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AN EXTENSION THEOREM FOR SIGNOTOPES

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Abstract. In 1926, Levi showed that, for every pseudoline arrangement $\mathcal{A}$ and two points in the plane, $\mathcal{A}$ can be extended by a pseudoline which contains the two prescribed points. Later extendability was studied for arrangements of pseudohyperplanes in higher dimensions. While the extendability of an arrangement of proper hyperplanes in $\mathbb{R}^d$ with a hyperplane containing d prescribed points is trivial, Richter-Gebert found an arrangement of pseudoplanes in $\mathbb{R}^3$ which cannot be extended with a pseudoplane containing two particular prescribed points.

In this article, we investigate the extendability of signotopes, which are a combinatorial structure encoding a rich subclass of pseudohyperplane arrangements. Our main result is that signotopes of odd rank are extendable in the sense that for two prescribed crossing points we can add an element containing them. Moreover, we conjecture that in all even ranks $r \geqslant 4$ there exist signotopes that are not extendable for two prescribed points. Our conjecture is supported by examples in ranks 4, 6, 8, 10, and 12 that were found with a SAT-based approach.

Keywords. Arrangement of pseudolines, extendability, Levi's extension lemma, arrangement of pseudohyperplanes, signotope, oriented matroid, partial order, Boolean satisfiability (SAT)

Mathematics Subject Classifications. 68R05, 51H99

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1. Introduction

Given a family of hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$, any d points in $\mathbb{R}^d$, not all on a common hyperplane of $\mathcal{H}$, define a hyperplane which is distinct from the hyperplanes in $\mathcal{H}$. For dimension $d = 2$, Levi [Lev26] proved in his pioneering article on pseudoline arrangements that the fundamental extendability of line arrangements also applies in the more general setting of pseudoline arrangements. A *pseudoline* is a Jordan curve in the Euclidean plane such that its removal from the plane results in two unbounded components, and a *pseudoline arrangement* is a family of pseudolines such that each pair of pseudolines intersects in exactly one point, where the two curves cross properly.

Theorem 1.1 (Levi's extension lemma for pseudoline arrangements [Lev26]). *Given an arrangement $\mathcal{A}$ of pseudolines and two points in $\mathbb{R}^2$, not on a common pseudoline of $\mathcal{A}$. Then $\mathcal{A}$ can be extended by an additional pseudoline which passes through the two prescribed points.*

Levi actually stated the extension lemma for projective arrangements. However, the two versions are equivalent. We present a proof of Levi's extension lemma in Section 3.1. The idea of this proof can be cast in the purely combinatorial language of signotopes, which can be adapted to prove an extension theorem with two prescribed points for signotopes in arbitrary even dimensions.

Generalizations of Levi's extension lemma to higher dimensions have been studied by Goodman and Pollack [GP81], they presented an arrangement of 8 pseudoplanes in $\mathbb{R}^3$ and a selection of three points such that there is no extension of the arrangement with a pseudoplane containing the points. Richter-Gebert [RG93] found an example of a rank 4 oriented matroid on 12 elements such that there is no one-element-extension with an element containing two prescribed cocircuits. By the representation theorem of Folkman and Lawrence [FL78] this translates to the non-extendability of an arrangement of 12 pseudoplanes in projective 3-space with two prescribed points. The existence of an extension theorem for oriented matroids with two or more prescribed points or or counterexamples in higher dimensions/ranks remains open.

Our new proof of Levi's extension lemma can be adapted to work in a purely combinatorial setting of signotopes. The result is a proof of 2-extendability, i.e., extendability with two prescribed points, of signotopes for all even dimensions d, that is, when the rank $r = d + 1$ is odd; see Theorem 3.3. Surprisingly, there are non-2-extendable examples in rank 4, 6, 8, 10, and 12. The examples have been found with the aid of SAT solvers. We describe the background in Section 7. We conjecture that there is no extension theorem for any even rank $r \geqslant 4$; see Conjecture 3.5. Moreover, we show that for all ranks r, there are non-4-extendable r-signotopes; see Proposition 9.1.

Before we formulate our extension theorem for r-signotopes, we introduce some notation and discuss the relation between pseudoline arrangements and 3-signotopes (in Section 2). This leads to a reformulation of Levi's extension lemma which will be investigated in the context of signotopes of odd rank in Section 3.

2. Signotopes

Signotopes are the elements of higher Bruhat orders, which were introduced by Manin and Schechtman [MS89] and further studied by Kapranov and Voevodsky [KV91] and by Ziegler [Zie93]. Rank 2 signotopes correspond to permutations. In rank 3, signotopes correspond to *simple* Euclidean pseudoline arrangements (i.e., no three pseudolines cross in a common point) in the plane [FW01]. In higher ranks, signotopes can be represented by Euclidean pseudohyperplane arrangements. A direct construction of an arrangement corresponding to a signotope has been shown by Felsner and Weil [FW01]. An alternative approach can be based on the fact that signotopes can be seen as a subclass of uniform oriented matroids (see Subsection 2.1) and the representation theorem of Folkman and Lawrence [FL78]. A new geometric representation of r-signotopes with points and curves in the plane was introduced by Miyata [Miy17] (see also [BFK15] for the rank 3 case).

For $r \geqslant 1$ a *signotope* of *rank* r (short: r-signotope) on n elements is a mapping σ from r-element subsets (r-*subsets*) of the linearly ordered ground set $[n] = \{1 < 2 < \ldots < n\}$ to "+" or "−", i.e., $\sigma : \binom{[n]}{r} \to \{+,-\}$, such that for every $(r+1)$-subset $P = \{x_1, \ldots, x_{r+1}\}$ ($(r+1)$-*packet*) of $[n]$ with $x_1 < x_2 < \ldots < x_{r+1}$ there is at most one sign change in the sign sequence

$$(\sigma(P\backslash\{x_1\})\,\sigma(P\backslash\{x_2\})\ \ldots\ \sigma(P\backslash\{x_{r+1}\}))\,.$$

Note that this sequence lists the signs of all induced r-subsets of P in reverse lexicographic order. For 3-signotopes, the following 8 sign patterns on 4-subsets are allowed:

$$(++++),\ (+++-),\ (++--),\ (+---),\ (----),\ (---+),\ (--++),\ (-+++).$$

We write $P = (x_1, \ldots, x_t)$ to denote a t-subset of $[n]$ with elements $x_1 < x_2 < \ldots < x_t$. For such a P let $P_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_t)$ denote the set without x_j. With the convention $- < +$, the condition about sign changes in r-signotopes can be written as a monotonicity condition for $(r+1)$-packets $P = (x_1, \ldots, x_{r+1})$:

$$\sigma(P_1) \leqslant \sigma(P_2) \leqslant \ldots \leqslant \sigma(P_{r+1}) \quad \text{or} \quad \sigma(P_1) \geqslant \sigma(P_2) \geqslant \ldots \geqslant \sigma(P_{r+1}).$$

Felsner and Weil [FW01] showed that there is a bijection between rank 3 signotopes and the equivalence classes of simple arrangements of pseudolines in $\mathbb{R}^2$ with a fixed north cell. Two pseudoline arrangements are *combinatorially equivalent* if the order of crossings along each pseudoline is the same, which is the essential property for the following construction. Felsner and Weil speak of *marked* arrangements where the marking consists of a designated *north cell* and an orientation of the pseudolines such that the north-cell is to their right. In such a representation, we label the pseudolines from top to bottom left of all crossings by $1, \ldots, n$. Since two pseudolines cross exactly once, the pseudolines appear in reversed order on the right. The corresponding 3-signotope σ is obtained as follows: The sign of $\sigma(a, b, c)$ for $a < b < c$ indicates the orientation of the triangle formed by the pseudolines a, b, c (see Figure 2.1). If the crossing of a and c is below b, it is $\sigma(a, b, c) = +$ and if the crossing of a and c is above b, it is $\sigma(a, b, c) = -$. In the following, we identify the crossings with the elements that cross, i.e. for 3-signotopes crossings are subsets of size 2. The 3-signotope σ gives information about the

Figure 2.1: Connection between pseudoline arrangements and 3-signotopes.

partial order of the crossings from left to right. If $\sigma(a,b,c) = +$, it holds $ab \prec ac \prec bc$ and if $\sigma(a,b,c) = -$, it is $bc \prec ac \prec ab$. The transitive hull of these relations is the partial order corresponding to the signotope σ

For $r \geqslant 4$, r-signotopes can be represented by a class of sweepable pseudohyperplane arrangements in $\mathbb{R}^{r-1}$. Given an r-signotope the $(r-1)$-subsets of the groundset are in bijection to the points of the corresponding hyperplane arrangement. As in the $r = 3$ case an r-signotope induces a partial order on the $(r-1)$-subsets which is combinatorially defined as follows. For an r-signotope σ and every r-subset $X = (x_1, \dots, x_r)$ define:

$$\begin{aligned} X_1 \succ X_2 \succ \cdots \succ X_r \quad &\text{if} \quad \sigma(x_1,\dots,x_r) = +, \qquad \text{and} \\ X_1 \prec X_2 \prec \cdots \prec X_r \quad &\text{if} \quad \sigma(x_1,\dots,x_r) = -. \end{aligned} \tag{$\star$}$$

Recall that we use the convention $x_1 < \ldots < x_r$ and $X_i = X \backslash \{x_i\}$. Taking the transitive closure of all relations obtained from r-subsets, we obtain a partial order $\prec_\sigma$ on the $(r-1)$-subsets corresponding to σ [FW01, Lemma 10].

2.1. Signotopes as a rich subclass of oriented matroids

Kapranov and Voevodsky [KV91] studied higher Bruhat orders and from their Theorem 4.9 it follows that signotopes are in bijection with marked oriented matroids. Another way to see this relation is the following. The *alternating extension* of an r-signotope σ is the function $\chi_\sigma : [n]^r \to \{-,0,+\}$ defined by

$$\chi_\sigma(x_1,\dots,x_r) = \begin{cases} 0 & \text{if } |\{x_1,\dots,x_r\}| < r, \\ \operatorname{sgn}(\pi) \cdot \sigma(x_{\pi(1)},\dots,x_{\pi(r)}) & \text{for a permutation } \pi \in S_r \\ & \text{if } x_{\pi(1)} < \cdots < x_{\pi(r)}. \end{cases}$$

Bergold [Ber23, Section 2.2.5] verified the 3-Term Grassmann–Plücker relations for χ_σ. Therefore χ_σ is a chirotope of a uniform rank r oriented matroid. An alternative proof has been given by Miyata [Miy17] who verified the cocircuit axioms of oriented matroids.

It is well known that the number of oriented matroids of rank r on n elements is $2^{\Theta(n^{r-1})}$ [BLS$^+$99, Corollary 7.4.3]. As shown by Balko [Bal19, Theorem 3], r-signotopes are a rich subclass of oriented matroids of rank r. In Section 8 we give a shorter proof of Balko's asymptotic estimate on the number of r-signotopes.

Proposition 2.1 (Balko [Bal19])**.** *For $r \geqslant 3$, the number of r-signotopes on $[n]$ is $2^{\Theta(n^{r-1})}$.*

In ranks 1 and 2 there are 2^n and $n!$ signotopes on $[n]$, respectively. Rank 1 signotopes are mappings from $[n]$ to $\{+,-\}$ without any additional property and 2-signotopes are permutations. For rank $r \geqslant 3$, the precise number of r-signotopes on $[n]$ has been computed for small values of r and n; see A6245 (rank 3) and A60595 to A60601 (rank 4 to rank 10) on the OEIS [OEI].

3. An Extension Theorem for Signotopes

In this section, we first prove an extension lemma for 3-signotopes. Taking advantage of the bijection between 3-signotopes and simple marked arrangements of pseudolines we can state and prove the lemma in an intuitive geometric context. In Section 3.2 we give the formal statement of our extension theorem for signotopes in odd ranks. The theorem is motivated by the pseudoline version and its proof works along the same lines. The techniques for the proof for general odd rank are presented in Section 4 and Section 5. The proof is completed in Section 6.

3.1. An extension lemma for 3-signotopes

In this subsection we first prove Lemma 3.1 which is a restricted version of Theorem 1.1 (Levi's extension lemma). We then show how the full generality of the theorem can be obtained from our lemma. Finally, we comment on other proofs of Levi's extension lemma.

Lemma 3.1. *Given a simple marked arrangement $\mathcal{A}$ of pseudolines and two crossing points x and x' of $\mathcal{A}$, not on a common pseudoline of $\mathcal{A}$. Then $\mathcal{A}$ can be extended by an additional pseudoline which passes through the two prescribed points.*

Proof. From the previous section we know that an r-signotope σ on $[n]$ induces a partial order $\prec_\sigma$ on the $(r-1)$-subsets of $[n]$. In the rank 3 case this partial order has a simple description in terms of the arrangement $\mathcal{A}$ corresponding to σ. On each pseudoline of $\mathcal{A}$ the order of the crossings is from left to right and the partial order is the transitive closure of the order of crossings on all pseudolines. In the following we denote the partial order $\prec_\sigma$ as $P_\mathcal{A}$.

The proof of Lemma 3.1 is in two steps:

(1) *Sweeping:* Show extendability of $\mathcal{A}$ in the case where x and x' are incomparable in $P_\mathcal{A}$.

(2) *Rotating:* The case where x and x' are comparable in $P_\mathcal{A}$ can be reduced to the case where they are incomparable.

If $\mathcal{A}'$ is an extension of a marked arrangement $\mathcal{A}$ of n pseudolines with an additional pseudoline ℓ_{n+1}, then the set of crossings of $\mathcal{A}$ which are left and above ℓ_{n+1} in $\mathcal{A}'$ is a down-set of $P_\mathcal{A}$. In [FW01], Felsner and Weil studied sweeps of marked arrangements. From their theory, it follows that for every down-set D of $P_\mathcal{A}$ there is an extension $\mathcal{A}'$ of $\mathcal{A}$ with an additional pseudoline ℓ_{n+1} such that the set of crossings of $\mathcal{A}$ which are left and above ℓ_{n+1} in $\mathcal{A}'$ is D.

