Skeletal generalizations of chip-firing games, parking functions, and Dyck paths

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Abstract. For $0 \leqslant k \leqslant n-1$, we introduce a family of k*-skeletal paths* which are counted by the nth Catalan number for each k, and specialize to Dyck paths when $k = n-1$. We similarly introduce k*-skeletal parking functions* which are equinumerous with spanning trees on the complete graph with $n+1$ vertices for each k, and specialize to classical parking functions for $k = n-1$. The preceding constructions are generalized to paths lying in a trapezoid with base $c > 0$ and southeastern diagonal of slope $1/m$; c and m need not be integers. We give bijections among these families when k varies with m and c fixed. Our constructions are motivated by chip firing and have connections to combinatorial representation theory and tropical geometry.

Keywords. Chip firing, skeletal objects, lattice paths, Dyck paths, parking functions, Catalan numbers, ballot numbers

Mathematics Subject Classifications. 05A15, 05A19, 05C57

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1. Introduction

Motivated by developments in chip firing, tropical geometry, and combinatorial representation theory, we introduce new families of combinatorial objects, called k-*skeletal paths* and k-*skeletal functions*, that depend on k and certain other parameters. These objects generalize Dyck paths, parking functions, and lattice paths inside trapezoids and thereby provide new combinatorial interpretations of Catalan numbers, ballot numbers, parking function counts, and q-analogues of these numbers. As an initial special case of these ideas, we describe our generalizations of Dyck paths of order n. Given k, n with $0 \leqslant k < n$, define a k-*skeletal path* to be a path from $(0,0)$ to (n,n) consisting of unit-length east steps and north steps satisfying these two conditions:

(K1) The last $k+1$ north steps start weakly to the left of the line $x = y$.

(K2) There do not exist $k+1$ consecutive rows in which the north steps all start strictly to the left of the line $x = y$.

A primary result of this paper is Theorem 3.2 (cf. Theorem 2.3), which constructs bijections between the set of k-skeletal paths and the set of k'-skeletal paths for all k, k' between 0 and $n-1$. Because the $(n-1)$-skeletal paths are the same as Dyck paths, we see that the number of k-skeletal paths is given by the nth Catalan number. These k-skeletal paths provide combinatorial interpretations of the Catalan numbers that we believe to be new (see Stanley's compilations of interpretations [SF99, Ex. 6.19] and [Sta13]).

Theorem 3.2 and our other results apply in a much more general setting. In lattice path enumeration theory, one may study classical Dyck paths (paths in a triangle with boundary $x = y$) or rational-slope Dyck paths (paths in a triangle with boundary $x = (a/b)y$) or trapezoidal paths (paths in the trapezoid bounded by $y = 0$, $y = n$, $x = 0$, and $x = my + c$). Some of the recent literature in this area includes [ALW16, ARW13, GM14, Loe03]. In particular, there are a number of papers that consider paths lying below lines of arbitrary (i.e., real) slope, especially from the viewpoint of generalizing the q,t-Catalan. For example, in [BHM+23] the authors present a generalization of the Shuffle Theorem, a subject in which the area and $\mathrm{dinv}/\mathrm{bounce}$ statistics are fundamental. Similarly, in [BM23], the authors consider myriad topics ranging from a generalization of Young's lattice to various q- and q,t-enumerations.

Section 3 presents our general notion of k-skeletal paths based on an additive subgroup $\mathcal{G}$ of $\mathbb{R}$ and parameters $c, m \in \mathcal{G}$ and $n \in \mathbb{Z}_{>0}$. Informally, these k-skeletal paths generalize paths in a trapezoid with height n, base c, and diagonal of slope $1/m$, where all east steps in the paths have lengths in $\mathcal{G}$. Classical lattice paths arise as the special case where $\mathcal{G} = \mathbb{Z}$ and m, c are integers. For that special case, Corollary 3.3 states that the number of k-skeletal paths (for each k between 0 and $n-1$) is given by an m-ballot number.

In Section 4, we extend the framework of Section 3 to consider k-skeletal functions, which can be viewed as lattice paths with north steps labeled according to certain rules. Our k-skeletal labeled lattice paths generalize various kinds of parking functions (determined by the parameters $n, m, c, \mathcal{G}$). For fixed choices of these parameters, the number of k-skeletal labeled paths is independent of k and equals the corresponding parking function count. See Theorem 4.4 for the precise statement.

As described in the above two paragraphs, Sections 3 and 4 introduce and explore various k-skeletal objects that generalize lattice paths and parking functions. While these objects can be understood and appreciated solely through the lattice-path framework, our motivation in defining them rests squarely within the framework of chip firing on graphs. As such, we precede these two sections with Section 2, which presents classical chip firing on graphs and a natural generalization to *$\mathcal{G}$-valued chip firing*, where $\mathcal{G}$ is any additive subgroup of $\mathbb{R}$. The k-skeletal chip configurations defined lead directly to their k-skeletal lattice-path counterparts.

The final two sections of this paper contain additional results in two different directions. Section 5 considers q-analogues of k-skeletal objects using statistics with close connections to various dinv statistics [Hag08] appearing in the theory of q, t-Catalan polynomials. These connections raise the possibility that our k-skeletal objects may yield useful new insights on q, t-Catalan polynomials, diagonal harmonics modules, and related constructions in combinatorial representation theory. Section 6 uses the well-known "first-return recursion" for m-Dyck paths to reprove the enumerative result of Corollary 3.3 in the special case $c = 1$.

The remainder of this introduction consists of two independent subsections giving more detailed background from chip firing to motivate our combinatorial results. Section 1.1 provides an introductory account of the general role chip firing plays in a story that, in Sections 3 and 4, is told primarily through the language of lattice paths. Section 1.2 dives deeper into the algebraic and topological developments in chip-firing theory that led us to the definitions of k-skeletal paths and functions given here. The rest of this paper is mostly independent of the next two subsections, which nevertheless provide additional context for understanding the connections between our results and other areas.

1.1. Motivation from classical chip firing

The chip-firing game [Big99, BLS91] (see [CP18, Kli18] for a general introduction, and additional references below) is a dynamical process that can be played on any loopless connected graph such as the complete graph K_{1+n} on the vertex set $\{0, 1, 2, \ldots, n\}$. In the game, we start with a *chip configuration* (or *divisor*) $D = (D(1), \ldots, D(n))^T$, encoded as a column vector and interpreted as the placement of $D(i) \geqslant 0$ chips on vertex i for each i between 1 and n. A vertex $v \geqslant 1$ *fires* by moving one chip from v to each of its n neighbors. Vertex 0 is distinguished as the *sink*. We typically focus on the *legal firings* — those for which no non-sink vertex ends up with a negative number of chips after the firing action. Configurations without any legal firings are *stable*. Several legal firings are illustrated in Figure 1.1.

Because every firing move on a complete graph sends one chip to the sink, any sequence of legal firings must eventually terminate. The terminal chip configurations resulting from initial configurations in which every vertex can fire are called *critical configurations* (also called *recurrent configurations*). Critical configurations play an important role in chip-firing theory, as outlined in Section 1.2. For $n = 3$, there are $(3 + 1)^{3-1} = 16$ different critical configurations. While we could list all sixteen as vertex-weighted graphs, it is convenient to introduce a method of associating an unlabeled lattice path with any chip configuration. Associate with the configuration D the path $\pi(D)$ that has j unit-length north steps in column i when j non-sink vertices have exactly i chips on them. These north steps are arranged vertically so that we obtain a path

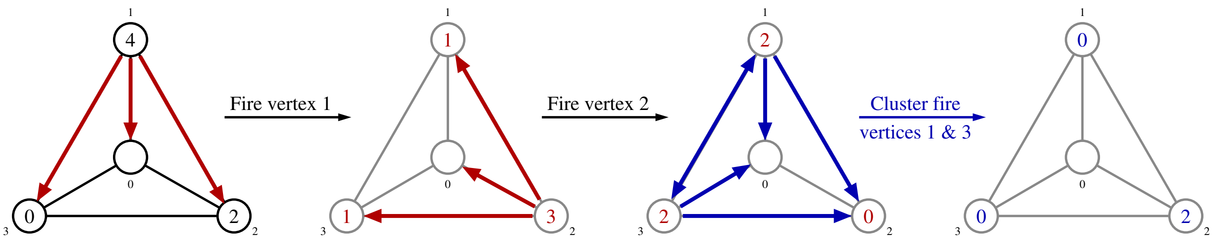

Figure 1.1: The leftmost graph illustrates K_{1+3} with a chip distribution of $D(1) = 4$, $D(2) = 2$ and $D(3) = 0$. Vertices 1 and 2 are fired in sequence. The resulting chip configuration is stable, but vertices 1 and 3 can be fired as a cluster to obtain the fourth distribution, which is superstable.

from $(0, 0)$ to (n, n) by connecting the runs of north steps in each column by unit-length east steps as necessary. The sixteen critical configurations for K_{1+3} give rise to the five unlabeled lattice paths shown in Figure 1.2(a). This mapping from arbitrary chip configurations to unlabeled paths is many-to-one, but we can modify it by adding labels to get a bijection between chip configurations and labeled lattice paths. To obtain a labeled lattice path, we label the north steps along each line $x = i$ with the elements of $D^{-1}(\{i\})$, sorted into increasing order from bottom to top. Figure 1.2(b) illustrates a collection of three configurations sharing the same underlying unlabeled path.

