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CYLINDRIC P-TABLEAUX FOR $(\mathbf{3}+\mathbf{1})$ -FREE POSETS

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Abstract. Tatsuyki Hikita recently proved the Stanley–Stembridge conjecture, showing that the e-coefficients of the chromatic symmetric function of an incomparability graph of a $(\mathbf{3}+\mathbf{1})$ -free poset are non-negative. It remains an open problem to find combinatorial interpretations of the e-coefficients. For a $(\mathbf{3}+\mathbf{1})$-free P, we define a hybrid of P-tableaux and cylindric tableaux called cylindric P-tableaux. The weight generating function of cylindric P-tableaux of shape $\lambda/\mu/d$ are shown to be P-analogs of cylindric Schur functions defined by a determinantal formula. We deduce that certain sums of the e-expansion coefficients of the chromatic symmetric function $X_{\mathrm{inc}(P)}$ are counted by the number of standard cylindric P-tableaux of the appropriate shape. We connect the P-analogs of symmetric functions to a theorem on Hecke algebra immanants due to Clearman–Hyatt–Shelton–Skandera.

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Mathematics Subject Classifications. 05E05

1. Introduction

The chromatic symmetric function $X_G(\mathbf{x})$ of a graph G is the sum $\sum_\kappa \mathbf{x}_\kappa$ over all proper colorings $\kappa : V(G) \to \mathbb{Z}_{>0}$ where $\mathbf{x}_\kappa = \prod_{v \in V(G)} x_{\kappa(v)}$. It is of particular interest when G is the incomparability graph $\mathrm{inc}(P)$ of a $(\mathbf{3}+\mathbf{1})$-free poset P. In this case, a theorem of Haiman showed that $X_{\mathrm{inc}(P)}(\mathbf{x})$ is Schur positive [Hai93], and a combinatorial formula for the Schur expansion was given by Gasharov [Gas96]. Stanley and Stembridge [Sta95, SS93] conjectured that $X_{\mathrm{inc}(P)}(\mathbf{x})$ is a positive sum of elementary symmetric functions. The Stanley–Stembridge conjecture has remained an active area of research, with a range of techniques developed to study chromatic symmetric functions. Combinatorial interpretations of the coefficient of e_λ are known when λ is a rectangular [CHSS16, Ste91], two-column [CHSS16, Hwa24, Wol97], or hook shape [Hwa24, Wol97]. Additionally, X_G was shown to be e-positive for several classes of graphs [CH22, CH19, Dah19, DvW18, GS01, HP19]. Tatsuyuki Hikita [Hik24] recently gave

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a probabilistic proof of the Stanley–Stembridge conjecture, showing that that the e-coefficients are non-negative. Finding a combinatorial interpretation of the e-coefficients remains a major open problem and is the subject of the current paper.

In 1998, Stanley [Sta98] introduced P-*analogs* of symmetric functions for studying chromatic symmetric functions. This is the starting point for our work here, so we review the basic setup. For a finite poset P, let $\mathbb{Z}[\mathbf{u}]$ be the polynomial ring in the commuting variables $\mathbf{u} = \{u_p : p \in P\}$. We define the P-*elementary function* $e_k^P(\mathbf{u}) \in \mathbb{Z}[\mathbf{u}]$ by

$$e_k^P(\mathbf{u}) = \sum_{i_1 <_P i_2 <_P \dots <_P i_k} u_{i_1} u_{i_2} \cdots u_{i_k}. \tag{1.1}$$

Let $\Lambda(\mathbf{x})$ denote the ring of symmetric functions in variables $\mathbf{x} = x_1, x_2, \dots$ For $f(\mathbf{x}) \in \Lambda(\mathbf{x})$, we define the P-analog $f^P(\mathbf{u})$ of $f(\mathbf{x})$ to be the image of $f(\mathbf{x})$ under the ring homomorphism

$$\psi \colon \Lambda(\mathbf{x}) \to \mathbb{Z}[\mathbf{u}], \quad e_k(\mathbf{x}) \mapsto e_k^P(\mathbf{u}). \tag{1.2}$$

The following theorem of Stanley connects P-analogs of symmetric functions to the chromatic symmetric function.

Theorem 1.1. *[Sta98, Corollary 2.2] The e-expansion of $X_{\mathrm{inc}(P)}(\mathbf{x})$ can be expressed in terms of the P-analogs $m_\lambda^P(\mathbf{u})$ of the monomial symmetric functions $m_\lambda(\mathbf{x})$ as*

$$X_{\mathrm{inc}(P)}(\mathbf{x}) = \sum_{\lambda} \langle \mathbf{u}_P \rangle m_\lambda^P(\mathbf{u})\, e_\lambda(\mathbf{x}), \tag{1.3}$$

where $\langle \mathbf{u}_P \rangle m_\lambda^P(\mathbf{u})$ is the coefficient of $\mathbf{u}_P = \prod_{p \in P} u_p$ in the $\mathbf{u}$-monomial expansion of $m_\lambda^P(\mathbf{u})$.

More recently, a non-commutative generalization of P-analogs were introduced by Hwang [Hwa24] and further developed by Blasiak–Eriksson–Pylyavskyy–Siegl [BEPS25]. The current paper is only concerned with the commutative setting.

Gasharov's result [Gas96] can be rephrased as giving a combinatorial interpretation for the $\mathbf{u}$-monomial expansion for $s_{\lambda/\mu}^P(\mathbf{u})$ in terms of P-tableaux of shape λ/μ (see Theorem 2.3). From Theorem 1.1, the goal of finding a combinatorial interpretation of the coefficients of $X_{\mathrm{inc}(P)}(\mathbf{x})$ in the e-basis is the same as finding a combinatorial interpretation of the coefficients of $m_\lambda^P(\mathbf{u})$ expanded in terms of $\mathbf{u}$-monomials. Hence a natural intermediate goal is to establish combinatorial interpretations for various P-analogs $\psi(f(\mathbf{x}))$ of symmetric functions $f(\mathbf{x})$ which "lie in between" monomial symmetric functions and Schur functions. One such class are the cylindric Schur functions $s_{\lambda/\mu/d}(\mathbf{x})$, where "lie in between" has the precise meaning that $s_{\lambda/\mu/d}(\mathbf{x})$ is an m-positive symmetric function such that the m-expansion coefficients are bounded by the m-expansion coefficients of $s_{\lambda/\mu}(\mathbf{x})$.

The cylindric Schur functions are based on cylindric partitions of Gessel and Krattenthaler [GK97], and were used by Postnikov to study the quantum cohomology of the Grassmannian [Pos05]. Lam showed that cylindric Schur functions are special cases of skew affine Schur functions [Lam06]. McNamara [McN06] conjectured and Lee [Lee19] proved that cylindric skew Schur functions expand positively in terms of cylindric Schur functions and that the coefficients of this expansion are the same as 3-point Gromov–Witten invariants.

Gessel–Krattenthaler [GK97] describe a method for expressing cylindric Schur functions as a sum of determinants and give an explicit formula in the case when λ is a rectangular shape and $d = 0$. Postnikov [Pos05, Eq. (11)] (see also [McN06, §6]) then gave the following Jacobi–Trudi-like identity for cylindric Schur functions, making the result of Gessel–Krattenthaler explicit for arbitrary λ and $d > 0$.

Theorem 1.2. *Let $\lambda/\mu/d$ be a cylindric shape. Then*

$$s_{\lambda/\mu/d}(\mathbf{x}) = \sum_{\substack{k_1+k_2+\cdots+k_{\lambda_1}=0\\ k_i\in\mathbb{Z}}} \det\left[e_{k_i(\lambda_1+d)+\lambda'_i-\mu'_j-i+j}(\mathbf{x})\right]. \tag{1.4}$$

We define P-analogs $s^P_{\lambda/\mu/d}(\mathbf{u}) := \psi(s_{\lambda/\mu/d}(\mathbf{x}))$ of the cylindric (skew) Schur functions in terms of (1.4). In Definition 2.4, we define sets $\mathrm{CT}_P(\lambda/\mu/d)$ of *cylindric P-tableaux*. Our main result shows that for $(\mathbf{3}+\mathbf{1})$-free posets P, every $s^P_{\lambda/\mu/d}(\mathbf{u})$ has a positive $\mathbf{u}$-monomial expansion in terms of cylindric P-tableaux.

Theorem 1.3. *For any $(\mathbf{3}+\mathbf{1})$-free poset P and cylindric shape $\lambda/\mu/d$,*

$$s^P_{\lambda/\mu/d}(\mathbf{u}) := \sum_{\substack{k_1+k_2+\cdots+k_r=0\\ k_i\in\mathbb{Z}}} \det\left[e^P_{k_i(\lambda_1+d)+\lambda'_i-\mu'_j-i+j}(\mathbf{u})\right] = \sum_{T\in\mathrm{CT}_P(\lambda/\mu/d)} \mathbf{u}^T. \tag{1.5}$$

The next corollary shows that the number of standard cylindric P-tableaux of a cylindric shape are counted by certain sums of the coefficients c^P_λ in $X_{\mathrm{inc}(P)}(\mathbf{x}) = \sum_\lambda c^P_\lambda e_\lambda(\mathbf{x})$. The proof follows directly from Theorem 1.3, Theorem 1.1, and the notation covered in Section 3.

