{k_{m-1}-1, k_m-1} (\phi_v(w_S^v)), \end{aligned}$$

if $\text{wt } w_S^v = \mathbf{K}$, and $w_S^v \mapsto 0$ otherwise. (The notation is again hopefully suggestive.)

Lemma 5.8. Let $\mathbf{K} \in \Pi_m^{\leq N}$ and let $\text{wt } x_S^h = \mathbf{K}$. We have the following explicit description of the action of the ladder operators:

$$\begin{aligned} \mathbf{E}_j \mathbf{1}_{\mathbf{K}} \cdot w_S^v &= \sum_{\substack{1 \leq i \leq N \\ (i,j) \notin S, (i,j+1) \in S}} (-1)^{\ell(S_j, \{i\}) + \ell(\{i, S_{j+1} \setminus \{i+1\})} w_{S \cup \{(i,j)\} \setminus \{(i,j+1)\}}^v, \\ \mathbf{F}_j \mathbf{1}_{\mathbf{K}} \cdot w_S^v &= \sum_{\substack{1 \leq i \leq N \\ (i,j) \in S, (i,j+1) \notin S}} (-1)^{\ell(S_j \setminus \{i\}, \{i\}) + \ell(\{i, S_{j+1}\})} w_{S \setminus \{(i,j)\} \cup \{(i,j+1)\}}^v, \\ \mathbf{e}_i \mathbf{1}_{\mathbf{K}} \cdot w_S^v &= \sum_{\substack{1 \leq i \leq N \\ (i,j) \notin S, (i,j+1) \notin S}} (-1)^{\ell(S_j, \{i\}) + \ell(\{i, S_{j+1}\})} w_{S \cup \{(i,j), (i,j+1)\}}^v, \\ \mathbf{f}_i \mathbf{1}_{\mathbf{K}} \cdot w_S^v &= \sum_{\substack{1 \leq i \leq N \\ (i,j) \in S, (i,j+1) \in S}} (-1)^{\ell(S_j \setminus \{i\}, \{i\}) + \ell(\{i, S_{j+1} \setminus \{i\}\})} w_{S \setminus \{(i,j), (i,j+1)\}}^v. \end{aligned}$$

Proof. This follows from the Definition of ϕ_v and the formulas

$$\begin{aligned} E_{i,j}^{i+1,j-1}(v_S \otimes v_T) &= \sum_{\substack{t \in T \\ t \notin S}} (-1)^{\ell(S, \{t\}) + \ell(\{t, T \setminus \{t\})} v_{S \cup \{t\}} \otimes v_{T \setminus \{t\}}, \\ F_{i,j}^{i-1,j+1}(v_S \otimes v_T) &= \sum_{\substack{s \in S \\ s \notin T}} (-1)^{\ell(S \setminus \{s\}, \{s\}) + \ell(\{s, T\})} v_{S \setminus \{s\}} \otimes v_{T \cup \{s\}}, \\ e_{i,i}^{i+1,i+1}(v_S \otimes v_T) &= \sum_{\substack{x \notin S \\ x \notin T}} (-1)^{\ell(S, \{x\}) + \ell(\{x, T\})} v_{S \cup \{x\}} \otimes v_{T \cup \{x\}}, \\ f_{i,i}^{i-1,i-1}(v_S \otimes v_S) &= \sum_{\substack{y \in S \\ y \in T}} (-1)^{\ell(S \setminus \{y\}, \{y\}) + \ell(\{y, T \setminus \{y\})} v_{S \setminus \{y\}} \otimes v_{T \setminus \{y\}}. \end{aligned}$$

That these formulas hold is a direct calculation. □

Proposition 5.9. (*Howe's actions diagrammatically.*) *The ladder operators above define an $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ -action that agrees with the $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ -action in [Definition 4.32](#).*

Proof. Using [Lemma 4.35](#) and [Lemma 5.8](#), it is easy to see the actions agree up to a sign. Since the action in [Definition 4.32](#) has no signs, it suffices to show that the signs in [Lemma 4.35](#) cancel the signs coming from [Lemma 4.24](#).

For example, to verify the two e_j actions agree, we have to check that

$$(-1)^{\ell(\sigma_h^v(S))} = (-1)^{\ell(\sigma_h^v(S \cup \{(i,j)\} \setminus \{(i,j+1)\}))} (-1)^{\ell(S_j, \{i\}) + \ell(\{i\}, S_{j+1} \setminus \{i+1\})}.$$

We leave the verification of the remaining cases to the reader. \square

We finally arrive at a proof, using webs, that the actions of $O(N)$ and $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ on the space $\Lambda^*(V \otimes \mathbb{A}_\heartsuit^m)$ commute.

Proof of [Proposition 4.16](#). From [Proposition 5.9](#), we see the action of the generators of $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ on $\Lambda^*(V \otimes \mathbb{A}_\heartsuit^m)$ is given in [Lemma 5.8](#) which are compositions of tensor products of the $O(N)$ intertwiners in [Definition 5.2](#), see [Definition 5.7](#). \square

5.2. Howe duality

It follows from [Lemma 7.9](#), see also [Remark 7.10](#), that the set of \mathfrak{so}_{2m} -weights appearing in $\Lambda^*(V \otimes \mathbb{A}_\heartsuit^m)$, denoted $\Pi_m^{\leq N}$ above, is a *saturated set*, see [Definition 7.2](#). Recall also that $\Pi_{m,+}^{\leq N}$ is a saturated set of dominant \mathfrak{so}_{2m} -weights, see [Definition 7.4](#).

Notation 5.10. Write $S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m})$ to denote the Schur algebra quotient of $\dot{U}_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ determined by the saturated set $\Pi_{m,+}^{\leq N}$. The reader unfamiliar with these is referred to the background in [subsubsection 7.1.4](#) below.

The algebra $S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m})$, is the quotient of $\dot{U}_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ by the two sided ideal generated by $\mathbf{1}_K$ such that $K \notin \Pi_m^{\leq N}$. Moreover, this algebra contains orthogonal idempotents $\mathbf{1}_K$, for $K \in \Pi_m^{\leq N}$, such that $\sum_{K \in \Pi_m^{\leq N}} \mathbf{1}_K = 1$.

One can think of $S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m})$ as an algebraic version of webs with m boundary points on the top and bottom of the diagram.

Notation 5.11. Similar to [Notation 2.17](#), for fixed m , we view $S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m})$ as a category with objects $\mathbf{1}_K$, for $K \in \Pi_m^{\leq N}$, and morphisms $\text{Hom}_{S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m})}(\mathbf{1}_K, \mathbf{1}_L) = \mathbf{1}_L S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m}) \mathbf{1}_K$. The composition of morphisms is multiplication in the algebra.

Lemma 5.12. (*A Schur functor.*) *The action of $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ on $\Lambda^*(V \otimes \mathbb{A}_\heartsuit^m)$ induces a functor*

$$\mathcal{S}_{\mathbb{A}_\heartsuit}^{\mathfrak{so}_{2m}} : S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m}) \rightarrow \mathbf{Fund}_{\mathbb{A}_\heartsuit}(O(N))$$

such that $\mathbf{1}_K \mapsto \Lambda^K$, for all $K \in \Pi_m^{\leq N}$.

Proof. Lemma A.10 implies that

$$\mathbf{1}_K \cdot \Lambda^*(V \otimes \mathbb{A}_\heartsuit^m) = \begin{cases} \Lambda^K & \text{if } K \in \Pi_m^{\leq N}, \\ 0 & \text{if } K \notin \Pi_m^{\leq N}. \end{cases}$$

Therefore, the action of $U_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})$ induces a functor $S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m}) \rightarrow \mathbf{Fund}_{\mathbb{A}_\heartsuit}(O(N))$. □

We have already discussed how $\Lambda^*(V \otimes \mathbb{A}_\heartsuit^m) \cong \Lambda^*(\mathbb{A}_\heartsuit^m)^{\otimes N}$. We also have:

Lemma 5.13. *The space $\Lambda^*(V \otimes \mathbb{F}_\heartsuit^m)$ is a full tilting representation for $S_{\mathbb{F}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m})$.*

Proof. Because of Lemma 4.15, it suffices to show that if $K \in \Pi_{m,+}^{\leq N}$, then the indecomposable tilting $U_{\mathbb{F}_\heartsuit}(\mathfrak{so}_{2m})$ -representation $T_{\mathbb{F}_\heartsuit}(K)$ (see section 7) is a summand of $\Lambda^*(V \otimes \mathbb{F}_\heartsuit^m)$, which is Lemma A.37. □

Proposition 5.14 (One-sided double commutant). *The functor $S_{\mathbb{F}_\heartsuit}^{\mathfrak{so}_{2m}}$ is fully faithful.*

Proof. Using Lemma 5.13, Lemma 7.13, and Proposition 4.16 we find that $S_{\mathbb{F}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m})$ injects into $\text{End}_{O(N)}(\Lambda^*(V \otimes \mathbb{F}_\heartsuit^m))$. It then follows from Equation 7.1, Proposition 4.37, and the equality $\text{rank}_{\mathbb{A}_\heartsuit} \Delta_{\mathbb{A}_\heartsuit}(K) = \dim_{\mathbb{Q}_\heartsuit} L_{\mathbb{Q}_\heartsuit}(K) = \dim_{\mathbb{F}_\heartsuit} \Delta_{\mathbb{F}_\heartsuit}(K)$, that

$$\dim_{\mathbb{F}_\heartsuit} S_{\mathbb{F}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m}) = \sum_{K \in \Pi_{m,+}^{\leq N}} (\text{rank}_{\mathbb{A}_\heartsuit} \Delta_{\mathbb{A}_\heartsuit}(K))^2 = \dim_{\mathbb{F}_\heartsuit} \text{End}_{O(V)}(\Lambda^*(V \otimes \mathbb{F}_\heartsuit^m)).$$

Injectivity then implies that $S_{\mathbb{F}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m}) \rightarrow \text{End}_{O(V)}(\Lambda^*(V \otimes \mathbb{F}_\heartsuit^m))$ is surjective. □

The other side of the double commutant theorem, Proposition 5.14 for the orthogonal action, can be done similarly. However, we do not need it in this work.

5.3. fully faithful

The full functor $\mathcal{H}_{\mathbb{A}_\heartsuit}^{so}(N, m)$ in Proposition 3.13 factors through the Schur quotient, inducing a full functor

$$\mathcal{H}S_{\mathbb{A}_\heartsuit}^{so}(N, m): S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m}) \twoheadrightarrow \mathbf{Web}_{\mathbb{A}_\heartsuit}(O(N)), \mathbf{1}_K \mapsto K_1 \otimes \dots \otimes K_m,$$

and both Equation 2.13 and Proposition 3.13 hold. Thus, we have the commuting diagram

$$\begin{array}{ccc} \dot{U}_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m}) & \xrightarrow{\mathcal{H}_{\mathbb{A}_\heartsuit}^{so}(N, m)} & \mathbf{Web}_{\mathbb{A}_\heartsuit}(O(N)) \\ & \searrow \text{quo.} & \uparrow \mathcal{H}S_{\mathbb{A}_\heartsuit}^{so}(N, m) \\ & & S_{\mathbb{A}_\heartsuit}^{\leq N}(\mathfrak{so}_{2m}) \end{array}$$

Proposition 5.15. *We have the following commuting diagram:*

$$\begin{array}{ccccc}
 \dot{\mathbf{U}}_{\mathbb{A}_\vartheta}(\mathfrak{so}_{2m}) & \xrightarrow{\mathcal{H}_{\mathbb{A}_\vartheta}^{so}(N,m)} & \mathbf{Web}_{\mathbb{A}_\vartheta}(O(N)) & \xrightarrow{\mathcal{P}_{\mathbb{F}_\vartheta}^{O(N)}} & \mathbf{Fund}_{\mathbb{A}_\vartheta}(O(N)) \\
 & \searrow \text{quo.} & \uparrow \mathcal{HS}_{\mathbb{A}_\vartheta}^{so}(N,m) & \nearrow \mathcal{S}_{\mathbb{F}_\vartheta}^{so_{2m}} & \\
 & & \mathbf{S}_{\mathbb{A}_\vartheta}^{\leq N}(\mathfrak{so}_{2m}) & &
 \end{array}$$

$$\mathcal{S}_{\mathbb{F}_\vartheta}^{so_{2m}} = \mathcal{P}_{\mathbb{F}_\vartheta}^{O(N)} \circ \mathcal{HS}_{\mathbb{A}_\vartheta}^{so}(N, m). \quad (5.1)$$

Proof. Note that both sides of Equation 5.1 send $\mathbf{1}_K$ to Λ^K . The desired claim then follows from Definition 5.7, Proposition 5.9, and Proposition 5.6. \square

Denote by $_{}^{\oplus, \mathbb{C}\oplus}$ the additive idempotent completion. For the following theorem recall the category of tilting $O(N)$ -representations $\mathbf{Tilt}_{\mathbb{F}_\vartheta}(O(N))$ as in subsection 7.2.

Theorem 5.16. *(The orthogonal web calculus.) The presentation functor*

$$\mathcal{P}_{\mathbb{F}_\vartheta}^{O(N)}: \mathbf{Web}_{\mathbb{F}_\vartheta}(O(N)) \rightarrow \mathbf{Fund}_{\mathbb{F}_\vartheta}(O(N))$$

is an equivalence of symmetric ribbon \mathbb{F}_ϑ -linear categories. Moreover,

$$\mathbf{Web}_{\mathbb{F}_\vartheta}(O(N))^{\oplus, \mathbb{C}\oplus} \cong \mathbf{Tilt}_{\mathbb{F}_\vartheta}(O(N))$$

as symmetric ribbon \mathbb{F}_ϑ -linear categories.

Proof. That the functor preserves the structures as in the statement follows immediately from the definitions. The functor is also essentially surjective by the construction of $\mathbf{Fund}_{\mathbb{F}_\vartheta}(O(N))$, so it remains to argue that $\mathcal{P}_{\mathbb{F}_\vartheta}^{O(N)}$ is fully faithful.

Suppose W is a morphism in $\mathbf{Web}_{\mathbb{F}_\vartheta}(O(N))$ which $\mathcal{P}_{\mathbb{F}_\vartheta}^{O(N)}$ maps to zero. Since $\mathcal{HS}_{\mathbb{A}_\vartheta}^{so}(N, m)$ is full by Proposition 3.13, there is a morphism u in $\dot{\mathbf{S}}_{\mathbb{F}_\vartheta}(\mathfrak{so}_{2m})^{\leq N}$ so that $W = \mathcal{HS}_{\mathbb{A}_\vartheta}^{so}(N, m)(u)$. Then Equation 5.1 implies $\mathcal{S}_{\mathbb{F}_\vartheta}^{so_{2m}}(u) = \mathcal{P}_{\mathbb{F}_\vartheta}^{O(N)}(W) = 0$. Proposition 5.14 says that $u = 0$, so $W = \mathcal{HS}_{\mathbb{A}_\vartheta}^{so}(N, m)(u) = 0$. Thus, $\mathcal{P}_{\mathbb{F}_\vartheta}^{O(N)}$ is faithful. A similar easy argument using Proposition 5.14 and Equation 5.1 shows that $\mathcal{P}_{\mathbb{F}_\vartheta}^{O(N)}$ is full.

The final claim follows then from the equivalence and Proposition 7.53. \square

We actually proved our main theorem:

Proof of Theorem 1.3. Theorem 5.16 is a more refined formulation of Theorem 1.3. \square

Proofs of Proposition 3.4 and Proposition 3.13, wrap-up. We use the result Theorem 5.16 to fill in some of the remaining statements from section 3.

The case of [Proposition 3.4](#). We note that we have the commuting diagram (we marked the functor we are interested in)

$$\begin{array}{ccc}
 \mathbf{Web}_{\mathbb{F}_\heartsuit}(\mathrm{GL}(N)) & \xrightarrow[\cong]{\mathcal{P}_{\mathbb{F}_\heartsuit}^{\mathrm{GL}(N)}} & \mathbf{Fund}_{\mathbb{F}_\heartsuit}(\mathrm{GL}(N)) \\
 \mathcal{E}_{\mathbb{A}_\heartsuit}^{\mathrm{GL} \rightarrow \mathrm{O}} \downarrow & & \downarrow \\
 \mathbf{Web}_{\mathbb{F}_\heartsuit}(\mathrm{O}(N)) & \xrightarrow[\cong]{\mathcal{P}_{\mathbb{F}_\heartsuit}^{\mathrm{O}(N)}} & \mathbf{Fund}_{\mathbb{F}_\heartsuit}(\mathrm{O}(N))
 \end{array}$$

Here $\mathbf{Fund}_{\mathbb{F}_\heartsuit}(\mathrm{GL}(N))$ is the analog of $\mathbf{Fund}_{\mathbb{F}_\heartsuit}(\mathrm{O}(N))$ but for the general linear group, and $\mathcal{P}_{\mathbb{F}_\heartsuit}^{\mathrm{GL}(N)}$ is the presentation functor from [\[CKM14\]](#). We note:

- Commutativity of this diagram follows by careful comparison of the definitions.
- The top and bottom functors are equivalences by [\[CKM14, Theorem 3.3.1\]](#) and [Theorem 5.16](#), respectively.
- The right functor is faithful since $\mathrm{O}(N)$ is a subgroup of $\mathrm{GL}(N)$.