Now consider a marked arrangement $\mathcal{A}$ with two crossing points x and x' which are incomparable in $P_\mathcal{A}$. In $P_\mathcal{A}$ the crossings x and x' generate a down-set D which has x and x' as maximal elements. Let $\mathcal{A}'$ be the extension of $\mathcal{A}$ by ℓ_{n+1} such that D is the set of crossings which are left and above ℓ_{n+1} in $\mathcal{A}'$. Let $\mathcal{A}''$ be obtained from $\mathcal{A}'$ by perturbing ℓ_{n+1} such that it contains x and x' instead of passing by these crossings. Figure 3.1 shows an example. The arrangement $\mathcal{A}''$ is the extension needed for the proof of (1).

For (2), we need the concept of *rotations*. Let $\mathcal{A}$ be marked arrangement of pseudolines with the standard labeling of the pseudolines. The rotation $\mathcal{A}_{\text{rot}}$ of $\mathcal{A}$ is obtained by moving the north-cell of $\mathcal{A}$ counterclockwise to the next unbounded cell and reversing the orientation of the first line. Figure 3.2 shows an example where in addition to the reorientation the pseudolines

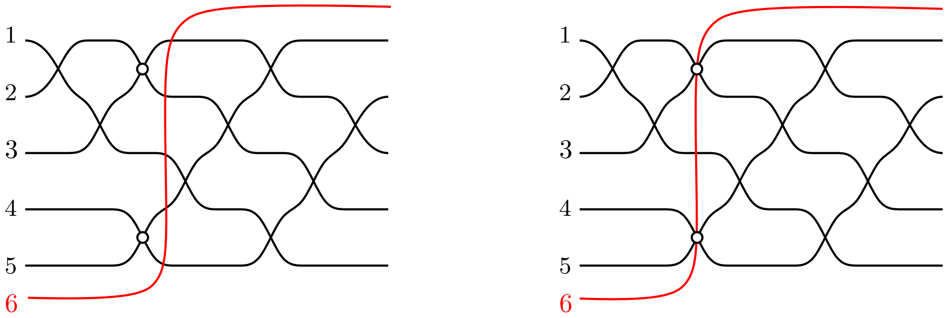

Figure 3.1: An arrangement $\mathcal{A}$ (black pseudolines) with two incomparable crossing points (white) and the extensions $\mathcal{A}'$ and $\mathcal{A}''$ with a (red) pseudoline ℓ_6 defined by the two crossings.

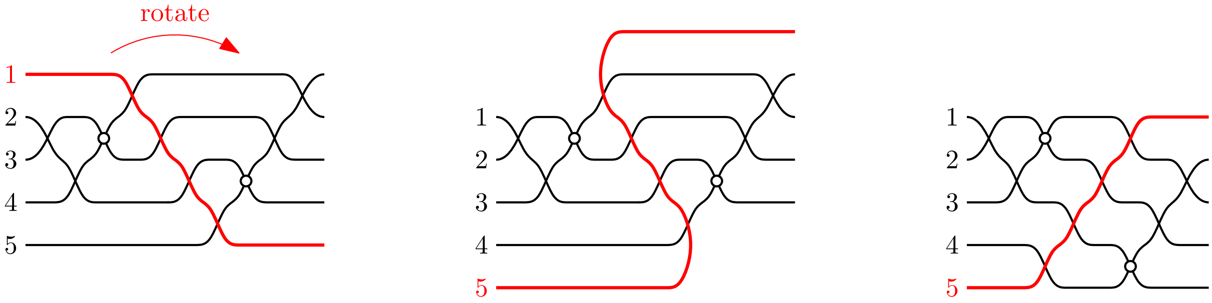

Figure 3.2: Illustration of a clockwise rotation of a pseudoline arrangement. The rotated pseudoline is highlighted in red.

have been relabeled and made x-monotone. Note that in the example the two white crossings are comparable in $P_{\mathcal{A}}$ and incomparable in $P_{\mathcal{A}_{\text{rot}}}$. We claim that if two crossings x and x' are comparable in $P_{\mathcal{A}}$, then there is some i such that in the arrangement $\mathcal{A}' = \mathcal{A}_{\text{rot}(i)}$ obtained after i rotations x and x' are incomparable.

To prove (2), we rotate the arrangement $\mathcal{A}$ with crossings x and x' which are comparable in $P_{\mathcal{A}}$ until in $\mathcal{A}' = \mathcal{A}_{\text{rot}(i)}$ crossings x and x' are incomparable. Now by part (1), there is an extension $\mathcal{A}''$ of $\mathcal{A}$ with an additional pseudoline ℓ which contains x and x'. By changing the marking of $\mathcal{A}''$ back to the original cell we obtain a marked arrangement $\mathcal{A}'''$ which is an extension of $\mathcal{A}$ with a pseudoline that contains x and x'.

To prove the claim, we first note that the effect of n rotations on an arrangement of n pseudolines is the reversal of the orientation of all pseudolines, i.e., $P_{\mathcal{A}_{\text{rot}(n)}}$ is the *dual (reversed)* of $P_{\mathcal{A}}$ and if $x \prec x'$ in $P_{\mathcal{A}}$, then $x' \prec x$ in $P_{\mathcal{A}_{\text{rot}(n)}}$. Hence, there is a smallest i with $1 \leqslant i \leqslant n$ such that $x \not\prec x'$ in $P_{\mathcal{A}_{\text{rot}(i)}}$. The pseudoline ℓ which is rotated when going from $P_{\mathcal{A}_{\text{rot}(i-1)}}$ to $P_{\mathcal{A}_{\text{rot}(i)}}$ partitions the crossings of $\mathcal{A}_{\setminus\ell} = \mathcal{A}_{\text{rot}(i-1)} \setminus \{\ell\} = \mathcal{A}_{\text{rot}(i)} \setminus \{\ell\}$ into a down-set D and an up-set U of $P_{\mathcal{A}_{\setminus\ell}}$. Since $x \prec x'$ in $P_{\mathcal{A}_{\text{rot}(i-1)}}$ and $x \not\prec x'$ in $P_{\mathcal{A}_{\text{rot}(i)}}$, every directed path from x' to x in $\mathcal{A}_{\text{rot}(i-1)}$ contains an edge on ℓ so that there is no directed path from x' to x in $P_{\mathcal{A}_{\setminus\ell}}$. By assumption at most one of x and x' is on ℓ, moreover, at most one of x' and x belongs to each of D and U. Together this implies that x and x' are incomparable in $P_{\mathcal{A}_{\text{rot}(i)}}$.

This completes the proof of (2) and the proof of the lemma. □

Generalizing the lemma. Theorem 1.1 is more general than the lemma in two respects. It has no restriction with respect to the prescribed points. They may be interior points in cells or be points on a pseudoline. Moreover, the arrangement $\mathcal{A}$ is not required to be simple, i.e., it may have crossings of higher multiplicity.

If a prescribed point p is not a crossing point, we replace p by the crossing x_p which is the leftmost crossing of the pseudoline segment (or cell) containing p. The extending pseudoline ℓ through x_p can be perturbed locally in the neighborhood of x_p to make p a point on ℓ.

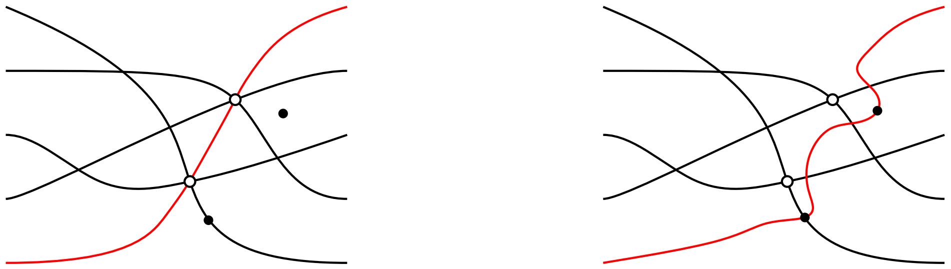

Figure 3.3: Perturbing an extending pseudoline to make it pass through the prescribed points.

If $\mathcal{A}$ is a non-simple arrangement, we can locally perturb crossing points of higher multiplicity to obtain a simple arrangement $\mathcal{A}'$. The perturbation transforms a crossing point x of degree k into a simple arrangement of k pseudolines with all crossings in a small disc centered at x. The simple arrangement obtained by the perturbations can be extended, and the crossing points of higher degree of the original arrangement can be recovered by contracting the small arrangements in the discs to points. The procedure is illustrated in Figure 3.4.

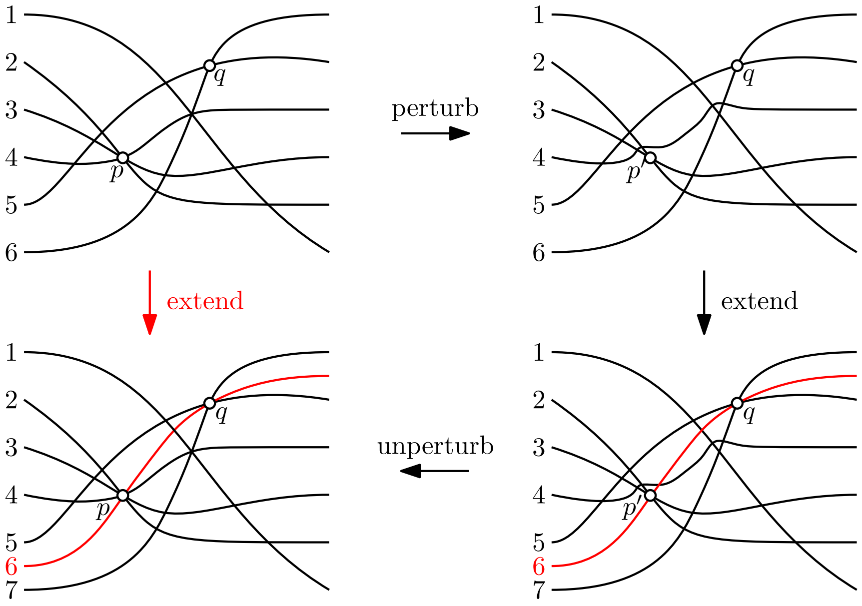

Figure 3.4: When perturbing the top-left arrangement, the multi-crossing point p (the intersection of 2, 3, and 4) is split into three simple crossing points, including the point p' (the intersection of 2 and 3). After the extension, the three crossing points are contracted again.

Other proofs of the theorem Theorem 1.1 is classically proved using the following statement: *If $\mathcal{A}$ is an arrangement and γ is a curve connecting p and q, then there exists a curve γ' connecting p and q such that γ' is crossing each pseudoline of $\mathcal{A}$ at most once.* Proofs of the statement are based on local deformations of the curve. This approach has been taken in the original paper of Levi [Lev26] as well as in [Grü72], [AMRS18], and [Sch19]. Proofs based on sweeping have been given in [SH91] and [FW01]. The idea in [FW01] is as follows: Use a projective transformation to move the first prescribed point p to infinity and make the cell of p the north-cell. A sweep of the marked arrangement will at some moment have the sweep-line in a position where it contains q. This sweep-line is an extending pseudoline containing p and q. Inversion of the initial projective transformation completes the proof. Snoeyink and Hershberger [SH91] proved an analog of Levi's lemma for arrangements of pseudocircles. In this case three points not on a common pseudocircle can be prescribed.

3.2. The extension theorem for signotopes

An extension of a simple arrangement $\mathcal{A}$ with Lemma 3.1 yields a non-simple arrangement. To make the extended arrangement simple we can use local perturbations. Such a perturbation yields a *triangular cell* with a segment of the extending pseudoline ℓ on one side and a crossing x of $\mathcal{A}$ as the opposite corner.

Triangular cells play an important role in the study of pseudoline arrangements, since it is possible to change the orientation of a triangle by moving one of the pseudolines over the crossing of the two others. Such a local perturbation is called a *triangle flip*. It does not change the orientation of any other triangle in the arrangement. The triangular cells of the arrangement $\mathcal{A}$ corresponding to a 3-signotope σ are in one-to-one correspondence with 3-subsets such that if we change the sign of this 3-subset in σ, we obtain a new signotope σ'. We call such a 3-subset a *fliple*. The notion of fliples generalizes to higher ranks. In an r-signotope σ on $[n]$, an r-subset $X \subseteq [n]$ is a *fliple* if both assignments $+$ and $-$ to $\sigma(X)$ result in a signotope. In [Ber23, Section 6.2] it is shown that all elements of an r-signotope contribute to a fliple and all but at most two contribute to at least two fliples. Therefore, an r-signotope on n elements contains at least $\frac{2n-2}{r}$ fliples. In the rank 3 case it is known [FK99] that there are at least $n-2$ triangular cells (fliples). It is worth noting that fliples in signotopes are the analog of mutations in oriented matroids. It is open whether every uniform oriented matroid contains at least one mutation [BLS$^+$99, Chapter 7.3].