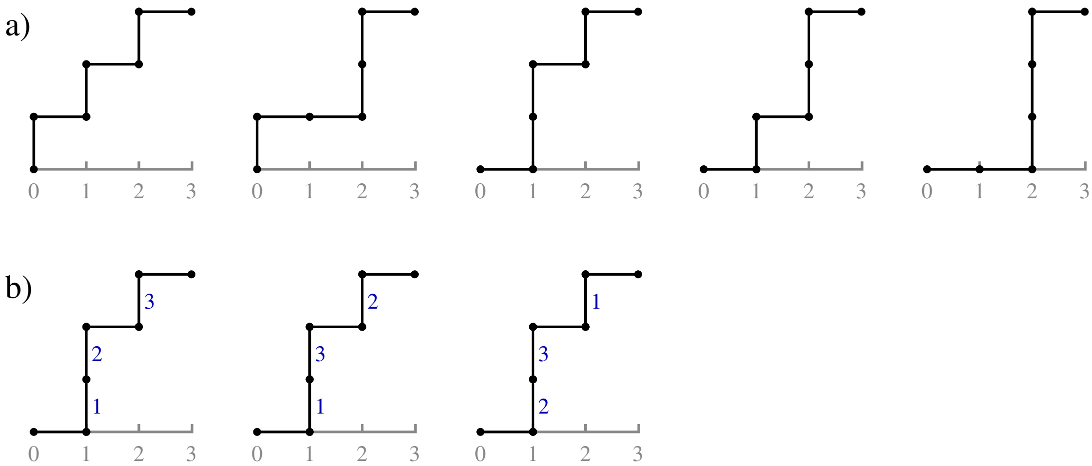

Figure 1.2: (a) The five unlabeled lattice paths representing the sixteen critical configurations for $n = 3$. In (b) we show the three labeled lattice paths sharing the same unlabeled lattice path corresponding to all configurations in which one non-sink vertex has two chips and the other two have one chip each.

What we have described so far — with only one vertex firing at a time — is the *abelian sandpile model* [BTW87, Dha90]. More generally, we can *cluster fire* a subset $S \subseteq \{1, 2, \ldots, n\}$ by firing all of the vertices in S simultaneously. In the *unconstrained firing model*, any nonempty subset of vertices is allowed to cluster fire; the cluster firing is *legal* if no vertex ends up with a negative number of chips. A *superstable* configuration is one in which no nonempty subset of the non-sink vertices can legally cluster fire. The third configuration in Figure 1.1 is stable but not superstable, as the two vertices 1 and 3 can legally be fired simultaneously. For $n = 3$, there are sixteen superstable configurations, which are represented by the unlabeled paths shown in Figure 1.3.

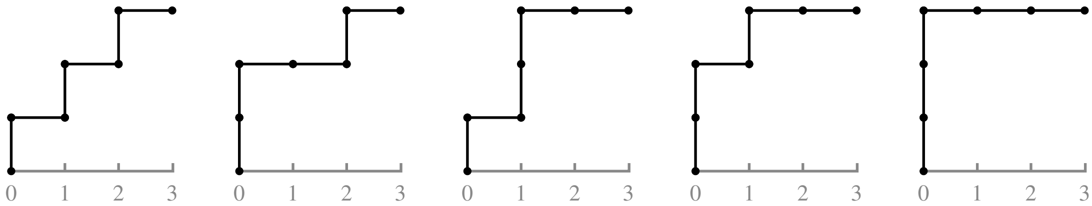

Figure 1.3: The five unlabeled paths corresponding to the sixteen superstable configurations for $n = 3$.

There are two aspects of the superstable configurations we wish to highlight. First, there is a duality between the critical and superstable configurations, as suggested by the fact that there are sixteen configurations of each type for $n = 3$. More precisely, let $D_{\max}$ denote the configuration in which each non-sink vertex of K_{1+n} has $n-1$ chips. This is the "maximal" chip configuration that is stable. It turns out that D is critical if and only if $D_{\max} - D$ is superstable. This result appears in the work of Gabrielov [Gab93] and has been further generalized [BKO+24, GK15]. We refer the reader to [Kli18, Thm. 2.6.19] for a proof. See also [AdDL16] for an algorithm for moving between these configurations in a fixed chip-firing class. The duality in the particular case $n = 3$ can be seen by examining Figures 1.2(a) and 1.3. Also see Remark 3.4 in Section 3.

The second aspect worth highlighting is that superstable configurations are parking functions. Under our labeling conventions, a function $f : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, n-1\}$ is a *parking function* if for all i between 0 and $n-1$, the number of j with $f(j) \geqslant i$ is at most $n - i$ (see [Hag08, Chap. 5] and Section 4). If we associate a labeled path with each parking function, as we did above for chip configurations, one can check that the conditions on the values of a parking function translate into the underlying unlabeled path being a Dyck path. In fact, this is just a rephrasing of the conditions for a configuration to be superstable: no subset S of i non-sink vertices, where $1 \leqslant i \leqslant n$, has at least $n - i + 1$ chips on each vertex. Parking functions play an important role in combinatorics and representation theory. (For instance, the Frobenius series for the diagonal coinvariants can be expressed as a weighted sum indexed by parking functions [ECAM18, HHL+05].) As such, any generalization of superstable configurations has the potential to illuminate questions in combinatorial representation theory.

Looking back at Conditions (K1) and (K2) defining the k-skeletal paths, we may interpret the two extreme cases using chip firing. On one hand, the 0-skeletal paths are the paths corresponding to critical configurations, arising naturally from the abelian sandpile model. On the other hand, the $(n-1)$-skeletal paths are the Dyck paths corresponding to the superstable configurations, arising naturally from the unconstrained firing model. Figure 3.3 illustrates all k-skeletal paths when $n = 3$ and k is 0, 1, or 2. Figure 3.2 illustrates the duality between 0-skeletal paths and $(n-1)$-skeletal paths for a more general choice of parameters.

In Section 2.2 below, we introduce Conditions (C0)–(C2) to define k-skeletal chip configurations. These configurations arise in the context of chip firing on a complete graph K_{1+n} in which edges between non-sink edges are weighted by m, edges touching the sink are weighted by c, and the chip count at each vertex belongs to an additive subgroup $\mathcal{G}$ of $\mathbb{R}$ rather than $\mathbb{Z}$. In Section 3.1 we see how these conditions provide a way to define k-skeletal paths associated with general trapezoidal or triangular regions, as specified by the parameters n, m, and c.

When $\mathcal{G} = \mathbb{Z}$ and $c, m \in \mathbb{Z}$, we obtain finite collections of k-skeletal paths and functions that are counted (respectively) by m-ballot numbers and generalized parking function counts. However, when $\mathcal{G}$ is any non-cyclic subgroup of $\mathbb{R}$, we obtain infinite collections of k-skeletal objects. For coprime positive integers a, b, the *rational Catalan number* $\frac{1}{a+b}\binom{a+b}{a,b}$ counts rational-slope Dyck paths contained in a triangle bounded by $x = (a/b)y$. It is natural to ask if there is some alternate version of our k-skeletal constructions that leads to new collections of paths that are counted (for all k) by the rational Catalan number. We address this problem in a forthcoming paper [BLW], which requires yet another novel variation of the classical chip-firing model.

1.2. Motivation from algebraic and geometric aspects of chip firing

The simple definition of chip firing on graphs belies the richness of this theory and its myriad connections with other areas of mathematics. These areas include statistical physics [BTW87, Dha90]; arithmetic geometry [Lor89, Lor91, Ray70]; poset theory [Mos72, Pre86, Pro02]; lattice theory [AM10, BLHN97, Big99]; tropical geometry [BN07, MZ08]; and commutative algebra [CRS02, PPW13, PS04]. Additionally, some authors have previously sought to find connections between chip-firing and topics close to combinatorial representation theory — see [CLB16, CPS16, DDI+24, DL13]. The present article can be seen as continuing in this vein of research as we focus on leveraging connections to the combinatorics of trapezoidal lattice paths, both unlabeled and labeled.

The chip-firing process is not limited to complete graphs. We may start with any connected, undirected graph $G = (V, E)$ with vertex set $V = \{v_1, \ldots, v_N\}$ and edge set $E = \{e_1, \ldots, e_M\}$. We assume G has no loops, but G may have multiple edges between two vertices. We choose a distinguished *sink* vertex $q \in V$ and set $V' = V \setminus \{q\}$. As before, a *chip configuration* D is an assignment of an integer number of chips to each vertex of G. The *degree* of a chip configuration D is $\sum_{v \in V} D(v)$. Given a chip configuration D of known degree d, we have $D(q) = d - \sum_{v \in V'} D(v)$. So when focusing on chip configurations of a fixed degree, we may safely ignore the chip count $D(q)$ at the sink.

The action of firing a vertex or a set of vertices can be expressed in terms of the *Laplacian of* G, the $N \times N$ matrix $\mathbf{L}$ with entries

$$\mathbf{L}_{i,j} = \begin{cases} \text{degree of } v_i, & \text{if } i = j; \\ -(\text{number of edges linking } v_i \text{ and } v_j), & \text{if } i \neq j. \end{cases}$$

For $S \subseteq V'$, let $\mathbf{e}_S$ be the column vector with 1s in those positions i for which $v_i \in S$ and 0s elsewhere. The configuration $D' = D - \mathbf{L}\mathbf{e}_S$ is defined as the configuration obtained from D by *cluster firing* the set S.

In the study of chip firing, we are often interested in certain distinguished chip configurations. As sketched in Section 1.1 for the sandpile model on $G = K_{1+n}$, these distinguished configurations include the *critical* (also called *recurrent*) configurations and the *superstable* configurations. In terms of the Laplacian, the superstable chip configurations D are those satisfying:

1. $D(v) \geqslant 0$ for all $v \in V'$; and
2. for every $\varnothing \neq S \subseteq V'$, there exists $v \in V'$ such that $(D - \mathbf{L}\mathbf{e}_S)(v) < 0$.