These together implies that the left functor is faithful.

The case of [Proposition 3.13](#). Directly from [Proposition 5.14](#) and [Theorem 5.16](#). □

6. Semisimplification

We first rapidly recall the notion of *semisimplification* of a rigid symmetric monoidal category such that the endomorphisms of the unit object is spanned by the identity. Then we prove [Theorem 1.4](#) by constructing an equivalence, in analogy with [\[BEAEO20, Section 3\]](#). The basic material below can be found in many sources such as [\[EGNO15\]](#) or [\[EO22\]](#).

6.1. A reminder on semisimplifications

All categories are assumed to be \mathbb{F}_\heartsuit -linear and have finite dimensional homomorphism spaces. Let \mathbf{C} be a rigid symmetric monoidal category with monoidal unit $\mathbb{1}$, such that $\mathrm{End}_{\mathbf{C}}(\mathbb{1}) = \mathbb{F}_\heartsuit \cdot \mathrm{id}_{\mathbb{1}}$.

Notation 6.1. Write $\mathrm{ev}_X : X^* \otimes X \rightarrow \mathbb{1}$, $\mathrm{coev} : \mathbb{1} \rightarrow X \otimes X^*$ for the unit and counit of adjunction realizing X^* as the right dual of X . Let $P_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ denote the symmetry.

Definition 6.2. The *trace* of an endomorphism $f : X \rightarrow X$ in \mathbf{C} , denoted $\mathrm{tr}_{\mathbf{C}}(f)$, is the element of \mathbb{F}_\heartsuit such that

$$\mathrm{ev}_X \circ P_{X,X^*} \circ (f \otimes \mathrm{id}) \circ \mathrm{coev}_X = \mathrm{tr}_{\mathbf{C}}(f) \cdot \mathrm{id}_{\mathbb{1}}.$$

We define the *categorical dimension* of X as $\mathrm{dim}_{\mathbf{C}}(X) = \mathrm{tr}_{\mathbf{C}}(\mathrm{id}_X)$.

Remark 6.3. The category of vector spaces over \mathbb{F}_φ , denoted $\mathbf{Vec}_{\mathbb{F}_\varphi}$, is a rigid symmetric monoidal category such that the endomorphisms of the monoidal identity are spanned by the identity endomorphism. The categorical dimension of a vector space is the usual dimension of the vector space under the map $\mathbb{Z} \rightarrow \mathbb{F}_\varphi$. More generally, if there is an \mathbb{F}_φ -linear symmetric monoidal functor $F : \mathbf{C} \rightarrow \mathbf{Vec}_{\mathbb{F}_\varphi}$, then $\dim_{\mathbf{C}}(X)$ is equal to the dimension of the vector space $F(X)$ when one views $\dim_{\mathbb{F}_\varphi} F(X)$ as an element of \mathbb{F}_φ .

We make the following simplifying assumption: every nilpotent endomorphism in \mathbf{C} has trace zero.

Lemma 6.4. *If there is a symmetric monoidal functor from \mathbf{C} to an abelian category, then \mathbf{C} satisfies the simplifying assumption.*

Proof. See [EO22, Remark 2.9]. □

Remark 6.5. All the categories we consider have the property in Lemma 6.4, and therefore satisfy the simplifying assumption.

Definition 6.6. The subcategory of *negligible morphisms* in \mathbf{C} , denoted $\mathcal{N}_{\mathbf{C}}$, is the category with the same objects as \mathbf{C} , and with morphisms

$$\mathrm{Hom}_{\mathcal{N}_{\mathbf{C}}}(X, Y) = \{f: X \rightarrow Y \mid \text{such that } \mathrm{tr}_{\mathbf{C}}(f \circ g) = 0, \text{ for all } g: Y \rightarrow X\}.$$

The *semisimplification* of \mathbf{C} is the quotient category

$$\overline{\mathbf{C}} = \mathbf{C}/\mathcal{N}_{\mathbf{C}},$$

which is defined as the category with the same objects as \mathbf{C} , and with morphisms

$$\mathrm{Hom}_{\overline{\mathbf{C}}}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(X, Y)/\mathrm{Hom}_{\mathcal{N}_{\mathbf{C}}}(X, Y).$$

Write $\pi_{\mathbf{C}}: \mathbf{C} \rightarrow \overline{\mathbf{C}}$ for the quotient functor.

Lemma 6.7. *The category $\overline{\mathbf{C}}$ is monoidal.*

Proof. It is well-known that $\mathcal{N}_{\mathbf{C}}$ is a \circ - \otimes -ideal, see e.g. [EO22, Theorem 2.6], which implies the claim. □

Example 6.8. Key examples of semisimplifications are the *Verlinde categories* in the spirit of [AP95], see also [EGNO15, Section 8.18.2] for background and references. These are constructed as follows. Let G be a simple algebraic group over \mathbb{Z} with Coxeter number h . Let \mathbb{F}_φ be an infinite field of characteristic $p \geq h$. Then the Verlinde category $\mathbf{Ver}_{\mathbb{F}_\varphi}(G)$ for G is the semisimplification of the category of tilting G -representations:

$$\mathbf{Ver}_{\mathbb{F}_\varphi}(G) := \overline{\mathbf{Tilt}_{\mathbb{F}_\varphi}(G)}.$$

This construction works more generally, e.g. also for $G = \mathrm{GL}(N)$ or $G = \mathrm{O}(N)$. The latter category $\mathbf{Ver}_{\mathbb{F}_\varphi}(\mathrm{O}(N))$ plays an important role in this paper.

For the purpose of this paper, a category is *semisimple* if it is abelian and every object is isomorphic to a finite direct sum of simple objects.

Lemma 6.9. *We have the following.*


1. *The category $\overline{\mathbf{C}}$ is rigid symmetric monoidal and $\text{End}_{\overline{\mathbf{C}}}(\mathbf{1}_{\overline{\mathbf{C}}}) = \mathbb{F}_\varphi \cdot \text{id}_{\mathbf{1}_{\overline{\mathbf{C}}}}$.*
2. *The category $\overline{\mathbf{C}}$ is semisimple.*
3. *An object \overline{X} is a simple objects in $\overline{\mathbf{C}}$ if and only if X is an indecomposable object in \mathbf{C} with $\dim_{\mathbf{C}}(X) \neq 0$. Two such simple objects are isomorphic in $\overline{\mathbf{C}}$ if and only if the corresponding indecomposable objects are isomorphic in \mathbf{C} .*
4. *Suppose \mathbf{D} is a rigid symmetric monoidal semisimple category. If $\mathcal{F}: \mathbf{C} \rightarrow \mathbf{D}$ is a full symmetric monoidal functor, then there is a fully faithful symmetric monoidal functor $\overline{\mathcal{F}}: \overline{\mathbf{C}} \rightarrow \mathbf{D}$ such that $\mathcal{F} = \overline{\mathcal{F}} \circ \pi_{\mathbf{C}}$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\mathcal{F}} & \mathbf{D} \\ \pi_{\mathbf{C}} \searrow & & \nearrow \overline{\mathcal{F}} \\ & \overline{\mathbf{C}} & \end{array} .$$

Proof. Because of the simplifying assumption, all of this is in [EO22, Section 2]. □

6.2. Colored Brauer diagrams

By a *two-part partition diagram* with b bottom and t top points we mean a diagram of the following form:

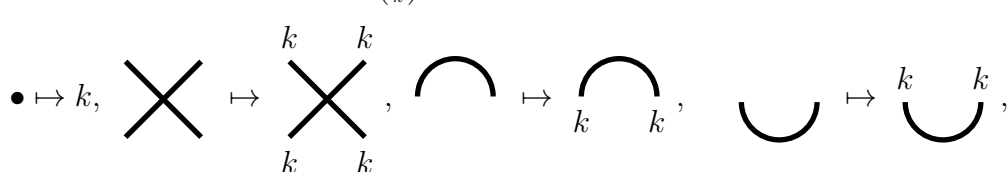


$$\text{where } b = 6, t = 8. \tag{6.1}$$

In words, a two-part partition diagram with b bottom and t top points is a diagram corresponding to a partition of $\{1, \dots, b\} \cup \{1, \dots, t\}$ where every block has two parts. A *Brauer diagram* with m bottom and N top points is then any representative of diagrams that represent the same partition. These assemble into the *Brauer category* Br_d [Bra37, LZ15].

This category can be thought of as the symmetric ribbon \mathbb{F}_φ -linear category \otimes -generated by a selfdual object \bullet of categorical dimension d . In other words, the category Br_d has a universal mapping property [LZ15, Theorem 2.6].

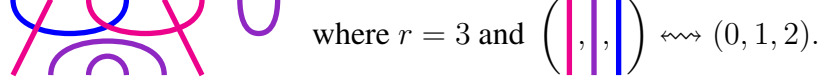
Proposition 6.10. *For $k = 1, \dots, N$, there is a symmetric ribbon \mathbb{F}_φ -linear functor*

$$\text{Br}_{\binom{N}{k}} \rightarrow \text{Web}_{\mathbb{F}_\varphi}(\text{O}(N)),$$


where we draw the defining structures of $\text{Br}_{\binom{N}{k}}$ in the standard way.

Proof. Using the Brauer categories universal mapping property, this follows from [Lemma 3.6](#). \square

For $r \in \mathbb{Z}_{\geq 1}$, a *colored Brauer diagram* with colors $\{0, \dots, r-1\}$ is a diagram of the form



where $r = 3$ and $\left(\begin{array}{c} | \\ | \\ | \end{array} \right) \rightsquigarrow (0, 1, 2)$.

Definition 6.11. Let $(d_i) = (d_0, \dots, d_{r-1}) \in \mathbb{Z}^r$. Define the *colored Brauer category* $\mathbf{Br}_{(d_i)}$ to be the rigid symmetric monoidal category with objects tensor products of the self-dual objects of dimension d_i for $i = 0, \dots, r-1$, and morphisms $\{0, \dots, r-1\}$ -colored Brauer diagrams.

Recall that $_{-}^{\oplus, \mathbb{C}\oplus}$ denotes the additive idempotent completion.

Lemma 6.12. *The category $\mathbf{Br}_{(d_i)}^{\oplus, \mathbb{C}\oplus}$ is an additive idempotent closed symmetric ribbon \mathbb{F}_{\heartsuit} -linear category such that the endomorphisms of $\mathbf{1}$ are the \mathbb{F}_{\heartsuit} -span of $\text{id}_{\mathbf{1}}$. Moreover, $\mathbf{Br}_{(d_i)}^{\oplus, \mathbb{C}\oplus}$ admits an abelian envelope which is also a symmetric ribbon \mathbb{F}_{\heartsuit} -linear category.*

Proof. The additive idempotent completion of a symmetric ribbon \mathbb{F}_{\heartsuit} -linear category is a symmetric ribbon \mathbb{F}_{\heartsuit} -linear category. To see that the endomorphisms of the unit object are all scalar multiples of the identity, note that the only Brauer diagram with empty bottom and top is the empty diagram. For the final claim we refer to [[Cou21](#), Theorem A]. \square

Note that [Lemma 6.9](#) implies that all the semisimplifications that we will see below are semisimple. Recall the notion of *Deligne tensor product* [[EGNO15](#), Section 1.11], which we denote by \boxtimes . The Deligne tensor product preserves the class of \mathbb{F}_{\heartsuit} -linear semisimple rigid symmetric monoidal categories [[EGNO15](#), Section 4.6].

Lemma 6.13. *There is an equivalence of symmetric ribbon \mathbb{F}_{\heartsuit} -linear categories*

$$\overline{\mathbf{Br}_{(d_i)}^{\oplus, \mathbb{C}\oplus}} \rightarrow \boxtimes_{i=0}^{r-1} \overline{\mathbf{Br}_{d_i}^{\oplus, \mathbb{C}\oplus}}.$$

Proof. We follow [[BEAEO20](#), Proof of Lemma 3.3]. The universal mapping property inherent in the definition of $\mathbf{Br}_{(d_i)}^{\oplus, \mathbb{C}\oplus}$ gives rise to a functor $\mathbf{Br}_{(d_i)}^{\oplus, \mathbb{C}\oplus} \rightarrow \boxtimes_{i=0}^{r-1} \overline{\mathbf{Br}_{d_i}^{\oplus, \mathbb{C}\oplus}}$. This functor is clearly full, so we can apply [Lemma 6.9](#) to deduce there is an equivalence $\overline{\mathbf{Br}_{(d_i)}^{\oplus, \mathbb{C}\oplus}} \rightarrow \boxtimes_{i=0}^{r-1} \overline{\mathbf{Br}_{d_i}^{\oplus, \mathbb{C}\oplus}}$. \square

After replacing $\mathbf{Br}_{(d_i)}^{\oplus, \mathbb{C}\oplus}$ by its abelian envelope from [Lemma 6.12](#), one could hope that [Lemma 6.13](#) holds without the semisimplification.

6.3. Bases at infinity

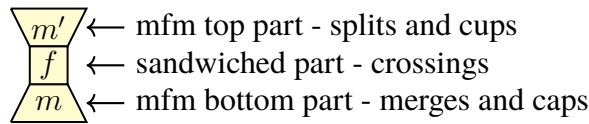
To prove that after semisimplification, the image of the colored Brauer diagrams in the webs $\mathbf{Web}_{\mathbb{F}_{\heartsuit}}(O(N))$ generate the category, we will use a spanning set for homomorphism spaces in the orthogonal web category which is analogous to the chicken foot diagrams in [[BEAEO20](#), Lemma 4.9].

For completeness, we define two spanning sets for the morphisms in $\mathbf{Web}_{\mathbb{A}_{\heartsuit}}(O(N))$. One of these is built from *many-to-few-to-many (mfm)* diagrams and the other from *few-to-many-to-few (fmf)* diagrams, with many and few referring to the total number of strings.

Remark 6.14. We expect that the spanning sets we are defining are actually bases whenever the total thickness of the strands is $\ll N$, but we will not need or prove this.

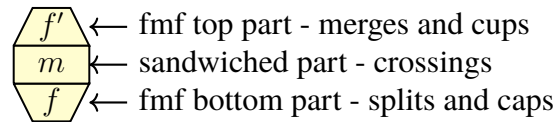
A web is a mfm bottom part if its associated partition contains no splits and caps and has a minimal number of crossings, its a mfm top part if its associated partition contains no merges and cups and has a minimal number of crossings. Dually, we define fmf bottom part and a fmf top part by swapping the roles of splits and merges. Finally, a web is a sandwiched part if it contains only crossings.

Definition 6.15. A web is called a *mfm sandwich diagram* if it is of the form



The set of all many-to-few sandwich diagrams from K to L by X_K^L .

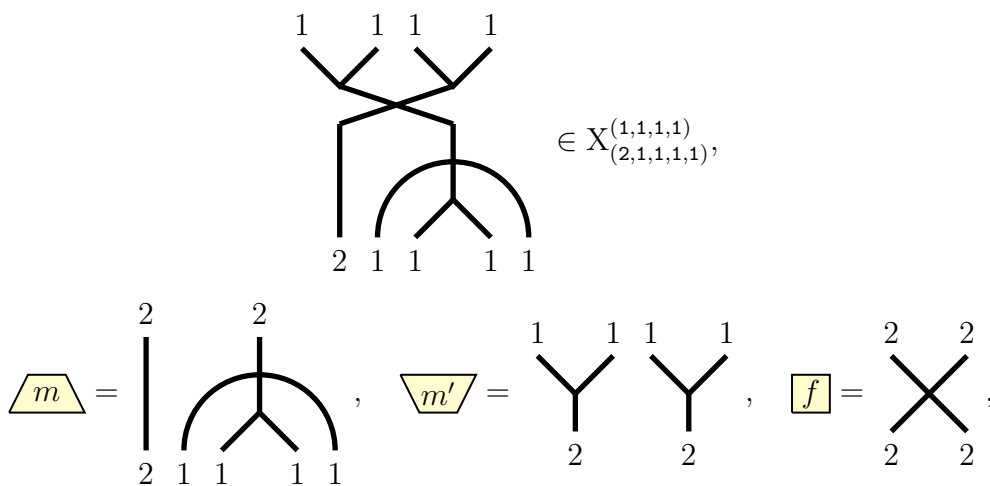
Similarly, a web is called a *fmf sandwich diagram* if it is of the form



For the through strands, say the thicknesses of strands add up to m (this number is the same at every generic horizontal cut). Then we also require the crossings in the middle have to be shortest coset representatives of types (b_1, \dots, b_k) and (t_1, \dots, t_l) in $S(m)$, where the b and t are the bottom and top endpoints of the through strands. The set of all few-to-many sandwich diagrams from K to L is denoted by O_K^L .