Let $\mathcal{A}$ be an arrangement of pseudolines, which are labeled $1,\dots,n$ from top to bottom on the left. If ℓ is a pseudoline extending $\mathcal{A}$, then the left endpoint of ℓ will be between two consecutive endpoints of pseudolines of $\mathcal{A}$. To re-establish the properties of the labeling, we have to set the label of ℓ accordingly and increase the label of every pseudoline that starts below ℓ by one. To cope with this relabeling-issue in terms of signotopes, we introduce the following notion. For $k \in [n]$ and a subset X of $[n]$, we define

$$X{\downarrow_k} = \{x \mid x \in X, x < k\} \cup \{x-1 \mid x \in X, x > k\}.$$

Note that the cardinality of X and $X{\downarrow_k}$ is the same if and only if $k \notin X$. If $k \in X$, then $|X{\downarrow_k}| = |X| - 1$. For an r-signotope σ on $[n]$, we define the *deletion* of an element $k \in [n]$

as the r-signotope $\sigma\downarrow_k$ on $[n-1]$ with

$$\sigma\downarrow_k(X\downarrow_k) = \sigma(X)$$

for all r-subsets $X \subseteq [n]$ with $k \notin X$. This is an r-signotope on $[n-1]$ because each $(r+1)$-packet comes from an $(r+1)$-packet for σ. The deletion of several elements $k_1 < \ldots < k_\ell$ at the same time is the repeated application of the deletion starting with the largest element to avoid additional index shifts. This is denoted by $\sigma\downarrow_{\{k_1,\ldots,k_\ell\}}$.

Definition 3.2. An r-signotope σ on $[n]$ is *extendable through* a set of pairwise disjoint $(r-1)$-subsets $I_1,\ldots,I_t$ of $[n]$ if there exists $k \in [n+1]$ and an r-signotope σ^* on $[n+1]$ with fliples $I_1^*,\ldots,I_t^*$ such that $\sigma^*\downarrow_k = \sigma$, and $I_i^*\downarrow_k = I_i$ for each $i \in [t]$.

An r-signotope σ on $[n]$ is *t-extendable* if it is extendable through every set of t pairwise disjoint $(r-1)$-subsets of $[n]$.

Note that a t-extendable r-signotope on $n > (r-1)t$ elements is also $(t-1)$-extendable. While 1-extendability for signotopes is quite easy to prove, the 2-extendability is highly nontrivial and will finally be proved in Section 6. In Section 9 we go further and consider t-extendability for $t \geqslant 3$.

For 1-extendability, the strategy in the setting of pseudolines is to add a pseudoline just below one of the two pseudolines crossing in the prescribed point and parallel to this pseudoline in the sense that they have exactly the same crossing points. At the prescribed point the extending pseudoline crosses the two pseudolines involved in the crossing and in the remaining part we again go parallel but just above the same pseudoline. For an illustration, see Figure 3.5. From the theory developed in the following, 1-extendability for signotopes of all ranks is an easy corollary, see Corollary 4.2.

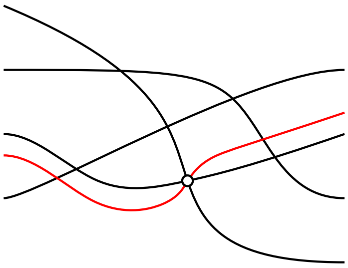

Figure 3.5: Adding a new pseudoline extending the arrangement (black) through the marked crossing point by adding a parallel pseudoline (red) to one of the pseudolines involved in the crossing.

In the following, the focus is on 2-extendability. We show that 2-extendability can be guaranteed for signotopes of odd rank.

Theorem 3.3 (Extension theorem for signotopes of odd rank)**.** *For every odd rank $r \geqslant 3$, every r-signotope is 2-extendable.*

The proof of Theorem 3.3 (see Section 6) generalizes to the more general setting, where the $(r-1)$-subsets I and J, which are fliples in the extension, may intersect.

Corollary 3.4. *For $r \geqslant 3$, let σ be an r-signotope on $[n]$, and $I, J \subseteq [n]$ two $(r-1)$-subsets such that $|I \cap J| + r$ is odd. Then σ is extendable to an r-signotope σ^* on $[n+1]$ with fliples I^*, J^* and an extending element $k \in [n+1]$ such that $\sigma^*{\downarrow_k} = \sigma$, and $I^*{\downarrow_k} = I$, and $J^*{\downarrow_k} = J$.*

For 2-signotopes, which are permutations, the prescribed sets are singletons. Hence we are in the setting of prescribed disjoint 1-subsets. Extendability through t elements corresponds to inserting an element next to the t prescribed elements in the permutation. Hence 2-signotopes are not 2-extendable. Note that with the same reason rank 2 oriented matroids are not 2-extendable. However, the statement of Corollary 3.4 is still true for $r = 2$ since the two sets I, J have to be equal.

The statement of Theorem 3.3 applies only to signotopes of odd rank. This is not just a defect of our proof because signotopes in even rank indeed behave differently. For ranks $r = 4, 6, 8, 10, 12$ we found r-signotopes on $n = 2r$ elements, which are not 2-extendable. In Section 7 we describe them in more detail and give properties for all even r which imply that an r-signotope with those properties is not 2-extendable. Based on these examples, we dare the conjecture:

Conjecture 3.5 (No extension theorem for signotopes of even rank)**.** For every even rank $r \geqslant 2$, there is an r-signotope which is not 2-extendable.

Above we have seen that 2-extendability (Theorem 3.3) can be generalized in terms of the parity of $r + |I \cap J|$ (Corollary 3.4). A similar generalization of non-2-extendability (Conjecture 3.5) is possible: *If the conjecture is true, then there are r-signotopes which are not extendable through two prescribed crossing points I and J where $|I \cap J| + r$ is even.* The important part to prove this statements is that the projection of an r-signotope to the intersection $I \cap J$ is a signotope of even rank in which the prescribed points are disjoint. Hence we can choose this projection to be a signotope σ' which admits no 2-extension through the two sets $I{\downarrow_{(I\cap J)}}$ and $J{\downarrow_{(I\cap J)}}$. This ensures that every lifting of σ' has no extension through I and J. In the following we make this rough sketch precise.

First we define the projection of a signotope to one of its elements. This operation is commonly used in matroid theory and geometry and also known as *contraction*. In the resulting signotope the rank and the number of elements are both decreased by one[1]. For an r-signotope σ on $[n]$, we define the *projection* to an element $k \in [n]$ as $\sigma \downdownarrows_k$ by

$$\sigma \downdownarrows_k (X{\downarrow_k}) = \sigma(X)$$

for all r-subsets $X \subseteq [n]$ with $k \in X$. This is an $(r-1)$-signotope on $[n-1]$ because each r-packet was part of an $(r+1)$-packet for σ. Similar as for the deletion, the projection on a subset of elements $k_1 < \ldots < k_\ell$ is defined by successively applying the projection starting with the largest element k_ℓ. This projection is denoted by $\sigma \downdownarrows_{\{k_1,\ldots,k_\ell\}}$. Clearly, a fliple of σ containing k yields a fliple in the projection $\sigma \downdownarrows_k$ and more generally:

---

[1] Since both, the rank and the number of elements are decreased, this operation is denoted by two down arrows. For the deletion, which is only one down arrow, only the number of elements is decreased while the rank stays the same.

Observation 3.6. *Let F be a fliple of the r-signotope σ and $K \subseteq F$. Then $F{\downarrow_K}$ is a fliple of $\sigma \downdownarrows_K$.*

A *lifting* of a signotope σ is a signotope σ' such that there is a set K such that the projection $\sigma' \downdownarrows_K$ equals σ. To construct the counterexamples, we show that every signotope has a lifting. The proof of the following lemma will be given in Subsection 6.3.

Lemma 3.7 (Lifting Lemma)**.** *For every r-signotope σ on $[n]$ there exists an $(r+1)$-signotope σ^+ on $[n+1]$ such that $\sigma^+ \downdownarrows_{n+1}= \sigma$.*

We are ready to prove the announced generalization of non-2-extendability.

Let σ be an r-signotope on $[n]$ which is *not* 2-extendable through the two disjoint $(r-1)$-subsets I, J. For $m \in \mathbb{N}$ let σ^+ be an m-fold lifting of σ, this is an r'-signotope on $[n']$ where $r' = r+m$ and $n' = n+m$. With $M = \{n+1, \dots, n+m\}$ let $I' = I \cup M$ and $J' = J \cup M$. The claim is that there is no r'-signotope σ^* on $[n'+1]$ with an element k and fliples I^*, J^* such that $\sigma^*{\downarrow_k} = \sigma'$, $I^*{\downarrow_k} = I'$, and $J^*{\downarrow_k} = J'$. Suppose σ^* exists, let $M^* = (I^* \cap J^*) \setminus \{k\}$ and let k' be the label of k in $\sigma^*{\downarrow_{M^*}}$. Observation 3.6 implies that $I^*{\downarrow_{I' \cap J'}}$ and $J^*{\downarrow_{I' \cap J'}}$ are fliples in the projection $\sigma^* \downdownarrows_{I' \cap J'}$. Note that $(I^*{\downarrow_{M^*}}){\downarrow_{k'}} = (I^*{\downarrow_k}){\downarrow_M} = I'{\downarrow_M} = I$ and analogously for J^*. This shows that $\sigma^* \downdownarrows_{M^*}$ is an extension of σ by the element k' such that $I \cup \{k'\}$ and $J \cup \{k'\}$ are fliples, a contradiction.

4. Extendability and Incomparable Elements

For the proof of the extension theorem for odd rank signotopes (cf. Section 6.1), there are two central ingredients that correspond to the two steps of the proof of Lemma 3.1. The first one is discussed in this section. If the two prescribed crossing points are incomparable elements in the partial order $\prec_\sigma$ corresponding to the r-signotope σ, then we can find an extension, by adding an element at the last position (recall the rank 3 case illustrated in Figure 3.1). For the definition of the partial order $\prec_\sigma$ on all $(r-1)$-subsets corresponding to a signotope σ see (Equation ($\star$)) in Section 2.

Proposition 4.1 (Extension for incomparable elements)**.** *Let σ be an r-signotope on $[n]$ and $\prec$ be its partial order on $\binom{[n]}{r-1}$. For every down-set $\mathcal{D} \subseteq \binom{[n]}{r-1}$ there exists an r-signotope σ^* on $[n+1]$ such that for all maximal elements M of $\mathcal{D}$, the r-subset of the form $M \cup \{n+1\}$ is a fliple of σ^* and $\sigma^*{\downarrow_{n+1}} = \sigma$.*

Proof. Define the extended r-signotope σ^* on $[n+1]$ for $x_1 < \ldots < x_r$ as follows

$$\sigma^*(x_1, \dots, x_r) := \begin{cases} \sigma(x_1, \dots, x_r) & \text{if } x_1, \dots, x_r \in [n]; \\ + & \text{if } x_r = n+1 \text{ and } \{x_1, \dots, x_{r-1}\} \in \mathcal{D}; \\ - & \text{if } x_r = n+1 \text{ and } \{x_1, \dots, x_{r-1}\} \notin \mathcal{D}. \end{cases}$$

Clearly it holds $\sigma^*{\downarrow_{n+1}} = \sigma$. In the following we show that σ^* is an r-signotope on $[n+1]$. For every $(r+1)$-subset $P = (x_1, \dots, x_{r+1})$, we show that the sequence

$$(\sigma^*(P_1)\ \sigma^*(P_2)\ \ldots\ \sigma^*(P_{r+1}))$$

has at most one sign change. If $x_{r+1} \leqslant n$, then all signs on the considered r-subsets are the same as for σ. Since σ is an r-signotope, there is at most one sign change in the sequence.

In the other case, we have $x_{r+1} = n+1$. For all $j \leqslant r$ we have $n+1 \in P_j$. Furthermore, $\sigma^*(P_{r+1}) = \sigma(P_{r+1})$ because $n+1 \notin P_{r+1}$. We consider two cases. First, if $\sigma(P_{r+1}) = +$ by definition of the partial order it is

$$P\backslash\{x_{r+1}, x_i\} = P_{r+1}\backslash\{x_i\} \succ P_{r+1}\backslash\{x_j\} = P\backslash\{x_{r+1}, x_j\} \qquad \text{for } i < j.$$

By the property of a down-set this means that, whenever $P\backslash\{x_{r+1}, x_i\} \in \mathcal{D}$, we also have $P\backslash\{x_{r+1}, x_j\} \in \mathcal{D}$ for $i < j$. Let i^* be the smallest integer such that $P\backslash\{x_{r+1}, x_{i^*}\} \in \mathcal{D}$. Then by definition of σ^* it is $\sigma^*(P_j) = +$ for all $j \geqslant i^*$ and $\sigma^*(P_j) = -$ for all $j < i^*$.

Similar arguments apply if $\sigma(P_{r+1}) = -$. Then we have

$$P\backslash\{x_{r+1}, x_i\} \prec P\backslash\{x_{r+1}, x_j\} \qquad \text{for } i < j.$$

This time let i^* be the smallest integer such that $P\backslash\{x_{r+1}, x_{i^*}\} \notin \mathcal{D}$. Then by definition of σ^* we have $\sigma^*(P_j) = +$ for all $j < i^*$ and $\sigma^*(P_j) = -$ for all $j \geqslant i^*$.

Let M be a maximal element of the down-set $\mathcal{D}$. Then by definition $\sigma^*(M \cup \{n+1\}) = +$. In every packet in which $M \cup \{n+1\}$ appears, $M \cup \{n+1\}$ is the maximal element in the lexicographic order such that the removal of one element is still in $\mathcal{D}$. By the analysis above it follows that $M \cup \{n+1\}$ is adjacent to a sign change in each packet. Hence it is a fliple. □

From this proposition and the fact that every element in a partial order defines a down-set in which it is the maximal element, it directly follows that all r-signotopes with $r \geqslant 2$ are 1-extendable by adding a new element at the last position.

Corollary 4.2 (1-extendability). *For $r \geqslant 2$ let σ be an r-signotope on $[n]$ and $I \subseteq [n]$ an $(r-1)$-subset. Then there is an extending r-signotope σ^* on $n+1$ elements such that $I \cup \{n+1\}$ is a fliple and $\sigma^*{\downarrow_{n+1}} = \sigma$.*

Proposition 4.1 states that for every down-set of the partial order of a signotope σ on n elements we can add a new element at position $n+1$ such that in the extended signotope σ^* we have $\sigma^*(I \cup \{n+1\}) = +$ if and only if I is in the down-set. The following proposition shows that the converse holds: if σ^* is an extension of σ with a new element $n+1$ at the last position, then the collection of all sets I with $\sigma^*(I \cup \{n+1\}) = +$ is a down-set of the partial order $\prec$ of σ.