Corollary 1.4. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset on $[n] = \{1, 2, \ldots, n\}$, and let $\lambda/\mu/d$ be a cylindric shape such that λ/μ has n cells. Letting a_ν denote the coefficients in the monomial expansion $s_{\lambda/\mu/d}(\mathbf{x}) = \sum_{\nu\vdash n} a_\nu m_\nu(\mathbf{x})$ and c^P_ν the coefficients in $X_{\mathrm{inc}(P)} = \sum_{\nu\vdash n} c^P_\nu e_\nu(\mathbf{x})$, we have*

$$\sum_{\nu\vdash n} a_\nu c^P_\nu = \langle \mathbf{u}_P\rangle\, s^P_{\lambda/\mu/d}(\mathbf{u}) \;=\; \#\{T \in \mathrm{CT}_P(\lambda/\mu/d) : \mathbf{u}^T = u_1u_2\cdots u_n\}. \tag{1.6}$$

In other words, $\sum_{\nu\vdash n} a_\nu c^P_\nu$ is the number of standard cylindric P-tableaux of shape $\lambda/\mu/d$.

In Section 2, we recall the necessary background on P-tableaux and cylindric partitions. In Section 3, we define cylindric P-tableaux, examine some of their properties, and prove Theorem 1.3. In Section 4, we look at special cases and extensions of Corollary 1.4. We use a monotonicity property of $s^P_{\lambda/\mu/d}(\mathbf{u})$ to strengthen Corollary 1.4. Furthermore, we recover a positive formula of Stembridge [Ste92, Theorem 2.8] and Clearman–Hyatt–Shelton–Skandera [CHSS16, Theorem 4.7 (v-b)] for the coefficient c^P_λ when λ is a rectangle.

2. Preliminaries

2.1. Preliminaries

For an integer partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l) \vdash n$, the (English style) Young diagram of shape λ is the set $\{(i, j) : 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant \lambda_i\}$, drawn as boxes labeled with matrix-style coordinates where the box (i, j) is in row i and column j. We often identify partitions with their corresponding Young diagrams such that, for partitions λ, μ, we write $\mu \subset \lambda$ to mean that the Young diagram of μ is contained in the Young diagram of λ. For $\mu \subset \lambda$, the skew shape λ/μ is the difference of Young diagrams $\lambda - \mu$. Let λ' denote the transpose partition of λ and $\ell(\lambda)$ for the number of nonzero parts of λ.

A semistandard Young tableau of shape λ/μ is a function $T : \lambda/\mu \to \mathbb{Z}_{>0}$ such that $T(i,j) < T(i+1,j)$ and $T(i,j) \leqslant T(i,j+1)$, i.e. a semistandard Young tableau is an assignment of positive integers to the boxes of λ/μ such that the entries strictly increase down columns and weakly increase across rows. Let $\mathrm{SSYT}(\lambda/\mu)$ be the set of semistandard Young tableaux of shape λ/μ. A semistandard Young tableau T is a *standard Young tableau* if the entries of T are $\{1, 2, \ldots, |\lambda/\mu|\}$ with each entry appearing exactly once.

2.2. Chromatic symmetric functions

For a graph $G = (V, E)$, Stanley [Sta95] defined the *chromatic symmetric function* $X_G(\mathbf{x}) = \sum_\kappa \prod_{v \in V} x_{\kappa(v)}$, where the sum is over proper colorings $\kappa : V \to \mathbb{Z}_{>0}$. Write $X_G(\mathbf{x}) = \sum_{\lambda \vdash |V(G)|} c_\lambda^G e_\lambda(\mathbf{x})$ for the expansion of $X_G(\mathbf{x})$ in the basis of elementary symmetric functions. When G is the incomparability graph of a poset P, we write $c_\lambda^P = c_\lambda^G$. The Stanley–Stembridge Conjecture is concerned with the case when G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset P, meaning that there is no set of four elements $\{x, a_1, a_2, a_3\}$ of P such that $a_1 <_P a_2 <_P a_3$ and x is incomparable to each a_i.

The e-expansion of the chromatic symmetric function $X_G(\mathbf{x})$ is closely related to acyclic orientations of the graph. Write $\mathrm{Sink}(G, j)$ for the number of acyclic orientations of G with j sinks. One of the fundamental results on the e-coefficients of the chromatic symmetric function is the following theorem of Stanley.

Theorem 2.1. *[Sta95, Theorem 3.3] Let G be a graph on n vertices. Then for $1 \leqslant j \leqslant n$,*

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = j}} c_\lambda^G = \mathrm{Sink}(G, j). \tag{2.1}$$

2.3. Reformulation of Gasharov's theorem

To set the stage for our main theorem on P-cylindric Schur functions, we discuss several precursors of this result. The well known Jacobi–Trudi identity gives a determinantal expansion of skew Schur functions in terms of the elementary symmetric functions

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} \mathbf{x}^T = \det[e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x})]_{i,j=1}^{\ell(\lambda')}, \tag{2.2}$$

where $\mathbf{x}^T = \prod_{b \in \lambda/\mu} x_{T(b)}$ and $x^T = 1$ if $\lambda = \mu$. Either side of the equality can be taken as the definition of the skew Schur function $s_{\lambda/\mu}(\mathbf{x})$. Thus, for a poset P, the P-analog of $s_{\lambda/\mu}(\mathbf{x})$ is given by

$$s_{\lambda/\mu}^P(\mathbf{u}) := \psi(s_{\lambda/\mu}(\mathbf{x})) = \det[e^P_{\lambda'_i - \mu'_j - i + j}(\mathbf{u})]_{i,j=1}^{\ell(\lambda')}. \tag{2.3}$$

Definition 2.2. For a poset P, a *P-tableau* T of shape λ/μ is a function $T : \lambda/\mu \to P$ if $T(i,j) <_P T(i+1,j)$ and $T(i,j) \not>_P T(i,j+1)$, i.e. an assignment of elements of P to the boxes of λ/μ is a P-tableau if it is increasing in P down columns and non-decreasing in P across rows. A P-tableau is *standard* if every element of P appears exactly once.

Theorem 2.3. *[Gas96, Theorem 3] For a $(\mathbf{3}+\mathbf{1})$-free poset P,*

$$s^P_{\lambda/\mu}(\mathbf{u}) = \sum_{T \in \mathrm{SSYT}_P(\lambda/\mu)} \mathbf{u}^T, \tag{2.4}$$

where $\mathrm{SSYT}_P(\lambda/\mu)$ denotes the set of P-tableaux of shape λ/μ and $\mathbf{u}^T = \prod_{b \in \lambda/\mu} u_{T(b)}$.

For straight shapes ($\mu = \varnothing$), this is a reformulation of Gasharov's theorem [Gas96, Theorem 3] into the language of P-analogs. It also follows directly from [BEPS25, Theorem 3.9]. For skew shapes, it will follow from the more general Theorem 1.3.

Definition 2.4. For a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$, define the *Young diagram of column shape* $\mathrm{col}(\alpha)$ to be the set $\{(i,j) : 1 \leqslant j \leqslant l, 1 \leqslant i \leqslant \alpha_j\}$ using matrix style coordinates. As before, we identify a composition shape with its Young diagram, and we let $\mathrm{col}(\alpha)$ denote the above set. For a partition β with $\mathrm{col}(\beta) \subset \mathrm{col}(\alpha)$, write $\mathrm{col}(\alpha, \beta)$ for $\mathrm{col}(\alpha) - \mathrm{col}(\beta)$. For a poset P, Gasharov [Gas96] defined a *P-array of shape* $\mathrm{col}(\alpha,\beta)$ to be a function $A : \mathrm{col}(\alpha,\beta) \to P$ that is increasing in P down columns. Note that unlike P-tableaux, P-arrays have no conditions relating entries in different columns. The P-tableaux of shape λ/μ are exactly P-arrays of shape $\mathrm{col}(\lambda', \mu')$ such that the rows have no P-descents. For a composition α and partition β, write $\mathrm{Array}_P(\alpha,\beta)$ for the set of P-arrays of shape $\mathrm{col}(\alpha,\beta)$. If $\mathrm{col}(\beta) \not\subset \mathrm{col}(\alpha)$, we take $\mathrm{Array}_P(\alpha,\beta)$ to be the empty set.

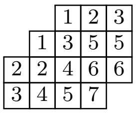
2.4. Cylindric Schur functions

Definition 2.5. For a Young tableau T of shape λ/μ and a nonnegative integer d, define T^d to be the filled diagram obtained by gluing a copy of the first column of T to the last column of T so that the copied column is shifted up by d cells. i.e. T^d has shape ν/θ where the conjugate shapes are $\nu' = (\lambda'_1 + d, \lambda'_2 + d, \dots, \lambda'_l + d, \lambda'_1)$ and $\theta' = (\mu'_1 + d, \mu'_2 + d, \dots, \mu'_l + d, \mu'_1)$. Note that ν' and θ' will be partitions only if $d \geqslant \lambda'_1 - \lambda'_l$ and $d \geqslant \mu'_1 - \mu'_l$. For a skew shape λ/μ and an integer d such that $d \geqslant \max(\lambda'_1 - \lambda'_l, \mu'_1 - \mu'_l)$, we say $\lambda/\mu/d$ is a *cylindric (skew) shape*. For simplicity, we do not distinguish between between cylindric straight shapes and cylindric skew shapes as the distinction is not as well-defined as in the classical case.