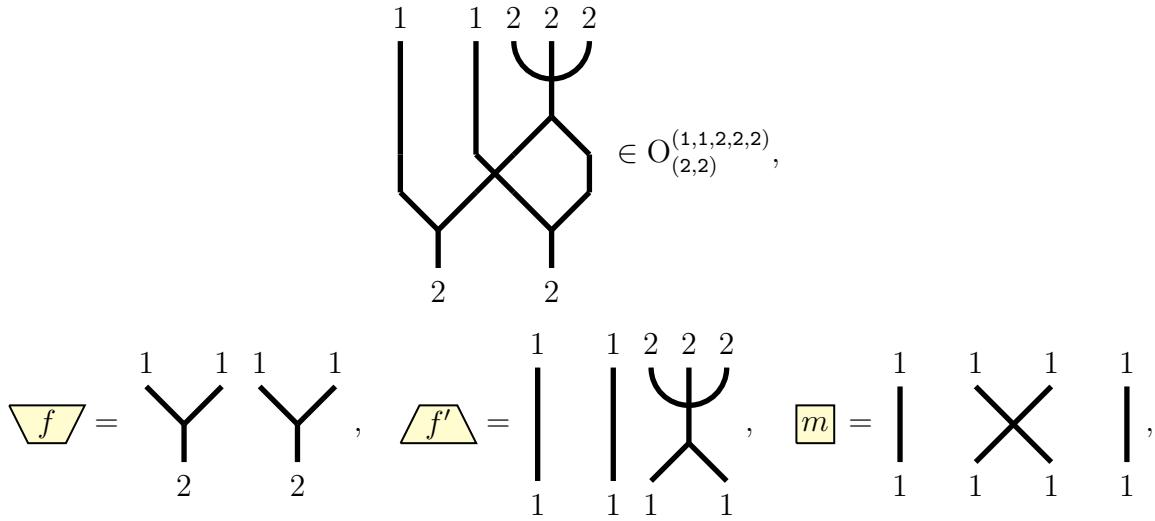
In [Definition 6.15](#) we sandwich a symmetric group in between merges, splits, caps and caps.

Example 6.16. The diagram



is an example of a many-to-few sandwich diagram that we also split into its defining pieces.

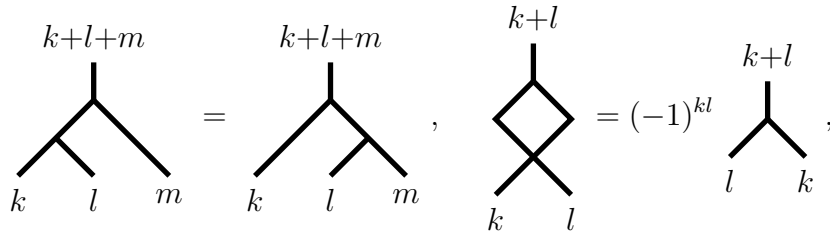
Moreover, the diagram



is an example of an element in $O_{(2,2)}^{(1,1,2,2,2)}$.

Proposition 6.17. *The sets X_K^L and O_K^L are $\mathbb{A}_\triangleright$ -linear spanning sets of $\text{Hom}_{\mathbf{Web}_{\mathbb{A}_\triangleright}(O(N))}(\mathbb{K}, L)$.*

Proof. By Lemma 3.6, $\mathbf{Web}_{\mathbb{A}_\triangleright}(O(N))$, we can push all trivalent vertices and Morse points (cups and caps) to wherever we want them to be. The relations



then ensure that the crossings that end up in the sandwiched part are given by shortest coset representatives for O_K^L . □

Remark 6.18. Our strategy to construct X_K^L is borrowed from semigroup and monoid theory where similar constructions known under the slogan of *Green relations or cells*, see for example [Tub24, Section 4]. Explicitly and in the spirit of sandwich cellularity, [Bro55] worked out a semisimple version of Proposition 6.17 for the Brauer algebra (which sits inside the category $\mathbf{Web}_{\mathbb{A}_\triangleright}(\text{GL}(N))$ by Proposition 6.10).

Constructions similar to O_K^L are sometimes known as *chicken feet bases* and have appeared in several different contexts in the literature, see e.g. [SW11, Definition 5.26], [RT16, Proof of Theorem 1.10], [BEAEO20, Section 4] or [DKM24, Section 3.5].

Neither of these should be confused with cellular or light-ladder-type bases as in [AST18], [Eli15] or [Bod22].

6.4. Orthogonal semisimplifications and colored Brauer diagrams

Recall that \mathbb{F}_ϑ is an infinite field over \mathbb{A}_ϑ . As usual in modular representation theory, let $p = \text{char } \mathbb{F}_\vartheta \in \{3, 5, 7, \dots\} \cup \{\infty\}$ with $p = \infty$ in case the characteristic of \mathbb{F}_ϑ is zero.

Lemma 6.19. *If $p > N$, then there is an equivalence of symmetric ribbon \mathbb{F}_ϑ -linear categories*

$$\overline{\text{Br}_N^{\oplus, \mathbb{C}^\oplus}} \rightarrow \overline{\text{Tilt}_{\mathbb{F}_\vartheta}(\text{O}(N))}.$$

Proof. Our hypothesis on p implies that Λ^k appears as a direct summand of $V^{\otimes k}$ for $k \in [0, N]$. Thus $\text{Fund}_{\mathbb{F}_\vartheta}(\text{O}(N))^{\oplus, \mathbb{C}^\oplus}$ is equivalent to the additive idempotent completion of the full monoidal subcategory generated by V .

The universal property of $\text{Br}_N^{\oplus, \mathbb{C}^\oplus}$ gives us a functor $\text{Br}_N^{\oplus, \mathbb{C}^\oplus} \rightarrow \text{Rep}_{\mathbb{F}_\vartheta}(\text{O}(N))$ sending the generating object to V . This functor is full, see [dCP76, Section 7], so Lemma 6.9 implies there is an equivalence between $\overline{\text{Br}_N^{\oplus, \mathbb{C}^\oplus}}$ and the additive idempotent completion of the full monoidal subcategory generated by V .

The result then follows from Proposition 7.53. □

Lemma 6.20. *If $p > d_i$ for $i \in \{0, \dots, r-1\}$, then*

$$\overline{\text{Br}_{(d_i)}^{\oplus, \mathbb{C}^\oplus}} \rightarrow \boxtimes_{i=0}^{r-1} \overline{\text{Br}_{d_i}^{\oplus, \mathbb{C}^\oplus}} \rightarrow \boxtimes_{i=0}^{r-1} \overline{\text{Tilt}_{\mathbb{F}_\vartheta}(\text{O}(d_i))}$$

are equivalences of symmetric ribbon \mathbb{F}_ϑ -linear categories.

Proof. Combine Lemma 6.13 and Lemma 6.19. □

Now we drop the assumption that $p > N$ and study the semisimplification for $\text{Tilt}_{\mathbb{F}_\vartheta}(\text{O}(N))$.

Definition 6.21. For $k \in \mathbb{Z}_{\geq 0}$ we use $(k)_p = (k_0, k_1, \dots)$ to denote the p -adic digits of k as in section 1.

Let $x, y \in \mathbb{Z}_{\geq 0}$. Then define $x \leq_p y$ if $(x)_p$ is less than or equal to $(y)_p$ entrywise, meaning $x_i \leq y_i$ for all $i \in \mathbb{Z}_{\geq 0}$.

Because of Proposition 7.53, $\text{Tilt}_{\mathbb{F}_\vartheta}(\text{O}(N))$ is the additive idempotent completion of the category $\text{Fund}_{\mathbb{F}_\vartheta}(\text{O}(N))$, which is monoidally generated by the exterior powers Λ^k for $k \in [0, N]$. In fact, after passing to the semisimplification something stronger is true.

Remark 6.22. The main player below is *Lucas' theorem*:

$$\binom{a}{b} \equiv \prod_{i \in \mathbb{Z}_{\geq 0}} \binom{a_i}{b_i}.$$

Here we again use p -adic digits (a_i) and (b_i) for $a, b \in \mathbb{Z}_{\geq 0}$.

Lemma 6.23. *The category $\overline{\text{Fund}_{\mathbb{F}_\vartheta}(\text{O}(N))^{\oplus, \mathbb{C}^\oplus}}$ is \otimes -generated by Λ^{p^i} for $i \geq 0$.*

Proof. For $p = \infty$ there is nothing to show, so assume $p < \infty$. The same argument in the proof of [BEAEO20, Lemma 3.4] shows that

1. If $k \not\leq_p N$, then Lucas' theorem implies that p divides $\dim_{\mathbb{F}_\varphi} \Lambda^k$ and therefore $\overline{\Lambda^k} \cong 0$.
2. If $k \leq_p N$, then Λ^k is a direct summand of $\bigotimes_{i \in \mathbb{Z}_{>0}} (\Lambda^{p^i})^{\otimes k_i}$.

We conclude that $\overline{\mathbf{Fund}_{\mathbb{F}_\varphi}(O(N))^{\oplus, \mathcal{C}_\oplus}}$ is \otimes -generated by the claimed exterior powers. \square

Lemma 6.23 tells us that every object in $\overline{\mathbf{Fund}_{\mathbb{F}_\varphi}(O(N))^{\oplus, \mathcal{C}_\oplus}}$ is a sum of summands of tensor products of p^i th exterior powers. The following lemma helps us understand morphisms between tensor products of p^i th exterior powers.

Lemma 6.24. *Let $a, b \in \mathbb{Z}_{>0}$ such that $a + b = p^i$. The morphisms $\mathbf{Y}_{p^i}^{a,b}$ and $\mathbf{X}_{a,b}^{p^i}$ are zero in $\overline{\mathbf{Fund}_{\mathbb{F}_\varphi}(O(N))^{\oplus, \mathcal{C}_\oplus}}$.*

Proof. Suffices to show the merge in split morphisms are in the negligible ideal. To this end, use [Proposition 3.4](#) and then we can use the same argument as for the proof of [\[BEAEO20, Lemma 4.16\]](#). \square

Lemma 6.25. *Using p -adic digits, in $\mathbf{Web}_{\mathbb{F}_\varphi}(O(N))$ we have*

$$k \bigcirc = N_i.$$

Proof. Another important consequence of Lucas' theorem is that

$$\dim_{\mathbf{Fund}_{\mathbb{F}_\varphi}(O(N))} \Lambda^{p^i} = N_i,$$

where we use the p -adic digits N_i . In particular, the p^i labeled circle in $\mathbf{Web}_{\mathbb{F}_\varphi}(O(N))$ is equal to N_i times the empty diagram. \square

It follows from [Lemma 6.25](#) that there is a symmetric ribbon \mathbb{F}_φ -linear functor

$$\mathbf{Br}_{(N)_p} \rightarrow \mathbf{Web}_{\mathbb{F}_\varphi}(O(N))$$

which sends the color i generating object – which has dimension N_i in $\mathbf{Br}_{(N)_p}$ – to the generating object in $\mathbf{Web}_{\mathbb{F}_\varphi}(O(N))$ labeled p^i . Crossings colored i and j are sent to crossings labeled p^i and p^j , while cups and caps colored i are sent to cups and caps labeled p^i . Composing with the functor

$$\mathbf{Web}_{\mathbb{F}_\varphi}(O(N)) \rightarrow \mathbf{Fund}_{\mathbb{F}_\varphi}(O(N)) \rightarrow \overline{\mathbf{Fund}_{\mathbb{F}_\varphi}(O(N))^{\oplus, \mathcal{C}_\oplus}},$$

then taking additive idempotent completion of $\mathbf{Br}_{(N)_p}$ we get the following.

Lemma 6.26. *The functor $\mathbf{Br}_{\mathbb{A}_\varphi}(N)_{(N)_p}^{\oplus, \mathcal{C}_\oplus} \rightarrow \overline{\mathbf{Fund}_{\mathbb{F}_\varphi}(O(N))^{\oplus, \mathcal{C}_\oplus}}$ is essentially surjective.*

Proof. Immediate from [Lemma 6.23](#). \square

We now prove our second main theorem:

Proof of Theorem 1.4. Our argument is analogous to the proof of [BEAEO20, Theorem 4.17]. It follows from Proposition 6.17 and Lemma 6.24 that the functor

$$\mathbf{Br}_{(N)_p}^{\oplus, \mathbb{C} \oplus} \rightarrow \overline{\mathbf{Fund}_{\mathbb{F}_\vartheta}(O(N))^{\oplus, \mathbb{C} \oplus}}$$

is full. It follows then from Lemma 6.26 and Lemma 6.9 that there is an equivalence

$$\overline{\mathbf{Br}_{(N)_p}^{\oplus, \mathbb{C} \oplus}} \xrightarrow{\cong} \overline{\mathbf{Fund}_{\mathbb{F}_\vartheta}(O(N))^{\oplus, \mathbb{C} \oplus}}.$$

Thus, we have a chain of equivalences

$$\boxtimes_{i=0}^{r-1} \overline{\mathbf{Tilt}_{\mathbb{F}_\vartheta}(O(N_i))} \xleftarrow{\text{Lemma 6.20}} \overline{\mathbf{Br}_{(N)_p}^{\oplus, \mathbb{C} \oplus}} \rightarrow \overline{\mathbf{Fund}_{\mathbb{F}_\vartheta}(O(N))^{\oplus, \mathbb{C} \oplus}} \xrightarrow{\text{Proposition 7.53}} \overline{\mathbf{Tilt}_{\mathbb{F}_\vartheta}(O(N))}.$$

The proof is complete. □

Remark 6.27. By Theorem 1.4, $\overline{\mathbf{Tilt}_{\mathbb{F}_\vartheta}(O(N_i))}$ has finitely many simple objects if and only if all p -adic digits are not 2. To see this note that $\mathbf{Rep}_{\mathbb{F}_\vartheta}(O(2))$ contributes infinitely many simple objects, while all other cases contribute finitely many simple objects.

7. Background: highest weight categories for orthogonal groups

This section summarizes the highest weight theory of the orthogonal group, and also of the special orthogonal group. The former is difficult to find in the literature since it is not simply connected, so we decided to give the details although the material is well-known to experts.

We will work over \mathbb{A} as in Notation 2.1 whose fraction field is \mathbb{Q} , and then switch to \mathbb{A}_ϑ and \mathbb{F}_ϑ for the orthogonal group.

7.1. Tilting representations in general

The following can be found in many works, e.g. [Don93] or [Rin91]. Also the appendix of [Don98] covers lot of material relevant for us, and so does [Jan03]. See also the additional material to [AST18], and the setting in [BS21] that we will use from time to time.

7.1.1 Integral representation theory for semisimple groups

Let \mathfrak{g} be a semisimple Lie algebra over \mathbb{Q} . We have the algebra $U_{\mathbb{A}} = U_{\mathbb{A}}(\mathfrak{g})$ which is the \mathbb{A} -subalgebra of $U_{\mathbb{Q}} = U_{\mathbb{Q}}(\mathfrak{g})$ generated by $e_\alpha^{(a)}$, $f_\alpha^{(a)}$, and $\binom{h_\alpha}{c}$ for all $\alpha \in \Delta$, and $a, b, c \in \mathbb{Z}_{\geq 0}$. Also associated to \mathfrak{g} is a simply connected semisimple group scheme $G_{\mathbb{A}}$. Write $\mathbf{Rep}_{\mathbb{A}} = \mathbf{Rep}(G_{\mathbb{A}})$ for its category of free finite rank $G_{\mathbb{A}}$ -representations (\mathbf{Rep} means in general free finite rank representations). Such a representation gives rise to a free \mathbb{A} -module of finite rank with an action of $U_{\mathbb{A}}$. This gives rise to a fully faithful monoidal functor $\mathbf{Rep}_{\mathbb{A}} \rightarrow \mathbf{Rep}_{\mathbb{A}} U_{\mathbb{A}}$.

The Chevalley involution $\omega: U_{\mathbb{Q}} \rightarrow U_{\mathbb{Q}}$, which swaps e_α and f_α , and negates h_α , also preserves $U_{\mathbb{A}} \subset U_{\mathbb{Q}}$. We also denote the restriction of ω to $U_{\mathbb{A}}$ by ω . Given a $U_{\mathbb{A}}$ -representation m , we obtain another $U_{\mathbb{A}}$ -representation, denoted M^ω , by twisting the action of $U_{\mathbb{A}}$ by ω ,

i.e. $\rho_{M^\omega} = \rho_m \circ \omega$. If m is a free finite rank \mathbb{A} -module, then $M^* = \text{Hom}_{\mathbb{A}}(M, \mathbb{Z})^\omega$ is too. Moreover, since $\omega^2 = 1$, the natural identification of m with its double dual gives a canonical isomorphism of $U_{\mathbb{A}}$ -representations: $M \cong M^{**}$.

Let $\mathfrak{a} \in X^+ = X^+(\mathfrak{g})$. Then, after choosing a Borel subalgebra B , $U_{\mathbb{A}}$ has *induced representations* $\nabla_{\mathbb{A}}(\mathfrak{a}) = \mathbf{Ind}_B^G(-\mathfrak{a})$, and *Weyl representations* $\Delta_{\mathbb{A}}(\lambda) = \nabla_{\mathbb{A}}(\mathfrak{a})^*$. Since $\nabla_{\mathbb{A}}(\mathfrak{a}) \cong \nabla_{\mathbb{A}}(\mathfrak{a})^{**} = \Delta_{\mathbb{A}}(\mathfrak{a})^*$, we also refer to induced representations as *dual Weyl representations*.