The first condition means that no non-sink vertex has a negative number of chips. The second condition means that there do not exist any non-trivial legal cluster firings. These superstable configurations are equivalent to the *G-parking functions* of Postnikov and Shapiro [PS04] and the *q-reduced divisors* of Baker and Norine [BN07]. The duality with critical configurations is closely related to Riemann–Roch duality for graphs [AB10, BN07, CL13] and Alexander duality for monomial ideals [MS13].

Parking functions of order n are in bijection with trees on $n + 1$ (labeled) vertices [Kre80]. It is a classical result that the number of parking functions of order n is $(n + 1)^{n-1}$ (due to Cayley [Cay89] in the context of labeled trees; see also [KW66, Rio69]).

Proposition 1.1 (see [AB10, GH16, KY08, PS04])**.** *If G is the complete graph on $n+1$ vertices, then the set of G-parking functions (and hence the set of superstable configurations and the set of q-reduced divisors of degree 0) can be naturally identified with the set of parking functions of order n.*

This connection holds more generally. The number of G-parking functions is counted by the number of spanning trees of G [PS04]. To explain this connection, it is helpful to describe how the critical configurations arise algebraically.

We say that chip configurations D and D^* are *chip-firing equivalent*, written $D \sim D^*$, if we can obtain D^* from D by a finite sequence of chip-firing moves. We have $D \sim D^*$ if and only if there is an integer vector $\mathbf{w} \in \mathbb{Z}^N$ with $D - D^* = \mathbf{L}\mathbf{w}$. Here we are not concerned with whether vertices have negative numbers of chips, so we do not need to distinguish between the cluster fire of a set S and the sequential firing of all vertices in S; we can stay within the abelian sandpile model.

The set of chip configurations on G is an additive group isomorphic to $\mathbb{Z}^N$ if we add chip configurations pointwise. The *chip-firing equivalence class* of D is $[D] = \{D^* : D^* \sim D\}$. Equivalent configurations have the same degree, so it makes sense to define $\deg([D]) = \deg(D)$. The collection of chip-firing equivalence classes forms an additive group $\mathrm{Pic}(G) = \mathbb{Z}^N/\mathbf{L}(\mathbb{Z}^N)$ called the *Picard group*. Let $\mathrm{Pic}^d(G)$ be the subset of $\mathrm{Pic}(G)$ consisting of classes $[D]$ where $\deg(D) = d$. The set $\mathrm{Pic}^0(G)$ is a subgroup of $\mathrm{Pic}(G)$, called the *critical group* (this group goes by different names depending on the context in which it is introduced: the *chip-firing group*, the *Jacobian*, and the *sandpile group*).

For a connected graph G, there is a natural isomorphism $\mathrm{Pic}(G) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(G)$. So $\mathrm{Pic}^0(G)$ can be identified with the torsion part of the $\mathbb{Z}$-cokernel of $\mathbf{L}$. The critical configurations central to this paper can be taken to be representatives of elements of $\mathrm{Pic}^0(G)$. We can take the number of chips on the sink to be the negative of the total number on the non-sink vertices, leading to a total degree of zero. As G is connected, it can be shown by elementary group theory that for any i, $\left|\mathrm{Pic}^0(G)\right| = |\det(\mathbf{L}_i)|$, where $\mathbf{L}_i$ is the matrix obtained by deleting ith row and column of $\mathbf{L}$. On the other hand, Kirchhoff's Matrix-Tree Theorem [Loe17, §3.18] states that $|\det(\mathbf{L}_i)|$ is the number of spanning trees of G, hence $\left|\mathrm{Pic}^0(G)\right|$ is equal to the number of spanning trees of a graph (see [BS13]). Several explicit bijections between $\mathrm{Pic}^0(G)$ and the set of spanning trees of G are known (see [ABKS14, Bac12, BCT10, CL03, Dha90, KY08, PYY17, Yue17]).

We have outlined some of the close connections among critical configurations, G-parking functions, $\mathrm{Pic}^0(G)$ and spanning trees of G. Statistical physicists Caracciolo, Paoletti, and Sportiello [CPS12] and (independently) Backman [Bac12] discovered a generalization of G-parking functions with respect to an abstract simplicial complex.[1] Recall that a simplicial complex Δ on V' is a collection of nonempty subsets of V' such that $\{v\} \in \Delta$ for all $v \in V'$, and whenever $\varnothing \neq T \subseteq S \in \Delta$, T also belongs to Δ. We allow the vertices of any subset in Δ to fire simultaneously. Regarding the objects introduced in [Bac12], a chip configuration $D : V \to \mathbb{Z}$ will be termed Δ-*critical* if these three conditions hold:

1. For each $v \in V'$, $D(v) \geqslant 0$.
2. For each $S \in \Delta$, there exists $v \in S$ with $(D - \mathbf{L}\mathbf{e}_S)(v) < 0$.
3. For each nonempty $S \subseteq V'$, the configuration $D+\mathbf{L}\mathbf{e}_S$ does not satisfy both Conditions (1) and (2).

In the case where the complex Δ consists of all one-element subsets of V', we recover the sandpile model; the Δ-critical configurations are the critical (recurrent) configurations. At the other extreme, when Δ is the full complex consisting of all nonempty subsets of V', we recover the unconstrained chip-firing model; the Δ-critical configurations are the superstable configurations.

One motivation to consider such simplicial complexes in the context of chip firing is a desire to develop a notion of divisor theory for tropical curves with respect to an open cover as well as a discrete version for graphs. For a fixed graph G, sink q, and simplicial complex Δ, the Δ-critical configurations interpolate between the recurrent configurations and the superstable configurations. All such interpolations are equinumerous with the set of spanning trees in the graph.

Theorem 1.2 ([Bac12, Theorem 1,Lemma 2],[CPS12, Section 3])**.** *For each fixed sink vertex q and simplicial complex Δ on V', every chip configuration D is equivalent to a unique Δ-critical configuration.*

Corollary 1.3. *The number of Δ-critical configurations is the number of spanning trees of G.*

An explicit bijection between Δ-critical configurations and spanning trees was provided in [Bac12].

The motivating idea for the present work was to specialize this construction to $G = K_{1+n}$ and take Δ to be the k-skeleton of the full complex (so Δ consists of all subsets of $\{1, 2, \ldots, n\}$ of

[1] The first author thanks Lionel Levine for first observing the connection between these works.

size between 1 and $k+1$). (See closely related work in [AdDL16, DL13].) The advantage of this setup is that Δ is invariant under S_n, so we are able to investigate not only the corresponding generalization of classical parking functions, but also the analogues of Dyck paths (the S_n-orbits of the parking functions), which connect to combinatorial representation theory and adjacent fields. These objects admit an intrinsic description without any reference to chip firing. As described in Theorems 3.2 and 4.4, our construction extends to generalizations of rational parking functions and trapezoidal lattice paths that go beyond the framework previously studied in [Bac12]. However, as noted at the end of Section 1.1, some choices of $\mathcal{G}$ lead to collections of paths that are infinite, illustrating that there are limits to the extent to which this framework can be viewed as generalizing rational parking functions. See [BLW]. We are hopeful that these objects will find applications in the study of q,t-Catalan combinatorics, where parking functions and Dyck paths have previously been utilized. Section 5 explains how the q,t-Catalan dinv statistics interact nicely with our k-skeletal constructions, allowing us to define k-skeletal versions of the $t=1$ specialization of the q,t-Catalan that are independent of k.

2. $\mathcal{G}$-Valued chip firing and skeletal chip configurations

This section introduces a generalization of chip firing based on an additive subgroup $\mathcal{G}$ of $\mathbb{R}$ (cf. [LP09]). This framework motivates our study of skeletal paths and skeletal functions in subsequent sections. After introducing the framework, defining skeletal chip configurations, and proving some basic properties, we discuss how the theory from Section 1.2 can be adapted to the setting of $\mathcal{G}$-valued chip configurations. While we focus on chip firing on the complete graph, Section 2.6 outlines how to extend the ideas of this section to more general graphs.

2.1. Chip-firing framework

We assume this setup: $\mathcal{G}$ is a fixed additive subgroup of $\mathbb{R}$, n is a positive integer, and $c,m \in \mathcal{G}$ are parameters with $c>0$ and $m \geqslant 0$. We study the following chip-firing model built from these parameters. Let K_{n+1} be the complete graph with vertex set $\{0,1,2,\dots,n\}$, where 0 is a special vertex called the *sink*. For each $i \neq j$ between 1 and n, the edge from i to j has *capacity* m. For each i between 1 and n, the edge from i to 0 has *capacity* c. A *chip configuration* on K_{n+1} with values in $\mathcal{G}$ is a function $D:[n]\to\mathcal{G}$, where $[n]=\{1,2,\dots,n\}$. We think of $D(i)$ as the chip count at vertex i, which might be negative or non-integral (depending on $\mathcal{G}$). Our chip configurations do not record the chip count at the sink vertex 0. The chip configuration D is *nonnegative* (written $D \geqslant 0$) iff $D(i) \geqslant 0$ for all $i \in [n]$.

We introduce an operation ϕ_i on chip configurations called *firing at vertex* i. By definition, $\phi_i(D)$ is the configuration obtained from D by decreasing $D(i)$ by $m(n-1)+c$ and increasing $D(j)$ by m for all $j \neq i$ in $[n]$. Intuitively, when vertex i fires, it sends m chips along each of the $n-1$ edges to other $j \in [n]$, and it sends c chips to the sink. We say a vertex $i \in [n]$ can *legally fire* iff $D(i) \geqslant m(n-1)+c$. If $D \geqslant 0$ and vertex i can legally fire, then $\phi_i(D) \geqslant 0$.