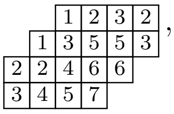
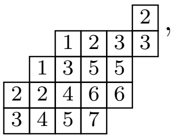
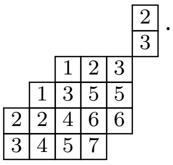
A Young tableau T of shape λ/μ is a *cylindric tableau* of shape $\lambda/\mu/d$ if T^d is a Young tableau. Write $\mathrm{CSSYT}(\lambda/\mu/d)$ for the set of semistandard cylindric Young tableaux of shape $\lambda/\mu/d$ and $\mathrm{CSYT}(\lambda/\mu/d)$ for the set of standard cylindric Young tableaux of shape $\lambda/\mu/d$.

For a tableau T of shape λ/μ, observe that if $d \geqslant \lambda'_1$, then cells in the copied column in T^d will not be adjacent to any cells in the previous column. Thus for $d \geqslant \lambda'_1$, we have $\mathrm{CSSYT}(\lambda/\mu/d) = \mathrm{SSYT}(\lambda/\mu)$.

Example 2.6. Consider the following tableau

$T =$  . (2.5)

Then T^2, T^3, and T^4 are

 ,  ,  . (2.6)

We have that $5554/21/2$, $5554/21/3$, and $5554/21/4$ are cylindric shapes. As T^2 is not weakly increasing across rows, $T \notin \mathrm{CSSYT}(5554/21/2)$. As T^3 and T^4 are semistandard Young tableaux, T is an element of $\mathrm{CSSYT}(5554/21/3)$ and $\mathrm{CSSYT}(5554/21/4)$.

Definition 2.7. For a cylindric (skew) shape $\lambda/\mu/d$ define the cylindric (skew) Schur function $s_{\lambda/\mu/d}(\mathbf{x})$ by

$$s_{\lambda/\mu/d}(\mathbf{x}) = \sum_{T \in \mathrm{CSSYT}(\lambda/\mu/d)} \mathbf{x}^T. \tag{2.7}$$

Cylindric tableaux are *monotone with respect to increasing* d in the following sense. One can verify that for any cylindric shape $\lambda/\mu/d$, $\mathrm{CSSYT}(\lambda/\mu/d) \subseteq \mathrm{CSSYT}(\lambda/\mu/d+1)$. Therefore, if $s_{\lambda/\mu/d+1}(\mathbf{x}) - s_{\lambda/\mu/d}(\mathbf{x}) = \sum_{\nu \vdash |\lambda/\mu|} a_\nu m_\nu(\mathbf{x})$ is the monomial expansion of the difference, then $a_\nu \geqslant 0$ for all $\nu \vdash |\lambda/\mu|$.

Remark 2.8. Definition 2.5 and Definition 2.7 are related to other definitions in the literature as follows.

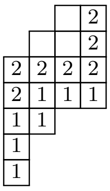
(i) Definition 2.5 is slightly more general than the definition used by Gessel–Krattenthaler [GK97]. For a cylindric shape $\lambda/\mu/d$ in our notation with $d > 0$, we convert to a cylindric shape $\nu/\eta/m$ in the notation of Gessel and Krattenthaler by taking

$$\nu = (\lambda_1 + \lambda_{1+d} + \lambda_{1+2d} + \cdots, \lambda_2 + \lambda_{2+d} + \lambda_{2+2d} + \cdots, \ldots, \lambda_d + \lambda_{2d} + \cdots), \tag{2.8}$$

$$\eta = \mu, \quad \text{and} \quad m = \lambda_1. \tag{2.9}$$

(ii) For $d > 0$, Definition 2.7 of $s_{\lambda/\mu/d}(\mathbf{x})$ is equivalent to the definition used by Postnikov [Pos05] and McNamara [McN06], but our conventions for cylindric shapes are different. Following [McN06, §4], we convert a cylindric shape $\lambda/\mu/d$ in our notation to a shape $\nu/m/\theta$ in the notation of Postnikov and McNamara in the following way: let $k = d$ and $n = \lambda_1 + d$. Then $\theta = \mu$, and ν is obtained from λ by removing m n-ribbons from the border of λ, where m is the smallest number such that ν has at most k rows.

For example, for $\lambda = (4, 4, 4, 4, 2, 1, 1)$, $\mu = (2, 1)$, and $d = 3$, to convert to the notation of Postnikov and McNamara, we repeatedly remove ribbons with 7 cells from the bottom edge of λ until the resulting shape has at most 3 rows. In the following figure, we label the cells removed in the first ribbon with 1's and label the cells in the second ribbon with 2's.



Therefore we have that the equivalent shape $\nu/m/\theta$ in the notation of Postnikov and McNamara is $\nu = (3, 3)$, $m = 2$, and $\theta = (2, 1)$.

3. Main results

In this section we give a proof of our main result Theorem 1.3. We define *cylindric P-tableaux*. In Lemma 3.4, we show that P-cylindric Schur functions can be written as a signed sum over certain P-arrays. We then define a sign-reversing involution to show that this sum reduces to a positive sum over cylindric P-tableaux. Our proof of Theorem 1.3 combines ideas of Gessel–Krattenthaler [GK97, Proposition 1] and Gasharov [Gas96, Theorem 3].

Definition 3.1. As with semistandard Young tableaux, for a P-tableau T and a nonnegative integer d, define T^d to be the diagram filled with elements of P obtained by gluing a copy of the first column of T to the last column of T so that the copied column is shifted up by d cells. For a cylindric shape $\lambda/\mu/d$, we say a P-tableau T is a *cylindric P-tableau* of shape $\lambda/\mu/d$ if T^d is a P-tableau. Write $\mathrm{CT}_P(\lambda/\mu/d)$ for the set of all cylindric P-tableaux. As with ordinary cylindric tableaux, one can again verify that

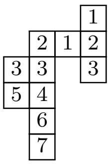
$$\mathrm{CT}_P(\lambda/\mu/d) \subseteq \mathrm{CT}_P(\lambda/\mu/d+1) \tag{3.1}$$

for any cylindric shape $\lambda/\mu/d$.

As with P-tableaux, for a P-array A let A^d be the P-array obtained by gluing the first column of A to the right of the last column of A so that the copied column is shifted up by d cells, i.e. A^d has shape $\mathrm{col}(\bar{\alpha}, \bar{\beta})$ where $\bar{\alpha} = (\alpha_1 + d, \alpha_2 + d, \ldots, \alpha_l + d, \alpha_1)$ and $\bar{\beta} = (\beta_1 + d, \beta_2 + d, \ldots, \beta_l + d, \beta_1)$.

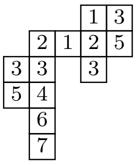
Example 3.2. For P the total order on $\mathbb{Z}_{>0}$, the P-array

$A =$ (3.2)

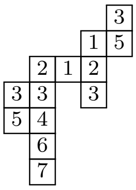


has shape $\mathrm{col}((4, 6, 2, 3), (2, 1, 1))$, and we have

$A^2 =$



and $A^3 =$



. (3.3)

The P-array A^2 has shape $\mathrm{col}((6,8,4,5,4),(4,3,3,2,2))$, and the P-array A^3 has shape $\mathrm{col}((7,9,5,6,4),(5,4,4,3,2))$.

Definition 3.3. Fix a cylindric shape $\lambda/\mu/d$. Write $\tilde{\mathbb{Z}}^{\lambda_1}$ for the set of $\mathbf{k} = (k_1, \dots, k_{\lambda_1}) \in \mathbb{Z}^{\lambda_1}$ such that $k_1 + k_2 + \cdots + k_{\lambda_1} = 0$. For a permutation $\pi \in S_{\lambda_1}$ and $\mathbf{k} \in \tilde{\mathbb{Z}}^{\lambda_1}$, let $\pi_d^{\mathbf{k}}(\lambda) = (\pi_d^{\mathbf{k}}(\lambda)_1, \pi_d^{\mathbf{k}}(\lambda)_2, \dots, \pi_d^{\mathbf{k}}(\lambda)_{\lambda_1})$ be the sequence of integers defined by

$$\pi_d^{\mathbf{k}}(\lambda)_i = k_i(\lambda_1 + d) + \lambda'_{\pi(i)} + i - \pi(i). \tag{3.4}$$

Note that when π is the identity permutation and $\mathbf{k} = \mathbf{0}$, we have $\pi_d^{\mathbf{k}}(\lambda) = \lambda'$. Let

$$\tilde{B} = \tilde{B}(P, \lambda/\mu/d) = \Big\{(\pi, \mathbf{k}, A) : \pi \in S_{\lambda_1},\ \mathbf{k} \in \tilde{\mathbb{Z}}^{\lambda_1},\ A \in \mathrm{Array}_P(\mathrm{col}(\pi_d^{\mathbf{k}}(\lambda), \mu'))\Big\}, \tag{3.5}$$

$$B = B(P, \lambda/\mu/d) = \big\{(\pi, \mathbf{k}, A) \in \tilde{B} : A^d \text{ is not a } P\text{-tableau}\big\}. \tag{3.6}$$

Note that if $\pi_d^{\mathbf{k}}(\lambda)_i < \mu'_i$ for some i and $\mathbf{k}$, there are no P-arrays of shape $\mathrm{col}(\pi_d^{\mathbf{k}}(\lambda), \mu')$. For $(\pi, \mathbf{k}, A) \in \tilde{B}$, define the *sign* of $(\pi, \mathbf{k}, A) \in \tilde{B}$ to be $\mathrm{sgn}(\pi)$, where sgn is the character of S_n such that $\mathrm{sgn}(\mathrm{id}) = 1$ and $\mathrm{sgn}(\pi \circ (i\ j)) = -\mathrm{sgn}(\pi)$ for any distinct $i, j \in [n]$. We can then write $s^P_{\lambda/\mu/d}(\mathbf{u})$ as a sum over elements in $\tilde{B}(P, \lambda/\mu/d)$.