Proposition 4.3. *For $r \geqslant 2$, let σ be an r-signotope on $[n]$ with partial order $\prec$. Let σ^* on $[n+1]$ be an extension of σ and let I and J be $(r-1)$-subsets of $[n]$ with $I \prec J$.*

$$\textit{If } \sigma^*(J \cup \{n+1\}) = +, \textit{ then } \sigma^*(I \cup \{n+1\}) = +, \textit{ and}$$
$$\textit{if } \sigma^*(I \cup \{n+1\}) = -, \textit{ then } \sigma^*(J \cup \{n+1\}) = -.$$

In particular $\mathcal{D}^{\sigma^*}_{n+1} = \{I \subset [n] : |I| = r-1, \sigma^*(I \cup \{n+1\}) = +\}$ is a down-set of $\prec$, and $\mathcal{U}^{\sigma^*}_{n+1} = \{I \subset [n] : |I| = r-1, \sigma^*(I \cup \{n+1\}) = -\}$ is an up-set of $\prec$.

Proof. By the definition of the partial order, there is a chain of $(r-1)$-subsets of $[n]$ such that $I = J_m \prec \ldots \prec J_2 \prec J_1 = J$ and for each pair of consecutive subsets the intersections $J_{i+1} \cap J_i$ consists of exactly $r-2$ elements. Assume that $\sigma^*(J_1 \cup \{n+1\}) = +$. We show by induction that $\sigma^*(J_i \cup \{n+1\}) = +$ for all all $i = 2, \ldots, m$. For a fixed $i \geqslant 2$, consider the $(r+1)$-packet $P = J_{i+1} \cup J_i \cup \{n+1\}$. Since $n+1$ is the largest element in this $(r+1)$-packet, the last set in the packet is $P_{r+1} = J_{i+1} \cup J_i$ and by definition of an extension $\sigma^*(J_{i+1} \cup J_i) = \sigma(J_{i+1} \cup J_i)$. From the definition of $\prec$ in ($\star$) and the relation $J_{i+1} \prec J_i$ we see that

$$\sigma(J_{i+1} \cup J_i) = \begin{cases} + & \text{if } J_{i+1} <_{\text{lex}} J_i \\ - & \text{if } J_{i+1} >_{\text{lex}} J_i. \end{cases}$$

If $J_{i+1} <_{\text{lex}} J_i$, then in the packet P we have $\sigma^*(J_i \cup \{n+1\})$ left of $\sigma^*(J_{i+1} \cup \{n+1\})$ left of $\sigma^*(J_{i+1} \cup J_i) = +$. Since by the inductive assumption $\sigma^*(J_i \cup \{n+1\}) = +$ we obtain $\sigma^*(J_{i+1} \cup \{n+1\}) = +$ from the monotonicity of σ^* on packets.

If $J_{i+1} >_{\text{lex}} J_i$, then in the packet P we have $\sigma^*(J_{i+1} \cup \{n+1\})$ left of $\sigma^*(J_i \cup \{n+1\})$ left of $\sigma^*(J_{i+1} \cup J_i) = -$. Since by the inductive assumption $\sigma^*(J_i \cup \{n+1\}) = +$ we obtain $\sigma^*(J_{i+1} \cup \{n+1\}) = +$ from the monotonicity of σ^* on packets.

We now have shown that $\mathcal{D}^{\sigma^*}_{n+1} = \{I \subset [n] : |I| = r-1, \sigma^*(I \cup \{n+1\}) = +\}$ is a down-set of $\prec$. Since the complement of a down-set is an up-set we immediately get that $\mathcal{U}^{\sigma^*}_{n+1} = \{I \subset [n] : |I| = r-1, \sigma^*(I \cup \{n+1\}) = -\}$ is an up-set of $\prec$. ☐

Lemma 4.4. *For $r \geqslant 2$, let σ be an r-signotope on $[n]$ and $\prec$ the corresponding partial order. For all $(r-1)$-subsets I and all r-signotopes σ^* on $[n+1]$ such that $\sigma^*{\downarrow_{n+1}} = \sigma$, $I \cup \{n+1\}$ is a fliple of σ^* and J is comparable to I in $\prec$, it is*

$$\sigma^*(J \cup \{n+1\}) = \begin{cases} +, & \text{if } J \prec I; \\ -, & \text{if } J \succ I. \end{cases}$$

Proof. Since $I \cup \{n+1\}$ is a fliple we can move from σ^* to the companion signotope with the sign of the fliple reverted. Applying Proposition 4.3 and its dual to these two extensions of σ yields the lemma. ☐

If there are two maximal elements in a down-set, they are incomparable. This gives a condition to ensure that a signotope is not extendable by an element added at the last position.

Proposition 4.5. *For $r \geqslant 3$, let σ be an r-signotope on $[n]$ and let $I_1, \ldots, I_t$ be disjoint $(r-1)$-subsets, $t \geqslant 2$. Then there is an r-signotope σ^* on $[n+1]$ with $\sigma^*{\downarrow_{n+1}} = \sigma$ such that for all $j = 1, \ldots, t$ the r-subsets $I_j \cup \{n+1\}$ are fliples if and only if $I_1, \ldots, I_t$ are pairwise incomparable in the partial order corresponding to σ.*

Proof. The existence of an extension if the prescribed subsets are pairwise incomparable follows from Proposition 4.1, just consider the down-set generated by $I_1, \ldots, I_t$.

For the other direction assume that among $I_1, \ldots, I_t$ there are two sets I_i and I_j with $I_i \prec I_j$ and there is an extension σ^*. The set I_i is in the down-set of I_j and $I_j \cup \{n+1\}$ is a fliple of

σ^*. Lemma 4.4 implies that $\sigma^*(I_i \cup \{n+1\} = +$. Since $I_i \cup \{n+1\}$ is a fliple we can change its sign to $-$ and obtain a valid signotope. Since we assume $r \geqslant 3$ and I_i and I_j are disjoint, the two r-subsets $I_i \cup \{n+1\}$ and $I_j \cup \{n+1\}$ do not appear in a common $(r+1)$-packet. Hence flipping $I_i \cup \{n+1\}$ from $+$ to $-$ does not affect whether $I_j \cup \{n+1\}$ is a fliple. In particular $I_j \cup \{n+1\}$ is still a fliple in the signotope in which $I_i \cup \{n+1\}$ was flipped to $-$. This is not possible by Lemma 4.4. A contradiction. □

If we want to prove 2-extendability we may not assume that the two prescribed $(r-1)$-subsets are incomparable. Still we can use Proposition 4.1 to proof the extendability result. As in the case of arrangements of pseudolines we will use *rotations* to turn comparability into incomparability. Rotations are subject of the following section.

5. Rotations

Let us start by looking at the rotation operation for marked arrangements of pseudolines in the language of 3-signotopes. If we rotate pseudoline 1 as in Figure 3.2, then the orientation of a triple is preserved if and only if pseudoline is not part of the triple. In the figure the triple (triangle) $\{2,3,4\}$ in the left arrangement is turned into the triple $\{1,2,3\}$ in the right arrangement and keeps its orientation, however, the triple $\{1,3,4\}$ ist turned into $\{2,3,5\}$ and changes its orientation. When rotating the first element of σ becomes the last one in the rotated signotope σ_{rot}. In terms of the 3-signotope σ the signs of the rotated signotope σ_{rot} are: $\sigma_{\text{rot}}(a,b,c) = \sigma(a+1,b+1,c+1)$ if $c \neq n$ and $\sigma_{\text{rot}}(a,b,n) = -\sigma(1,a+1,b+1)$.

For general r, we define the *clockwise rotated* signotope σ_{rot} of a given r-signotope σ as:

$$\sigma_{\text{rot}}(x_1,\ldots,x_r) = \begin{cases} \sigma(x_1+1,\ldots,x_r+1) & \text{if } x_1 < x_2 < \ldots < x_r < n, \\ -\sigma(1,x_1+1,\ldots,x_{r-1}+1) & \text{if } x_1 < x_2 < \ldots < x_r = n. \end{cases}$$

For notational convenience we define $x_{\text{rot}} = x-1$ if $x \neq 1$, and $1_{\text{rot}} = n$, and for a subset $X = (x_1,\ldots,x_k)$ of $[n]$ with $x_1 < \ldots < x_k$:

$$X_{\text{rot}} = \{x_{\text{rot}} : x \in X\} = \begin{cases} (x_1-1, x_2-1,\ldots,x_k-1) & \text{if } x_1 > 1, \\ (x_2-1,\ldots,x_k-1,n) & \text{if } x_1 = 1. \end{cases}$$

This allows us to write $\sigma_{\text{rot}}(X_{\text{rot}}) = \sigma(X)$ if $1 \notin X$ and $\sigma_{\text{rot}}(X_{\text{rot}}) = -\sigma(X)$ if $1 \in X$. The *k-fold clockwise rotation* is denoted by $\sigma_{\text{rot}(k)}$. Note that the rotation operation depends on the number n of elements of a signotope. While n is known when rotating a signotope itself, we have to be careful when rotating a subset $X \subseteq [n]$. As the following lemma shows, the rotated signotope is indeed an r-signotope.

Lemma 5.1. *Let σ be an r-signotope on $[n]$. Then σ_{rot} is an r-signotope on $[n]$.*

Proof. Consider an $(r+1)$-packet $P' = (x'_1,\ldots,x'_{r+1})$. Since rotation induces a bijection on the $(r+1)$-packets of $[n]$ there is a packet $P = (x_1,\ldots,x_{r+1})$ such that $P' = P_{\text{rot}}$.

If the rotated element 1 is not in P, i.e., $x'_{r+1} < n$, then $x_i = x'_i + 1$ for all $i = 1, \dots, r+1$ and the signs of the r-subsets of packet P' have to be considered in the same order

$$\begin{array}{rccccc} & (\ \sigma_{\text{rot}}(P'_1) & \sigma_{\text{rot}}(P'_2) & \dots & \sigma_{\text{rot}}(P'_r) & \sigma_{\text{rot}}(P'_{r+1})\ ) \\ = & (\ \sigma(P_1) & \sigma(P_2) & \dots & \sigma(P_r) & \sigma(P_{r+1})\ ). \end{array}$$

as for P. The latter has at most one sign change since σ is an r-signotope.

If the rotated element 1 is in P, that is, $x'_{r+1} = n$ and $x_1 = 1$, then we have $x_{i+1} = x'_i + 1$ for all $i = 1, \dots, r$. Note that $n \in P'_i$ for $i = 1, \dots, r$ and hence $1 \in P_j$ for $j = 2, \dots, r+1$. The sign sequence of P' is

$$\begin{array}{rlllll} & (\sigma_{\text{rot}}(P'_1) & \sigma_{\text{rot}}(P'_2) & \dots & \sigma_{\text{rot}}(P'_r) & \sigma_{\text{rot}}(P'_{r+1})) \\ = & (\sigma_{\text{rot}}(P' \backslash \{x_2 - 1\}) & \sigma_{\text{rot}}(P' \backslash \{x_3 - 1\}) & \dots & \sigma_{\text{rot}}(P' \backslash \{x_{r+1} - 1\}) & \sigma_{\text{rot}}(P' \backslash \{n\})) \\ = & (-\sigma(P_2) & -\sigma(P_3) & \dots & -\sigma(P_{r+1}) & \sigma(P \backslash \{1\})) \\ = & (-\sigma(P_2) & -\sigma(P_3) & \dots & -\sigma(P_{r+1}) & \sigma(P_1)), \end{array}$$

which has at most one sign change because $(\sigma(P_1)\ \sigma(P_2)\ \dots\ \sigma(P_r)\ \sigma(P_{r+1}))$ has at most one sign change due to the signotope property of σ. □

The following lemma shows that up to the relabeling the rotated signotope σ_{rot} has the same fliples as σ.

Lemma 5.2. *Let σ be an r-signotope and let F be a fliple of σ. Then F_{rot} is a fliple in the clockwise rotated signotope σ_{rot}.*

Proof. To prove that an r-subset F_{rot} is a fliple, we need to check all $(r+1)$-packets P' with $F_{\text{rot}} \subset P'$. Let P' be such a packet and let P be such that $P' = P_{\text{rot}}$. Since F is a fliple in σ we know that if we change the sign of $\sigma(F)$ there is still at most one sign change in the sequence $\sigma(P_1), \sigma(P_2), \dots, \sigma(P_r), \sigma(P_{r+1})$, we abbreviate this by saying that F is *flipable* in P.

If $1 \notin P$, then $\sigma(P_i) = \sigma_{\text{rot}}(P'_i)$ for all i. Moreover if j is such that $F = P_j$ then $F_{\text{rot}} = P'_j$. Since the signs in the sequence stay the same F_{rot} is flipable in P'.