More generally, suppose S is a nonempty subset of $[n]$ of size $|S|$. The operation ϕ_S (*firing at vertex set* S) acts on any D by decreasing $D(i)$ by $m(n-|S|)+c$ for each $i \in S$ and

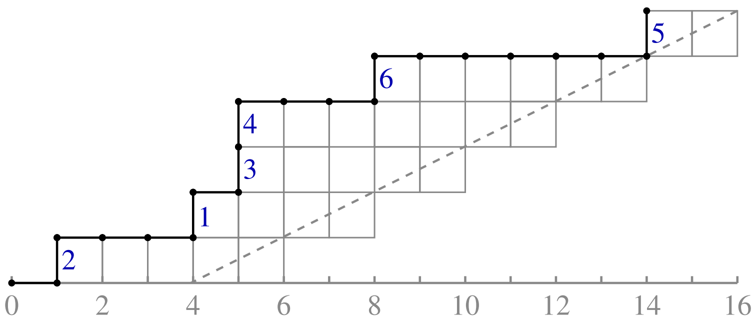

Figure 2.1: The labeled lattice path for the configuration D in Example 2.1.

increasing $D(j)$ by $m|S|$ for all $j \in [n] \setminus S$. We say S can *legally fire* in configuration D iff $D(i) \geqslant m(n-|S|)+c$ for all $i \in S$. If $D \geqslant 0$ and subset S can legally fire, then $\phi_S(D) \geqslant 0$.

Next, let T be a nonempty subset of $[n]$ of size $|T|$. The operation β_T (*borrowing at vertex set* T) acts on any D by increasing $D(i)$ by $m(n-|T|)+c$ for each $i \in T$ and decreasing $D(j)$ by $m|T|$ for all $j \in [n] \setminus T$. Evidently, β_T is the two-sided inverse of ϕ_T when acting on all (not necessarily nonnegative) chip configurations. We say T can *legally borrow* in configuration D iff $D(j) \geqslant m|T|$ for all $j \in [n] \setminus T$. If $D \geqslant 0$ and subset T can legally borrow, then $\beta_T(D) \geqslant 0$.

Consider any chip configuration $D : [n] \to \mathcal{G}$. Given $v, z \in [n]$, we say that v is *poorer* than z (and z is *richer* than v) to mean: $D(v) < D(z)$, or $D(v) = D(z)$ and $v < z$. We can visualize D as a labeled path in the plane, as follows. Place the vertices in rows 1 through n (where row i is between the lines $y = i-1$ and $y = i$) in order from poorest to richest. Let each vertex $v \in [n]$ with chip count $D(v)$ be the label attached to a north step on the line $x = D(v)$. By our definition, if several vertices have equal chip count, they appear in the same column with labels increasing from bottom to top. Finally, connect the n north steps with east steps to get a path proceeding northeast. If $D \geqslant 0$, then this path may start at the origin. For any chip configuration D, let $\mathrm{LP}(D)$ be the associated labeled path. Let $x_i(D)$ be the x-coordinate of the north step of $\mathrm{LP}(D)$ starting on the line $y = i$. Equivalently, $x_i(D)$ is the chip count at the $(i+1)$th poorest vertex of D.

Example 2.1. Let $\mathcal{G} = \mathbb{Z}$, $n = 6$, $m = 2$, $c = 4$, and $D = (D(1), \ldots, D(6)) = (4, 1, 5, 5, 14, 8)$. We may identify $\pi(D)$ with the lattice path ENEEENENNEEENEEEEEN, where the north steps are labeled $2, 1, 3, 4, 6, 5$ from bottom to top. See Figure 2.1. Note that we also have $(x_0(D), x_1(D), \ldots, x_5(D)) = (1, 4, 5, 5, 8, 14)$. In D, vertex 5 can fire since it has 14 chips. Firing vertex 5 would change D to $\phi_5(D) = (6, 3, 7, 7, 0, 10)$. In D, no 2-element subset can legally fire since both vertices in the subset would need at least 12 chips. Similarly, for all $f \geqslant 2$, no f-element subset can legally fire in D.

2.2. Skeletal chip configurations

For $k \in \{0, 1, \ldots, n-1\}$, a chip configuration D is called k-*skeletal* iff these conditions hold:

(C0) $D \geqslant 0$.

(C1) For all $S \subseteq [n]$, if $0 < |S| \leqslant k+1$, then S cannot legally fire in configuration D.

(C2) For all nonempty $T \subseteq [n]$, if T can legally borrow in configuration D, then there exists $S \subseteq [n]$ such that $0 < |S| \leqslant k+1$ and S can legally fire in configuration $\beta_T(D)$.

Condition (C2) says that (C1) fails for every configuration reachable from D by a legal borrow move.[2] Let SKC_k be the set of k-skeletal chip configurations.

Remark 2.2. To motivate these conditions, consider the classical cases where $\mathcal{G} = \mathbb{Z}$, $c = m = 1$, and $k \in \{0, n-1\}$. In the case $k = n-1$, condition (C1) says that D is superstable (no nonempty subset can legally fire) and condition (C2) automatically holds (take $S = T$). In the case $k = 0$, condition (C1) says that D is stable (no single vertex can legally fire) while condition (C2) amounts to Dhar's Criterion for configuration D to be recurrent [Dha90].

Our enumerative result for k-skeletal chip configurations is the following.

Theorem 2.3. *For fixed $\mathcal{G}, n, c, m$, $|\,\mathrm{SKC}_k\,|$ is independent of k (possibly infinite). There are canonical bijections between the various sets SKC_k. In the case $\mathcal{G} = \mathbb{Z}$, $|\,\mathrm{SKC}_k\,| = c(mn+c)^{n-1}$ for $k = 0, 1, \ldots, n-1$.*

We prove this theorem in Section 4.5 by reducing it to an analogous theorem for k-skeletal functions, which are defined later. To prepare for that proof, and to motivate the definition of k-skeletal paths, here we derive conditions equivalent to (C1) and (C2) that are simpler to check from the labeled lattice path representing D. This discussion culminates in Theorem 2.10 below.

2.3. Analysis of condition (C1)

Lemma 2.4. *Let $D : [n] \to \mathcal{G}$ be a nonnegative chip configuration. D satisfies condition (C1) iff the labeled path $\mathrm{LP}(D)$ satisfies this condition:*

(C1′) The last $k+1$ north steps of $\mathrm{LP}(D)$ all start strictly left of the line $x = my + c$.

Proof. Given $D \geqslant 0$ and a fixed integer f with $0 < f \leqslant n$, the following statements are equivalent:

(a) There exists an f-element subset S of $[n]$ that can legally fire in D.

(b) The set of f richest vertices in $[n]$ can legally fire in D.

[2] Inspired by conversations with the first author and Sam Hopkins, Dochtermann [Doc17] investigated functions which satisfy (C0) and (C1), but not (C2). When one drops condition (C2) the resulting objects are determined by monomial ideals and thus amenable to techniques from commutative algebra. This line of inquiry was further pursued by Dochtermann and King [DK21] as well as by Kumar, Lathar, and Roy [KLS21, KLR22, Roy20].

(c) The fth richest vertex in D has at least $m(n-f)+c$ chips.

(d) $x_{n-f}(D) \geqslant m(n-f)+c$.

(e) The point $(x_{n-f}(D), n-f)$ lies weakly right of the line $x = my + c$.

Negating each statement, we deduce: no f-element subset S of $[n]$ can legally fire in configuration D iff the point $(x_{n-f}(D), n-f)$ lies strictly left of the line $x = my + c$. Applying this to $f = 1, 2, \ldots, k+1$, we see that Conditions (C1) and (C1′) are equivalent. □

The following notation will be convenient. Given $D\colon [n] \to \mathcal{G}$, define $g_i(D) = mi + c - x_i(D)$ for $0 \leqslant i < n$. We call $G(D) = (g_0(D), \ldots, g_{n-1}(D))$ the *area vector* of D. The configuration in Example 2.1 has area vector $(3, 2, 3, 5, 4, 0)$. With this notation, Condition (C1′) for $\mathrm{LP}(D)$ is equivalent to this condition:

(C1″) The last $k+1$ entries in $G(D)$ are strictly positive.

2.4. Analysis of condition (C2)

We next develop a simplified version of Condition (C2). First we need a lemma characterizing when there exists a p-element subset T of $[n]$ such that T can legally borrow in D.

Lemma 2.5. *Let $D : [n] \to \mathcal{G}$ be nonnegative. For all $p \in \{1, 2, \ldots, n\}$,*

$$\text{(some } p\text{-element subset } T \text{ of } [n] \text{ can legally borrow in } D\text{) iff } (p = n \text{ or } g_p(D) \leqslant c). \tag{2.1}$$

In (2.1) and below, we only evaluate the second clause of the OR-statement when $p < n$.

Proof. In the case $p = n$, the set $T = [n]$ can always legally borrow. In the case $1 \leqslant p < n$, the following statements are equivalent:

(a) There exists $T \subseteq [n]$ with $|T| = p$ such that T can legally borrow in D.

(b) There exist $n - p$ vertices in $[n]$ that each have at least mp chips in D.

(c) The richest $n - p$ vertices in D each have at least mp chips.

(d) The $(p+1)$th poorest vertex in D has at least mp chips.

(e) $x_p(D) \geqslant mp$.

(f) $mp + c - g_p(D) \geqslant mp$.

(g) $g_p(D) \leqslant c$. □

Condition (e) says that the point $(x_p(D), p)$ on $\mathrm{LP}(D)$ lies weakly right of the line $x = my$.

Example 2.6. For the configuration D in Example 2.1, we can find subsets of size 1, 2, 4, 5, and 6 that can legally borrow. But no such subset exists of size $p = 3$, since $g_3(D) = 5 > 4 = c$. Indeed, vertices 1, 2, 3, and 4 each have fewer than 6 chips, so T would need to include all of these vertices for β_T to be legal. But then T has size larger than 3.