Lemma 3.4. *Let $\lambda/\mu/d$ be a cylindric shape. Then*

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{(\pi,\mathbf{k},A)\in\tilde{B}} \mathrm{sgn}(\pi)\mathbf{u}^A, \tag{3.7}$$

where $\mathbf{u}^A = \prod_{a\in A} u_a$ is the product over the entries of A.

Proof. By definition, we have that

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \det\left[e^P_{k_i(\lambda_1+d)+\lambda'_i-\mu'_j-i+j}(\mathbf{u})\right]. \tag{3.8}$$

As the determinant of an $n \times n$ matrix M is given by $\det M = \sum_{\pi\in S_n} \mathrm{sgn}(\pi) \prod_{i=1}^n M_{\pi(i),i}$, we can write

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \sum_{\pi\in S_{\lambda_1}} \mathrm{sgn}(\pi) \prod_{i=1}^{\lambda_1} e^P_{k_{\pi(i)}(\lambda_1+d)+\lambda'_{\pi(i)}-\mu'_i-\pi(i)+i}(\mathbf{u}) \tag{3.9}$$

$$= \sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \sum_{\pi\in S_{\lambda_1}} \mathrm{sgn}(\pi) \prod_{i=1}^{\lambda_1} e^P_{k_i(\lambda_1+d)+\lambda'_{\pi(i)}-\mu'_i-\pi(i)+i}(\mathbf{u}), \tag{3.10}$$

since the symmetric group S_{λ_1} acts on $\tilde{\mathbb{Z}}^{\lambda_1}$ by permuting indices. Observe that for a composition α and partition β such that $\mathrm{col}(\beta) \subset \mathrm{col}(\alpha)$, we have

$$\sum_{A\in\mathrm{Array}_P(\alpha,\beta)} \mathbf{u}^A = \prod_{i=1}^{\ell(\alpha)} e^P_{\alpha_i-\beta_i}(\mathbf{u}). \tag{3.11}$$

Thus for $\pi \in S_{\lambda_1}$ and $\mathbf{k} \in \tilde{\mathbb{Z}}^{\lambda_1}$,

$$\sum_{A\in \mathrm{Array}_P(\pi^{\mathbf{k}}_d(\lambda),\mu')} \mathbf{u}^A = \prod_{i=1}^{\lambda_1} e^P_{k_i(\lambda_1+d)+\lambda'_{\pi(i)}+i-\pi(i)-\mu'_i}(\mathbf{u}) = \prod_{i=1}^{\lambda_1} e^P_{\pi^{\mathbf{k}}_d(\lambda)_i-\mu'_i}. \tag{3.12}$$

Combining equations (3.10) and (3.12), we have the desired equality

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \sum_{\pi\in S_{\lambda_1}} \mathrm{sgn}(\pi) \sum_{A\in \mathrm{Array}_P(\pi^{\mathbf{k}}_d(\lambda),\mu')} \mathbf{u}^A = \sum_{(\pi,\mathbf{k},A)\in\tilde{B}} \mathrm{sgn}(\pi)\mathbf{u}^A. \tag{3.13}$$

□

Remark 3.5. In the proof of Lemma 3.4, we used the fact that S_{λ_1} acts on $\tilde{\mathbb{Z}}^{\lambda_1}$ to show that

$$\begin{aligned}&\sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \sum_{\pi\in S_{\lambda_1}} \mathrm{sgn}(\pi) e^P_{k_i(\lambda_1+d)+\lambda'_{\pi(i)}+i-\pi(i)-\mu'_i}(\mathbf{u})\\ &\qquad = \sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \sum_{\pi\in S_{\lambda_1}} \mathrm{sgn}(\pi) e^P_{k_{\pi(i)}(\lambda_1+d)+\lambda'_{\pi(i)}+i-\pi(i)-\mu'_i}(\mathbf{u}).\end{aligned}$$

Therefore we obtain two equivalent determinantal formulas for $s^P_{\lambda/\mu/d}(\mathbf{u})$ by replacing k_i with k_j,

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \det[e^P_{k_i(\lambda_1+d)+\lambda'_i-\mu'_j-i+j}(\mathbf{u})] = \sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \det[e^P_{k_j(\lambda_1+d)+\lambda'_i-\mu'_j-i+j}(\mathbf{u})]. \tag{3.14}$$

Let us look closely at the properties of B and $\tilde{B}$.

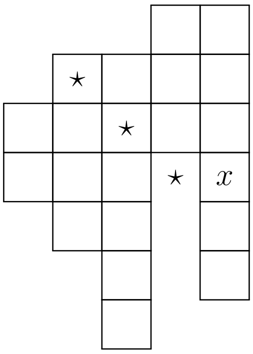
Definition 3.6. For a P-array $A = [a_{s,t}]_{(s,t)\in\mathrm{col}(\alpha,\beta)}$ of shape $\mathrm{col}(\alpha,\beta)$, columns $i < j$ *intersect* in A if there exists some $(m, j) \in \mathrm{col}(\alpha,\beta)$ such that

1. $m - j + i + 1 > \alpha_i$, or
2. $(m - j + i + 1, i) \in \mathrm{col}(\alpha,\beta)$ and $a_{m-j+i+1,i} >_P a_{m,j}$.

We say (m, j) is an *intersection point* of columns i and j, and that $(m - j + i + 1, i)$ is a *witness* to the intersection point.

Remark 3.7. When P is a total order, Gessel–Viennot [GV85] and Gessel–Krattenthaler [GK97] interpreted each column of a P-array as a path drawn in the plane and on a cylinder respectively. Definition 3.6 then gives a criteria for when two such paths intersect and extends the definition to P-arrays, which have no such geometric interpretation. We use the word *intersection* in the current setting to reflect this history.

Example 3.8. For the Young diagram of shape $\mathrm{col}(45736, 211)$, an intersection point in the cell labelled x will have potential witnesses in the southeast diagonal directly below the diagonal containing x. We label the possible witnesses for x with stars.



(3.15)

Example 3.9. Let P be the total order on $\mathbb{Z}_{>0}$, and let A be the P-array from Example 3.2. The diagram below shows A with underlined entries in the intersection points.

$$A = \begin{array}{cccc} & & & 1 \\ & 2 & \underline{1} & 2 \\ 3 & 3 & & \underline{3} \\ 5 & \underline{4} & & \\ & \underline{6} & & \\ & \underline{7} & & \end{array} \tag{3.16}$$

We have that $(3,4)$ is an intersection point with witness $(3,3)$, as $(3,3)$ is on the southeast diagonal directly to the left of $(3,4)$ and $3 > \alpha_3 = 2$. We have that $(2,3)$ is an intersection point with witness $(2,2)$, as $a_{2,2} = 2 >_P 1 = a_{2,3}$.

The following lemma applied to A^d shows that B consists of all triples $(\pi, \mathbf{k}, A) \in \tilde{B}$ such that A^d has an intersection point.

Lemma 3.10. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset and A be a P-array of shape* $\mathrm{col}(\alpha, \beta)$ *for some composition α and partition β. The following are equivalent:*

(i) A is a P-tableau,

(ii) A has no adjacent intersecting columns,

(iii) A has no intersecting columns.

Proof. By Definiton 2.2, A P-tableau is a P-array of skew partition shape that is nondescending in P across rows. A P-array A is of skew partition shape λ/μ only if there is no $c > 1$ and $r > \lambda'_{c-1}$ such that $(r, c) \in \lambda/\mu$. But that is exactly condition (1) of the definition of intersecting columns applied to $i = c-1$ and $j = c$. Likewise, the nondecreasing row condition of a P-tableau is equivalent to saying that no adjacent columns satisfy condition (2) of the definition of intersecting columns. Therefore (i) and (ii) are equivalent.

If A has adjacent intersecting columns, then A has intersecting columns, so (iii) $\implies$ (ii).