If $L_{\mathbb{Q}}(\mathfrak{a})$ is the simple $U_{\mathbb{Q}}$ -representation with fixed highest weight vector $v_{\mathfrak{a}}^+$, of weight \mathfrak{a} , then $\Delta_{\mathbb{A}}(\mathfrak{a}) \cong U_{\mathbb{A}} \cdot v_{\mathfrak{a}} \subset L_{\mathbb{Q}}(\mathfrak{a})$. In fact, $\Delta_{\mathbb{A}}(\mathfrak{a})$ is a free \mathbb{A} -module which is a direct sum of its weight spaces, and therefore has a *character*. This character is equal to the character of $L_{\mathbb{Q}}(\mathfrak{a})$, which in turn is given by *Weyl's character formula*. From $\nabla_{\mathbb{A}}(\mathfrak{a}) = \Delta_{\mathbb{A}}(\mathfrak{a})^*$, we find that $\nabla_{\mathbb{A}}(\mathfrak{a})$ is also a free \mathbb{A} -module, which is a direct sum of its weight spaces, and has character given by the Weyl character formula.

For each $\mathfrak{a} \in X^+$, there is a unique $U_{\mathbb{A}}$ -representation homomorphism, $\text{hs}_{\mathfrak{a}}: \Delta_{\mathbb{A}}(\mathfrak{a}) \rightarrow \nabla_{\mathbb{A}}(\mathfrak{a})$ such that $v_{\mathfrak{a}}^+ \mapsto (v_{\mathfrak{a}}^+)^*$. We may write hs in place of $\text{hs}_{\mathfrak{a}}$ if the weight is clear from context. Let $\mathfrak{a}, \mathfrak{b} \in X^+$. Then we have *Ext-vanishing*:

$$\text{Ext}^i(\Delta_{\mathbb{A}}(\mathfrak{a}), \nabla_{\mathbb{A}}(\mathfrak{b})) = \begin{cases} \mathbb{A} \cdot \text{hs} & \text{if } i = 0 \text{ and } \mathfrak{a} = \mathfrak{b}, \\ 0 & \text{otherwise.} \end{cases}$$

Write $\mathbf{Fil}_{\mathbb{A}}^{\Delta}$ to denote the full subcategory of objects in $\mathbf{Rep}_{\mathbb{A}}$ which admit a filtration by Weyl representations. Similarly, write $\mathbf{Fil}_{\mathbb{A}}^{\nabla}$ for the full subcategory with object admitting filtrations by dual Weyl representations. Define

$$\mathbf{Tilt}_{\mathbb{A}} = \mathbf{Tilt}_{\mathbb{A}}(G) = \mathbf{Fil}_{\mathbb{A}}^{\Delta} \cap \mathbf{Fil}_{\mathbb{A}}^{\nabla}.$$

The objects of this category are *tilting representations*.

Remark 7.1. Using Lusztig's work on canonical bases for quantum groups [Lus10, Part IV], [Kan98] shows that each of the subcategories $\mathbf{Fil}_{\mathbb{A}}^{\Delta}$, $\mathbf{Fil}_{\mathbb{A}}^{\nabla}$, and $\mathbf{Tilt}_{\mathbb{A}}$, is closed under tensor product. Over a field this results is Paradowski's [Par94].

7.1.2 Highest weight theory for reductive groups

Let \mathbb{F} be a field. We can define all the notions from above by scalar extension from \mathbb{A} to \mathbb{F} , and we also get a fully faithful monoidal functor $\mathbf{Rep}_{\mathbb{F}} \rightarrow \mathbf{Rep}_{\mathbb{F}}^{U_{\mathbb{F}}}$.

All the results of the previous section still hold over \mathbb{F} . But now, there is more since we can talk about *simple representations*. For each $\mathfrak{a} \in X^+$, there is a finite dimensional simple representation $L_{\mathbb{F}}(\mathfrak{a})$, which is the unique simple quotient representation of $\Delta_{\mathbb{F}}(\mathfrak{a})$ and the unique simple subrepresentation of $\nabla_{\mathbb{F}}(\mathfrak{a})$. This implies that the map hs factors nontrivially through $L_{\mathbb{F}}(\mathfrak{a})$. Moreover, the set $\{L_{\mathbb{F}}(\mathfrak{a}) | \mathfrak{a} \in X^+\}$ is a complete and irredundant set of simple objects in $\mathbf{Rep}_{\mathbb{F}}$.

Recall that the usual partial order on weights X is defined by $\mathfrak{b} \leq \mathfrak{a}$ if $\mathfrak{a} - \mathfrak{b}$ is an $\mathbb{Z}_{\geq 0}$ -linear combination of positive roots. If $\mathfrak{a} \in X$, then we write

$$X(\leq \mathfrak{a}) = \{\mathfrak{b} \in X | \mathfrak{b} \leq \mathfrak{a}\},$$

and $X^+(\leq a) = X(\leq a) \cap X^+$. Here X^+ means dominant (integral) weights.

For $\pi \subset X^+$ and M in $\mathbf{Rep}_{\mathbb{F}}$, let M_{π} be the largest subrepresentation with all composition factors isomorphic to $L_{\mathbb{F}}(b)$, where $b \in \pi$. Define $\mathbf{Rep}_{\mathbb{F}}(\pi)$ to be the full subcategory of $\mathbf{Rep}_{\mathbb{F}}$ with objects $M = M_{\pi}$. In the case that $\pi = X^+(\leq a)$, we simply write $M_{\leq a}$ and $\mathbf{Rep}_{\mathbb{F}}(\leq a)$.

Fix $a \in X^+$. Since $L_{\mathbb{F}}(b) = L_{\mathbb{F}}(b)_{\leq b}$ and Weyl and dual Weyl representations have the same character as $L_{\mathbb{Q}}(a)$, it is easy to see that $\Delta_{\mathbb{F}}(a) = \Delta_{\mathbb{F}}(a)_{\leq a}$ and $\nabla_{\mathbb{F}}(a) = \nabla_{\mathbb{F}}(a)_{\leq a}$. One can show that

$$\mathrm{Ext}_{\mathbf{Rep}_{\mathbb{F}}(\leq a)}^{>0}(\Delta_{\mathbb{F}}(a), L_{\mathbb{F}}(a)) = 0, \quad \mathrm{Ext}_{\mathbf{Rep}_{\mathbb{F}}(\leq a)}^{>0}(L_{\mathbb{F}}(a), \nabla_{\mathbb{F}}(a)).$$

Since $\Delta_{\mathbb{F}}(a)$ has $L_{\mathbb{F}}(a)$ as its unique simple quotient, we can say that $\Delta_{\mathbb{F}}(a)$ is a projective cover of $L_{\mathbb{F}}(a)$ in $\mathbf{Rep}_{\mathbb{F}}(\leq a)$. Similarly, $\nabla_{\mathbb{F}}(a)$ is an injective hull of $L_{\mathbb{F}}(a)$ in $\mathbf{Rep}_{\mathbb{F}}(\leq a)$.

Thus, the category $\mathbf{Rep}_{\mathbb{F}}$ is a (semi-infinity) *highest weight category*. It follows from that we have the following classification of indecomposable objects in $\mathbf{Tilt}_{\mathbb{F}}$:

- (i) For each $a \in X^+$, there is an *indecomposable tilting representations*, denoted $T_{\mathbb{F}}(a)$, which has as part of its Weyl filtration a subrepresentation $\Delta_{\mathbb{F}}(a) \rightarrow T_{\mathbb{F}}(a)$.
- (ii) If $a \neq b$, then $T_{\mathbb{F}}(a) \not\cong T_{\mathbb{F}}(b)$.
- (iii) Every indecomposable object in $\mathbf{Tilt}_{\mathbb{F}}$ is of the form $T_{\mathbb{F}}(a)$ for some $a \in X^+$.

7.1.3 Saturated sets

It is not difficult to check that $\mathbf{Rep}_{\mathbb{F}}(\leq a)$ is also a highest weight category. We want to generalize this property for other subsets $\pi \subset X^+$. The key is the following:

Definition 7.2. A set $S \subset X$ is *saturated* if for all $a \in X$ and for all $\alpha \in \Phi^+$, then

$$a - i\alpha \in S \quad \text{when} \quad \begin{cases} 0 \leq i \leq \alpha^{\vee}(a) & \text{if } \alpha^{\vee}(a) \geq 0, \\ \alpha^{\vee}(a) \leq i \leq 0 & \text{if } \alpha^{\vee}(a) < 0. \end{cases}$$

Here Φ^+ denotes the set of positive roots.

Lemma 7.3. *Saturated sets are invariant under unions, intersections, and under the action of the Weyl group W associated to \mathfrak{g} .*

Proof. The first two claims are immediate. Moreover, since W is generated by reflections, the claim follows from considering the formula for the reflection perpendicular to $\alpha \in \Phi$, $s_{\alpha}(a) = a - \alpha^{\vee}(a) \cdot \alpha$, while noting that $s_{\alpha} = s_{-\alpha}$. \square

The prototypical example of a saturated set of weights is the set of weights in a Weyl representation. Moreover, from [Ste98, Theorem 1.9] we have

$$\mathrm{wt} V(a) = W \cdot X^+(\leq a).$$

Here $\mathrm{wt} L_{\mathbb{Q}}(a)$ denotes the weights of the simple highest weight $\mathfrak{g}(\mathbb{Q})$ -representation $L_{\mathbb{Q}}(a)$. This suggests the following definition.

Definition 7.4. A set of dominant weights $\pi \subset X^+$ is *saturated* if for all $\mathbf{a} \in \pi$, we have $X^+(\leq \mathbf{a}) \subset \pi$.

Example 7.5. The prototypical example of a set of dominant weights which is saturated is $X^+(\leq \mathbf{a})$.

Lemma 7.6. *If $S \subset X$ is a saturated set of weights, then $S \cap X^+$ is a saturated set of dominant weights.*

Proof. See [Ste98, Proof of Lemma 1.8]. □

Remark 7.7. The definition of a set of weights being saturated is classical, see e.g. [Bou02, Exercises VI.1.23-24 and VI.2.5]. The notion of a set of dominant weights being saturated came later, see e.g. [Don98, Definition A3].

Proposition 7.8. *If $\pi \subset X^+$ is saturated, then $\mathbf{Rep}_{\mathbb{F}}(\pi)$ is a highest weight category, with indexing set π and partial order induced from X^+ by $\pi \subset X^+$.*

Proof. This is [Don98, Proposition A3.4]. □

Finally, we state a Lemma which makes it easy to verify certain sets are saturated. This lemma is comparable to [DGS06, Proposition 1.3.2].

Lemma 7.9. *Suppose V is any finite dimensional $\mathfrak{g}(\mathbb{Q})$ -representation, then $\text{wt } V$ is saturated.*

Proof. Since V is completely reducible, we have

$$\text{wt } V = \text{wt} \bigoplus_{\mathbf{a} \in X^+} L_{\mathbb{Q}}(\mathbf{a})^{\oplus [V:L_{\mathbb{Q}}(\mathbf{a})]} = \bigcup_{\mathbf{a} \in X^+, [V:L_{\mathbb{Q}}(\mathbf{a})] \neq 0} \text{wt } L_{\mathbb{Q}}(\mathbf{a}),$$

The claim follows from observing that each $\text{wt } L_{\mathbb{Q}}(\mathbf{a})$ is saturated and being saturated is closed under unions, cf. Lemma 7.3. □

Remark 7.10. This criterion is particularly useful when we have a finite dimensional representation in $\mathbf{Rep}_{\mathbb{F}}$ which comes from a representation over $U_{\mathbb{A}}$, since then we can extend scalars from \mathbb{A} to \mathbb{Q} to verify the weight spaces are saturated.

7.1.4 Schur algebras

Proposition 7.8 suggests the following definition.

Definition 7.11. The *generalized Schur algebra* associated to a saturated set of dominant weights $\pi \subset X^+$, denoted $S_{\mathbb{A}}^{\leq N}(\mathfrak{g})$, or $S_{\mathbb{A}}^{\leq N}$ if \mathfrak{g} is understood, is defined as the quotient of $\dot{U}_{\mathbb{A}}$ by the ideal generated by $\mathbf{1}_{\chi}$ for all $\chi \notin W \cdot \pi$.

The algebra $S_{\mathbb{A}}^{\leq N}$ is an associative algebra with unit $\mathbf{1}_{\pi} = \sum_{\chi \in W \cdot \pi} \mathbf{1}_{\chi}$.

Remark 7.12. If V is a $U_{\mathbb{F}}$ -representation, then V_{π} is naturally a representation over $S_{\mathbb{F}}^{\leq N}$. In fact, using [Don98, Proposition A3.2(ii)], one finds there is an equivalence of additive \mathbb{F} -linear categories $\mathbf{Rep}_{\mathbb{F}}(\pi) \cong \mathbf{Rep}_{\mathbb{F}} S_{\mathbb{F}}^{\leq N}$.

The canonical basis \mathbb{B} for $\dot{U}_{\mathbb{A}}$ descends to a canonical basis $\mathbb{B}[\pi] = \prod_{\mathbf{a} \in \pi} \mathbb{B}[\mathbf{a}]$, where $\mathbb{B}[\mathbf{a}]$ is as defined in [Lus10, 29.1]. This renders $S_{\mathbb{A}}^{\leq N}$ a *based module*, as a left representation over $\dot{U}_{\mathbb{A}}$, and therefore $S_{\mathbb{A}}^{\leq N}$ has a filtration by Weyl representations, see [Lus10, Section 27.1.7].

A representation with a Weyl filtration will always embed into a tilting representations, cf. [BT25, Lemma 5B.11]. In particular, $S_{\mathbb{F}}^{\leq N}$ embeds in a tilting representations.

We learned the following key lemma from [AR96]:

Lemma 7.13. *A full tilting representations for $S_{\mathbb{F}}^{\leq N}$ is faithful.*

Proof. See [BT25, Proposition 5B.13]. □

Complete reducibility of finite dimensional representations over $U_{\mathbb{Q}}$ implies that

$$S_{\mathbb{Q}}^{\leq N} \cong \prod_{\mathbf{a} \in \pi} \text{End}(L_{\mathbb{Q}}(\mathbf{a})).$$

Since $L_{\mathbb{Q}}(\mathbf{a}) \cong \Delta_{\mathbb{Q}}(\mathbf{a})$, and $\Delta_{\mathbb{A}}(\mathbf{a})$ has the same formal character as $\Delta_{\mathbb{Q}}(\mathbf{a})$, it follows that $S_{\mathbb{A}}^{\leq N}$ has Weyl character

$$(S_{\mathbb{A}}^{\leq N} : \Delta_{\mathbb{A}}(\mathbf{a})) = \begin{cases} \text{rank}_{\mathbb{A}} \Delta_{\mathbb{A}}(\mathbf{a}) & \text{if } \mathbf{a} \in \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

The analog equality then follows for $S_{\mathbb{F}}^{\leq N}$, since $S_{\mathbb{A}}^{\leq N}$ is a free \mathbb{A} -module with basis $\mathbb{B}[\pi]$.

7.2. Tilting representations for orthogonal groups

The orthogonal group is disconnected, with identity component the special orthogonal group and component group $\mathbb{Z}/2\mathbb{Z}$. However, the usual theory of tilting representations for connected reductive groups can be modified as follows. First, following [AHR20] (taking a slightly different perspective in some places), we describe how to think about representations of the orthogonal group as a highest weight category. General theory from [BS21] then implies the existence of tilting representations for orthogonal groups.

7.2.1 Representations of $O(N)$

Following the conventions from before:

Notation 7.14. *We will write $\text{Rep}_{\mathbb{A}_{\heartsuit}}(O(N))$ and $\text{Rep}_{\mathbb{A}_{\heartsuit}}(\text{SO}(N))$ for the respective categories of finite dimensional representations over \mathbb{A}_{\heartsuit} . We also use similar notation that should be easy to guess from the context.*

The following defines an involutive algebra automorphism as one easily checks:

Definition 7.15. We define a map $\sigma : U_{\mathbb{A}_{\heartsuit}}(\mathfrak{so}_N) \rightarrow U_{\mathbb{A}_{\heartsuit}}(\mathfrak{so}_N)$ by:

1. When $N = 2n + 1$ we let

$$\begin{aligned} e_i &\mapsto e_i, f_i \mapsto f_i, h_i \mapsto h_i \text{ for } i \in [1, N - 1], \\ e_N &\mapsto -e_N, f_N \mapsto -f_N, h_N \mapsto h_N. \end{aligned}$$

2. When $N = 2n$ we let

$$\begin{aligned} e_i &\mapsto e_i, f_i \mapsto f_i, h_i \mapsto h_i \text{ for } i \in [1, N-2], \\ e_{N-1} &\mapsto e_N, f_{N-1} \mapsto f_N, h_{N-1} \mapsto h_N. \end{aligned}$$

In the even case σ is the automorphism induced by the type D Dynkin diagram automorphism swapping the fishtail vertices.

Write $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)^\sigma$ to denote the \mathbb{A}_∇ -algebra generated by $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$ and $\mathbb{A}_\nabla[\sigma]/(\sigma^2)$ subject to the relation

$$\sigma X \sigma^{-1} = \sigma(X) \quad \text{for all } X \in U_{\mathbb{A}_\nabla}(\mathfrak{so}_N). \quad (7.2)$$

Lemma 7.16. *As a right $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$ -representation $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)^\sigma$ is freely generated by 1 and σ .*

Proof. Boring and omitted. \square

We can view a finite dimensional representation of $\mathrm{SO}(N)$ as a finite dimensional $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$ -representation, with a weight decomposition, such that the dominant weights are contained in $X^+(\mathrm{SO}(N))$ [Jan03, Sections 7.14–7.17]. From this perspective, a finite dimensional representation of $\mathrm{O}(N)$ can be viewed as a finite dimensional $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)^\sigma$ -representation, with a weight decomposition, such that the dominant weights are contained in $X^+(\mathrm{SO}(N))$.