The next result gives a simplification of Condition (C2) reminiscent of Dhar's Criterion in the classical case.

Lemma 2.7. *For any chip configuration $D \geqslant 0$, Condition (C2) is equivalent to this condition:*

(C2′) For all $p \in \{1, 2, \ldots, n\}$, if ($p = n$ or $g_p(D) \leqslant c$) and $g_{p-1}(D) > c$ and T is the set of p poorest vertices of D, then configuration $\beta_T(D)$ fails Condition (C1).

Proof. Fix $p \in \{1, 2, \ldots, n\}$. The idea of the proof is to reduce the number of p-element subsets T that we must consider when checking Condition (C2). This condition involves the following IF-statement: "If β_T is legal for D, then there exists $S \subseteq [n]$ with $0 < |S| \leqslant k+1$ where S can legally fire in $\beta_T(D)$." This IF-statement is automatically true in certain cases, listed next.

(i) In the case $p < n$ and $g_p(D) > c$: β_T is not legal (by (2.1)), so the IF-statement is true.

(ii) In the case where β_T is legal and some $v \in T$ has $D(v) \geqslant m(p-1)$: borrowing at T increases the chip count at v to be at least $m(n-1)+c$. So the IF-statement can be fulfilled by taking $S = \{v\}$.

(iii) In the case where β_T is legal and $g_{p-1}(D) \leqslant c$: the pth poorest vertex in D has chip count $x_{p-1}(D) = m(p-1) + c - g_{p-1}(D) \geqslant m(p-1)$. As T has size p, some vertex in T must have at least $m(p-1)$ chips. Thus, the preceding case (ii) applies, and the IF-statement in (C2) is true.

(iv) In the case where β_T is legal and $g_{p-1}(D) > c$ and T is not the set of p poorest vertices in D: T must contain some v outside this set, so $p < n$, $g_p(D) \leqslant c$ (by (2.1)), and $D(v) \geqslant x_p(D) = mp + c - g_p(D) \geqslant mp \geqslant m(p-1)$. So Case (ii) applies, and the IF-statement in (C2) is true.

Removing all these cases, and letting p vary, we conclude that Conditions (C2) and (C2′) are equivalent. □

2.5. Effect of β_T on area vectors

In the situation where the hypotheses of Condition (C2′) hold, the borrow move β_T has a simple effect on the area vector of D.

Lemma 2.8. *Suppose D satisfies Conditions (C0) and (C1), and p and T satisfy the hypothesis of (C2′). If D has area vector $(g_0, g_1, \ldots, g_{n-1})$, then $\beta_T(D)$ has area vector $(g_p, \ldots, g_{n-1}, g_0 - c, \ldots, g_{p-1} - c)$.*

Proof. We are assuming T is the set of p poorest vertices in D. Let $T' = [n] \setminus T$ be the set of $n-p$ richest vertices in D. For all $v \in T$, $D(v) \geqslant 0$. Borrowing at T shifts all chip counts in T up by $m(n-p)+c$, so $\beta_T(D)(v) \geqslant m(n-p)+c$ for all $v \in T$. On the other hand, for all $z \in T'$, $D(z) \leqslant x_{n-1}(D) < m(n-1)+c$ because D satisfies Condition (C1′). Borrowing at T shifts all chip counts in T' down by mp, so $\beta_T(D)(z) < m(n-p-1)+c$ for all $z \in T'$.

It follows that the list of vertices in $\beta_T(D)$, from poorest to richest, consists of T' (ordered as in D) followed by T (ordered as in D).

For brevity, write g_i for $g_i(D)$, x_i for $x_i(D)$, g_i^* for $g_i(\beta_T(D))$, and x_i^* for $x_i(\beta_T(D))$. We must prove $(g_0^*, \dots, g_{n-1}^*) = (g_p, \dots, g_{n-1}, g_0 - c, \dots, g_{p-1} - c)$. For i in the range $0 \leqslant i < n-p$, $x_i^* = x_{p+i} - mp$, so

$$g_i^* = mi + c - x_i^* = m(p+i) + c - x_{p+i} = g_{p+i}.$$

For i in the range $n - p \leqslant i < n$, $x_i^* = x_{i-(n-p)} + m(n-p) + c$, so

$$g_i^* = mi + c - x_i^* = m(i - (n-p)) - x_{i-(n-p)} = g_{i-(n-p)} - c.$$

□

Example 2.9. In Example 2.1, we can take $p = 4$, $T = \{2, 1, 3, 4\}$, and apply β_T to D to get $\beta_T(D) = (12, 9, 13, 13, 6, 0)$. This converts the area vector $(3, 2, 3, 5, 4, 0)$ for D to the area vector $(4, 0, -1, -2, -1, 1)$ for $\beta_T(D)$ by cyclically shifting $p = 4$ steps left, then subtracting $c = 4$ from the last 4 entries.

The next theorem summarizes our results so far.

Theorem 2.10. *Suppose $0 \leqslant k \leqslant n-1$ and $D : [n] \to \mathcal{G}$ is a nonnegative chip configuration with area vector $G(D) = (g_0, g_1, \dots, g_{n-1})$. The configuration D is k-skeletal if and only if:*

- *the last $k+1$ entries of $G(D)$ are > 0; and*
- *for every $p \in [n]$ satisfying ($p = n$ or $g_p \leqslant c$) and $g_{p-1} > c$, one of the last $k+1$ entries in $(g_p, \dots, g_{n-1}, g_0 - c, \dots, g_{p-1} - c)$ is $\leqslant 0$.*

This reduces the study of k-skeletal chip configurations to the study of area vectors (or equivalently, unlabeled paths), which we pursue in Section 3. This ultimately leads to a bijective proof of Theorem 2.3 and its analogues for unlabeled and labeled paths.

2.6. General $\mathcal{G}$-valued chip firing

So far, we have studied chip configurations with values in $\mathcal{G}$ only for certain weighted complete graphs with a sink vertex. Here we briefly indicate how to extend the general setup of Section 1.2 to the setting of $\mathcal{G}$-valued chip configurations.

Let $\mathcal{G}$ be an additive subgroup of $\mathbb{R}$. A *$\mathcal{G}$-weighted graph* is a simple graph $G = (V, E)$ together with a *weight function* $\mathrm{wt} : E \to \mathcal{G} \cap \mathbb{R}_{>0}$ that assigns a positive weight in $\mathcal{G}$ to each edge of G. Let $(v_1, v_2, \dots, v_N)$ be a fixed total ordering of V. The *Laplacian* of the $\mathcal{G}$-weighted graph G is the $N \times N$ matrix $\mathbf{L}$ with entries

$$\mathbf{L}_{i,j} = \begin{cases} \sum_{e:\ e \text{ touches } v_i} \mathrm{wt}(e) & \text{if } i = j; \\ -\mathrm{wt}(e) & \text{if } i \neq j \text{ and } e = \{v_i, v_j\} \in E. \end{cases}$$

The restriction to simple graphs is no real loss of generality. To model graphs with multiple edges between the same two vertices, we combine all those edges into a single edge whose weight is the sum of the original edge weights. This modification produces a simple graph with the same Laplacian as the original graph.

A $\mathcal{G}$-*valued chip configuration* is a function $D : V \to \mathcal{G}$. Chip configurations D and D^* are *chip-firing equivalent* if $D - D^* \in \mathbf{L}(\mathbb{Z}^N)$. The *chip-firing group* of the edge-weighted graph G is $\mathrm{Pic}(G, \mathrm{wt}) = \mathcal{G}^N/\mathbf{L}(\mathbb{Z}^N)$. This group, which may no longer be discrete, is the $\mathbb{Z}$-cokernel of the Laplacian. Define $\mathrm{Pic}^0(G, \mathrm{wt})$ to be the subgroup of $\mathrm{Pic}(G, \mathrm{wt})$ consisting of classes represented by degree-zero chip configurations. Given a sink $q \in V$ and an abstract simplicial complex Δ on $V' = V \setminus \{q\}$, we define Δ-*critical configurations* exactly as in Section 1.2. It would be interesting to see if Theorem 1.2 extends to the setting of $\mathcal{G}$-valued chip configurations. Our analysis of k-skeletal objects proves this result in the special cases where G is a complete graph (with sink and weights as described earlier) and Δ is the $(k+1)$-skeleton of V', namely the set of all nonempty subsets of V' of size $k + 1$ or less. This is the origin of the term "k-skeletal" for the various combinatorial collections considered here.

3. Skeletal paths

This section can be read independently of Section 2, although that section provides motivation for the concepts defined here.

3.1. Definitions and main result

We first introduce certain paths that generalize lattice paths. Throughout, n is a fixed positive integer and $\mathcal{G}$ is an additive subgroup of $\mathbb{R}$ (typically $\mathcal{G} = \mathbb{Z}$ or $\mathcal{G} = \mathbb{R}$). Formally, we define a *path of height* n *with values in* $\mathcal{G}$ to be a set $\pi = \{(x_i, i) : i = 0, 1, 2, \ldots, n-1\}$ of n points in $\mathbb{R}^2$ such that $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{n-1}$ and all $x_i \in \mathcal{G}$. Informally, we make the *picture of the path* π by drawing n unit-length north steps from (x_i, i) to $(x_i, i + 1)$ for $0 \leqslant i \leqslant n - 1$, and drawing east steps connecting $(x_i, i + 1)$ to $(x_{i+1}, i + 1)$ for $0 \leqslant i < n - 1$. The points (x_i, i) are called the *vertices* of the path π. The formal definition focuses on these vertices (the starting points of the north steps) to make connections to parking functions and chip-firing configurations more transparent and to avoid ambiguities involving initial east steps and final east steps.