It remains to show that for A having shape $\mathrm{col}(\alpha,\beta)$, if two columns of A intersect then two adjacent columns intersect. To do this, choose a pair $i < j$ of intersecting columns of A such that $j-i$ is as small as possible. We then show that $j-i=1$. Suppose $j-i>1$. Observe that if $\alpha_t < \alpha_{t+1}$, then columns t and $t+1$ intersect, so we may assume α is a partition. Let (m,j) be an intersection point between columns i and j. As α is a partition, the witness $(m-j+i+1,i)$ is in $\mathrm{col}(\alpha,\beta)$, and as β is a partition, we have that

$$\{(s,t) \mid m-j+i+1 \leqslant s \leqslant m, i \leqslant t \leqslant j\} \subseteq \mathrm{col}(\alpha,\beta). \tag{3.17}$$

As (m,j) is an intersection point between columns i and j, we have that $a_{m-1,j} <_P a_{m,j} <_P a_{m-j+i+1,i}$. If $a_{m,j-1} <_P a_{m-1,j-1}$, then $(m-1,j)$ is an intersection point between columns $j-1$ and j, contradicting our assumption that $j-i$ is as small as possible. If $a_{m-1,j-1} <_P a_{m-j+i+1,i}$, then $(m-1,j-1)$ is an intersection point between columns i and $j-1$, again contradicting the assumption that $j-i$ is as small as possible. If $a_{m-1,j} \not<_P a_{m-1,j-1}$ and $a_{m-1,j-1} \not<_P a_{m-j+i+1,i}$, then $a_{m-1,j}, a_{m,j}, a_{m-j+i+1,i}$, and $a_{m-1,j-1}$ form an induced $(\mathbf{3}+\mathbf{1})$, which contradicts the fact that P is $(\mathbf{3}+\mathbf{1})$-free.

Therefore, if A is a P-array with a pair of intersecting columns, then A must have an adjacent pair of intersecting columns, as desired. □

The next lemma shows that if P is a $(\mathbf{3}+\mathbf{1})$-free poset and $\lambda/\mu/d$ is a cylindric shape, $\tilde{B}-B$ consists of exactly the triples $(\mathrm{id},\mathbf{0},A)$ such that A is a cylindric P-tableau of shape $\lambda/\mu/d$.

Lemma 3.11. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset and $\lambda/\mu/d$ a cylindric shape with $\lambda_1 \geqslant 1$. Let $(\pi,\mathbf{k},A) \in \tilde{B}(P,\lambda/\mu/d)$. If A^d is a P-tableau, then π is the identity permutation and $\mathbf{k}=\mathbf{0}$.*

Proof. For $(\pi,\mathbf{k},A) \in \tilde{B}(P,\lambda/\mu/d)$, we know that A is a P-array of shape $\mathrm{col}(\pi^{\mathbf{k}}_d(\lambda),\mu')$. Then A^d is a P-array of shape $\mathrm{col}(\alpha,\beta)$, where $\alpha = (\pi^{\mathbf{k}}_d(\lambda)_1+d, \pi^{\mathbf{k}}_d(\lambda)_2+d, \dots, \pi^{\mathbf{k}}_d(\lambda)_{\lambda_1}+d, \pi^{\mathbf{k}}_d(\lambda)_1)$ and $\beta = (\mu'_1+d, \mu'_2+d, \dots, \mu'_{\lambda_1}+d, \mu'_1)$. Assume A^d is a P-tableau. Then no pair of columns of A^d intersect by Lemma 3.10. As column j cannot intersect column 1 in A^d, if $(m,j) \in \mathrm{col}(\alpha,\beta)$ we have that $(m-j+2,1)$ is not a witness to an intersection in A^d, so $m-j+2 \leqslant \alpha_1$. In particular, if $\alpha_j > \beta_j$ then $(\alpha_j,j) \in \mathrm{col}(\alpha,\beta)$ and

$$\alpha_j - j + 2 \leqslant \alpha_1. \tag{3.18}$$

As column i cannot intersect column λ_1+1 in A^d, we have that $(\alpha_{\lambda_1+1},\lambda_1+1)$ is not an intersection point. So similarly,

$$\alpha_{\lambda_1+1} - (\lambda_1+1) + i + 1 \leqslant \alpha_i. \tag{3.19}$$

Substituting $\alpha_{\lambda_1+1} = \pi^{\mathbf{k}}_d(\lambda)_1$, $\alpha_i = \pi^{\mathbf{k}}_d(\lambda)_i + d$, and $\pi^{\mathbf{k}}_d(\lambda)_i = k_i(\lambda_1+d) + \lambda'_{\pi(i)} + i - \pi(i)$ for $i < \lambda_1+1$, we have that inequalities (3.18) and (3.19) are equivalent to

$$(k_i - k_1)(\lambda_1 + d) \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} + \pi(i) - (\pi(1) + 1), \tag{3.20}$$

$$\lambda'_{\pi(1)} - \lambda'_{\pi(i)} + \pi(i) - \pi(1) + 1 \leqslant (k_i - k_1 + 1)(\lambda_1 + d). \tag{3.21}$$

By (3.20) and (3.21), we have the inequalities

$$(k_i - k_1)(\lambda_1 + d) \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} + \pi(i) - (\pi(1) + 1) < (k_i - k_1 + 1)(\lambda_1 + d). \tag{3.22}$$

We now show that for all i satisfying $\pi(i) > \pi(1)$, we have $k_i = k_1$. As $\pi \in S_{\lambda_1}$, we have $0 < \pi(1) < \pi(i) < \lambda_1 + 1$ and

$$0 \leqslant \pi(i) - (\pi(1) + 1) < \lambda_1. \tag{3.23}$$

As $\lambda/\mu/d$ is a cylindric shape, we have $\lambda'_1 \geqslant \lambda'_2 \geqslant \ldots \geqslant \lambda'_{\lambda_1}$ and $d \geqslant \lambda'_1 - \lambda'_{\lambda_1}$, so

$$0 \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} \leqslant \lambda'_1 - \lambda'_{\lambda_1} \leqslant d. \tag{3.24}$$

Adding these inequalities we have

$$0 \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} + \pi(i) - (\pi(1) + 1) < \lambda_1 + d. \tag{3.25}$$

By (3.22), this gives

$$(k_i - k_1)(\lambda_1 + d) \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} + \pi(i) - (\pi(1) + 1) < \lambda_1 + d. \tag{3.26}$$

Therefore $k_i - k_1 < 1$. Furthermore from (3.22), we have

$$0 \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} + \pi(i) - (\pi(1) + 1) < (k_i - k_1 + 1)(\lambda_1 + d), \tag{3.27}$$

so $k_i - k_1 > -1$, and we have that $k_i = k_1$.

We claim that for all i satisfying $\pi(i) < \pi(1)$, we have $k_i = k_1 - 1$. As $\pi \in S_{\lambda_1}$, we have $0 < \pi(i) < \pi(1) \leqslant \lambda_1$, so

$$-\lambda_1 \leqslant \pi(i) - (\pi(1) + 1) < 0. \tag{3.28}$$

As $\lambda/\mu/d$ is a cylindric shape, $\lambda'_1 \geqslant \lambda'_2 \geqslant \ldots \geqslant \lambda'_{\lambda_1}$ and $d \geqslant \lambda'_1 - \lambda'_{\lambda_1}$, so

$$-d \leqslant \lambda'_{\lambda_1} - \lambda'_1 \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} \leqslant 0. \tag{3.29}$$

Adding inequalities (3.28) and (3.29), we have

$$-(\lambda_1 + d) \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} + \pi(i) - (\pi(1) + 1) < 0. \tag{3.30}$$

Then from (3.22), $(k_i - k_1)(\lambda_1 + d) < 0$. As $\lambda_1 + d > 0$, we have $k_1 > k_i$. From (3.22),

$$-(\lambda_1 + d) \leqslant \lambda'_{\pi(1)} - \lambda'_{\pi(i)} + \pi(i) - (\pi(1) + 1) < (k_i - k_1 + 1)(\lambda_1 + d), \tag{3.31}$$

so $-1 < k_i - k_1 + 1$. Therefore we have that $k_1 = k_i + 1$.

We claim that $\mathbf{k} = \mathbf{0}$. From the above argument, we have that $k_i = k_1$ if $\pi(i) > \pi(1)$ and $k_i = k_1 - 1$ if $\pi(i) < \pi(1)$. As $\mathbf{k} \in \widetilde{\mathbb{Z}}^{\lambda_1}$,

$$k_1 + k_2 + \cdots + k_{\lambda_1} = \lambda_1 k_1 - \#\{i \in [\lambda_1] : \pi(i) < \pi(1)\} = 0. \tag{3.32}$$

Since $0 \leqslant \#\{i \in [\lambda_1] : \pi(i) < \pi(1)\} < \lambda_1$ and k_1 is an integer, (3.32) gives $k_1 = 0$ and $\#\{i : \pi(i) < \pi(1)\} = 0$, so $k_1 = k_2 = \ldots = k_{\lambda_1} = 0$.