The Chevalley involution $\omega: U_{\mathbb{A}_\nabla}(\mathfrak{so}_N) \rightarrow U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$ swaps e_α and f_α and negates h_α .

Lemma 7.17. *The Chevalley involution commutes with σ and preserves the relations $\sigma^2 = 1$ and $\sigma X \sigma^{-1} = \sigma(X)$, for $X \in U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$. Thus, we can extend ω to an automorphism of $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)^\sigma$.*

Proof. Easy and omitted. \square

As usual, we can use the Chevalley involution to define the dual $\mathrm{O}(N)$ -representation by $U^* = \mathrm{Hom}_{\mathbb{A}_\nabla}(U, \mathbb{A}_\nabla)^\omega$ where $U \in \mathbf{Rep}_{\mathbb{A}_\nabla}(\mathrm{O}(N))$. Moreover, suppose that W is a finite dimensional $\mathrm{SO}(N)$ -representation. View W as a $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$ -representation. Then we define induction and restriction

$$\mathcal{I}_{\mathrm{SO}}^{\mathrm{O}}(-): \mathbf{Rep}_{\mathbb{A}_\nabla}(\mathrm{O}(N)) \rightleftarrows \mathbf{Rep}_{\mathbb{A}_\nabla}(\mathrm{SO}(N)): \mathcal{R}_{\mathrm{O}}^{\mathrm{SO}}(-)$$

as follows. Before doing so, note that, given a finite dimensional representation U of $\mathrm{O}(N)$, we obtain a $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)^\sigma$ -representation structure on U . We then define:

$$\begin{aligned} \mathcal{I}_{\mathrm{SO}}^{\mathrm{O}}(W) &= U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)^\sigma \otimes_{U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)} W, \\ \mathcal{R}_{\mathrm{O}}^{\mathrm{SO}}(U) &= \mathcal{R}_{U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)^\sigma}^{U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)}(U). \end{aligned}$$

Lemma 7.18. *Induction and restriction are exact.*

Proof. Since $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)^\sigma$ is free as a right $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$ -representation, $\mathcal{I}_{\mathrm{SO}}^{\mathrm{O}}$ is exact. A similar argument works for $\mathcal{R}_{\mathrm{O}}^{\mathrm{SO}}$. \square

7.2.2 Dominant weights for $O(N)$

We will write

$$n = \begin{cases} \frac{N-1}{2} & \text{if } N \text{ is odd,} \\ \frac{N}{2} & \text{if } N \text{ is even.} \end{cases}$$

Let W be a finite dimensional $O(N)$ -representation. Then W is naturally an $SO(N)$ -representation, by restriction, and therefore decomposes into a direct sum of weight spaces indexed by $X(SO(N)) = X(\mathfrak{so}_N) \cap \bigoplus_{i=1}^N \mathbb{Z}\epsilon_i$. If $w \in W$ is an $SO(N)$ -weight vector, then we write $\text{wt}_{SO}(w)$ for the corresponding element in $X(SO(N))$. The dominant weights of simple $SO(N)$ -representations are parameterized by the set

$$X^+(SO(N)) = \{a_1\epsilon_1 + \dots + a_n\epsilon_n \mid a_i \in \mathbb{Z}, a_1 \geq \dots \geq a_n, \begin{matrix} a_n \geq 0 & \text{if } N \text{ is odd} \\ a_{n-1} \geq |a_n| & \text{if } N \text{ is even} \end{matrix}\} \subset X^+(\mathfrak{so}_N).$$

Weights for $O(N)$ are pairs of data, the $SO(N)$ -weight, and a “weight” for σ :

Definition 7.19. If U is a finite dimensional $O(N)$ -representation and $u \in U$ is a σ eigenvector with $\sigma \cdot u = \epsilon \cdot u$, then we write $\epsilon_\sigma(u) = \epsilon$. If $u \in U$ is not a σ eigenvector, then we write $\epsilon_\sigma(u) = 0$. If u is also a weight vector for $SO(N)$, then we write

$$\text{wt}_O(u) = (\text{wt}_{SO}(u), \epsilon_\sigma(u)),$$

and call it the $O(N)$ -weight of u .

The following partial order is taken from [AR96, Section 1].

Definition 7.20. Let $X(O(N))$ be the set of all pairs (\mathbf{a}, ϵ) which appear as weights $\text{wt}_O(u)$ for $u \in U$, where U ranges over all finite dimensional $O(N)$ -representations. Let $X^+(O(N))$ denote the dominant weights, that is pairs of the form

$$\begin{aligned} &(\mathbf{a}, \pm 1), \text{ for } \mathbf{a} \in X^+(SO(N)), \text{ such that } \sigma(\mathbf{a}) = \mathbf{a}, \\ &(\mathbf{a}, 0), \text{ for } \mathbf{a} \in X^+(SO(N)), \text{ such that } \sigma(\mathbf{a}) \neq \mathbf{a}. \end{aligned}$$

The partial order that we will use is: $(\mathbf{a}, \epsilon) < (\mathbf{b}, \epsilon')$ if and only if $\mathbf{a} < \mathbf{b}$ or $\mathbf{a} < \sigma(\mathbf{b})$.

Let σ denote the generator of the group of automorphisms of the Dynkin diagram for \mathfrak{so}_N . This is trivial when N is odd and a nontrivial involution when N is even. The Dynkin diagram automorphism induces maps, which we also denote σ , on all objects which are determined by the \mathfrak{so}_N Dynkin diagram. In particular, σ acts on $X(\mathfrak{so}_N)$, preserving the subset $X^+(\mathfrak{so}_N)$. Note that σ is the identity when N is odd. When N is even, σ acts on $X(\mathfrak{so}_N)$ by $(a_1, \dots, a_{n-1}, a_n) \mapsto (a_1, \dots, a_{n-1}, -a_n)$.

Definition 7.21. Let Λ_+ be the set of all partitions, that is weakly decreasing sequences of elements in $\mathbb{Z}_{\geq 0}$. We identify partitions with their Young diagram. Taking the transpose of the Young diagram determines an involution of Λ_+ , denoted by $\lambda \mapsto \lambda^T$ (the transpose diagram). Define the dominant $O(N)$ -weights to be

$$\Lambda_+^{O(N)} = \{\lambda \in \Lambda_+ \mid (\lambda^T)_1 + (\lambda^T)_2 \leq N\}.$$

Let further $\mathcal{Y}: X^+(SO(N)) \rightarrow \Lambda_+$ be defined by $\sum_{i=1}^N a_i \epsilon_i \mapsto (a_1, \dots, a_{n-1}, |a_n|)$.

Remark 7.22. If $\mathbf{a} \in X^+(\mathrm{SO}(N))$, then $\mathcal{Y}(\mathbf{a}) = \mathcal{Y}(\sigma(\mathbf{a}))$.

Remark 7.23. The image of the map \mathcal{Y} is contained in $\Lambda_+^{\mathrm{O}(N)}$. We saw in [Remark 7.22](#) that \mathcal{Y} is not injective when N is even, and \mathcal{Y} is injective when N is odd. Moreover, \mathcal{Y} is not surjective. The image of \mathcal{Y} is the subset of $\lambda \in \Lambda_+^{\mathrm{O}(N)}$ such that $(\lambda^T)_1 \leq n$.

The set $\Lambda_+^{\mathrm{O}(N)}$ is not closed under taking the transpose, but there is another involution on this set.

Definition 7.24. Define the *twisting involution* on $\Lambda_+^{\mathrm{O}(N)}$, denoted $\lambda \mapsto \lambda^{tw}$, by $\lambda^{tw} = (N - (\lambda^T)_1, (\lambda^T)_2, \dots)^T$.

In words: the twist of λ has the same Young diagram, except the first column is replaced with $N - (\lambda^T)_1$ boxes.

Remark 7.25. The fixed points of $\lambda \mapsto \lambda^{tw}$ are exactly the λ such that $(\lambda^T)_1 = (\lambda^T)_2$. In particular, if N is odd, then the twisting involution on $\Lambda_+^{\mathrm{O}(N)}$ does not have any fixed points.

Lemma 7.26. *We have the following.*

1. The map $\tau: X^+(\mathrm{O}(N)) \rightarrow \Lambda_+^{\mathrm{O}(N)}$

$$\begin{aligned} \tau: X^+(\mathrm{O}(N)) &\rightarrow \Lambda_+^{\mathrm{O}(N)}, \\ (\mathbf{a}, +1) &\mapsto (\mathbf{a}_1, \dots, \mathbf{a}_N), \quad (\mathbf{a}, -1) \mapsto (\mathbf{a}_1, \dots, \mathbf{a}_N)^{tw}, \text{ for } \sigma(\mathbf{a}) = \mathbf{a}, \\ (\mathbf{a}, 0) &\mapsto (\mathbf{a}_1, \dots, \mathbf{a}_{n-1}, |\mathbf{a}_N|) \text{ otherwise,} \end{aligned}$$

is a bijection.

2. The image of the map

$$\mathcal{Y}: X^+(\mathrm{SO}(N)) \rightarrow \Lambda_+^{\mathrm{O}(N)}$$

is a fundamental domain for the twisting involution acting on $\Lambda_+^{\mathrm{O}(N)}$, and $\sigma(\mathbf{a}) \neq \mathbf{a}$ if and only if $\mathcal{Y}(\mathbf{a})^{tw} = \mathcal{Y}(\mathbf{a})$.

Proof. Not difficult and omitted. □

Thus, we get:

Remark 7.27. There are three ways to encode a dominant weight for the orthogonal group in the literature and that we use in this paper:

1. $(\mathbf{a}, \epsilon) \in X^+(\mathrm{SO}(N)) \times \{0, \pm 1\}$,
2. $(\mathcal{Y}(\mathbf{a}), \epsilon) \in \Lambda_+ \times \{0, \pm 1\}$, and
3. $\tau(\mathbf{a}, \epsilon) \in \Lambda_+^{\mathrm{O}(N)}$.

Which one is more convenient depends on the context.

Notation 7.28. For a finite dimensional $O(N)$ -representation U , and $(\mathbf{a}, \epsilon) \in X(O(N))$, we write

$$[(\mathbf{a}, \epsilon)]U = \{u \in U \mid \text{wt}_{O(N)}(u) = (\mathbf{a}, \epsilon)\}$$

for the (\mathbf{a}, ϵ) -weight space of U . For $\lambda \in \Lambda_+^{O(N)}$, there is a corresponding $(\mathbf{a}, \epsilon) \in X^+(O(N))$, and we will write

$$[\lambda]U = [(\mathbf{a}, \epsilon)]U,$$

for any finite dimensional $O(N)$ -representation U .

We define the $O(N)$ partial order on $\Lambda_+^{O(N)}$ as follows.

Definition 7.29. Suppose $\lambda, \mu \in \Lambda_+^{O(N)}$ correspond to (\mathbf{a}, ϵ) and (\mathbf{b}, ϵ') , respectively. Let

$$\mu <_{O(N)} \lambda \text{ if } \mathbf{b} < \mathbf{a}, \text{ or } \sigma(\mathbf{b}) < \mathbf{a},$$

where $<$ on $X^+(\text{SO}(N))$ is the restriction of the usual partial order on $X(\mathfrak{so}_N)$.

Lemma 7.30. Definition 7.29 defines a partial order.

Proof. This follows from [AHR20, Lemma 3.1]. □

Example 7.31. Consider $\lambda = (0^N)$ and $\mu = (1^N)$. In this case, both λ and μ correspond via \mathcal{Y} to $0 \in X^+(\text{SO}(N))$. However, since $\lambda \neq \mu$, the partitions are not comparable with respect to $<_{O(N)}$. On the other hand, we have $(0^N) <_{O(N)} (1^2, 0^{N-2})$ and $(1^N) <_{O(N)} (1^2, 0^{N-2})$.

Lemma 7.32. We have $\mu < \lambda$ if and only if one of the following holds:

$$\mathbf{b} < \mathbf{a}, \text{ or } \mathbf{b} < \sigma(\mathbf{a}), \text{ or } \sigma(\mathbf{b}) < \mathbf{a}, \text{ or } \sigma(\mathbf{b}) < \sigma(\mathbf{a}).$$

Proof. Since σ is a Dynkin diagram automorphism, it preserves the usual partial order on the set $X(\mathfrak{so}_N)$, and therefore

$$\mathbf{b} < \mathbf{a} \text{ if and only if } \sigma(\mathbf{b}) < \sigma(\mathbf{a}).$$

Since σ is an involution, we have

$$\sigma(\mathbf{b}) < \mathbf{a} \text{ if and only if } \mathbf{b} < \sigma(\mathbf{a}).$$

Thus, the claim follows. □

Definition 7.33. Given two partitions \mathcal{Y} and \mathcal{Y}' , we say that $\mathcal{Y} \trianglelefteq \mathcal{Y}'$ if $\sum_{i=1}^k \mathcal{Y}_i \leq \sum_{i=1}^k \mathcal{Y}'_i$ for all $k \geq 0$.

Note that $(1^N) <_{O(N)} (1^2, 0^{N-2})$, but $(1^2, 0^{N-2}) \triangleleft (1^N)$. Thus, the order on partitions from Definition 7.33 is not adapted to $\Lambda_+^{O(N)}$. It is however useful to use \trianglelefteq to compare \mathbf{a} and \mathbf{b} in $X^+(\text{SO}(N))$ when considering $\mathcal{Y}(\mathbf{a})$ and $\mathcal{Y}(\mathbf{b})$.

Lemma 7.34. *Let $\mathbf{a}, \mathbf{b} \in X^+(\mathrm{SO}(N))$. If $\mathbf{a} \leq \mathbf{b}$, then $\mathcal{Y}(\mathbf{a}) \leq \mathcal{Y}(\mathbf{b})$.*

Proof. We prove this for $n = 4$, the general case when $N = 2n$ is an exercise and when $N = 2n + 1$ is an easier exercise.

Suppose $\mathbf{a}, \mathbf{b} \in X^+(\mathrm{SO}(8)) \subset \bigoplus_{i=1}^4 \mathbb{Z}\epsilon_i$ and that $\mathbf{a} \leq \mathbf{b}$. Since a $\mathbb{Z}_{\geq 0}$ -linear combination of positive roots is a $\mathbb{Z}_{\geq 0}$ -linear combination of simple roots, we have $\mathbf{b} - \mathbf{a} = w\alpha_1 + x\alpha_2 + y\alpha_3 + z\alpha_4$, where $w, x, y, z \in \mathbb{Z}_{\geq 0}$. Thus, $\mathbf{b} = (\mathbf{a}_1 + w, \mathbf{a}_2 - w + x, \mathbf{a}_3 - x + y + z, \mathbf{a}_4 - y + z)$ and

- $\mathbf{b}_1 - \mathbf{a}_1 = w \geq 0$,
- $(\mathbf{b}_1 + \mathbf{b}_2) - (\mathbf{a}_1 + \mathbf{a}_2) = x \geq 0$,
- $(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3) - (\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3) = y + z \geq 0$, and
- $(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + |\mathbf{b}_4|) - (\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + |\mathbf{a}_4|) = y + z + |\mathbf{a}_4 - y + z| - |\mathbf{a}_4| \geq 0$.

The last inequality follows from noticing that $|z - y| \leq |z| + |y| = z + y$, so

$$|\mathbf{a}_4| = |\mathbf{a}_4 + (z - y) - (z - y)| \leq |\mathbf{a}_4 + z - y| + |z - y| \leq |\mathbf{a}_4 + z - y| + z + y.$$

The proof is complete. \square

Definition 7.35. Let $\lambda, \mu \in \Lambda_+^{\mathrm{O}(N)}$ correspond to (\mathbf{a}, ϵ) and (\mathbf{b}, ϵ') , respectively. We define a partial order on $\Lambda_+^{\mathrm{O}(N)}$ by declaring $\lambda < \mu$ if $\mathcal{Y}(\mathbf{a}) \triangleleft \mathcal{Y}(\mathbf{b})$.

Lemma 7.36. *If $\lambda \leq_{\mathrm{O}(N)} \mu$, then $\lambda \leq \mu$.*

Proof. Follows from [Definition 7.29](#), [Lemma 7.34](#), and [Remark 7.22](#). \square

7.2.3 Standard representations for $\mathrm{O}(N)$

Since U is a finite dimensional $\mathrm{O}(N)$ -representation. It in particular has a weight space decomposition as a representation of $\mathrm{SO}(N)$.