We say the path $\pi = \{(x_0, 0), \ldots, (x_{n-1}, n - 1)\}$ is *nonnegative* if $x_0 \geqslant 0$. For such a path, we often draw an initial east step from $(0, 0)$ to $(x_0, 0)$. For sets of paths where there is a known upper bound M for all x_i, we may also draw a final east step from (x_{n-1}, n) to (M, n). When $\mathcal{G} = \mathbb{Z}$, nonnegative paths can be identified with classical *lattice paths*, which are sequences of unit-length north steps and unit-length east steps starting at the origin. We write $P_n = P_n(\mathcal{G})$ for the set of all paths of height n with values in $\mathcal{G}$.

Example 3.1. Figure 3.1 shows the nonnegative path $\{(1/2, 0), (3/2, 1), (3/2, 2), (3, 3)\} \in P_4(\mathbb{Q})$ with its vertices marked by dots.

Fix parameters $c, m \in \mathcal{G}$ with $c > 0$ and $m \geqslant 0$. The *reference line* for these parameters is the line with equation $x = my + c$. For each integer $k \in \{0, 1, \ldots, n - 1\}$, a path $\pi \in P_n$ is called a k-*skeletal path (for parameters* c *and* m*)* if and only if these conditions hold:

(P0) π is nonnegative.

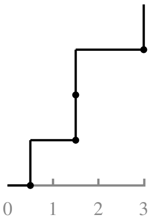

Figure 3.1: Picture of the nonnegative path $\{(1/2, 0), (3/2, 1), (3/2, 2), (3, 3)\}$.

(P1) The last $k + 1$ north steps of π all start strictly left of the line $x = my + c$.

(P2) There do not exist $k + 1$ consecutive rows such that the north steps of π in those rows all start strictly left of the line $x = my$.

In terms of the vertices (x_i, i) of π, Condition (P0) requires that $x_0 \geqslant 0$. Condition (P1) requires that $x_i < mi + c$ for all i in $\{n - 1, n - 2, \ldots, n - (k + 1)\}$. In the special case $\mathcal{G} = \mathbb{Z}$, we could equivalently require $x_i \leqslant mi + c - 1$, but the general case requires the strict inequality. Condition (P2) requires that for any i in $\{0, 1, \ldots, n - (k + 1)\}$, there must exist j in $\{i, i + 1, i + 2 \ldots, i + k\}$ with $x_j \geqslant mj$. Let $\mathrm{SKP}_k = \mathrm{SKP}_k(\mathcal{G}, n; c, m)$ be the set of k-skeletal paths of height n with parameters c and m and values in $\mathcal{G}$. The main theorem of this section is the following:

Theorem 3.2. *For all $k, k' \in \{0, 1, \ldots, n - 1\}$, there is a canonical bijection from SKP_k to $\mathrm{SKP}_{k'}$.*

The proof of Theorem 3.2 appears in Sections 3.2 through 3.5. In Section 3.2 we introduce the notion of an area vector g associated with a path and rephrase Conditions (P0), (P1), and (P2) in terms of area vectors. In Section 3.3 we introduce certain equivalence classes of area vectors that, by Theorem 3.10 of that section, contain exactly one k-skeletal area vector for each k between 0 and $n - 1$. The bijection of Theorem 3.2 thereby arises by mapping any given k-skeletal area vector to the corresponding k'-skeletal area vector in its equivalence class. Thus, the proof of Theorem 3.2 is reduced to that of Theorem 3.10, which is carried out in Sections 3.4 and 3.5.

Before continuing on to the proof of Theorem 3.2, we consider its enumerative consequences for the case of $\mathcal{G} = \mathbb{Z}$.

Corollary 3.3. *When $\mathcal{G} = \mathbb{Z}$, the number of k-skeletal paths is given by an m-ballot number: for all k between 0 and $n - 1$,*

$$|\,\mathrm{SKP}_k\,| = \frac{c}{(m+1)n + c}\binom{(m+1)n + c}{n}. \tag{3.1}$$

Proof. For $k = n - 1$, equation (3.1) is a classical result. Condition (P1) restricts attention to paths whose north steps start at points $(x, y) \in \mathbb{Z}^2$ with $x < my + c$, or equivalently $x \leqslant my + c - 1$. Condition (P2) is automatically satisfied since the first north step of any such path starts at $(x_0, 0)$ with $x_0 \geqslant 0$. We can identify these paths with lattice paths from $(0, 0)$ to $(mn + c - 1, n)$ contained in the trapezoidal region

$$\mathcal{T} = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant n,\ 0 \leqslant x \leqslant my + c - 1\}.$$

The stated formula for the number of such lattice paths can be shown by induction (see, for example, [Loe17, Theorem 2.27]).

For $k < n - 1$, the claimed enumeration follows from the case of $k = n - 1$ along with the bijection from Theorem 3.2. □

Remark 3.4. When $k = 0$, we can interpret SKP_0 as counting lattice paths in a rotated version of the trapezoid $\mathcal{T}$ referenced in the proof of Corollary 3.3. By adjusting east steps we obtain a lattice path from $(-m, 0)$ to $(mn - m + c - 1, n)$. Under this bijection, SKP_0 is the set of lattice paths contained in the region $\{(x, y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant n,\ my - m \leqslant x \leqslant mn - m + c - 1\}$. This is $\mathcal{T}$ rotated 180 degrees about the point $\frac{1}{2}(mn - m + c - 1, n)$. Figure 3.2 gives an example where $n = 2$, $c = 1$, and $m = 2$.

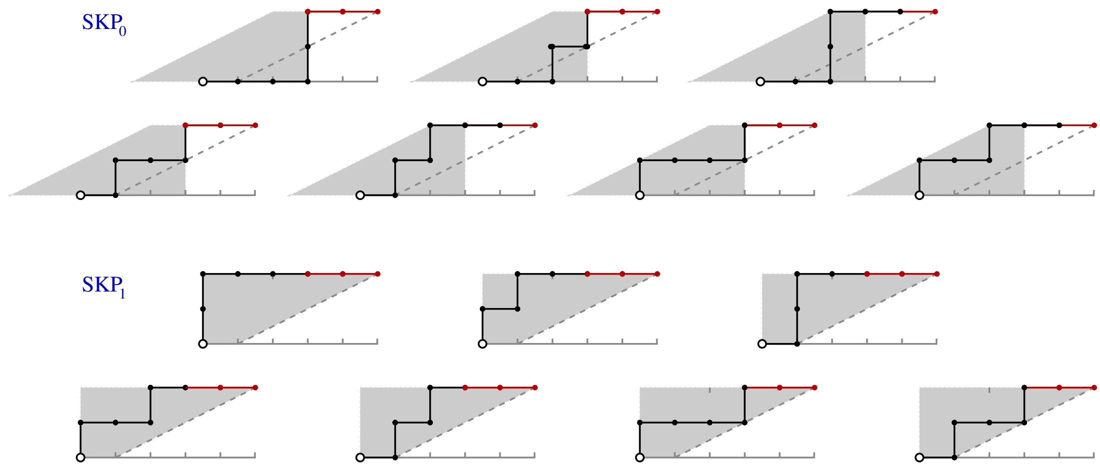

Figure 3.2: Illustration of the duality between SKP_0 and SKP_{n-1} in the case of $n = 2$, $m = 2$, and $c = 1$ as described in Remark 3.4. The black portion of each path in SKP_0 can be found, after a 180-degree rotation, as the black portion of the corresponding path in SKP_1.

Example 3.5. Let $\mathcal{G} = \mathbb{Z}$, $n = 3$, and $c = m = 1$. The reference line is $x = y + 1$. Here, we can view skeletal paths as lattice paths from $(0, 0)$ to $(3, 3)$ with three unit-length north steps and three unit-length east steps. We find

$$\begin{aligned}
\mathrm{SKP}_0 &= \{\mathrm{NENENE}, \mathrm{ENENNE}, \mathrm{NEENNE}, \mathrm{ENNENE}, \mathrm{EENNNE}\},\\
\mathrm{SKP}_1 &= \{\mathrm{NENENE}, \mathrm{NENNEE}, \mathrm{NNEENE}, \mathrm{ENNENE}, \mathrm{ENNNEE}\},\\
\mathrm{SKP}_2 &= \{\mathrm{NENENE}, \mathrm{NENNEE}, \mathrm{NNEENE}, \mathrm{NNENEE}, \mathrm{NNNEEE}\},
\end{aligned}$$

as illustrated in Figure 3.3. The set SKP_2 consists of Dyck paths in the triangle bounded by $x = 0$, $y = 3$, and $x = y$. Note these paths can touch the line $x = y$ but are strictly left of the reference line $x = y + 1$. Adjusting east steps in SKP_0 to get paths from $(-1, 0)$ to $(2, 3)$ as described in Remark 3.4, we get

$$\{\text{ENENEN}, \text{ENEENN}, \text{EEENNN}, \text{EENENN}, \text{EENNEN}\},$$

which is the set of rotated Dyck paths in the triangle bounded by $x = y - 1$, $y = 0$, and $x = 2$.