Otherwise we have $1 \in P$, and as observed in the proof of Lemma 5.1 we have

$$\begin{array}{rccccc} & (\ \sigma_{\text{rot}}(P'_1) & \sigma_{\text{rot}}(P'_2) & \dots & \sigma_{\text{rot}}(P'_r) & \sigma_{\text{rot}}(P'_{r+1})\ ) \\ = & (\ -\sigma(P_2) & -\sigma(P_3) & \dots & -\sigma(P_{r+1}) & \sigma(P_1)\ ). \end{array}$$

If $F = P_1$ then the sequence $(\sigma(P_2)\ \dots\ \sigma(P_{r+1}))$ is constant. This implies that the sign of $\sigma_{\text{rot}}(F_{\text{rot}}) = \sigma_{\text{rot}}(P'_{r+1}) = \sigma(P_1)$ can be flipped. If $F = P_{r+1}$ then the sign sequence $(\sigma(P_1)\ \dots\ \sigma(P_r))$ is constant. This shows that the sign of $\sigma_{\text{rot}}(F_{\text{rot}}) = \sigma_{\text{rot}}(P'_r)$ is adjacent to different signs and can thus be flipped. If $F = P_2$ then $\sigma(P_1) \neq \sigma(P_3)$ and the signs $\sigma_{\text{rot}}(P'_i)$ for $2 \leqslant i \leqslant r+1$ are the same. Hence $F_{\text{rot}} = P'_1$ is flipable in P'. If $F = P_j$ with $3 \leqslant j \leqslant r$, then $F_{\text{rot}} = P'_{j-1}$ is clearly flipable in P'. This shows that F_{rot} is flipable in all packets containing it and hence a fliple of σ_{rot}. □

5.1. Rotations and the partial order

For a signotope σ and its rotation σ_{rot}, the partial orders are denoted by $\prec$ and $\prec_{\text{rot}}$, respectively. Moreover, if two elements x, y are incomparable in $\prec$ (respectively $\prec_{\text{rot}}$), this is denoted by $x \| y$ (respectively $x \|_{\text{rot}} y$).

In this subsection we study the interaction between the rotation operator and the partial order. The following lemma states that for two $(r-1)$-subsets I and J which have a common intersection of $r-2$ elements and both contain the rotated element, the relation is reversed.

Lemma 5.3. *Let σ be an r-signotope with partial order $\prec$ and σ_{rot} the rotated signotope with corresponding partial order $\prec_{\text{rot}}$. For two $(r-1)$-subsets I, J such that $|I \cap J| = r-2$:*

$$\begin{aligned} &\textit{if } 1 \notin I \cup J, && \textit{then } I \prec J \iff I_{\text{rot}} \prec_{\text{rot}} J_{\text{rot}}, \textit{ and} \\ &\textit{if } 1 \in I \cap J, && \textit{then } I \prec J \iff I_{\text{rot}} \succ_{\text{rot}} J_{\text{rot}} \end{aligned}$$

Proof. If $1 \notin I \cup J$, then $\sigma(I \cup J) = \sigma_{\text{rot}}(I_{\text{rot}} \cup J_{\text{rot}})$ by definition. Furthermore $I <_{\text{lex}} J$ if and only if $I_{\text{rot}} <_{\text{lex}} J_{\text{rot}}$. Hence $I \prec J$ if and only if $I_{\text{rot}} \prec_{\text{rot}} J_{\text{rot}}$ as in the proof of Proposition 4.3.

If $1 \in I$ and $1 \in J$ the lexicographic order of I and J is the same as the lexicographic order of I_{rot} and J_{rot} but in the sign of the r-subset gets in σ is reversed, i.e., $\sigma(I \cup J) = -\sigma_{\text{rot}}(I_{\text{rot}} \cup J_{\text{rot}})$. Thus the order between I and J, and I_{rot} and J_{rot}, respectively is reversed. □

We now introduce two partitions of the family of all $(r-1)$-subsets. With respect to the first element 1, we partition the $(r-1)$-subsets $\binom{[n]}{r-1}$ into the following three sets:

$$\begin{aligned} \mathcal{H}_1^\sigma &= \{\, I \subset [n] : |I| = r-1,\ 1 \in I \,\}; \\ \mathcal{U}_1^\sigma &= \{\, I \subset [n] : |I| = r-1,\ 1 \notin I,\ \sigma(I \cup \{1\}) = + \,\}; \\ \mathcal{D}_1^\sigma &= \{\, I \subset [n] : |I| = r-1,\ 1 \notin I,\ \sigma(I \cup \{1\}) = - \,\}. \end{aligned}$$

Similarly, with respect to the last element n, we partition $\binom{[n]}{r-1}$ into the following three sets:

$$\begin{aligned} \mathcal{H}_n^\sigma &= \{\, I \subset [n] : |I| = r-1,\ n \in I \,\}; \\ \mathcal{U}_n^\sigma &= \{\, I \subset [n] : |I| = r-1,\ n \notin I,\ \sigma(I \cup \{n\}) = - \,\}; \\ \mathcal{D}_n^\sigma &= \{\, I \subset [n] : |I| = r-1,\ n \notin I,\ \sigma(I \cup \{n\}) = + \,\}. \end{aligned}$$

Note the sign change in the definition, that is, every $I \in \mathcal{U}_1^\sigma$ has the sign $\sigma(I \cup \{1\}) = +$ while every $I \in \mathcal{U}_n^\sigma$ fulfills $\sigma(I \cup \{n\}) = -$.

Lemma 5.4. *The sets $\mathcal{D}_n^\sigma$ and $\mathcal{U}_n^\sigma$ are a down-set and an up-set respectively of the partial order $\prec$ corresponding to the r-signotope σ.*

Proof. Let $\sigma' = \sigma{\downarrow_n}$ and view σ as an extension of σ' with the new element n. In Proposition 4.3 we have shown that $\{I \subset [n-1] : |I| = r-1, \sigma(I \cup \{n\}) = +\}$ is a down-set of $\prec_{\sigma'}$, and $\{I \subset [n-1] : |I| = r-1, \sigma(I \cup \{n\}) = -\}$ is an up-set of $\prec_{\sigma'}$. On $(r-1)$ subsets of $[n-1]$ the partial orders $\prec_{\sigma'}$ and $\prec_\sigma$ are equal, hence,

$$\{I \subset [n-1] : |I| = r-1, \sigma(I \cup \{n\}) = -\} = \{I \subset [n] : |I| = r-1, n \notin I, \sigma(I \cup \{n\}) = -\} = \mathcal{D}_n^\sigma$$

is a down-set in $\prec_\sigma$, and

$$\{I \subset [n{-}1] : |I|{=}r{-}1, \sigma(I{\cup}\{n\}){=}{-}\} = \{I \subset [n] : |I|{=}r{-}1, n \notin I, \sigma(I{\cup}\{n\}){=}{-}\} = \mathcal{U}_n^\sigma$$

is an up-set in $\prec_\sigma$. □

We now show that the two partitions are related by rotation. Let $\mathcal{X}_{\text{rot}} = \{X_{\text{rot}} \colon X \in \mathcal{X}\}$ denote the clockwise rotated sets of a set-system $\mathcal{X}$.

Lemma 5.5. *It holds* $(\mathcal{H}_1^\sigma)_{\text{rot}} = \mathcal{H}_n^{\sigma_{\text{rot}}}$, $(\mathcal{U}_1^\sigma)_{\text{rot}} = \mathcal{U}_n^{\sigma_{\text{rot}}}$, *and* $(\mathcal{D}_1^\sigma)_{\text{rot}} = \mathcal{D}_n^{\sigma_{\text{rot}}}$.

Proof. An $(r-1)$-subset I contains the first element 1 if and only if its clockwise rotation I_{rot} contains the last element n. Therefore, we have

$$(\mathcal{H}_1^\sigma)_{\text{rot}} = \mathcal{H}_n^{\sigma_{\text{rot}}} \text{ and } (\mathcal{U}_1^\sigma \cup \mathcal{D}_1^\sigma)_{\text{rot}} = \mathcal{U}_n^{\sigma_{\text{rot}}} \cup \mathcal{D}_n^{\sigma_{\text{rot}}}.$$

To show $(\mathcal{U}_1^\sigma)_{\text{rot}} = \mathcal{U}_n^{\sigma_{\text{rot}}}$ and $(\mathcal{D}_1^\sigma)_{\text{rot}} = \mathcal{D}_n^{\sigma_{\text{rot}}}$, it suffices to prove $(\mathcal{U}_1^\sigma)_{\text{rot}} \subseteq \mathcal{U}_n^{\sigma_{\text{rot}}}$ and $(\mathcal{D}_1^\sigma)_{\text{rot}} \subseteq \mathcal{D}_n^{\sigma_{\text{rot}}}$.

For the first of these subset relations $(\mathcal{U}_1^\sigma)_{\text{rot}} \subseteq \mathcal{U}_n^{\sigma_{\text{rot}}}$, let $I \in \mathcal{U}_1^\sigma$, i.e., $\sigma(I \cup \{1\}) = +$. After rotating the element 1, we obtain

$$+ = \sigma(I \cup \{1\}) = -\sigma_{\text{rot}}((I \cup \{1\})_{\text{rot}}) = -\sigma_{\text{rot}}(I_{\text{rot}} \cup \{n\}).$$

Hence $\sigma_{\text{rot}}(I_{\text{rot}} \cup \{n\}) = -$ which implies $I_{\text{rot}} \in \mathcal{U}_n^{\sigma_{\text{rot}}}$. An analogous argument shows $(\mathcal{D}_1^\sigma)_{\text{rot}} \subseteq \mathcal{D}_n^{\sigma_{\text{rot}}}$. □

Lemma 5.6. *The sets $\mathcal{D}_1^\sigma$ and $\mathcal{U}_1^\sigma$ are a down-set and an up-set, respectively, of the partial order $\prec$ corresponding to the r-signotope σ.*

Proof. Sets $I, J \in \mathcal{U}_1^\sigma \cup \mathcal{D}_1^\sigma$ do not contain element 1, hence, from Lemma 5.3 we get that $I \prec J$ if and only if $I_{\text{rot}} \prec_{\text{rot}} J_{\text{rot}}$. Since $\mathcal{D}_n^{\sigma_{\text{rot}}}$ is a down-set with respect to $\prec_{\text{rot}}$ (Lemma 5.4) we obtain that $\mathcal{D}_1^\sigma$ is a down-set respect to $\prec$. The argument for $\mathcal{U}_1^\sigma$ being an up-set is analogous. □

It is worth noting that for $I, J \in \mathcal{H}_1^\sigma$ (i.e., $1 \in I \cap J$) with $I \prec J$ Lemma 5.6 implies that any chain $I = I_1 \prec \ldots \prec I_k = J$ lies entirely in $\mathcal{H}_1^\sigma$ (i.e. $I_1, \ldots, I_k \in \mathcal{H}_1^\sigma$). Since a clockwise rotation converts comparability of elements containing the element 1, we have $I_{\text{rot}} = (I_1)_{\text{rot}} \succ_{\text{rot}} \ldots \succ_{\text{rot}} (I_k)_{\text{rot}} = J_{\text{rot}}$.

Proposition 5.7. *Let σ be an r-signotope on $[n]$ with partial order $\prec$. For two $(r-1)$-subsets I, J with $I \prec J$ and $1 \notin I \cap J$, it holds $I_{\text{rot}} \|_{\text{rot}} J_{\text{rot}}$ or $I_{\text{rot}} \prec_{\text{rot}} J_{\text{rot}}$.*

Proof. Suppose for a contradiction that I, J are two $(r-1)$-subsets with $I \prec J$ and $I_{\text{rot}} \succ_{\text{rot}} J_{\text{rot}}$.

If $J \in \mathcal{D}_1^\sigma$, then by Lemma 5.6, $I \in \mathcal{D}_1^\sigma$. If $I \in \mathcal{D}_1^\sigma$, then by Lemma 5.5, $I_{\text{rot}} \in \mathcal{D}_n^{\sigma_{\text{rot}}}$, Lemma 5.4 and the assumption that $I_{\text{rot}} \succ_{\text{rot}} J_{\text{rot}}$ we get $J_{\text{rot}} \in \mathcal{D}_n^{\sigma_{\text{rot}}}$. Applying Lemma 5.5 again yields $J \in \mathcal{D}_1^\sigma$. Hence $I \in \mathcal{D}_1^\sigma$ if and only if $J \in \mathcal{D}_1^\sigma$. Symmetrically, $J \in \mathcal{U}_1^\sigma$ if and only if $I \in \mathcal{U}_1^\sigma$.

Since $1 \notin I \cap J$ not both I and J can be in $\mathcal{H}_1^\sigma$. Hence I and J are both in $\mathcal{D}_1^\sigma$ or both in $\mathcal{U}_1^\sigma$ and $1 \notin I \cap J$. This together with $I \prec J$ and Lemma 5.3 implies $I_{\text{rot}} \prec_{\text{rot}} J_{\text{rot}}$. This is in contradiction to $I_{\text{rot}} \succ_{\text{rot}} J_{\text{rot}}$. □

6. 2-Extendability

In the three subsections of this section we prove Theorem 3.3, Corollary 3.4, and the lifting lemma (Lemma 3.7).

6.1. Extension theorem for odd rank (Theorem 3.3)

In this section, we first show that for each pair of disjoint crossing points, respectively $(r-1)$-subsets, odd rank signotopes admit a rotation in which the crossing points are incomparable. Together with Proposition 4.1 and Proposition 6.1 this allows us to prove Theorem 3.3.

Proposition 6.1. *Let $r \geqslant 3$ be an odd integer, let σ be an r-signotope on $[n]$ and let I, J be two disjoint $(r-1)$-subsets. There is a $k \leqslant n-1$ auch that after k clockwise rotations, σ, I, and J are transformed into $\sigma_{\text{rot}(k)}$, $I_{\text{rot}(k)}$, and $J_{\text{rot}(k)}$, respectively, such that $I_{\text{rot}(k)} \|_{\text{rot}(k)} J_{\text{rot}(k)}$.*

Proof. Assume I and J are comparable in the partial order $\prec$ corresponding to the r-signotope σ, otherwise $k=0$ is the desired rotation. Without loss of generality assume $I \prec J$. We show that after n clockwise rotations, i.e., when every element was rotated once, all signs of σ are reversed. Hence the partial order $\prec_{\text{rot}(n)}$ of $\sigma_{\text{rot}(n)}$ is the reverse of the relation $\prec$.

The sign of an r-subset $(z_1, \ldots, z_r)$ of $[n]$ changes with a rotation if and only if the rotated element is contained in $(z_1, \ldots, z_r)$. Since after rotating n times every z_i was rotated once, the sign of each r-subset has changed r times. Since r is odd, the sign after rotating n times is reversed. The resulting signotope $\sigma_{\text{rot}(n)}$ is the reverse of the original signotope σ and the corresponding partial order is also the reverse.