Lemma 7.37. *As $\mathrm{SO}(N)$ -representations we have*

$$\mathcal{R}_{\mathrm{O}}^{\mathrm{SO}} \mathcal{I}_{\mathrm{SO}}^{\mathrm{O}}(\Delta_{\mathbb{A}_{\heartsuit}}(\mathbf{a})) \cong \Delta_{\mathbb{A}_{\heartsuit}}(\mathbf{a}) \oplus \Delta_{\mathbb{A}_{\heartsuit}}(\sigma(\mathbf{a})).$$

Proof. It is easy to see from [Equation 7.2](#) that if $u \in U_{\lambda}$, then $\sigma(u) \in U_{\sigma(\lambda)}$, and that if $u \in U$ is annihilated by $U_{\mathbb{A}_{\heartsuit}}(\mathfrak{so}_N)^+$, then so is $\sigma(u)$. It follows that

$$1 \otimes v_{\mathbf{a}}^+ \mapsto (v_{\mathbf{a}}^+, 0), \quad \sigma \otimes v_{\mathbf{a}}^+ \mapsto (0, v_{\sigma(\mathbf{a})}^+),$$

is the desired isomorphism. \square

Suppose that $\sigma(\mathbf{a}) = \mathbf{a}$. Then we let

$$v_{(\mathbf{a},+1)}^+ = \frac{1}{2} (1 \otimes v_{\mathbf{a}}^+ + \sigma \otimes v_{\mathbf{a}}^+), \quad v_{(\mathbf{a},-1)}^+ = \frac{1}{2} (1 \otimes v_{\mathbf{a}}^+ - \sigma \otimes v_{\mathbf{a}}^+).$$

Lemma 7.38. *Each $v_{(\mathbf{a}, \pm 1)}$ generates an $O(N)$ -subrepresentation of $\mathcal{I}_{SO}^O(\Delta_{\mathbb{A}_\nabla}(\mathbf{a}))$. Moreover,*

$$U_{\mathbb{A}_\nabla}(\mathfrak{so}_N) \cdot v_{(\mathbf{a}, +1)} \oplus U_{\mathbb{A}_\nabla}(\mathfrak{so}_N) \cdot v_{(\mathbf{a}, -1)} = \mathcal{R}_O^{\text{SO}} \mathcal{I}_{SO}^O(\Delta_{\mathbb{A}_\nabla}(\mathbf{a})) \xrightarrow{\cong} \Delta_{\mathbb{A}_\nabla}(\mathbf{a}) \oplus \Delta_{\mathbb{A}_\nabla}(\mathbf{a})$$

induces isomorphisms of $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$ -representations, $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N) \cdot v_{(\mathbf{a}, \epsilon)} \cong \Delta_{\mathbb{A}_\nabla}(\mathbf{a})$, for $\epsilon \in \{\pm 1\}$.

Proof. The vectors $v_{(\mathbf{a}, \pm 1)}$ are highest weight vectors for $U_{\mathbb{A}_\nabla}(\mathfrak{so}_N)$ and eigenvectors, with eigenvalues ± 1 respectively, for σ . It follows from Equation 7.2, that for $\epsilon \in \{\pm 1\}$, the $O(N)$ -subrepresentation generated by $v_{(\mathbf{a}, \epsilon)}^+$ is the ϵ eigenspace of $\mathcal{I}_{SO}^O(\Delta_{\mathbb{A}_\nabla}(\mathbf{a}))$.

The second claim can then easily be checked. □

Using these lemmas we can make the following definition.

Definition 7.39. We define the *Weyl representation* for $O(N)$ with highest weight $\lambda \in \Lambda_+^{O(N)}$ as

$$\Delta_{\mathbb{A}_\nabla}(\lambda) = \begin{cases} U_{\mathbb{A}_\nabla}(\mathfrak{so}_N) \cdot v_{(\mathbf{a}, +1)} & \text{if } \sigma(\mathbf{a}) = \mathbf{a} \text{ and } \lambda = \mathcal{Y}(\mathbf{a}), \\ U_{\mathbb{A}_\nabla}(\mathfrak{so}_N) \cdot v_{(\mathbf{a}, -1)} & \text{if } \sigma(\mathbf{a}) = \mathbf{a} \text{ and } \lambda = \mathcal{Y}(\mathbf{a})^{tw} \\ \mathcal{I}_{SO}^O(\Delta_{\mathbb{A}_\nabla}(\mathbf{a})) & \text{if } \sigma(\mathbf{a}) \neq \mathbf{a}. \end{cases}$$

We define the *dual Weyl representation* with highest weight $\lambda \in \Lambda_+^{O(N)}$ as the dual space $\nabla_{\mathbb{A}_\nabla}(\lambda) = \Delta_{\mathbb{A}_\nabla}(\lambda)^*$.

There is another natural definition of dual Weyl representation, paralleling our definition of Weyl representation. That is as a summand of a dual Weyl representation for $SO(N)$ induced to $O(N)$. But in fact one arrives at the same definition this way.

Lemma 7.40. *For $\mathbf{a} \in X^+(SO(N))$, we have*

$$\begin{aligned} \mathcal{I}_{SO}^O(\Delta_{\mathbb{A}_\nabla}(\mathbf{a})) &\cong \begin{cases} \Delta_{\mathbb{A}_\nabla}(\mathcal{Y}(\mathbf{a})) \oplus \Delta_{\mathbb{A}_\nabla}(\mathcal{Y}(\mathbf{a})^{tw}) & \text{if } \sigma(\mathbf{a}) = \mathbf{a} \\ \Delta_{\mathbb{A}_\nabla}(\mathcal{Y}(\mathbf{a})) & \text{if } \sigma(\mathbf{a}) \neq \mathbf{a}. \end{cases} \\ \mathcal{I}_{SO}^O(\nabla_{\mathbb{A}_\nabla}(\mathbf{a})) &\cong \begin{cases} \nabla_{\mathbb{A}_\nabla}(\mathcal{Y}(\mathbf{a})) \oplus \nabla_{\mathbb{A}_\nabla}(\mathcal{Y}(\mathbf{a})^{tw}) & \text{if } \sigma(\mathbf{a}) = \mathbf{a} \\ \nabla_{\mathbb{A}_\nabla}(\mathcal{Y}(\mathbf{a})) & \text{if } \sigma(\mathbf{a}) \neq \mathbf{a}. \end{cases} \end{aligned}$$

Proof. This is essentially immediate from definitions. □

Lemma 7.41. *Let M denote either a Weyl or a dual Weyl representation, and let $\lambda, \mu \in \Lambda_+^{O(N)}$ with $\lambda \neq \mu$.*

$$\mathcal{R}_O^{\text{SO}}(M(\lambda)) \cong \begin{cases} M(\mathbf{a}) & \text{if } \mathcal{Y}(\mathbf{a}) \in \{\lambda, \lambda^{tw}\} \text{ and } \lambda \neq \lambda^{tw}, \\ M(\mathbf{a}) \oplus M(\sigma(\mathbf{a})) & \text{if } \mathcal{Y}(\mathbf{a}) = \lambda \text{ and } \lambda = \lambda^{tw}. \end{cases}$$

Moreover, if $M(\lambda)[\mu] \neq 0$, then $\mu <_{O(N)} \lambda$.

Proof. Because of the discussion above, it suffices to analyze the decomposition of the $SO(N)$ -representation $\mathcal{R}_O^{\text{SO}} \mathcal{I}_{SO}^O(M(\mathbf{a}))$, for $\mathbf{a} \in X^+(SO(N))$. This is analogous to Mackey theory for finite groups, and we leave it to the reader to fill in the details.

The second claim is [AHR20, Proposition 3.4]. □

Definition 7.42. For $\lambda \in \Lambda_+^{O(N)}$ define a map of $O(N)$ -representations $\text{hs}_\lambda : \Delta_{\mathbb{A}_\heartsuit}(\lambda) \rightarrow \nabla_{\mathbb{A}_\heartsuit}(\lambda)$ as follows. Suppose $\mathcal{Y}(\mathbf{a}) = \lambda$. If $\lambda \neq \lambda^{tw}$, then $\text{hs}_\lambda := \text{hs}_{\mathbf{a}}$, and if $\lambda \neq \lambda^{tw}$, then $\text{hs}_\lambda = \mathcal{I}_{\text{SO}}^O(\text{hs}_{\mathbf{a}})$.

Lemma 7.43. For $\lambda \in \Lambda_+^{O(N)}$, we have $\text{Hom}_{O(N)}(\Delta_{\mathbf{a}}(\lambda), \nabla_{\mathbb{A}_\heartsuit}(\lambda)) = \mathbb{A}_\heartsuit \cdot \text{hs}_\lambda$.

Proof. Note that if $\lambda \neq \lambda^{tw}$, then $\text{hs}_\lambda = \mathcal{R}_O^{\text{SO}}(\text{hs}_{\mathbf{a}})$, and if $\lambda \neq \lambda^{tw}$, then $\mathcal{R}_O^{\text{SO}}(\text{hs}_\lambda) = \text{hs}_{\mathbf{a}} \oplus \text{hs}_{\sigma(\mathbf{a})}$. Observing that we have $\sigma(v_{\mathbf{a}}^+) = v_{\sigma(\mathbf{a})}^+$, the claim follows from the fact that $\text{hs}_{\mathbf{a}}$ is spanning $\text{Hom}_{\text{SO}(N)}(\Delta_{\mathbb{A}_\heartsuit}(\mathbf{a}), \nabla_{\mathbb{A}_\heartsuit}(\mathbf{a}))$. \square

7.2.4 Simple representations for $O(N)$

We can view finite dimensional representations of $\text{SO}(N)$ over \mathbb{F}_\heartsuit as $U_{\mathbb{F}_\heartsuit}(\mathfrak{so}_N)$ -representations, with weight space decompositions, such that the weight spaces are contained in $X^+(\text{SO}(N))$. Similarly, we view representations of $O(N)$ as such $U_{\mathbb{F}_\heartsuit}(\mathfrak{so}_N)$ -representations, with a compatible action of σ .

Lemma 7.44. The $O(N)$ -representation $\Delta_{\mathbb{F}_\heartsuit}(\lambda)$ has a unique maximal $O(N)$ -subrepresentation, consisting of the sum of subrepresentations U such that $U \cap \Delta_{\mathbb{F}_\heartsuit}(\lambda)_\lambda = 0$.

Proof. The usual Yoga. \square

Using the previous lemma, we define $L_{\mathbb{F}_\heartsuit}(\lambda)$ as the unique simple quotient of $\Delta_{\mathbb{F}_\heartsuit}(\lambda)$. It then follows by duality that $\nabla_{\mathbb{F}_\heartsuit}(\lambda)$ has a simple socle which is isomorphic to $L_{\mathbb{F}_\heartsuit}(\lambda)$.

Lemma 7.45. The set $\{L_{\mathbb{F}_\heartsuit}(\lambda)\}_{\lambda \in \Lambda_+^{O(N)}}$ is a complete and irredundant list of the finite dimensional simple $O(N)$ -representations.

Proof. Let S be an simple $O(N)$ -representation. A standard argument coming from Clifford theory shows that $\mathcal{R}_O^{\text{SO}}(S)$ is completely reducible. Thus, it is a direct sum of simple representations for $\text{SO}(N)$. Choosing a direct sum decomposition into simple subrepresentations, we then obtain a map to a direct sum of dual Weyl representations. By Lemma 7.40, Frobenius reciprocity yields a non-zero map from S to a direct sum of dual Weyl representations for $O(N)$. Since S is simple, it follows that S is isomorphic to a summand of the socle of this direct sum of dual Weyl representations. Hence, $S \cong L_{\mathbb{F}_\heartsuit}(\lambda)$ for some $\lambda \in \Lambda_+^{O(N)}$. We leave it as an exercise, using highest weights and the action of σ , to argue that $L_{\mathbb{F}_\heartsuit}(\lambda) \cong L_{\mathbb{F}_\heartsuit}(\mu)$ implies $\lambda = \mu$. \square

Lemma 7.46. Let $\lambda \in \Lambda_+^{O(N)}$, then

$$\mathcal{R}_O^{\text{SO}}(L_{\mathbb{F}_\heartsuit}(\lambda)) \cong \begin{cases} L_{\mathbb{F}_\heartsuit}(\mathbf{a}) & \text{if } \mathcal{Y}(\mathbf{a}) \in \{\lambda, \lambda^{tw}\}, \text{ and } \lambda \neq \lambda^{tw}, \\ L_{\mathbb{F}_\heartsuit}(\mathbf{a}) \oplus L_{\mathbb{F}_\heartsuit}(\sigma(\mathbf{a})) & \text{if } \xi(\mathbf{a}) = \lambda = \lambda^{tw}. \end{cases}$$

Proof. Since $L_{\mathbb{F}_\heartsuit}(\lambda)$ is isomorphic to the socle of $\nabla_{\mathbb{F}_\heartsuit}(\lambda)$, we get an injective map

$$\mathcal{R}_O^{\text{SO}}(L_{\mathbb{F}_\heartsuit}(\lambda)) \hookrightarrow \mathcal{R}_O^{\text{SO}}(\nabla_{\mathbb{F}_\heartsuit}(\lambda)).$$

Another standard Clifford theory argument shows that $\mathcal{R}_O^{\text{SO}}(L_{\mathbb{F}_\vartheta}(\lambda))$ is a completely reducible finite dimensional $\text{SO}(N)$ -representation. It follows that $\mathcal{R}_O^{\text{SO}}(L_{\mathbb{F}_\vartheta}(\lambda))$ is a non-zero subrepresentation of the socle of $\mathcal{R}_O^{\text{SO}}(\nabla_{\mathbb{F}_\vartheta}(\lambda))$, which by [Lemma 7.46](#) is isomorphic to $L_{\mathbb{F}_\vartheta}(\mathfrak{a})$, if $\mathcal{Y}(\mathfrak{a}) \in \{\lambda, \lambda^{tw}\}$ and $\lambda \neq \lambda^{tw}$, or $L_{\mathbb{F}_\vartheta}(\mathfrak{a}) \oplus L_{\mathbb{F}_\vartheta}(\sigma(\mathfrak{a}))$, if $\mathcal{Y}(\mathfrak{a}) = \lambda = \lambda^{tw}$. If $\lambda \neq \lambda^{tw}$, the desired result is immediate. If $\lambda = \lambda^{tw}$, so $\sigma(\mathfrak{a}) \neq \mathfrak{a}$, then the desired result follows by looking at the σ action on weight spaces. \square

7.2.5 The orthogonal highest weight category

Recall that by, for example, [\[BS21, Section 6.4\]](#), the category $\mathbf{Rep}(\text{SO}(N))$, equipped with the poset $(X^+(\text{SO}(N)), \leq)$, is an upper finite highest weight category.

Lemma 7.47. *We have Ext-vanishing, i.e.:*

$$\text{Ext}_{O(N)}^i(\Delta_{\mathbb{A}_\vartheta}(\lambda), \nabla_{\mathbb{A}_\vartheta}(\mu)) \cong \begin{cases} 0 & \text{if } \lambda \neq \mu \text{ or } i > 0, \\ \mathbb{A}_\vartheta \cdot \text{hs} & \text{if } \lambda = \mu \text{ and } i = 0. \end{cases}$$

Proof. Since 2 is invertible in \mathbb{A}_ϑ , one can argue, see [\[Ben98, Corollary 3.6.18\]](#), that restriction induces an injective map of \mathbb{A}_ϑ -modules

$$\text{Ext}_{O(N)}^i(\Delta_{\mathbb{A}_\vartheta}(\lambda), \nabla_{\mathbb{A}_\vartheta}(\mu)) \rightarrow \text{Ext}_{\text{SO}(N)}^i\left(\mathcal{R}_O^{\text{SO}}(\Delta_{\mathbb{A}_\vartheta}(\lambda)), \mathcal{R}_O^{\text{SO}}(\nabla_{\mathbb{A}_\vartheta}(\mu))\right).$$

The usual Ext-vanishing implies that $\text{Ext}_{\text{SO}(N)}^{>0}(-, -) = 0$ whenever the first entry has a Weyl representation filtration and the second entry has a dual Weyl filtration. It follows from the statement [Lemma 7.41](#) that $\text{Ext}_{O(N)}^i(\Delta_{\mathbb{A}_\vartheta}(\lambda), \nabla_{\mathbb{A}_\vartheta}(\mu)) \cong 0$ for all $\lambda, \mu \in \Lambda_+^{O(N)}$. Since $\text{Ext}^0 = \text{Hom}$, and also $\text{Hom}_{\text{SO}(N)}(\Delta_{\mathbb{A}_\vartheta}(\mathfrak{a}), \nabla_{\mathbb{A}_\vartheta}(\mathfrak{b})) \cong 0$, whenever $\mathfrak{a} \neq \mathfrak{b}$, it suffices to show that

$$\begin{aligned} \text{Hom}_{O(N)}(\Delta_{\mathbb{A}_\vartheta}(\lambda), \nabla_{\mathbb{A}_\vartheta}(\lambda^{tw})) &\cong 0 \text{ when } \lambda \neq \lambda^{tw}, \\ \text{Hom}_{O(N)}(\Delta_{\mathbb{A}_\vartheta}(\lambda), \nabla_{\mathbb{A}_\vartheta}(\lambda)) &\cong \mathbb{A}_\vartheta \cdot \text{hs}_\lambda \text{ when } \lambda = \lambda^{tw}. \end{aligned}$$