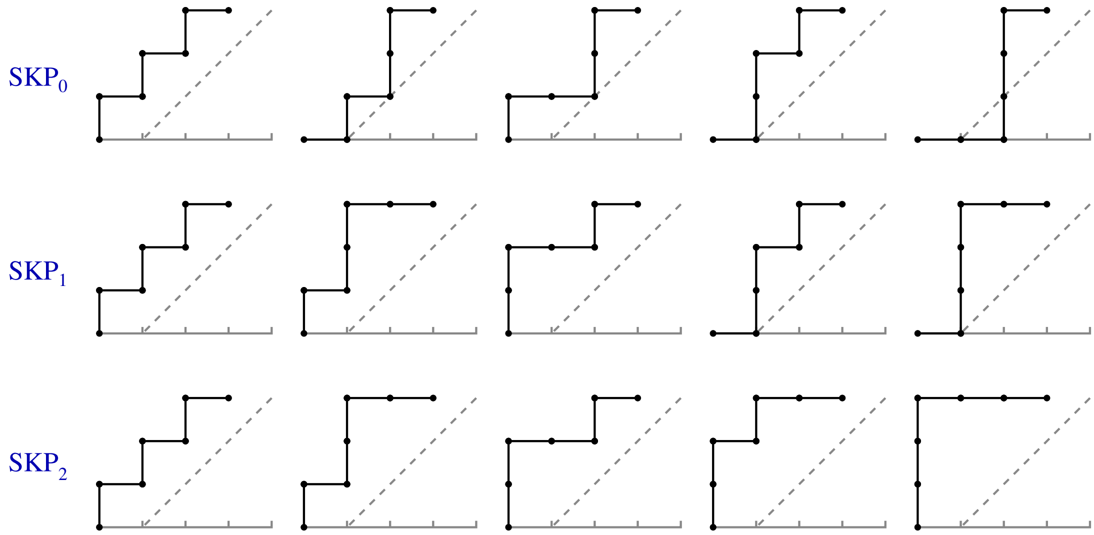

Figure 3.3: The three collections of paths from Example 3.5.

When $\mathcal{G}$ is a non-cyclic subgroup of $\mathbb{R}$, the collections of k-skeletal paths are infinite, but Theorem 3.2 still applies. We could obtain finite subcollections by imposing the extra requirement that all north steps of paths have integer x-coordinates. However, Theorem 3.2 no longer holds in this setting, as seen in the next example.

Example 3.6. Let $\mathcal{G} = \mathbb{Q}$, $n = 2$, $m = 3/2$ and $c = 1/2$. When $k = 0$, there are three lattice paths (consisting of unit-length east steps and north steps) that satisfy Condition (P1), namely ENNEE, NENEE, and NNEEE. None of these paths satisfy Condition (P2). But for $k = 1$, the two lattice paths NENEE and NNEEE satisfy both conditions. Compare to Example 3.11 below, which finds the (non-integral) 0-skeletal objects corresponding to these two paths.

3.2. Area vectors

To study skeletal paths, we develop a bijection between paths of height n and the area vectors defined next. An *area vector* is an n-tuple $g = (g_0, g_1, \ldots, g_{n-1}) \in \mathcal{G}^n$ such that $g_{i+1} \leqslant g_i + m$ for $0 \leqslant i < n - 1$. Let AV_n be the set of all such area vectors (AV_n also depends on m and $\mathcal{G}$). We call $g \in \mathrm{AV}_n$ a *Dyck* vector iff all $g_i > 0$.

For a path $\pi = \{(x_i, i)\} \in P_n$, define the *area vector of* π to be

$$G(\pi) = (g_0, g_1, \ldots, g_{n-1}), \text{ where } g_i = mi + c - x_i \text{ for } 0 \leqslant i < n.$$

Each g_i is the signed horizontal distance from vertex (x_i, i) to the reference line $x = my + c$. Since m, c belong to the subgroup $\mathcal{G}$, x_i is in $\mathcal{G}$ iff g_i is in $\mathcal{G}$. The inequality $x_i \leqslant x_{i+1}$ is equivalent to $g_{i+1} \leqslant g_i + m$. It follows that $G : P_n \to \mathrm{AV}_n$ is a bijection. From the definition of g_i, we also deduce: $x_0 \geqslant 0$ iff $g_0 \leqslant c$; $x_i < mi + c$ iff $g_i > 0$; $x_i < mi$ iff $g_i > c$. Thus, we can rephrase the definition of skeletal paths in terms of area vectors, as follows.

Proposition 3.7. *Let $g = (g_0, g_1, \ldots, g_{n-1})$ be the area vector of a path π. The path π is a k-skeletal path (for parameters c and m) if and only if these conditions hold:*

(A0) $g_0 \leqslant c$.

(A1) The last $k+1$ entries $g_{n-1}, \ldots, g_{n-(k+1)}$ of g are all strictly positive.

(A2) There do not exist $k+1$ consecutive entries $g_i, \ldots, g_{i+k}$ in g that all strictly exceed c.

We call $g \in \mathrm{AV}_n$ a k*-skeletal* area vector iff g satisfies (A0), (A1), and (A2). Let SKV_k be the set of such area vectors.

Example 3.8. Let $G = \mathbb{Z}$, $n = 3$, and $c = m = 1$. The k-skeletal paths found in Example 3.5 correspond to the following area vectors:

$$\begin{aligned}\mathrm{SKV}_0 &= \{(1,1,1), (0,0,1), (1,0,1), (0,1,1), (-1,0,1)\},\\ \mathrm{SKV}_1 &= \{(1,1,1), (1,1,2), (1,2,1), (0,1,1), (0,1,2)\},\\ \mathrm{SKV}_2 &= \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (1,2,3)\}.\end{aligned}$$

Example 3.9. Let $G = \mathbb{Z}$, $n = 14$, $c = 4$, and $m = 1$. Consider the path $\pi = \{(x_i, i)\}$, where

$$(x_0, x_1, \ldots, x_{13}) = (0,0,0,4,4,4,5,9,13,13,13,13,13,14).$$

Viewing π as a lattice path from $(0,0)$ to $(14,14)$ as shown in Figure 3.4, we have

$$\pi = \text{NNNEEEENNNENEEEENEEEENNNNNEN}.$$

The area vector of π is $G(\pi) = (4,5,6,3,4,5,5,2,-1,0,1,2,3,3)$. Since $G(\pi)$ ends in exactly four positive entries, Condition (A1) holds for all $k \leqslant 3$. Since $G(\pi)$ contains subsequences $5, 6$ and $5, 5$ but no longer consecutive subsequence of entries exceeding 4, Condition (A2) holds for all $k \geqslant 2$. Thus, the path π and area vector $G(\pi)$ are k-skeletal for $k = 2, 3$ (only).

3.3. Cycling operator on area vectors

Define the *cycling operator* $C : \mathcal{G}^n \to \mathcal{G}^n$ by

$$C(g_0, g_1, \ldots, g_{n-1}) = (g_1, \ldots, g_{n-1}, g_0 - c).$$

C is a bijection with inverse $C^{-1}(h_0, h_1, \ldots, h_{n-1}) = (h_{n-1} + c, h_0, h_1, \ldots, h_{n-2})$. We write $C^2 = C \circ C$, $C^{-2} = C^{-1} \circ C^{-1}$, and so on.

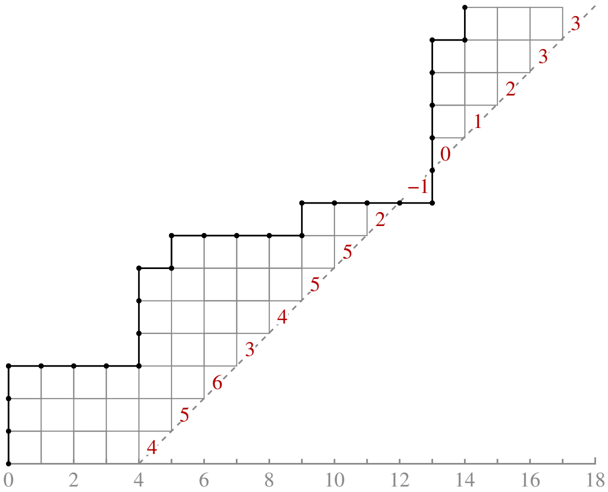

Figure 3.4: The path π from Example 3.9.

For $g = (g_0, g_1, \dots, g_{n-1}) \in \mathcal{G}^n$, define $|g| = g_0 + g_1 + \cdots + g_{n-1}$. We see that $|C(g)| = |g| - c$. For $g, h \in \mathcal{G}^n$, write $g \succeq h$ to mean $|g| \geqslant |h|$. Define $\mathrm{pos}(g) \in \{0, 1, \dots, n\}$ to be the largest integer ℓ such that the last ℓ entries of g are strictly positive. Observe that g is a Dyck vector iff $\mathrm{pos}(g) = n$.

Define $S = \{g \in \mathrm{AV}_n : g_0 \leqslant c \text{ and } g_{n-1} > 0\}$. Elements of S are the area vectors of nonnegative paths π whose last vertex is strictly left of the reference line. For any path π satisfying Conditions (P0) and (for some k) (P1), $G(\pi)$ belongs to S. Define an equivalence relation $\sim$ on S as follows: for $g, h \in S$, $g \sim h$ means $h = C^j(g)$ for some $j \in \mathbb{Z}$. Informally, this says that we can go from g to h by applying C or C^{-1} finitely many times, noting that some intermediate vectors along the way might not belong to S. Since applying C to g decreases $|g|$, each equivalence class of $\sim$ is totally ordered by the relation $\succeq$.

Theorem 3.10. *For each $k \in \{0, 1, \dots, n-1\}$, each equivalence class T of $\sim$ in S contains exactly one k-skeletal area vector, namely the least g in T (relative to $\succeq$) satisfying $\mathrm{pos}(g) > k$.*

Theorem 3.2 follows immediately from Theorem 3.10:

Proof of Theorem 3.2. We define a bijection $\mathrm{SKP}_k \to \mathrm{SKP}_{k'}$ by mapping each k-skeletal path π to the unique k'-skeletal path π' with $G(\pi') \sim G(\pi)$. □

Example 3.11. Let $\mathcal{G} = \mathbb{Q}$, $n = 2$, $m = 3/2$ and $c = 1/2$. Two of the infinitely many 1-skeletal area vectors for these parameters are $g = (1/2, 2)$ and $g' = (1/2, 1)$. These area vectors correspond to the two lattice paths NNEEE and NENEE from Example 3.6. Applying C to g six times leads to the 0-skeletal area vector $(-1, 1/2)$ in the same equivalence class as g. Applying C to g' twice leads to the 0-skeletal area vector $(0, 1/2)$.