By Proposition 5.7 we cannot reverse the order of a pair of disjoint $(r-1)$-subsets in a single rotation. Hence there will be a $k < n$ such that the two disjoint sets are incomparable in $\prec_{\text{rot}(k)}$. □

From the count of sign changes it follows that for all ranks r a sequence of $2n$ rotations returns to the original signotope.

We are ready now to prove the theorem which for convenience is restated.

Theorem 3.3 (Extension theorem for signotopes of odd rank)**.** *For every odd rank $r \geqslant 3$, every r-signotope is 2-extendable.*

Proof. Let σ be an r-signotope on $[n]$ and let I, J be a pair of disjoint $(r-1)$-subsets. By Proposition 6.1 there exists $k \in \{0, \ldots, n-1\}$ such that the k-fold rotated $(r-1)$-subsets $I_{\text{rot}(k)}, J_{\text{rot}(k)}$ are incomparable in the k-rotated signotope $\sigma_{\text{rot}(k)}$.

To extend the signotope $\sigma_{\text{rot}(k)}$, we use the down-set $\mathcal{D}$ consisting of $I_{\text{rot}(k)}$, $J_{\text{rot}(k)}$, and all $(r-1)$-subsets which are smaller in $\prec_{\text{rot}(k)}$. In this down-set $I_{\text{rot}(k)}$ and $J_{\text{rot}(k)}$ are maximal elements since they are incomparable. Hence we can apply Proposition 4.1 in order to add a new element at position $n+1$ in the rotated signotope $\sigma_{\text{rot}(k)}$ such that $I_{\text{rot}(k)} \cup \{n+1\}$ and $J_{\text{rot}(k)} \cup \{n+1\}$ are fliples. Let $\sigma^*_{\text{rot}(k)}$ denote the extended signotope with $\sigma^*_{\text{rot}(k)}\downarrow_{n+1} = \sigma_{\text{rot}(k)}$.

Finally, we need a rotation of $\sigma^*_{\mathrm{rot}(k)}$ which contains the original signotope σ. For this we apply $2(n+1)-(k+1)=2n+1-k$ rotations to $\sigma^*_{\mathrm{rot}(k)}$ and denote the resulting signotope by σ^*.

After $n+1$ rotations the label of the extending element is again $n+1$ and after additional $n-k$ rotations it is $k+1$. The extending element was the rotation element only once and we made $2n+1$ rotations in total, hence, every element of σ was the rotating element twice. Hence, the sign on each r-subset of elements of σ was reversed $2r$ times, and $\sigma(X{\downarrow_{k+1}}) = \sigma^*(X)$ for all r-subsets X of $[n+1]$ with $k+1 \notin X$.

Since $I_{\mathrm{rot}(k)} \cup \{n+1\}$ and $J_{\mathrm{rot}(k)} \cup \{n+1\}$ are fliples of $\sigma^*_{\mathrm{rot}(k)}$ and fliples are preserved under rotations, see Lemma 5.2, the signotope σ^*has fliples I^* and J^* containing $k+1$ with $I^*{\downarrow_{k+1}} = I$ and $J^*{\downarrow_{k+1}} = J$. ☐

6.2. Extendability with intersection (Corollary 3.4)

Corollary 3.4. *For $r \geqslant 3$, let σ be an r-signotope on $[n]$, and I, $J \subseteq [n]$ two $(r-1)$-subsets such that $|I \cap J| + r$ is odd. Then σ is extendable to an r-signotope σ^* on $[n+1]$ with fliples I^*, J^* and an extending element $k \in [n+1]$ such that $\sigma^*{\downarrow_k} = \sigma$, and $I^*{\downarrow_k} = I$, and $J^*{\downarrow_k} = J$.*

Proof. To prove Corollary 3.4, we proceed similar as in the proof of Theorem 3.3. By Proposition 4.1, it suffices to show that after some rotations the $(r-1)$-subsets corresponding to I and J are incomparable.

Let $m = |I \cap J|$. Since Theorem 3.3 covers the case $m = 0$, we may assume $m \geqslant 1$. We consider the following two cases.

First, assume that r is odd and m is even. For odd rank r, we have already seen that after n rotations, the signotope is reversed and hence the corresponding partial order is reversed. For even m, the relation between I and J is reversed m times during the n rotations, whenever we rotate one element $x \in I \cap J$, see Lemma 5.3. If we rotate an element $x \notin I \cap J$, the relation cannot be reversed, see Proposition 5.7. Since m is even and the order is reversed after n rotations, the corresponding $(r-1)$-subsets must be incomparable in between.

If r is even and m is odd, the n-fold rotation leaves the signotope unchanged and hence the partial order is the same. Since m is odd, we reverse the orientation of I and J exactly m times in a single rotation step. Hence they must be incomparable in between. The rest of the proof is essentially the same as the proof of Theorem 3.3. ☐

6.3. The lifting lemma

Lemma 3.7 (Lifting Lemma). *For every r-signotope σ on $[n]$ there exists an $(r+1)$-signotope σ^+ on $[n+1]$ such that $\sigma^+ {\downdownarrows_{n+1}} = \sigma$.*

Proof. This proof will be based on a piece of signotope-theory. The signotope order $S_r(n)$ is a partial order on the set of all r-signotopes on $[n]$, it is defined as the inclusion order on the preimage $\sigma^{-1}(+)$. As shown by Manin and Schechtman [MS89] (see also Felsner and Weil [FW01]) the following two statements hold:

(a) There is a surjective mapping from the set of maximum chains in $S_r(n)$ to the set of $(r+1)$-signotopes on $[n]$.

(b) Every r-signotope on $[n]$ is contained in a maximum chain of $S_r(n)$.

Let γ be a maximum chain $\sigma_0 \prec \ldots \prec \sigma_{\binom{n}{r}}$ of r-signotopes containing σ at position i, i.e., $\sigma = \sigma_i$. Such a chain exists by (b). Moreover σ_0 is the constant $-$ function and $\sigma_{\binom{n}{r}}$ the constant $+$ function. From σ_j to σ_{j+1} there is exactly one r-subset flipped from $-$ to $+$. Let $A_1, \ldots, A_{\binom{n}{r}}$ be the order of flipped subsets. Hence in $\sigma = \sigma_i$ the r-subsets $A_1, \ldots, A_i$ are mapped to $+$ and the remaining r-subsets $A_{i+1}, \ldots, A_{\binom{n}{r}}$ are mapped to $-$.

By (a), there is an $(r+1)$-signotope σ_γ on $[n]$ corresponding to the maximum chain γ for which $\sigma_1, \ldots, \sigma_{\binom{n}{r}}$ is a sweep. In particular $A_1, \ldots, A_{\binom{n}{r}}$ is a linear extension of the partial order $\prec_\gamma$ corresponding to σ_γ. A prefix of a linear extension is a down-set of the order, hence, $A_1, \ldots, A_i$ is a down-set of $\prec_\gamma$. Proposition 4.1 yields an extension σ^+ of σ_γ such that

$$\sigma^+(X) = \begin{cases} \sigma_\gamma(X) & \text{if } X \subset [n]; \\ \sigma(X{\downarrow}_{n+1}) & \text{if } n+1 \in X \subset [n+1]. \end{cases}$$

The properties of σ^+ are as needed, in particular $\sigma^+ \downdownarrows_{n+1} = \sigma$. Figure 6.1 illustrates the construction for $r = 2$. ☐

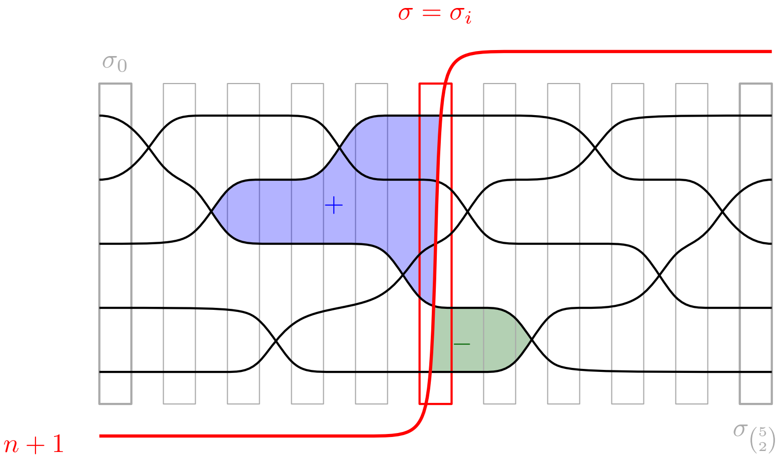

Figure 6.1: Illustration of the construction. The assignment for the 3-signotope depends on the maximum chain of permutations. The 3-signotope projected to $n+1$ (red) yields the starting 2-signotope σ.

7. Examples in Even Rank: SAT Attack and Properties

Theorem 3.3 yields extensions for odd ranks. So what about even ranks? As discussed before, Goodman and Pollack and Richter-Gebert already presented non-extendable oriented matroids of rank 4 with 3 respectively 2 prescribed points. For both examples, we tested via computer that there is no ordering of the elements such that they are signotopes. For signotopes of rank 4, we used a complete computer supported enumeration of all signotopes with $n \leqslant 8$ and tested 2-extendability for each signotope. On 6 and 7 elements all 4-signotopes are 2-extendable. On 8 elements we found non-2-extendable 4-signotopes. For both, the enumeration and the 2-extendability test, we modeled SAT instances which were then solved using the python interfaces `pycosat` [S$^+$] and `pysat` [IMM18] to run the SAT solver `picosat`, version 965, [Bie08] and `cadical`, version 1.0.3 [Bie19], respectively.

Using this two-level-SAT approach we managed to find the first examples of 4-signotopes which are not 2-extendable. In order to keep symmetries and similarities of the nicely structured example of rank 4, we restricted the search space to examples in rank r on $2r$ elements. While for rank 4 all signotopes on 8 elements can be enumerated within a few seconds, the complete enumeration in higher ranks is unpractical as the number of r-signotopes on $2r$ elements grows faster than doubly exponential in r (cf. Proposition 2.1). To be able to approach higher ranks, we further analyzed the structure of the non-2-extendable rank 4 examples and of the subsequently found rank 6 examples. These made it possible to find a recursive construction. See Section 7.3 for more details.

With properties observed in the rank 4 example and the subsequently found rank 6 examples as additional constraints, we further restricted the search space so that only “reasonable” candidates were enumerated. With these restrictions, we managed to find examples for rank 8, 10, and 12 which are not 2-extendable.

7.1. SAT model for enumeration

We give a short description of the encoding of r-signotopes on n elements in terms of a SAT instance. Such an instance consists of a Boolean formula which has a valid assignment if and only if there is a signotope with the specified properties. In particular we model the instance with a Boolean formula in *conjunctive normal form* (short: CNF), which is a conjunction of clauses. Each *clause* is a disjunction of variables and their negation (called literals).

To model a signotopes σ on $[n]$ and its fliples as a CNF formula, we define Boolean variables S_X for every r-subset $X \in \binom{[n]}{r}$ and interpret the value $\sigma(X) = +$ as $S_X = \textsc{True}$ and $\sigma(X) = -$ as $S_X = \textsc{False}$. Moreover, we have to ensure the monotonicity on $(r+1)$-packets. For this we list all possibilities of valid sign sequences, i.e., sign sequences of length $r+1$ with only one sign change. There are exactly $2r+2$ possible assignment of this sequence. Let $\mathcal{T}$ be the list of all those types. To encode which packet corresponds to which sequence and to ensure that every packet has exactly one of the sequences, we introduce auxiliary variables $T_{P,t}$ for every $P \in \binom{[n]}{r+1}$ and $t \in \mathcal{T}$ which we synchronize with the values of the corresponding r-subsets. Let $t(i)$ be the sign of t at position i. For $t(i) = -$ and a Boolean variable X we say $t(i) \cdot X$ is $\neg X$. Moreover, if $t(i) = +$, then $t(i) \cdot X$ is X. The variable $T_{P,t}$ is $\textsc{True}$ if and only if the sign sequence of the $(r+1)$-packet P is the same as the sign sequence $t \in \mathcal{T}$. In particular $T_{P,t} \Leftrightarrow \bigwedge_{j=1,\ldots,r+1} t(j) \cdot S_{P_j}$. For the CNF we have to add the clauses

$$\begin{aligned} &\neg T_{P,t} \ \vee \ t(j) \cdot S_{P_j} \text{ for all } j = 1, \ldots, r+1; \text{ and} \\ &T_{P,t} \ \vee \bigvee_{j=1,\ldots,r+1} \neg\, t(j) \cdot S_{P_j}. \end{aligned}$$

Note that the first direction of the implication is modeled by the $r+1$ clauses while the reverse direction is modeled by the clause in the second line.

An important part of the extendability are fliples. In a first step we define variables $F_{X,P}$ for every $(r+1)$-packet $P \in \binom{[n]}{r+1}$ and every r-subset $X \in \binom{P}{r}$ to indicate whether X is a fliple when σ is restricted to P. If $F_{X,P} = \textsc{True}$, the r-subset X is flipable in the packet P, i.e., is next to a sign change or at the beginning, respectively end, of a constant sign sequence. The information about the sign change is already encoded in $T_{P,t}$. Assume X is at position $j \in \{1, \dots, r+1\}$ of the packet P. Then X is flipable in P if and only if the sign sequence t of P has the sign change between position $j-1$ and j or between position j and $j+1$. For this let $\mathcal{T}_j$ be the set of sign sequences such that the j-th sign can be flipped without violating the monotonicity condition. Now it is $F_{X,P} \Leftrightarrow \bigvee_{t \in \mathcal{T}_j} T_{P,t}$. Hence for the CNF, we add the clauses

$$\neg F_{X,P} \vee \bigvee_{t \in \mathcal{T}_j} T_{P,t}; \text{ and}$$
$$F_{X,P} \vee \neg T_{P,t} \text{ for all } t \in \mathcal{T}_j.$$

Using the $F_{X,P}$ variables, we can assert the variables $F_X = \bigwedge_{P \in \binom{[n]}{r+1}:\, X \subset P} F_{X,P}$ for every $X \in \binom{[n]}{r}$ to indicate whether X forms a fliple in the signotope. Again we add the following clauses to the CNF

$$\neg F_X \vee F_{X,P} \text{ for all } P \supset X; \text{ and}$$
$$F_X \vee \bigvee_{P \supset X} \neg F_{X,P}.$$

Last but not least, we introduce variables $L_{X,k}$ to indicate whether X is the k-th fliple in the lexicographically ordered list of all r-subsets from $\binom{[n]}{r}$. By assigning the $L_{X,k}$ variables for the lexicographically largest r-subset X, we can then enumerate only configurations with a prescribed number of fliples.