Next we claim that π must be the identity permutation. As A^d has no intersecting columns, by Lemma 3.10, we have that α must be a partition and $\alpha_i \geqslant \alpha_j$ whenever $i < j$. If π is not the identity permutation, there is some i so that $\pi(i) > \pi(i+1)$. As $\mathbf{k} = \mathbf{0}$, we have that

$$\alpha_i = k_i(\lambda_1 + d) + \lambda'_{\pi(i)} + i - \pi(i) + d = \lambda'_{\pi(i)} + i - \pi(i) + d, \tag{3.33}$$

$$\alpha_{i+1} = k_{i+1}(\lambda_1 + d) + \lambda'_{\pi(i+1)} + i + 1 - \pi(i+1) + d = \lambda'_{\pi(i+1)} + i + 1 - \pi(i+1) + d. \tag{3.34}$$

As λ is a partition and $\pi(i) > \pi(i+1)$, we have $\lambda'_{\pi(i)} \leqslant \lambda'_{\pi(i+1)}$ and

$$\alpha_i = \lambda'_{\pi(i)} + i - \pi(i) + d < \lambda'_{\pi(i+1)} + i + 1 - \pi(i+1) + d = \alpha_{i+1}. \tag{3.35}$$

But then α is not a partition, and π must be the identity permutation. □

Lemma 3.12. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset, and let A be a P-array of shape* $\mathrm{col}(\alpha, \beta)$ *such that columns i and j intersect for some $i < j$. Let m be the smallest index so that (m, j) is an intersection point between columns i and j. If $m < \alpha_j$ and $m - j + i > \beta_i$, then $a_{m-j+i,i} <_P a_{m+1,j}$.*

Proof. As $m < \alpha_j$, we have that $(m+1, j) \in \mathrm{col}(\alpha, \beta)$. As β is a partition and $i < j$ and $m - j + i > \beta_i$, we have $m - 1 \geqslant m - j + i > \beta_i \geqslant \beta_j$, so $(m-1, j) \in \mathrm{col}(\alpha, \beta)$. As m is the smallest index so that (m, j) is an intersection point of columns i and j and $m - j + i > \beta_i$, we have that $(m - j + i, i) \in \mathrm{col}(\alpha, \beta)$ and $a_{m-1,j} \not<_P a_{m-j+i,i}$. As $a_{m-1,j} <_P a_{m,j} <_P a_{m+1,j}$ and P is $(\mathbf{3}+\mathbf{1})$-free, we must have $a_{m-j+i,i} <_P a_{m+1,j}$ as desired. □

Definition 3.13. Let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ be a composition, and let β be a partition such that $\mathrm{col}(\beta) \subset \mathrm{col}(\alpha)$. For a P-array A of shape $\mathrm{col}(\alpha, \beta)$ with columns i and j intersecting, we define a *swap* at i and j by letting $\mathrm{swap}(A, i, j)$ be the P-array obtained in the following way.

1. Let (m, j) be the intersection point between columns i and j such that m is as small as possible.

2. Define $L_{i,j}$ and $R_{i,j}$ to be the sets

$$L_{i,j} = \{a_{r,i} : r \geqslant m - j + i + 1\} \quad \text{and} \quad R_{i,j} = \{a_{r,j} : r > m\}.$$

3. Define $\mathrm{swap}(A, i, j)$ to be the P-array obtained by moving $R_{i,j}$ to column i, moving $L_{i,j}$ to column j maintaining their relative order, and fixing the remaining cells of A.

Given a non-negative integer d, $\beta_1 - \beta_l \leqslant d$, and columns i and j such that i and j intersect in A^d, we define the *cylindric swap* at i and j by letting $\operatorname{swap}(A, i, j, d)$ be $\operatorname{swap}(A, i, j)$ if $j \neq l + 1$ and the P-array obtained from the following procedure if $j = l + 1$.

(1′) Let (m, j) be the intersection point between columns i and j so that m is as small as possible.

(2′) Define the set $L_{i,j}$ as in step (2) above and $R_{i,j}$ by

$$R_{i,j} = \{a_{r,j} : r > m\} = \{a_{r,1} : r > m + d\}$$

(3′) Define $\operatorname{swap}(A, i, j, d)$ to be the P-array obtained from A by moving $R_{i,j}$ from column 1 to column i, moving $L_{i,j}$ to column 1, and fixing the rest of the cells of A.

In other words, if $A' = [a'_{s,t}] = \operatorname{swap}(A, i, j, d)$, then

$$\text{if } j \neq \ell(\lambda) + 1, \qquad a'_{s,t} = \begin{cases} a_{s-j+i,i} & t = j,\ s > m \\ a_{s+j-i,j} & t = i,\ s > m - j + i \\ a_{s,t} & \text{otherwise,} \end{cases}$$

$$\text{if } j = \ell(\lambda) + 1, \qquad a'_{s,t} = \begin{cases} a_{s-j+i+d,i} & t = 1,\ s > m - d \\ a_{s+j-i,j} & t = i,\ s > m - j + i \\ a_{s,t} & \text{otherwise.} \end{cases}$$

Observe that the case of $i = 1$ and $j = \ell(\lambda) + 1$ is not possible as $(m - \ell(\lambda) + 1, 1) \in \operatorname{col}(\alpha, \beta)$ and $x_{m,\ell(\lambda)+1} = x_{m+d,1} \not<_P x_{m-\ell(\lambda)+1,1}$ whenever $(m, \ell(\lambda) + 1) \in \operatorname{col}(\alpha, \beta)$. Note that the cylindric swap is an operation on A rather than on A^d, even though we use an intersection point of A^d to define it. Lemma 3.12 ensures that the columns of $\operatorname{swap}(A, i, j)$ and $\operatorname{swap}(A, i, j, d)$ are P-increasing and therefore well-defined P-arrays.

Example 3.14. Let P be the poset on $\mathbb{Z}_{>0}$ such that $i <_P j$ if $i < j$ and $j - i > 1$, and consider the following P-arrays A and A^1

$$A = \begin{array}{ccc} & 2 & 3 \\ 1 & 4 & 5 \\ 4 & 9 & 7 \\ 6 & & \end{array}, \qquad A^1 = \begin{array}{cccc} & 2 & 3 & \underline{1} \\ 1 & 4 & 5 & 4 \\ 4 & 9 & \underline{7} & 6 \\ 6 & & & \end{array}. \tag{3.36}$$

The underlined entries indicate the intersection points of A^1. We then have

$$\operatorname{swap}(A, 2, 3, 1) = \begin{array}{ccc} & 2 & 3 \\ 1 & 4 & 5 \\ 4 & & 7 \\ 6 & & 9 \end{array} \quad \text{and} \quad \operatorname{swap}(A, 3, 4, 1) = \begin{array}{ccc} & 2 & 4 \\ 1 & 4 & 6 \\ 3 & 9 & \\ 5 & & \\ 7 & & \end{array}. \tag{3.37}$$

Lemma 3.15. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset and let $\lambda/\mu/d$ be a cylindric shape. If $(\pi, \mathbf{k}, A) \in B(P, \lambda/\mu/d)$ such that columns i and j intersect in A^d, then $(\sigma, \mathbf{k}', A') \in B(P, \lambda/\mu/d)$ where*

$$\sigma = \begin{cases} \pi \circ (i\ j) & \text{if } j \neq \lambda_1 + 1 \\ \pi \circ (1\ i) & \text{if } j = \lambda_1 + 1, \end{cases} \tag{3.38}$$

$$\mathbf{k}' = \begin{cases} (k_1, \dots, k_j, \dots, k_i, \dots, k_{\lambda_1}) & \text{if } j \neq \lambda_1 + 1 \\ (k_i + 1, k_2, \dots, k_{i-1}, k_1 - 1, k_{i+1}, \dots, k_{\lambda_1}) & \text{if } j = \lambda_1 + 1, \end{cases} \tag{3.39}$$

and $A' = \operatorname{swap}(A, i, j, d)$. Furthermore, if m is the smallest index such that (m, j) is an intersection point between columns i and j in A^d, then (m, j) is an intersection point between columns i and j in $(A')^d$.

Proof. Let $(\pi, \mathbf{k}, A) \in B(P, \lambda/\mu/d)$ such that columns i and j of A^d intersect. Write $A' = [a'_{s,t}] = \operatorname{swap}(A, i, j, d)$ and $\operatorname{col}(\alpha, \mu')$ for the shape of A'. Let m be the smallest index so that (m, j) is an intersection point for columns i and j of A^d. Then entry (m, j) is the same in A^d and $(A')^d$, and positions $(m+1, j)$ and $(m - j + i + 1, i)$ are swapped. Therefore (m, j) is an intersection point between columns i and j of $(A')^d$.