The first equality follows from noting that $\text{hs}_\mathfrak{a}$ spans the space of $\text{SO}(N)$ -homomorphisms over \mathbb{A}_ϑ , here $\mathcal{Y}(\mathfrak{a}) \in \{\lambda, \lambda^{tw}\}$, and $\text{hs}_\mathfrak{a}$ does not commute with σ . The second equality is [Lemma 7.43](#). \square

Lemma 7.48. *In $\mathbf{Rep}_{\mathbb{F}_\vartheta}(O(N))_{\leq \lambda}$: The $O(N)$ -representation $\Delta_{\mathbb{F}_\vartheta}(\lambda)$, respectively $\nabla_{\mathbb{F}_\vartheta}(\lambda)$, is the projective cover, respectively the injective hull, of $L_{\mathbb{F}_\vartheta}(\lambda)$.*

Proof. The second claim follows from the first by duality. To show the first claim, it suffices to show that $\text{Ext}_{O(N)}^{>0}(\Delta_{\mathbb{F}_\vartheta}(\lambda), L_{\mathbb{F}_\vartheta}(\nu)) = 0$ for all $\nu \leq \lambda$. Again, noting that \mathbb{F}_ϑ is a field over \mathbb{A}_ϑ , so $2 \in \mathbb{F}_\vartheta^\times$, we can use [\[Ben98, Corollary 3.6.18\]](#) to observe that restriction induces an injection

$$\text{Ext}_{O(N)}^i(\Delta_{\mathbb{F}_\vartheta}(\lambda), L_{\mathbb{F}_\vartheta}(\nu)) \hookrightarrow \text{Ext}_{\text{SO}(N)}^i(\mathcal{R}_O^{\text{SO}}(\Delta_{\mathbb{F}_\vartheta}(\lambda)), \mathcal{R}_O^{\text{SO}}(L_{\mathbb{F}_\vartheta}(\nu))).$$

Since $\mathbf{Rep}(\text{SO}(N))$ is well-known to be a highest weight category, we can observe that if $\mathfrak{a} \in X^+(\text{SO}(N))$, then $\Delta_{\mathbb{F}_\vartheta}(\mathfrak{a})$ is a projective cover of $L_{\mathbb{F}_\vartheta}(\mathfrak{a})$ in $\mathbf{Rep}(\text{SO}(N))_{\leq \mathfrak{a}}$. Thus, $\text{Ext}_{\text{SO}(N)}^{>0}(\Delta_{\mathbb{F}_\vartheta}(\mathfrak{a}), L_{\mathbb{F}_\vartheta}(\mathfrak{b})) \cong 0$ for all $\mathfrak{b} \leq \mathfrak{a}$. The claim then follows from [Lemma 7.41](#) and [Lemma 7.32](#). \square

By [BS21, Lemma 4.1 and Theorem 4.2], it follows that

$$\mathbf{Tilt}_{\mathbb{F}_\vartheta}(\mathcal{O}(N)) = \mathbf{Fil}_{\mathbb{F}_\vartheta}^{\Delta}(\mathcal{O}(N)) \cap \mathbf{Fil}_{\mathbb{F}_\vartheta}^{\nabla}(\mathcal{O}(N))$$

is an additive category, with isomorphism classes of indecomposable in bijection with $\Lambda_+^{\mathcal{O}(N)}$. For $\lambda \in \Lambda_+^{\mathcal{O}(N)}$, we write $\mathbf{T}_{\mathbb{F}_\vartheta}(\lambda)$ for the indecomposable tilting representations with subrepresentation $\Delta_{\mathbb{F}_\vartheta}(\lambda)$.

Proposition 7.49. *The category $\mathbf{Rep}_{\mathbb{F}_\vartheta}(\mathcal{O}(N))$ equipped with $(\Lambda_+^{\mathcal{O}(N)}, \leq)$ is a(n upper finite) highest weight category.*

Proof. We use [BS21, Corollary 3.64] and the above discussion. \square

7.2.6 Combinatorial orthogonal category

The following is an important property:

Lemma 7.50. *The tensor product of two Weyl representations (respectively dual Weyl representations) in $\mathbf{Rep}_{\mathbb{F}_\vartheta}(\mathcal{O}(N))$ has a filtration by Weyl representations (respectively dual Weyl representations).*

Proof. Because of compatibility of \otimes and $(-)^*$, along with exactness of $(-)^*$, it suffices to prove the result for Weyl representations. Let $\lambda, \mu \in \Lambda_+^{\mathcal{O}(N)}$. There are $\mathbf{a}, \mathbf{b} \in X^+(\mathrm{SO}(N))$ such that $\Delta_{\mathbb{F}_\vartheta}(\lambda)$, respectively $\Delta_{\mathbb{F}_\vartheta}(\mu)$, is a direct summand of $\mathcal{I}_{\mathrm{SO}}^{\mathcal{O}}(\Delta_{\mathbb{F}_\vartheta}(\mathbf{a}))$, respectively of $\mathcal{I}_{\mathrm{SO}}^{\mathcal{O}}(\Delta_{\mathbb{F}_\vartheta}(\mathbf{b}))$. Then $\Delta_{\mathbb{F}_\vartheta}(\lambda) \otimes \Delta_{\mathbb{F}_\vartheta}(\mu)$ is a direct summand of

$$\mathcal{I}_{\mathrm{SO}}^{\mathcal{O}}(\Delta_{\mathbb{F}_\vartheta}(\mathbf{a})) \otimes \mathcal{I}_{\mathrm{SO}}^{\mathcal{O}}(\Delta_{\mathbb{F}_\vartheta}(\mathbf{b})) \cong \mathcal{I}_{\mathrm{SO}}^{\mathcal{O}}\left(\Delta_{\mathbb{F}_\vartheta}(\mathbf{a}) \otimes \mathcal{R}_O^{\mathrm{SO}}(\mathcal{I}_{\mathrm{SO}}^{\mathcal{O}}(\Delta_{\mathbb{F}_\vartheta}(\mathbf{b})))\right).$$

We use [Don98, Proposition A2.2(vi)] as follows. Since a summand of Weyl filtered representations is Weyl filtered, inductions and restrictions of Weyl representations are Weyl filtered, and tensor products of Weyl $\mathrm{SO}(N)$ -representations are Weyl filtered, the claim follows from exactness of $\mathcal{I}_{\mathrm{SO}}^{\mathcal{O}}$ and $\mathcal{R}_O^{\mathrm{SO}}$. \square

Note that in the proof above we work over \mathbb{F}_ϑ . This is because we are using that summand of a Weyl filtered representation is Weyl filtered, which follows from highest weight category theory, and is therefore not present over \mathbb{A}_ϑ .

Proposition 7.51. *The category $\mathbf{Tilt}_{\mathbb{F}_\vartheta}(\mathcal{O}(N))$ is a symmetric ribbon \mathbb{F}_ϑ -linear category.*

Proof. Lemma 7.50 implies that $\mathbf{Fil}_{\mathbb{F}_\vartheta}^{\Delta}(\mathcal{O}(N))$ and $\mathbf{Fil}_{\mathbb{F}_\vartheta}^{\nabla}(\mathcal{O}(N))$ are closed under tensor products. Being pivotal and symmetric is inherited from $\mathbf{Rep}_{\mathbb{F}_\vartheta}(\mathcal{O}(N))$, and so is the ribbon property. \square

Lemma 7.52. *The Weyl representations $\Delta_{\mathbb{A}_\vartheta}(1^i)$ are isomorphic to $\Lambda^i = \Lambda^i(\mathbb{A}_\vartheta^N)$ and are simple tilting representations, for $i = 0, 1, \dots, N$.*

Proof. Recall that if N is even, then

$$\Delta_{\mathbb{A}_\heartsuit}(\varpi_i) \cong \mathcal{R}_O^{\text{SO}}(\Lambda^i) \text{ for } i \in [1, n-1], \quad \Delta_{\mathbb{A}_\heartsuit}(2\varpi_N) \cong \mathcal{R}_O^{\text{SO}}(\Lambda^N).$$

Moreover, if N is odd, then

$$\begin{aligned} \Delta_{\mathbb{A}_\heartsuit}(\varpi_i) &\cong \mathcal{R}_O^{\text{SO}}(\Lambda^i) \text{ for } i \in [1, n-2], \quad \Delta_{\mathbb{A}_\heartsuit}(\varpi_{n-1} + \varpi_N) \cong \mathcal{R}_O^{\text{SO}}(\Lambda^{n-1}), \\ \Delta_{\mathbb{A}_\heartsuit}(2\varpi_{n-1}) \oplus \Delta_{\mathbb{A}_\heartsuit}(2\varpi_N) &\cong \mathcal{R}_O^{\text{SO}}(\Lambda^n). \end{aligned}$$

Note that each highest weight above is fixed by σ except for $2\varpi_{n-1}$ and $2\varpi_N$ which are permuted by σ .

It then follows from [JMW16, Sections 3.6.2 and 3.6.4] that each Weyl $\text{SO}(N)$ -representation appearing above is tilting. In particular, each of these Weyl representations is isomorphic to its dual over \mathbb{A}_\heartsuit .

One then easily argues that independent of whether N is even or odd, we have $\Lambda^i \cong \Delta_{\mathbb{A}_\heartsuit}(1^i)$ for $i \in [0, n]$, and $\Lambda^i \mathbb{F}_\heartsuit^N$ is in $\mathbf{Tilt}_{\mathbb{F}_\heartsuit}(\text{O}(N))$ for $i \in [0, n]$. \square

Let $\mathbf{Fund}_{\mathbb{F}_\heartsuit}(\text{O}(N)) \subset \mathbf{Rep}_{\mathbb{F}_\heartsuit}(\text{O}(N))$ be the full subcategory spanned by the representations as in Lemma 7.52.

Proposition 7.53. *There is an equivalence of pivotal symmetric ribbon categories*

$$\mathbf{Fund}_{\mathbb{F}_\heartsuit}(\text{O}(N))^{\oplus, \subset \oplus} \rightarrow \mathbf{Tilt}_{\mathbb{F}_\heartsuit}(\text{O}(N)).$$

Proof. Recall from Definition 7.20 that dominant weights for $\text{O}(N)$ are given by pairs of a dominant $\text{SO}(N)$ weight and $\epsilon \in \{\pm 1, 0\}$. Tensoring with the determinant $\text{O}(N)$ -representation corresponds to swapping the sign in ϵ . Thus, since the determinant $\text{O}(N)$ -representation is an exterior power, the results follows from the same statement about $\text{SO}(N)$ -representations. \square

Remark 7.54. One could expect that there is an equivalence of symmetric ribbon \mathbb{F}_\heartsuit -linear categories $\mathbb{F}_\heartsuit \otimes_{\mathbb{A}_\heartsuit} \mathbf{Fund}_{\mathbb{A}_\heartsuit}(\text{O}(N)) \rightarrow \mathbf{Fund}_{\mathbb{F}_\heartsuit}(\text{O}(N))$.

A. Some combinatorial facts for Howe’s duality

We now fill in some details regarding section 6.

A.1. A short overview

Remark A.1. Nothing in this section is new. And since it can be pieced together from the literature, we will be very brief.

Recall that $\Lambda^*(V \otimes \mathbb{A}_\heartsuit^m)$ is an $\text{O}(N)$ - $\text{U}_{\mathbb{A}_\heartsuit}(\mathfrak{so}_{2m})^{op}$ -birepresentation by Proposition 4.16, and we will only use the associated two actions.

There is a basis, different than the w_S basis, which is a weight basis for both actions. These vectors will necessarily also be indexed by $S \subset \square_{N \times m}$, so for each box and dot diagram, we will associate a $\text{O}(N)$ -weight and an \mathfrak{so}_{2m} -weight.

In the case of $\text{O}(N)$ and \mathfrak{so}_{2m} , dominant weights can be encoded three different ways:

- (i) As a *sequence* of positive integers which are immediately read off of the box and dot diagram, see [Definition 7.21](#) and [Definition A.3](#).
- (ii) As a simple representation of a *maximal torus* (that is in terms of the usual notion of weight, generalized to disconnected groups), see [Definition 7.21](#).
- (iii) As a pair of a *Young diagram* (a.k.a. integer partitions) and an integer in $\{-1, 0, 1\}$, see [Remark 7.27](#) and [Definition A.12](#).

A.2. $U_{\mathbb{A}_\diamond}(\mathfrak{so}_{2m})$ -weights in $\Lambda^*(V \otimes \mathbb{A}_\diamond^m)$

It turns out that there is a simpler convention for writing the weight vectors in [Lemma 4.35](#) instead of using the ϵ_j basis. Note that the operators $e_{j,j}^0 = x_j \partial_j$ acting on $\Lambda^*(\mathbb{A}_\diamond^m)$ can be made to act on $\Lambda^*(\mathbb{A}_\diamond^m)^{\otimes N}$ by derivations, and then we can transport this action to $\Lambda^*(V \otimes \mathbb{A}_\diamond^m)$ with the isomorphism ϕ_h . If we instead think of the weight of w_S^h in terms of the eigenvalues of the elements $e_{j,j}^0$, then it is possible to recover how the h_j act from [Definition 4.6](#). We will give the details now.

Notation A.2. Let Π_m be the set of compositions of length m , i.e. tuples $K = (K_1, \dots, K_m)$ such that $K_j \in \mathbb{Z}_{\geq 0}$. We write Π_m^N to denote the subset of Π_m consisting of K such that $K_j \leq N$ for $j \in [1, m]$, or in other words, the compositions that fit into a N - m rectangle.

Definition A.3. Given $S \subset \square_{N \times m}$, we define $\text{wt}(w_S^h) = (|S_1|, \dots, |S_m|) \in \Pi_m^{\leq N}$.

Example A.4. Here is an example:

$$S = \begin{array}{|c|c|c|c|c|c|} \hline \bullet & \bullet & & & & \\ \hline \bullet & & & & & \bullet \\ \hline \bullet & & & & \bullet & \\ \hline \end{array} \rightsquigarrow \text{wt}(w_S^h) = (3, 1, 0, 0, 1, 1).$$

Indeed, in terms of dot diagrams [Definition A.3](#) counts the number of dots in columns.

Recall that $\Pi_m^{\leq N}$ denotes the set of \mathfrak{so}_{2m} -weights appearing in $\Lambda^*(V \otimes \mathbb{A}_\diamond^m)$.

Lemma A.5. We have $\{\text{wt}(w_S^h) \mid S \subset \square_{N \times m}\} = \Pi_m^{\leq N}$.

Proof. Note that $\text{wt}(w_S^h) \in \Pi_m^{\leq N}$. Given $K \in \Pi_m^{\leq N}$, let

$$S^K = \cup_{1 \leq j \leq m} \{(1, j), (2, j), \dots, (K_j, j)\}.$$

Since $1 \leq K_j \leq N$ for $j \in [1, m]$, we have $S \subset \square_{N \times m}$. Then from [Definition A.3](#) we see $\text{wt}(w_{S^K}^h) = K$. \square

Lemma A.6. Suppose $\text{wt}(w_S^h) = (K_1, \dots, K_m)$. Then we have $e_{j,j}^o \cdot w_S^h = |S_j| \cdot w_S^h$ and therefore

$$h_j \cdot w_S^h = (|S_j| - |S_{j+1}|) \cdot w_S^h, \quad h_m \cdot w_S^h = (|S_{m-1}| + |S_m| - N) \cdot w_S^h,$$

where $j \in [1, m-1]$.

Proof. Since $d_j(S) = |S_j| - (N - |S_j|)$, we have

$$\begin{aligned} \frac{1}{2}(d_j(S) - d_{j+1}(S)) &= |S_j| - |S_{j+1}|, \text{ for } j \in [1, m - 1], \\ \frac{1}{2}(d_{m-1}(S) + d_m(S)) &= |S_{m-1}| + |S_m| - N. \end{aligned}$$

Since $\alpha_j = \epsilon_j - \epsilon_{j+1}$ and $\alpha_m = \epsilon_{m-1} + \epsilon_m$, the claim then follows from [Lemma 4.35](#). □

It follows from [Lemma A.6](#) that

$$(K_1, \dots, K_m) \mapsto \sum_{i=1}^m (K_i - \frac{N}{2}) \epsilon_i \tag{A.1}$$

converts between weights $\text{wt}(w_S^h) \in \Pi_m^{\leq N}$ and $\text{wt}_{\mathfrak{so}_{2m}}(w_S^h) \in X(\mathfrak{so}_{2m}) \subset \bigoplus_{i=1}^m \mathbb{Z} \frac{\epsilon_i}{2}$ (the notation $X(\mathfrak{so}_{2m})$ was specified in [Notation 4.8](#)). Note that the N in the $-N/2$ factor depends on $S \subset \square_{N \times m}$. Thus, we will not refer to Π_m in what follows, only $\Pi_m^{\leq N}$.

Example A.7. For $m = 6, N = 1$ and $S = \{1, 3, 4\}$ we have

$$\begin{array}{|c|c|c|c|c|c|} \hline \bullet & & \bullet & \bullet & & \\ \hline \end{array} \rightsquigarrow \text{wt}(w_S^h) = (1, 0, 1, 1, 0, 0) \mapsto \text{wt}_{\mathfrak{so}_{2m}}(x_S) = \frac{1}{2}(1, -1, 1, 1, -1, -1),$$

See also [Example 4.12](#).