7.2. SAT model for testing 2-extendability

We are now ready to formulate a SAT instance to decide whether a given signotope σ on $[n]$ and given disjoint $(r-1)$-tuples I, J can be extended by an additional element $n+1$ such that $I \cup \{n+1\}$ and $J \cup \{n+1\}$ are fliples in the extension σ^*. This is sufficient to test extendability since whenever there is an extension, there is a rotation such that the signotope is extendable by an element at the last position. As in Section 7.1, we create a SAT instance to find an $(n+1)$-element signotope but we add constraints to fix σ and to assert that $I \cup \{n+1\}$ and $J \cup \{n+1\}$ are fliples in σ^*.

For a given signotope σ on elements $[n]$ we can now iterate over all disjoint $(r-1)$-tuples I, J and test whether there is a rotation of σ and I, J such that in the extension σ^* by the element $n+1$ the r-tuples $I \cup \{n+1\}$ and $J \cup \{n+1\}$ are fliples. If for some I, J no such rotations exists, we have certified that σ is not 2-extendable.

7.3. Structure of the examples supporting Conjecture 3.5

In order to find the first witnessing examples for Conjecture 3.5 in rank 4, we used the two-step SAT approach as described in Sections 7.1 and 7.2. To make higher ranks accessible, we had to get a better understanding of the examples found in rank 4. Hence we filtered those with regularities and symmetries to come up with a generalization of the observed properties and analyzed their structure. Our aim was to find connections between examples in different ranks, for example using projection and deletion arguments. For this we investigated the structure of rank 4 and rank 6 examples.

One of the first and crucial observations was that there exist signotopes such that for every choice of even indices $I \subset E_r := \{2, 4, \dots, 2r\}$ and every choice of odd indices $J \subset O_r := \{1, 3, \dots, 2r-1\}$ there is no such extension. In fact, for such examples it is sufficient to check $I = \{2, 4, \dots, 2r-2\}$ and $J = \{1, 3, \dots, 2r-3\}$ to verify the non-2-extendability. This observation not only allowed us to restrict the search space, but also to speed up the extendability-test by a factor of $\Theta(r^2)$ since not all pairs of $(r-1)$-tuples I, J need to be tested.

We experimented with further properties. Here is collection of properties, which we decided to require from examples in rank r with $n = 2r$ elements. In the following we denote by $X = (x_1, x_2, \dots, x_r)$ an r-tuple and use the notation $(-)^i = +$ if i is even and $(-)^i = -$ if i is odd.

(a) $\sigma = \sigma_{\mathrm{rot}(4)}$, where $\sigma_{\mathrm{rot}(4)}$ is obtained by the 4-fold rotation of σ.

(b) $\sigma(2, 4, \dots, 2r) = -$ and $\sigma(1, 3, \dots, 2r-1) = +$.

(c) If there is only one even or only one odd element in X, then the sign $\sigma(X)$ depends only on the position of that element in X. More specifically: If x_i is the only even element in X, then $\sigma(X) = (-)^i$. If x_i is the only odd element in X, then it is $\sigma(X) = (-)^{i+1}$.

(d) If $x_1, \dots, x_i \in E_r$ and $x_{i+1}, \dots, x_r \in O_r$ with $2 \leqslant i \leqslant r-2$, then the sign of X is $\sigma(X) = (-)^{i+1}$.

(e) If $x_1, \dots, x_i \in O_r$ and $x_{i+1}, \dots, x_r \in E_r$ for $2 \leqslant i < r-2$, then if $x_r < 2r$ the sign of X is $\sigma(X) = -$ and if $x_j = 2j$ for all $j = i+1, \dots, r$ the sign is $\sigma(X) = +$.

Furthermore, we fix the following set of 8 fliples for rank 4.

$$\begin{aligned} F_4 = \{&(1,3,5,7), (2,4,6,8), (2,3,7,8), (1,3,4,8), \\ &(1,2,4,7), (3,5,6,8), (4,5,7,8), (3,4,6,7)\} \end{aligned}$$

Together with the automorphism $\mathrm{rot}(4)$ it is sufficient to mention only some of them:

$$\widehat{F}_4 = \{(1,3,5,7), (2,4,6,8), (4,5,7,8), (3,4,6,7), (1,2,4,7)\}$$

In rank 4, there are only four sets whose sign is not determined by the above properties:

$$(1,3,4,8), \qquad (4,5,7,8), \qquad (2,3,7,8), \qquad (3,4,6,7)$$

By the $\mathrm{rot}(4)$ symmetry, the sign of $(1,3,4,8)$ equals the sign of $(4,5,7,8)$. The third and fourth tuple also have the same sign. Hence, there are precisely 4 signotopes in rank 4 which fulfill the above properties. We fix one of them (the choice does not play a role) and refer to it as σ_4 in the following.

In order to find examples in higher ranks, we use the following property.

(f) Let σ_{r-2} be an example of rank $r-2$ on $2r-4$ elements. For an r-tuple $X \subseteq [2r]$ with $1,3 \notin X$ and $2,4 \in X$, we define the sign

$$\sigma_r(X) = \sigma_{r-2}(X{\downarrow_{1,2,3,4}}).$$

Altogether, if we start with one exampleok from rank 4 and recursively construct examples in higher ranks with the desired properties and further prescribe $(r/2)^2 + (r/2) + 2$ fliples for rank r, it finally turned out that there is a unique example in each of the ranks $r = 6, 8, 10, 12$. All examples and the source code to verify their correctness are available as supplementary data [BFS].

As shown in [Ber23, Section 3.6.3] for even r, every r-signotope with the above described properties is non-2-extendable. However the existence of such signotopes for $r \geqslant 14$ remains open. Even though we conjecture that there is an infinite family, we want to mention that we found examples in rank 4 and 6 which do not have the above properties and hence the assumptions might also be too strong.

7.3.1 Rank 4

We explicitly give the rank 4 example and give a visualization as pseudohyperplane arrangement. We fix σ_4 as the examples, which is not 2-extendable and has the properties (a)–(f). The missing values are assigned to $-$. Representing the signotope with a string of its signs in reversed lexicographic order of its 4-subsets, the complete signotope has the signs

$$\begin{aligned}\sigma_4 = {} & {+}{+}{+}{-}{-}{+}{+}{+}{-}{+}{+}{-}{+}{+}{+}{+}{+}{-}{-}{+}{-}{-}{+}{+}{+}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-} \\ & {+}{+}{+}{-}{+}{+}{-}{+}{+}{+}{+}{+}{-}{+}{+}{+}{-}{-}{-}{-}{+}{+}{-}{+}{+}{+}{+}{-}{+}{-}{+}{+}{+}{+}{+}.\end{aligned}$$

The representation of σ_4 is illustrated in Figure 7.2. An interactive visualization of the 3-dimensional object is available as supplemental data [BFS]. This is generated using a SageMath program which computes the sweep of the signotope and for every rank 3 signotope a wiring diagram of fixed length. In particular, there is only one crossing at a time in the wiring diagram. This helps to make the boundaries of the 3-dimensional visualization nice. The single wiring diagrams are given as in Figure 7.1 with the same colors assigned to the elements. We start with the reversed cyclic arrangement at the top left position and then continue line by line until we reach the cyclic arrangement. In each step we highlight the triangular cells, which are flipped either from the previous arrangement or to the following arrangement.

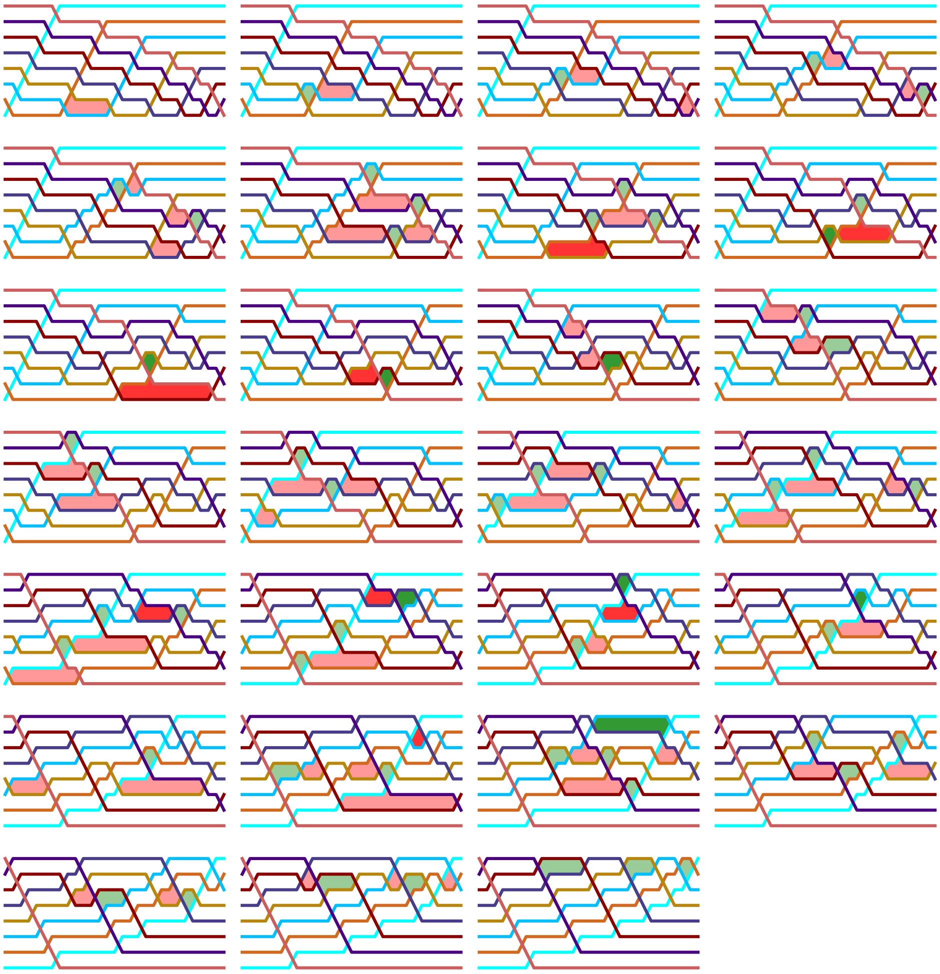

Figure 7.1: The sweep of pseudolines corresponding to σ_4 starting with the reversed cyclic arrangement in the top left. The odd elements $1, 3, 5, 7$ have a color from the red color family, whereas, the even elements $2, 4, 6, 8$ are colored with shades of blue and purple. Triangular cells which get flipped in comparison to the next signotope are marked red, if they have been flipped from the previous one they are green. Moreover we highlight the triangles consisting only of odd or only of even elements.

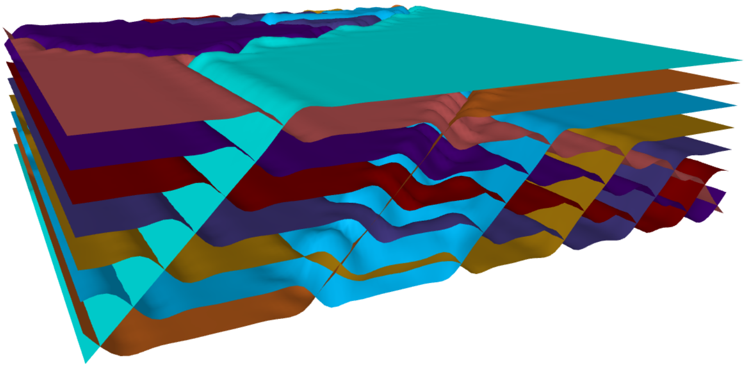

Figure 7.2: A 3-dimensional pseudohyperplane arrangement representing σ_4. In the front, we start with the reversed cyclic arrangement. The odd elements $1, 3, 5, 7$ have a color from the red color family, whereas, the even elements $2, 4, 6, 8$ are colored with shades of blue and purple.

8. Asymptotic Number of Signotopes

In this section, we give a short proof for Proposition 2.1.

Proposition 2.1 (Balko [Bal19]). *For $r \geqslant 3$, the number of r-signotopes on $[n]$ is $2^{\Theta(n^{r-1})}$.*

8.1. Proof of the upper bound

The upper bound follows immediately using the fact, that r-signotopes on n elements are rank r oriented matroids and their number is upper bounded by $2^{c(n^{r-1})}$ [BLS+99, Chapter 7.4]. For completeness, however, we include an inductive proof.

For rank 3, there exists a constant $c > 0$ such that for every n there are at most $2^{cn^2(1+o(1))}$ signotopes on n elements. The currently best bound $c = 0.657$ is due to Felsner and Valtr [FV11].