To show that $(\sigma, \mathbf{k}', A') \in B(P, \lambda/\mu/d)$, it suffices to show that A' is of shape $\operatorname{col}(\sigma_d^{\mathbf{k}'}(\lambda), \mu')$. We first consider the case when $1 \leqslant i < j \leqslant \lambda_1$. Performing $\operatorname{swap}(A, i, j)$ moves $\pi_d^{\mathbf{k}}(\lambda)_j - m$ entries of column j to column i and $\pi_d^{\mathbf{k}}(\lambda)_i - m + j - i$ entries from column i to column j. We then have that $\operatorname{swap}(A, i, j)$ has shape $\operatorname{col}(\alpha, \mu')$ where

$$\alpha_s = \begin{cases} \pi_d^{\mathbf{k}}(\lambda)_j + i - j & s = i \\ \pi_d^{\mathbf{k}}(\lambda)_i + j - i & s = j \\ \pi_d^{\mathbf{k}}(\lambda)_s & \text{otherwise.} \end{cases} \tag{3.40}$$

Substituting in for $\pi_d^{\mathbf{k}}(\lambda)_i$ and $\pi_d^{\mathbf{k}}(\lambda)_j$ from (3.4), we have

$$\alpha_i = k_j(\lambda_1 + d) + \lambda'_{\pi(j)} - \pi(j) + i, \tag{3.41}$$

and

$$\alpha_j = k_i(\lambda_1 + d) + \lambda'_{\pi(i)} - \pi(i) + j. \tag{3.42}$$

Let $\sigma = \pi \circ (i\ j)$ and $\mathbf{k}' = (k_1, \dots, k_j, \dots, k_i, \dots, k_{\lambda_1})$. Then

$$\alpha_i = k_j(\lambda_1 + d) + \lambda'_{\pi(j)} - \pi(j) + i = k_j(\lambda_1 + d) + \lambda'_{\sigma(i)} + i - \sigma(i) = \sigma_d^{\mathbf{k}'}(\lambda)_i, \tag{3.43}$$

$$\alpha_j = k_i(\lambda_1 + d) + \lambda'_{\pi(i)} - \pi(i) + j = k_i(\lambda_1 + d) + \lambda'_{\sigma(j)} + j - \sigma(j) = \sigma_d^{\mathbf{k}'}(\lambda)_j, \tag{3.44}$$

and for $s \neq i, j$,

$$\alpha_s = k_s(\lambda_1 + d) + \lambda'_{\pi(s)} + s - \pi(s) = k_s(\lambda_1 + d) + \lambda'_{\sigma(s)} + s - \sigma(s) = \sigma_d^{\mathbf{k}'}(\lambda)_s \tag{3.45}$$

so $\operatorname{swap}(A, i, j, d)$ is of shape $\sigma_d^{\mathbf{k}'}(\lambda)$ as desired.

Now consider the case when $j = \lambda_1 + 1$. Then swap(A, i, j, d) is obtained by moving $\pi^{\mathbf{k}}_d(\lambda)_1 - m$ entries of column 1 to column i and moving $\pi^{\mathbf{k}}_d(\lambda)_i - m + \lambda_1 + 1 - i + d$ entries from column i to column 1, leaving the rest of the columns unchanged. The swap(A, i, j, d) has shape col(α, μ') where

$$\alpha_s = \begin{cases} \pi^{\mathbf{k}}_d(\lambda)_i + \lambda_1 + 1 - i + d & s = 1 \\ \pi^{\mathbf{k}}_d(\lambda)_1 - \lambda_1 - 1 + i - d & s = i \\ \pi^{\mathbf{k}}_d(\lambda)_s & \text{otherwise.} \end{cases} \tag{3.46}$$

Let $\sigma = \pi \circ (1\ i)$ and $\mathbf{k}' = (k_i + 1, k_2, \ldots, k_{i-1}, k_1 - 1, k_{i+1}, \ldots, k_{\lambda_1})$. Then

$$\alpha_1 = \pi^{\mathbf{k}}_d(\lambda)_i - i + 1 + \lambda_1 + d = (k_i + 1)(\lambda_1 + d) + \lambda'_{\pi(i)} + 1 - \pi(i) \tag{3.47}$$

$$= k'_1(\lambda_1 + d) + \lambda'_{\sigma(1)} + 1 - \sigma(1) = \sigma^{\mathbf{k}'}_d(\lambda)_1, \tag{3.48}$$

$$\alpha_i = \pi^{\mathbf{k}}_d(\lambda)_1 + i - 1 - \lambda_1 - d = (k_1 - 1)(\lambda_1 + d) + \lambda'_{\pi(1)} + i - \pi(1) \tag{3.49}$$

$$= k'_i(\lambda_1 + d) + \lambda'_{\sigma(i)} + i - \sigma(i) = \sigma^{\mathbf{k}'}_d(\lambda)_i, \tag{3.50}$$

and for $s \neq 1, i$,

$$\alpha_s = \pi^{\mathbf{k}}_d(\lambda)_s = k_s(\lambda_1 + d) + \lambda'_{\pi(s)} + s - \pi(s) = k'_s(\lambda_1 + d) + \lambda'_{\sigma(s)} + s - \sigma(s) = \sigma^{\mathbf{k}'}_d(\lambda)_s. \tag{3.51}$$

Thus swap(A, i, j, d) is of shape $\sigma^{\mathbf{k}'}_d(\lambda)$ as desired. □

Definition 3.16. Let P be a poset. For a P-array A, we say an intersection point (m, j) is *P-minimal* if $a_{m,j} \not>_P a_{m',j'}$ for all intersection points (m', j') of A.

Definition 3.17. Let P be a $(\mathbf{3} + \mathbf{1})$-free poset and $\lambda/\mu/d$ be a cylindric shape. Define the map $\Phi : B(P, \lambda/\mu/d) \to B(P, \lambda/\mu/d)$ by

$$\Phi(\pi, \mathbf{k}, A) = (\sigma, \mathbf{k}', \text{swap}(A, i, j, d)), \tag{3.52}$$

where σ and $\mathbf{k}'$ are as in Lemma 3.15, (m, j) is the rightmost P-minimal intersection point of A^d, and i is the rightmost column such that columns i and j of A^d intersect with intersection point (m, j). Lemma 3.15 shows that this map is well-defined.

We want to show that Φ is a sign-reversing weight-preserving involution with no fixed points, where the *weight* of $(\pi, \mathbf{k}, A) \in B(P, \lambda/\mu/d)$ is the multiset of entries of A. From Definition 3.13, swap preserves the multiset of entries of A, so Φ is weight-preserving. Lemma 3.15 shows that Φ is sign-reversing and has no fixed points, since the permutations of $(\pi, \mathbf{k}, A)$ and $\Phi(\pi, \mathbf{k}, A)$ have opposite signs. We show that Φ is an involution in the following lemma.

Lemma 3.18. *Let P be a $(\mathbf{3} + \mathbf{1})$-free poset, and let $\lambda/\mu/d$ be a cylindric shape. Then $\Phi : B(P, \lambda/\mu/d) \to B(P, \lambda/\mu/d)$ is an involution.*

Proof. Assume $(\pi, \mathbf{k}, A) \in B(P, \lambda/\mu/d)$, and let $\Phi(\pi, \mathbf{k}, A) = (\sigma, \mathbf{k}', A')$. Suppose (m, j) is the rightmost P-minimal intersection point of A^d, and that column i is the rightmost column such that (m, j) is an intersection point between columns i and j. To show that Φ is an involution, it suffices to show that (m, j) is the rightmost P-minimal intersection point of $(A')^d$ and that i is the rightmost column such that (m, j) is an intersection point between columns i and j of $(A')^d$. Let $\mathrm{col}(\alpha, \beta)$ be the shape of A^d, and let $\mathrm{col}(\gamma, \beta)$ be the shape of $(A')^d$. Let $A^d = [x_{s,t}]_{(s,t)\in\mathrm{col}(\alpha,\beta)}$ and $(A')^d = [y_{s,t}]_{(s,t)\in\mathrm{col}(\gamma,\beta)}$.

From Lemma 3.15, we have that (m, j) is an intersection point of $(A')^d$. Furthermore, as columns $i+1, i+2, \ldots, j-1$ are unchanged by $\mathrm{swap}(A, i, j, d)$, i is the rightmost column such that (m, j) is an intersection point between columns i and j in $(A')^d$.

Next we show that (m, j) is a P-minimal intersection point of $(A')^d$. Suppose there is an intersection point (m', j') of $(A')^d$ such that $y_{m',j'} <_P y_{m,j}$. Observe that if $y_{s,t} \neq x_{s,t}$, then $y_{m,j} <_P x_{s,t}$. Therefore $y_{m',j'} = x_{m',j'}$. Furthermore, if (m', j') is an intersection point between columns i' and j' of $(A')^d$, either

1. $m' - j' + i' + 1 > \alpha_{i'}$,
2. $x_{m'-j'+i'+1,i'} = y_{m'-j'+i'+1,i'}$,
3. or $x_{m',j'} <_P x_{m,j} <_P x_{m'-j'+i'+1,i'}$.

In each case, (m', j') is an intersection point of A^d, which contradicts our assumption that (m, j) is a P-minimal intersection point of A^d.

To show that (m, j) is the rightmost P-minimal intersection point, suppose that (m', j') is an intersection point of $(A')^d$ with $j' > j$ and that $y_{m',j'}$ is incomparable to $y_{m,j}$. We must then have that $j < \ell(\lambda) + 1$, and the set of entries on the southeast diagonal below (m', j') in $(A')^d$ is the same as the set of entries on that diagonal in A^d. Therefore (m', j') must also be an intersection point in A^d, which contradicts our assumption that (m, j) is the rightmost P-minimal intersection of A^d. □

We now combine the previous lemmas to prove Theorem 1.3.