Lemma A.8. Let $S \subset \square_{N \times m}$. We have

$$\begin{aligned} \text{wt}_{\mathfrak{so}_{2m}}(w_S^h) \in X^+(\mathfrak{so}_{2m}) \\ \iff \\ \text{wt}(w_S^h) = K \in \Pi_m^{\leq N} \text{ is such that } K_1 - \frac{N}{2} \geq \dots \geq K_{m-1} - \frac{N}{2} \geq |K_m - \frac{N}{2}| \geq 0, \end{aligned} \tag{A.2}$$

$$\begin{aligned} \text{wt}_{\mathfrak{so}_{2m}}(w_S^h) \in X^-(\mathfrak{so}_{2m}) \\ \iff \\ \text{wt}(w_S^h) = K \in \Pi_m^{\leq N} \text{ is such that } K_1 - \frac{N}{2} \leq \dots \leq K_{m-1} - \frac{N}{2} \leq -|K_m - \frac{N}{2}| \leq 0, \end{aligned} \tag{A.3}$$

where $X^-(\mathfrak{so}_{2m})$ means antidominant \mathfrak{so}_{2m} -weights.

Proof. Use that $\mathfrak{a} \in X^+(\mathfrak{so}_{2m})$ if and only if $\alpha_i^\vee(\mathfrak{a}) \in \mathbb{Z}_{\geq 0}$, for $i \in [1, m]$, to deduce

$$\mathfrak{a} \in X^+(\mathfrak{so}_{2m}) \iff \mathfrak{a}_1 \geq \mathfrak{a}_2 \geq \dots \geq \mathfrak{a}_{m-1} \geq |\mathfrak{a}_m|.$$

Then apply [Equation A.1](#). □

Notation A.9. Write $\Pi_{m,+}^{\leq N}$ for the set of K such that [Equation A.2](#) holds and $\Pi_{m,-}^{\leq N}$ for the set of K such that [Equation A.3](#) holds.

For $K \in \Pi_m^{\leq N}$, we write $\Lambda^K = \Lambda^{k_1}(V) \otimes \dots \otimes \Lambda^{k_m}(V) \subset \Lambda^*(V)^{\otimes m}$. Note that since each Λ^{k_i} is a direct summand of $\Lambda^*(V)$, Λ^K is a summand of $\Lambda^*(V)^{\otimes m}$.

Lemma A.14. *If $\mathbf{a} \leq \mathbf{b}$, then $\mathcal{Y}(\mathbf{a}) \leq \mathcal{Y}(\mathbf{b})$.*

Proof. Standard, see the proof of [Lemma 7.34](#) for the $\text{SO}(N)$ version. □

Notation A.15. *Suppose that $K, L \in \Pi_{m,+}^{\leq N}$ correspond to $\mathbf{a}, \mathbf{b} \in X^+(\mathfrak{so}_{2m})$. If $\mathbf{a} \leq \mathbf{b}$, then we write $K \leq_{\mathfrak{so}_{2m}} L$.*

Definition A.16. Let $\mathbf{a}, \mathbf{b} \in X^+(\mathfrak{so}_{2m})$. Define a partial order $(\mathcal{Y}(\mathbf{a}), \epsilon(\mathbf{a})) < (\mathcal{Y}(\mathbf{b}), \epsilon(\mathbf{b}))$ if and only if $\mathcal{Y}(\mathbf{a}) \triangleleft \mathcal{Y}(\mathbf{b})$.

Notation A.17. *Suppose that $K, L \in \Pi_{m,+}^{\leq N}$ correspond to $\mathbf{a}, \mathbf{b} \in X^+(\mathfrak{so}_{2m})$. If $(\mathcal{Y}(\mathbf{a}), \epsilon(\mathbf{a})) < (\mathcal{Y}(\mathbf{b}), \epsilon(\mathbf{b}))$, then we write $K < L$.*

Lemma A.18. *Let $K, L \in \Pi_{m,+}^{\leq N}$. If $K \leq_{\mathfrak{so}_{2m}} L$, then $K \leq L$.*

Proof. Follows from [Lemma A.14](#). □

A.3. $O(N)$ -weights in $\Lambda^*(V \otimes \mathbb{A}_{\mathbb{Q}}^m)$

We follow the conventions and notation of [subsection 7.2.1](#). We use that $\text{SO}(N) \subset O(N)$ acts on $\Lambda^*(V \otimes \mathbb{A}_{\mathbb{Q}}^m)$ by restriction.

As we observed in [Remark 4.30](#), the basis v_i is not a weight basis with respect to our choice of $T \subset \text{SO}(N)$ from [Definition 4.29](#). Thus, neither is the basis w_S^v for $\Lambda^*(V \otimes \mathbb{A}_{\mathbb{Q}}^m)$.

Definition A.19. We write

$$\begin{aligned} z_{ij} &= a_i \otimes x_j, z_{N-i+1,j} = b_i \otimes x_j \text{ for } i \in [1, n], \\ z_{n+1,j} &= u \otimes x_j \text{ if } N = 2n + 1. \end{aligned}$$

For $S \subset \square_{N \times m}$, we also write z_S for the product of z_{ij} such that $(i, j) \in S$ ordered by the vertical reading, see [Definition 4.18](#), of S . We do not consider the horizontal reading for this basis.

Lemma A.20. *The set $\{z_S | S \subset \square_{N \times m}\}$ is an $\mathbb{A}_{\mathbb{Q}}$ -basis of $\text{SO}(N)$ -weight vectors.*

Proof. Since $\text{SO}(N) \subset O(N)$ acts on $\Lambda^*(V \otimes \mathbb{A}_{\mathbb{Q}}^m) \cong \Lambda^*(V)^{\otimes m}$ by the usual tensor product rule, this is easy to check and omitted. □

Lemma A.21. *Let $S \subset \square_{N \times m}$. Then*

$$\text{wt}_{\text{SO}}(z_S) = \sum_{i=1}^N (|S_i| - |S_{N-i+1}|) \epsilon_i.$$

Proof. Immediate from description of the T -action in [Definition 4.29](#). □

We now will describe the σ -action on the basis z_S^v .

Notation A.22. Suppose $N = 2n$. For $S \subset \square_{N \times m}$, write $\sigma(S) \subset \square_{N \times m}$ to denote the set determined by the conditions:

$$\begin{aligned} \sigma(S)_i &= S_i \text{ for } i \neq n, n+1, \\ \sigma(S)_N &= \{(t, n) | (t, n+1) \in S_{n+1}\} \text{ and } \sigma(S)_{n+1} = \{(t, n+1) | (t, n) \in S_N\}. \end{aligned}$$

Here σ is as in [Definition 4.31](#).

Lemma A.23. We have

$$\sigma \cdot z_S = \begin{cases} (-1)^{|S_{n+1}|} z_S & \text{if } N = 2n + 1, \\ (-1)^{|\{x | (x, n), (x, n+1) \in S\}|} z_{\sigma(S)} & \text{if } N = 2n. \end{cases}$$

Proof. Boring and omitted. □

Thus, if $N = 2n + 1$, then z_S is always a σ -eigenvector. While if $N = 2n$, then z_S is a σ -eigenvector if and only if $\sigma(S) = S$, i.e. the $N, n + 1$ horizontal strip of S only contains empty columns or double dot columns.

Remark A.24. Two different subsets of $\square_{N \times m}$ can give rise to two distinct basis vectors with the same $O(N)$ -weight. For example, suppose that $N = 2$ and $m = 2$. Then

$$\text{wt}_O(z) = (0, 1) = \text{wt}_O(z_{[1, N] \times [1, m]}),$$

as one easily checks.

Lemma A.25. If $(\mathbf{a}, \epsilon) \in X(O(N))$, then $[(\mathbf{a}, \epsilon)] \Lambda^*(V \otimes \mathbb{A}_{\mathbb{Q}}^m)$ is an $U_{\mathbb{A}_{\mathbb{Q}}}(\mathfrak{so}_{2m})$ -direct summand of $\Lambda^*(V \otimes \mathbb{A}_{\mathbb{Q}}^m)$.

Proof. This follows from [Proposition 4.16](#). □

A.4. $O(N)$ - $U_{\mathbb{A}_{\mathbb{Q}}}(\mathfrak{so}_{2m})$ -weights in $\Lambda^*(V \otimes \mathbb{A}_{\mathbb{Q}}^m)$

The formulas for the action of the raising and lowering operators in $U_{\mathbb{A}_{\mathbb{Q}}}(\mathfrak{so}_{2m})$ in terms of the z_S basis are different than in the w_S basis. However, it is still the case that z_S is a weight vector for $U_{\mathbb{A}_{\mathbb{Q}}}(\mathfrak{so}_{2m})$.

Lemma A.26. We have the following.

1. Let $S \subset \square_{N \times m}$, then $\text{wt}_{\mathfrak{so}_{2m}}(z_S) = \text{wt}_{\mathfrak{so}_{2m}}(w_S^h)$.
2. Let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$ correspond to $(\mathbf{a}, \epsilon) \in X^+(O(N))$. Consider a set $S \subset \square_{N \times m}$ which has box and dot diagram with λ_i dots in the i th row. Then $\text{wt}_O(z_S) = (\mathbf{a}, \epsilon)$.
3. Let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$ and consider a set $S \subset \square_{N \times m}$ which has box and dot diagram with λ_i dots in the i th row, and so that all dots are as far right as possible. Then $\text{wt}_{z_S} \in \Pi_{m, -}^{\leq N}$.

Proof. A calculation. □

Notation A.27. Let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$. We write S_λ to denote the set S as in Lemma A.26.(c).

Lemma A.28. Let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$. The vector z_{S_λ} is a highest weight vector for $SO(N)$ and a lowest weight vector for $U_{\mathbb{A}_\nabla}(\mathfrak{so}_{2m})$.

Proof. Another boring calculation. □

We prefer to not label representations by their lowest weight.

Remark A.29. Recall that the simple with lowest weight \mathfrak{b} has highest weight $w_0(\mathfrak{b})$. Since $\mathfrak{b} \in X_-(\mathfrak{so}_{2m})$, it follows that z_{S_λ} generates a subrepresentation of $\mathbb{Q}_\nabla \otimes_{\mathbb{A}_\nabla} \Lambda^*(V \otimes \mathbb{A}_\nabla^m)$ isomorphic to $L_{\mathbb{Q}_\nabla}(\mathfrak{a}, \epsilon) \boxtimes L_{\mathbb{Q}_\nabla}(w_0(\mathfrak{b}))$.

Definition A.30. Let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$ correspond to (\mathfrak{a}, ϵ) . Let $\mathfrak{b} = \text{wt}_{\mathfrak{so}_{2m}}(z_{S_\lambda}) \in X_-(\mathfrak{so}_{2m})$. Define $(\mathfrak{a}, \epsilon)^\dagger = w_0(\mathfrak{b})$.

We now set out to understand the dagger operation combinatorially.

Definition A.31. Write $N = 2n + 1$, if N is odd, and $N = 2n$, if N is even. Given a Young diagram \mathcal{Y} with at most N rows and at most m columns, i.e. the diagram fits on an N by m checkerboard. The complement of \mathcal{Y} , is defined as the Young diagram with $m - \mathcal{Y}_i$ boxes in the $n + 1 - i$ th row. Define \mathcal{Y}^{ct} to be the Young diagram obtained by taking the transpose of the complement of \mathcal{Y} .

Lemma A.32. If $\mathcal{Y}_1 \trianglelefteq \mathcal{Y}_2$, then $\mathcal{Y}_2^{\text{ct}} \trianglelefteq \mathcal{Y}_1^{\text{ct}}$

Proof. Taking complements preserves the partial order on Young diagrams. The claim follows from noting that transpose reverses the partial order. □

Lemma A.33. Let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$ correspond to (\mathfrak{a}, ϵ) and let $\mathfrak{b} = \text{wt}_{\mathfrak{so}_{2m}}(z_{S_\lambda})$. Then $-\mathfrak{b} \in X^+(\mathfrak{so}_{2m})$ corresponds to $(\mathcal{Y}(\mathfrak{a})^{\text{ct}}, \epsilon)$.

Proof. Omitted. □

Note that $w_0(\mathfrak{b}) = -w_0 \cdot (-\mathfrak{b})$. If m is even, then w_0 acts by -1 , so $-w_0$ is the identity, while if m is odd, then $-w_0$ is the Dynkin diagram automorphism, which multiplies the ϵ_m coordinate by -1 . Since $-\mathfrak{b}$ corresponds to $(\mathcal{Y}(\mathfrak{a})^{\text{ct}}, \epsilon)$, it follows that

$$w_0(\mathfrak{b}) \longleftrightarrow \begin{cases} (\mathcal{Y}(\mathfrak{a})^{\text{ct}}, \epsilon) & \text{if } m \text{ is even,} \\ (\mathcal{Y}(\mathfrak{a})^{\text{ct}}, -\epsilon) & \text{if } m \text{ is odd.} \end{cases}$$

Definition A.34. Let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$ correspond to (\mathfrak{a}, ϵ) . Define $\lambda^\dagger \in \Pi_{m,+}^{\leq N}$ as the weight corresponding to

$$(\mathcal{Y}(\mathfrak{a}), \epsilon)^\dagger = \begin{cases} (\mathcal{Y}(\mathfrak{a})^{\text{ct}}, \epsilon) & \text{if } m \text{ is even,} \\ (\mathcal{Y}(\mathfrak{a})^{\text{ct}}, -\epsilon) & \text{if } m \text{ is odd.} \end{cases}$$

Let $\lambda, \mu \in \Lambda_{+, \leq m}^{O(N)}$ correspond to $(\mathcal{Y}(\mathbf{a}), \epsilon)$ and $(\mathcal{Y}(\mathbf{b}), \epsilon')$. If $(\mathcal{Y}(\mathbf{a}), \epsilon) < (\mathcal{Y}(\mathbf{b}), \epsilon')$, then $\mathcal{Y}(\mathbf{a}) \triangleleft \mathcal{Y}(\mathbf{b})$, so by [Lemma A.32](#), $\mathcal{Y}(\mathbf{b})^{\text{ct}} \triangleleft \mathcal{Y}(\mathbf{a})^{\text{ct}}$, and therefore $(\mathcal{Y}(\mathbf{b}), \epsilon')^\dagger < (\mathcal{Y}(\mathbf{a}), \epsilon)^\dagger$.

Proposition A.35. *The map $\dagger: \Lambda_{+, \leq m}^{O(N)} \rightarrow \Pi_{m, +}^{\leq N}$ is an order reversing bijection.*

Proof. A calculation. □

Lemma A.36. *Let $\lambda \in \Lambda_{+, \leq m}^{O(N)}$ and $K \in \Pi_{m, +}^{\leq N}$. If $[\lambda] \Lambda^*(V \otimes \mathbb{A}_\nu^m)[K] \neq 0$, then $K \leq \lambda^\dagger$.*

Proof. Since $L_{\mathbb{Q}_\nu}(\mu)_\lambda \neq 0$ implies $\lambda <_{O(N)} \mu$ by [Lemma 7.41](#), [Lemma 7.36](#) gives

$$[\lambda]_{\mathbb{Q}_\nu} \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)[K] \subset \bigoplus_{\lambda \leq \mu} L_{\mathbb{Q}_\nu}(\mu) \boxtimes L_{\mathbb{Q}_\nu}(\mu^\dagger).$$

It follows from $[\lambda]_{\mathbb{Q}_\nu} \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)[K] \neq 0$, that there is some $\mu \in \Lambda_{+, \leq m}^{O(N)}$, such that $\lambda \leq \mu$ and $L_{\mathbb{Q}_\nu}(\lambda)(\mu^\dagger)[K] \neq 0$. Thus, $K \leq \mu^\dagger \leq \lambda^\dagger$. □

Lemma A.37. *If $\lambda \in \Lambda_{+, \leq m}^{O(N)}$, then $T_{\mathbb{F}_\nu}(\lambda^\dagger)$ is a direct summand of $[\lambda]_{\mathbb{F}_\nu} \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)$.*

Proof. By [Lemma 4.15](#), $\mathbb{F}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)$ is a tilting representations for $U_{\mathbb{F}_\nu}(\mathfrak{so}_{2m})$. The weight space $[\lambda]_{\mathbb{F}_\nu} \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)$ is a $U_{\mathbb{F}_\nu}(\mathfrak{so}_{2m})$ direct summand of $\mathbb{F}_\nu \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)$, so is a tilting representations for $U_{\mathbb{F}_\nu}(\mathfrak{so}_{2m})$. Thus, $[\lambda]_{\mathbb{F}_\nu} \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)$ is a direct sum of tilting representations of the form $T_{\mathbb{F}_\nu}(\lambda)(K)$, and by [Lemma A.36](#) $K \leq \lambda^\dagger$. Since $z_{S_\lambda} \in [\lambda]_{\mathbb{F}_\nu} \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)[\lambda^\dagger] \neq 0$, and since $T_{\mathbb{F}_\nu}(\lambda)(K)[L] \neq 0$ implies $L \leq K$, we conclude that $T_{\mathbb{F}_\nu}(\lambda)(\lambda^\dagger)$ is a summand of $[\lambda]_{\mathbb{F}_\nu} \otimes_{\mathbb{A}_\nu} \Lambda^*(V \otimes \mathbb{A}_\nu^m)$. □

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