For rank $r \geqslant 4$, we proceed by induction. Given an r-signotope σ on $[n]$, we compute its sequence of projections. For each $i \in [n]$, we project σ to i and obtain an $(r-1)$-signotope $\sigma \downdownarrows_i$ on $n-1$ elements. Since two distinct r-signotopes yield different sequences $(\sigma \downdownarrows_i)_{i \in [n]}$ of projections, we can bound the number of r-signotopes as

$$s(n, r) \leqslant (s(n-1, r-1))^n \leqslant \left(2^{c(n-1)^{r-2}}\right)^n \leqslant 2^{cn^{r-1}}.$$

8.2. Proof of the lower bound

For convenience we assume $n = rm$ for some $m \in \mathbb{N}$. We partition $[n] = \bigcup_{k=1}^{r} N_k$ into r intervals $N_k = [(k-1)m + 1, km]$ of size m.

For every r-subset $(x_1, \ldots, x_r)$ we define the weight $\phi(x_1, \ldots, x_r) = \left(\sum_{k=1}^{r-1} x_k\right) - x_r$. Note that, for r-subsets $X = (x_1, \ldots, x_{r+1})$ with $x_1 < \ldots < x_{r+1}$ as usual, it holds $\phi(X_1) > \ldots > \phi(X_r)$ and $\phi(X_r) < \phi(X_{r+1})$.

For a threshold T we now define a collection $\mathcal{S}_T$ of signotopes on $[n]$. A signotope σ is in $\mathcal{S}_T$ if σ has the following signs, where ± indicates that the sign can be chosen arbitrarily from $\{+,-\}$.

$$\sigma(x_1,\dots,x_r) = \begin{cases} - & \text{if } x_{r-1} \notin N_r,\, x_r \in N_r, \text{ and } \phi(x_1,\dots,x_r) > T \\ \pm & \text{if } x_{r-1} \notin N_r,\, x_r \in N_r, \text{ and } \phi(x_1,\dots,x_r) = T \\ + & \text{otherwise.} \end{cases}$$

Note that for $r = 3$, the construction as a pseudoline arrangement is illustrated in [Mat02, p. 134]. For the lower bound for general r we show two properties:

(i) The elements of $\mathcal{S}_T$ are indeed signotopes.

(ii) For fixed r and a suitable chosen T there are $2^{\Omega(n^{r-1})}$ elements in $\mathcal{S}_T$.

To show (i), we check the monotonicity of all $(r+1)$-subsets $X = (x_1,\dots,x_{r+1})$, that is, there is at most one sign-change in the sequence

$$(\sigma(X_1)\ \dots\ \sigma(X_{r+1})).$$

If $x_{r+1} \notin N_r$, then $\sigma(X_j) = +$ for all $j = 1,\dots,r+1$ and there is no sign change on the packet. Otherwise, there is some $k \in [r+1]$ such that $x_1,\dots,x_{k-1} \notin N_r$ and $x_k,\dots,x_{r+1} \in N_r$.

If $k < r$, then $x_{r-1}, x_r, x_{r+1} \in N_r$. Hence each X_j contains at least two elements from N_r, which implies $\sigma(X_j) = +$ for all $j = 1,\dots,r+1$ and there is no sign change on the packet.

If $k = r$, then each X_j with $j < r$ contains two elements from N_r and thus $\sigma(X_j) = +$ for $j < r$. We also know that $\phi(X_r) < \phi(X_{r+1})$, therefore, if $\sigma(X_r) = -$ then $\sigma(X_{r+1}) = -$ as well. Hence, there is at most one sign change on the packet.

Finally, if $k = r+1$, then $\sigma(X_{r+1}) = +$. Since $\phi(X_1) > \phi(X_2) > \dots > \phi(X_r)$ the sign sequence $(\sigma(X_1)\ \sigma(X_2)\ \dots\ \sigma(X_r))$ is a sequence of $-$ signs followed by a sequence of $+$ signs possibly one $\pm$ in between. Again there is at most one sign change on the packet.

This completes the proof of (i) that all elements of $\mathcal{S}_T$ are signotopes.

It remains to show (ii), i.e., for some T the set $\mathcal{S}_T$ contains sufficiently many elements. An r-subsets $(x_1,\dots,x_r)$ is *splitted* if $x_k \in N_k$ for all $k = 1,\dots,r$. If there exist a_T splitted r-subsets with $\phi(x_1,\dots,x_r) = T$, then $|\mathcal{S}_T| \geqslant 2^{a_T}$.

With a splitted r-subsets $(x_1,\dots x_r)$ we associate the r-tuple $(y_1,\dots,y_r) \in [m]^r$ where $y_k = x_k - (k-1)m$ for all $k \in [r]$. Indeed this is a bijection between splitted r-subsets and $[m]^r$.

There are $\binom{\ell-1}{r-2}$ possible compositions of an integer ℓ as a sum of $r-1$ positive integers, i.e., there are $\binom{\ell-1}{r-2}$ solutions of $\sum_{k=1}^{r-1} y_k = \ell$ with positive integer variables $y_1,\dots,y_{r-1}$.

Now consider the number of solutions of

$$\sum_{k=1}^{r-1} y_k = y_r. \tag{8.1}$$

with $y_1,\dots,y_r \in [m]$. Since y_r is in $[m]$, the number of solutions is $\sum_{\ell=1}^{m} \binom{\ell-1}{r-2}$. Using standard identities and estimates for binomial coefficients we obtained:

$$\sum_{\ell=1}^{m} \binom{\ell-1}{r-2} = \sum_{\ell=r-2}^{m-1} \binom{\ell}{r-2} = \binom{m}{r-1} = \Theta(m^{r-1}) = \Theta(n^{r-1}).$$

We now determine a suitable T such that $\phi(x_1, \dots, x_r) = \sum_{k=1}^{r-1} x_k - x_r = T$ if and only if the corresponding $y_1, \dots, y_r$ fulfill equation (8.1).

$$\begin{aligned}
0 &= \sum_{k=1}^{r-1} y_k - y_r \\
&= \sum_{k=1}^{r-1}(x_k - (k-1)m) - (x_r - (r-1)m) = \sum_{k=1}^{r-1} x_k - x_r - \sum_{k=1}^{r-1} m(k-1) + (r-1)m \\
&= \sum_{k=1}^{r-1} x_k - x_r - m \cdot \frac{(r-4)(r-1)}{2}.
\end{aligned}$$

Hence $T = m \cdot \frac{(r-4)(r-1)}{2}$ is a value which gives the lower bound.

9. 4-Extendability

In the previous section, we studied 1- and 2-extendability. Moreover, we developed techniques to investigate general t-extendability. In this section, we discuss examples which show that signotopes with rank $r \geqslant 4$ are not 4-extendable. Moreover, we know of signotopes which are not 3-extendable for $r = 3$ and $n = 6$, see Figure 9.1, and for $r = 5$ and $n = 12$.

Line arrangements are trivially not extendable by a line through three prescribed points. However, it might be possible that a line arrangement is extendable by a pseudoline. Is it not hard to see, that this is in general not the case. An example for this is the cyclic arrangement, see Figure 9.1. It is not possible to extend the arrangement with a pseudoline containing the three marked crossing points. If there was a pseudoline going through all three points, this new

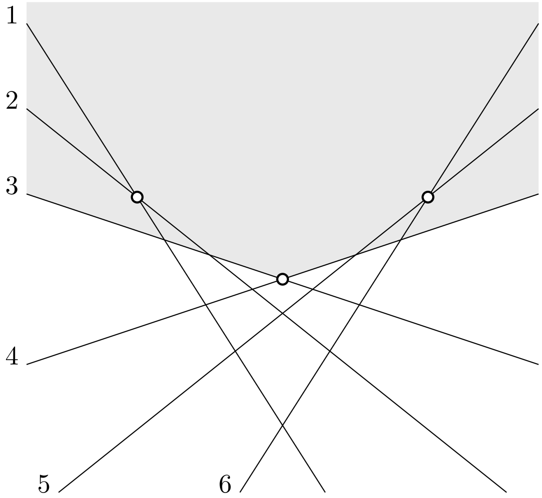

Figure 9.1: The cyclic line arrangement with 6 lines. There is no extension to a pseudoline arrangements with a pseudoline passing through the three marked points. A quadrant of the middle point containing the other two is shaded gray.

pseudoline would visit the three points in some order. One of the crossing points c has to be between the other two on the extending pseudoline. It follows that the other two points have to be on different sides of each pseudoline of the arrangement which contains c. This criterion shows that in the example none of the three points can be the middle one. Hence, the rank 3 signotope on 6 elements where every triple is mapped to $+$ is not 3-extendable.

The above arguments can be phrased using the partial order on $(r-1)$-sets corresponding to an r-signotope. Let $\sigma_{\max}$ be the constant $+$ signotope for given rank r and number of elements n. Generalizing the idea to ranks $r \geqslant 4$ we have the following proposition.

Proposition 9.1. *For $r \geqslant 3$ and $n = 4(r-1)$ the signotope $\sigma_{\max}$ is not 4-extendable.*

With Proposition 4.5 we already have a condition for the extendability by an element at the last position. We extend this characterization by going through all rotations and obtain the following corollary of Proposition 4.5.

Corollary 9.2. *An r-signotope σ on $[n]$ is t-extendable for $t \geqslant 2$ if and only if for all pairwise disjoint $(r-1)$-subsets $I_1, \ldots, I_t$ there exists a rotation in which they are pairwise incomparable.*

The strategy to prove Proposition 9.1 is to prescribe four $(r-1)$-subsets such that in each rotation at least two of them are comparable. For $\sigma_{\max}$, we consider four $(r-1)$-subsets which are disjoint and all elements contained in I_i are smaller than all elements of I_j for $i < j$. A possible choice of I_1, I_2, I_3, I_4 is

$$I_i = ((i-1)\cdot(r-1)+1, \ldots, i\cdot(r-1)).$$

We consider all rotations, and investigate which of the subsets are incomparable.

The next lemma shows that the comparability between two $(r-1)$-subsets can be deduced from the comparability of their images in a suitable restriction.

Lemma 9.3. *Let σ be an r-signotope on $[n]$ and I, J two disjoint $(r-1)$-subsets of $[n]$. Furthermore let $\sigma' = \sigma\downarrow_{[n]\setminus(I\cup J)}$ be the restriction to the elements of $I \cup J$, i.e., an r-signotope on $n' = 2(r-1) \leqslant n$ elements. Furthermore, let $I' = I\downarrow_{[n]\setminus(I\cup J)}$ and $J' = J\downarrow_{[n]\setminus(I\cup J)}$. For the partial orders $\prec$ of σ and $\prec'$ of σ', it holds: If $I' \prec' J'$, then $I \prec J$.*

Proof. If I' and J' are comparable in σ', i.e., $I' \prec' J'$, then there is a chain of $(r-1)$-subsets $I' = I'_0 \prec I'_1 \prec' \ldots \prec' I'_k = J'$ such that $|I'_i \cap I'_{i+1}| = r-2$ and $I_i \subseteq [n]$. For each of the I'_i there is a corresponding set I_i of σ such that $(I_i)\downarrow_{[n]\setminus(I\cup J)} = I'_i$. The relation of I_i and I_{i+1} in $\prec$ only depends on their lexicographic order and $\sigma(I_i \cup I_{i+1})$. The corresponding statement for σ' and the observation that deletion preserves the lexicographic order and the sign imply $I_i \prec I_{i+1}$ for all i. This implies the statement. ☐

This allows us to determine the comparabilities between the prescribed $(r-1)$-subsets I_1, I_2, I_3, I_4 in the order of $\sigma_{\max}$.

Lemma 9.4. *For $i < j$, it holds $I_i \prec I_j$ in the partial order $\prec$ corresponding to $\sigma_{\max}$.*

Proof. For each pair I_i, I_j with $1 \leqslant i < j \leqslant 4$ the restriction to $I_i \cup I_j$ maps the pair to $I_1 = (1, \ldots, r-1)$ and $I_2 = (r, \ldots, 2(r-1))$. Hence, by Lemma 9.3 it is enough to verify that $I_1 \prec I_2$ in the r-signotope $\sigma_{\max}$ on $[2(r-1)]$. Since the sign of all r-subsets $X^i = (i, \ldots, i+r)$ is $+$ we have $X^i \prec X^{i+1}$, i.e., $(i, \ldots, i+r-1) \prec (i+1, \ldots, i+r)$. From this relation for all i from 1 to $r-1$ and the transitivity of $\prec$, the claim follows. □

We are now ready to prove Proposition 9.1.

Proof of Proposition 9.1. For rank $r \geqslant 4$, we consider $I_1^{(k)}$, $I_2^{(k)}$, $I_3^{(k)}$, and $I_4^{(k)}$ and show that in each rotation at least one of $I_3^{(k)}$, $I_4^{(k)}$ and $I_1^{(k)}$, $I_2^{(k)}$ is comparable. Indeed for k between 1 and $2r-2$ the signs of r-subsets of $I_3^{(k)} \cup I_4^{(k)}$ are not affected so that $I_3^{(k)} \prec I_4^{(k)}$. For $k > 2r-2$ all signs of r-subsets of $I_1^{(k)} \cup I_2^{(k)}$ have been reversed r times so that for r even $I_1^{(k)} \prec I_2^{(k)}$ and for r odd $I_1^{(k)} \succ I_2^{(k)}$. □

10. Discussion

We have shown that signotopes of odd rank are 2-extendable. While this is not true for all signotopes with even rank, we proved that there are signotopes for every rank r which are not 4-extendable. While this manuscript was under review, Yan Alves Radtke announced a proof for the non-2-extendability conjecture concerning even rank signotopes. Using the 2-dimensional geometric model of r-signotopes as $(r-2)$-*intersecting pseudoconfigurations of points* introduced by Miyata [Miy17] he proves the non-2-extendability conjecture (Conjecture 3.5) in his Masters Thesis [Alv24]. However, these counterexamples for low rank do not coincide with the counterexamples presented in this paper and use far more elements.

Moreover it remains open whether for general pseudohyperplane arrangements, i.e. oriented matroids, there is a similar extendability result depending on the parity of the rank.

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