Proof of Theorem 1.3. We wish to show that

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{\mathbf{k}\in\tilde{\mathbb{Z}}^{\lambda_1}} \det[e^P_{k_i(\lambda_1+d)+\lambda'_i-\mu'_j-i+j}(\mathbf{u})] = \sum_{T\in\mathrm{CT}_P(\lambda/\mu/d)} \mathbf{u}^T. \tag{3.53}$$

In the case where $\lambda_1 = 0$, $s^P_{\lambda/\mu/d}(\mathbf{u}) = 1$ and $\mathrm{CT}_P(\lambda/\mu/d)$ consists only of the empty P-tableau, so the claim holds. So from here on we can assume $\lambda_1 > 0$. By Lemma 3.4,

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{(\pi,\mathbf{k},A)\in\tilde{B}} \mathrm{sgn}(\pi)\mathbf{u}^A. \tag{3.54}$$

By Lemma 3.15 and Lemma 3.18, Φ is a sign-reversing weight-preserving involution on $B = B(P, \lambda/\mu/d)$ with no fixed-points, so

$$\sum_{(\pi,\mathbf{k},A)\in B} \mathrm{sgn}(\pi)\mathbf{u}^A = 0. \tag{3.55}$$

We then have

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{(\pi,\mathbf{k},A)\in\tilde{B}-B} \operatorname{sgn}(\pi)\mathbf{u}^A. \tag{3.56}$$

By Lemma 3.11, $\tilde{B} - B$ consists of exactly the triples $(\mathrm{id}, \mathbf{0}, A)$ such that A is a cylindric P-tableau of shape $\lambda/\mu/d$. Thus we have

$$s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{A\in\mathrm{CT}_P(\lambda/\mu/d)} \mathbf{u}^A. \tag{3.57}$$

□

4. Special cases and consequences

In this section, we examine a monotonicity property of cylindric P-tableaux and two cases of cylindric Schur functions with particularly nice monomial expansions. Using Theorem 1.1 with Theorem 1.3, we obtain combinatorial interpretations for certain sums of e-coefficients of $X_{\mathrm{inc}(P)}(\mathbf{x}) = \sum_{\nu\vdash|P|} c^P_\nu e_\nu(\mathbf{x})$. We then state two open problems suggested by the results of this paper.

Recall from Section 2 that the m-coefficients of $s_{\lambda/\mu/d}(\mathbf{x})$ increase monotonically as d increases, so the difference $s_{\lambda/\mu/d+1}(\mathbf{x}) - s_{\lambda/\mu/d}(\mathbf{x})$ is m-positive. Similarly, from Equation (3.1) we have that for any poset P and any cylindric shape $\lambda/\mu/d$, $\mathrm{CT}_P(\lambda/\mu/d) \subseteq \mathrm{CT}_P(\lambda/\mu/d+1)$. Therefore, we say cylindric P-tableaux are *monotone* with respect to increasing d.

Corollary 4.1. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset on $[n]$, and let $\lambda/\mu/d$ be a cylindric shape with n cells. Then, $s^P_{\lambda/\mu/d+1}(\mathbf{u}) - s^P_{\lambda/\mu/d}(\mathbf{u})$ is $\mathbf{u}$-positive, and*

$$\sum_{\nu\vdash n} a_\nu c^P_\nu = \#\{T \in \mathrm{CT}_P(\lambda/\mu/d+1) - \mathrm{CT}_P(\lambda/\mu/d) : T \textit{ is standard}\}, \tag{4.1}$$

where a_ν is the coefficient of $m_\nu(\mathbf{x})$ in $s_{\lambda/\mu/d+1}(\mathbf{x}) - s_{\lambda/\mu/d}(\mathbf{x})$.

Proof. By Theorem 1.3 and the fact that cylindric P-tableaux are monotone with respect to increasing d,

$$s^P_{\lambda/\mu/d+1}(\mathbf{u}) - s^P_{\lambda/\mu/d}(\mathbf{u}) = \sum_{T\in\mathrm{CT}_P(\lambda/\mu/d+1)} \mathbf{u}^T - \sum_{T\in\mathrm{CT}_P(\lambda/\mu/d)} \mathbf{u}^T \tag{4.2}$$

$$= \sum_{T\in\mathrm{CT}_P(\lambda/\mu/d+1)-\mathrm{CT}_P(\lambda/\mu/d)} \mathbf{u}^T. \tag{4.3}$$

Therefore, by Theorem 1.1,

$$\sum_{\nu\vdash n} a_\nu c^P_\nu = \langle \mathbf{u}_P\rangle (s^P_{\lambda/\mu/d+1}(\mathbf{u}) - s^P_{\lambda/\mu/d}(\mathbf{u})) \tag{4.4}$$

$$= \#\{T \in \mathrm{CT}_P(\lambda/\mu/d+1) - \mathrm{CT}_P(\lambda/\mu/d) : T \text{ is standard}\}, \tag{4.5}$$

as desired. □

Corollary 4.2. *For a* $(\mathbf{3}+\mathbf{1})$*-free poset* P *on* $[n]$ *and positive integers* r, a *such that* $ra = n$*, the* e*-coefficient* $c^P_{(r^a)}$ *in the expansion of* $X_{\mathrm{inc}(P)}(\mathbf{x})$ *is the number of standard cylindric* P*-tableaux of shape* $(r^a)/\varnothing/0$*.*

Proof. From Theorem 1.1, $c^P_{(r^a)} = \langle \mathbf{u}_P \rangle m^P_{r^a}(\mathbf{u})$. Theorem 1.3 shows that $\langle \mathbf{u}_P \rangle s_{(r^a)/\varnothing/0}$ is the number of standard cylindric P-tableaux of shape $(r^a)/\varnothing/0$. Therefore it suffices to show that $s_{(r^a)/\varnothing/0}(\mathbf{x}) = m_{(r^a)}(\mathbf{x})$. Let $T \in \mathrm{CSSYT}((r^a)/\varnothing/0)$. Then for each $i \in [a]$, $T_{i,1} \leqslant T_{i,2} \leqslant \ldots \leqslant T_{i,r} \leqslant T_{i,1}$. Hence $T_{i,1} = T_{i,2} = \ldots = T_{i,r}$, and $s_{(r^a)/\varnothing/0}(\mathbf{x}) = m_{(r^a)}(\mathbf{x})$. □

Remark 4.3. Corollary 4.2 recovers a result of Clearman–Hyatt–Shelton–Skandera [CHSS16, Theorem 4.7 (v-b)] which gave a combinatorial interpretation to a theorem of Stembridge [Ste92, Theorem 2.8].

Corollary 4.4. *For a* $(\mathbf{3}+\mathbf{1})$*-free poset* P *on* $[n]$ *and non-negative integers* r, a, b *such that* $n = ar + b$ *and* $b < r$*, we have*

$$\sum_{\substack{\lambda \vdash |P| \\ \lambda_1 \leqslant r}} c^P_\lambda = \#\big\{T \in \mathrm{CT}_P((r^a, b)/\varnothing/1) : \ T \text{ is standard}\big\}. \tag{4.6}$$

Proof. From Theorems 1.1 and 1.3, it suffices to show that

$$s_{(r^a,b)/\varnothing/1} = \sum_{\substack{\lambda \vdash n \\ \lambda_1 \leqslant r}} m_\lambda(\mathbf{x}) \tag{4.7}$$

To see this, let $T \in \mathrm{CSSYT}((r^a, b)/\varnothing/1)$ and let $t_1, t_2, \ldots, t_n$ be the entries of T read left-to-right top-to-bottom. We then have that $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$ and $t_i < t_{i+r}$ for $1 \leqslant i \leqslant n - r$. Therefore, given any multiset of n positive numbers such that no number is repeated more than r times, we can fill the diagram of shape $(r^a, b)/\varnothing/1$ in exactly one way. Hence

$$s_{(r^a,b)/\varnothing/1} = \sum_{\substack{\lambda \vdash n \\ \lambda_1 \leqslant r}} m_\lambda(\mathbf{x}). \tag{4.8}$$

□

Corollary 4.4 gives a combinatorial interpretation for the sum of e-coefficients corresponding to partitions λ with $\lambda_1 \leqslant r$. A possible extension of this result would be to find a combinatorial interpretation for the sum of e-coefficients corresponding to partitions λ with λ_1 *equal* to r. From Theorem 1.1, this corresponds to showing that, if a, b, a', b', r are non-negative integers such that $n = ar + b = a'(r-1) + b'$, $b < r$, and $b' < r - 1$, the difference

$$s^P_{(r^a,b)/\varnothing/1}(\mathbf{u}) - s^P_{((r-1)^{a'},b')/\varnothing/1}(\mathbf{u}) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = r}} m^P_\lambda(\mathbf{u}) \tag{4.9}$$

is $\mathbf{u}$-positive. Theorem 1.3 suggests the following approach.

Problem 4.5. Let a, b, a', b', r be non-negative integers such that $n = ar + b = a'(r-1) + b'$, $b < r$, and $b' < r - 1$, and let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Find an injection

$$\iota : \mathrm{CT}_P(((r-1)^{a'}, b')/\varnothing/1) \to \mathrm{CT}_P((r^a, b)/\varnothing/1). \tag{4.10}$$

Corollary 4.4 is also reminiscent of Theorem 2.1. In particular, the partitions appearing in Theorem 2.1 are conjugate shapes of those appearing in Corollary 4.4. A natural goal then is to combine Corollary 4.4 and Theorem 2.1.

Problem 4.6. Let P be a $(\mathbf{3}+\mathbf{1})$-free poset on n elements, and let r, j be non-negative integers. Find a combinatorial interpretation for the sum

$$\sum_{\substack{\lambda \vdash n \\ \lambda_1 \leqslant r \\ \ell(\lambda) = j}} c_\lambda^P. \tag{4.11}$$

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