The remaining two subsections contain the proof of Theorem 3.10.

3.4. Preliminary lemmas

Lemma 3.12. *For all $g = (g_0, g_1, \dots, g_{n-1}) \in S$ and all $j \in \mathbb{Z}$, $C^j(g)$ belongs to AV_n.*

Proof. Write $j = ne + p$, where $e \in \mathbb{Z}$ and $0 \leqslant p < n$. If $p = 0$, then

$$C^j(g) = C^{ne}(g) = (g_0 - ec, g_1 - ec, \dots, g_{n-1} - ec). \tag{3.2}$$

Because $g \in S$, we know $g_{i+1} \leqslant g_i + m$ for $0 \leqslant i < n-1$. Adding $-ec$ to both sides of these inequalities, we see that $C^j(g)$ is an area vector. If $0 < p < n$, then

$$C^j(g) = C^{ne+p}(g) = (g_p - ec, \dots, g_{n-1} - ec; g_0 - (e+1)c, \dots, g_{p-1} - (e+1)c). \tag{3.3}$$

Each pair of consecutive entries separated by a comma satisfies the needed inequality to be an area vector, as we see by adding $-ec$ or $-(e+1)c$ to one of the inequalities $g_{i+1} \leqslant g_i + m$. We must also check that the pair of entries separated by a semicolon satisfies $g_0 - (e+1)c \leqslant g_{n-1} - ec + m$, or equivalently $g_0 - m \leqslant c + g_{n-1}$. Because $g \in S$, this last inequality follows by adding the inequalities $g_0 \leqslant c$ and $-m \leqslant 0 < g_{n-1}$. We even have the *strict* inequality $g_0 - (e+1)c < (g_{n-1} - ec) + m$, a fact that we will need later. □

Lemma 3.13. *Each equivalence class of $\sim$ in S is finite.*

Proof. Consider the equivalence class of some $g \in S$. Since $c > 0$, there exists $j_0 \in \mathbb{Z}_{>0}$ so that for all $j \geqslant j_0$, all entries in $C^j(g)$ are nonpositive. This is clear from formulas (3.2) and (3.3). Similarly, there exists $i_0 \in \mathbb{Z}_{<0}$ so that for all $i \leqslant i_0$, all entries in $C^i(g)$ exceed c. By definition of S, the equivalence class of g must be a subset of the finite set $\{C^r(g) : i_0 < r < j_0\}$. □

The next lemma proves the $k = n-1$ case of Theorem 3.10.

Lemma 3.14. *Each equivalence class T of $\sim$ contains exactly one Dyck vector.*

Proof. Let T be a fixed equivalence class in S. We first show T contains at most one Dyck vector. Suppose $g = (g_0, g_1, \dots, g_{n-1}) \in T$ is a Dyck vector. Then $g_0 \leqslant c$ (since $g \in S$), so $C(g) = (g_1, \dots, g_{n-1}, g_0 - c)$ has last entry $\leqslant 0$. If we continue to apply C, this entry will move left through the vector, and eventually it cycles back to the right and becomes even smaller. We see that all vectors following any Dyck vector g in the totally ordered set T are not Dyck vectors. Thus T cannot contain two different Dyck vectors.

Next we show that T does contain a Dyck vector. Start with any $g = (g_0, g_1, \dots, g_{n-1})$ in T. If g is a Dyck vector, then there is nothing to prove. Otherwise, choose the least integer $e > 0$ such that

$$g^* = C^{-en}(g) = (g_0 + ec, g_1 + ec, \dots, g_{n-1} + ec) \in \mathcal{G}^n$$

has all positive entries. Note that g^* may not belong to S. By minimality of e, there must exist j with $0 < g_j + ec \leqslant c$. Choose the least such index j, and let

$$g^+ = C^j(g^*) = (g_j + ec, g_{j+1} + ec, \dots, g_{n-1} + ec, g_0 + (e-1)c, g_1 + (e-1)c, \dots, g_{j-1} + (e-1)c).$$

In this new vector, the first entry $g_j + ec$ is $\leqslant c$ by choice of j. All entries (including the last one) are strictly positive, by choice of e and j. Also g^+ is an area vector by Lemma 3.12. So g^+ is in S and is a Dyck vector in T. □

Example 3.15. Let $\mathcal{G}=\mathbb{Z}$, $n=14$, $c=4$, $m=1$, and $g=(3,4,5,5,2,-1,0,1,2,3,3,0,1,2)\in S$. Following the proof of the lemma, we take $e=1$ to get

$$g^* = C^{-14}(g) = (7,8,9,9,6,3,4,5,6,7,7,4,5,6),$$

which has all positive entries but is not in S. We find $j=5$ and set

$$g^+ = C^5(g^*) = (3,4,5,6,7,7,4,5,6,3,4,5,5,2).$$

This is the unique Dyck vector in S equivalent to g. Starting at g^+ and applying C repeatedly, the equivalence class of g is $\{g^+, g', g, g''\}$, where

$$g' = C^6(g^+) = (4,5,6,3,4,5,5,2,-1,0,1,2,3,3); \quad g = C^9(g^+);$$

$$g'' = C^{13}(g^+) = (2,-1,0,1,2,3,3,0,1,2,-1,0,1,1).$$

We have $\mathrm{pos}(g^+) = 14 = n$, $\mathrm{pos}(g') = 4$, $\mathrm{pos}(g) = 2$, and $\mathrm{pos}(g'') = 2$. The definition shows that g^+ is k-skeletal for $4 \leqslant k < 14$, g' is k-skeletal for $2 \leqslant k < 4$, g is not k-skeletal for any k, and g'' is k-skeletal for $0 \leqslant k < 2$. This agrees with the conclusion of Theorem 3.10.

Given $g \in S$, the next lemma determines which subsequent objects $C^j(g)$ also belong to S.

Lemma 3.16. *Suppose $g = (g_0, g_1, \ldots, g_{n-1})$ is in S.*
(a) For each $e > 0$, $C^{en}(g) \in S$ if and only if $g_0 \leqslant (e+1)c$ and $g_{n-1} > ec$.
(b) For all e, p with $e \geqslant 0$ and $0 < p < n$, $C^{en+p}(g) \in S$ if and only if $g_p \leqslant (e+1)c$ and $g_{p-1} > (e+1)c$.

Proof. Because $g \in S$, Lemma 3.12 shows that all vectors $C^j(g)$ are area vectors. Assertions (a) and (b) follow at once from (3.2) and (3.3) and the definition of S. □

3.5. Analysis of k-skeletal conditions

To finish proving Theorem 3.10, we reformulate the k-skeletal Conditions (A0), (A1), and (A2) in several ways.

Proposition 3.17. *An area vector $g = (g_0, g_1, \ldots, g_{n-1})$ is k-skeletal iff these conditions hold:*

(A0) $g_0 \leqslant c$.

(A1′) $\mathrm{pos}(g) > k$.

(A2′) For all $p \in \{1, 2, \ldots, n\}$, if ($p = n$ or $g_p \leqslant c$) and $g_{p-1} > c$, then $\mathrm{pos}(C^p(g)) \leqslant k$.

Proof. Condition (A1) says that the last $k+1$ entries of g (and perhaps more entries) are strictly positive, which is equivalent to $\mathrm{pos}(g) > k$, as stated in (A1′). Next, assume $g \in \mathrm{AV}_n$ fails Condition (A2). Choose i and k so that $g_i, g_{i+1}, \ldots, g_{i+k}$ all exceed c and either $i + k = n - 1$ or $g_{i+k+1} \leqslant c$. In the case $i + k = n - 1$, let $p = n$. Then $g_{p-1} = g_{n-1} > c$ and

$$C^p(g) = (g_0 - c, g_1 - c, \ldots, g_{n-1} - c)$$

has $\mathrm{pos}(C^p(g)) > k$, which means that Condition (A2′) fails. In the case $i + k < n - 1$ and $g_{i+k+1} \leqslant c$, let $p = i + k + 1$. Then $g_p \leqslant c$, $g_{p-1} > c$, and

$$C^p(g) = (g_p, \ldots, g_{n-1}, g_0 - c, g_1 - c, \ldots, g_{p-1} - c)$$

has $\mathrm{pos}(C^p(g)) > k$, so Condition (A2′) fails. Similarly, reversing the reasoning in the two cases shows that the failure of (A2′) implies the failure of (A2). So Conditions (A2) and (A2′) are logically equivalent. □

Proposition 3.18. *An area vector $g = (g_0, g_1, \ldots, g_{n-1})$ is k-skeletal iff these conditions hold:*

(A0) $g_0 \leqslant c$.

(A1′) $\mathrm{pos}(g) > k$.

(A2″) For all $p \in \{1, 2, \ldots, n\}$ such that $C^p(g) \in S$, $\mathrm{pos}(C^p(g)) \leqslant k$.

Proof. Assume $g \in \mathrm{AV}_n$ satisfies (A0) and (A1′), so $g \in S$. We need only confirm that the hypothesis "if ($p = n$ or $g_p \leqslant c$) and $g_{p-1} > c$" in (A2′) is logically equivalent to the condition $C^p(g) \in S$. Consider the case $p = n$. The hypothesis simplifies to $g_{n-1} > c$. Taking $e = 1$ in Lemma 3.16(a), $C^n(g) \in S$ iff $g_0 \leqslant 2c$ and $g_{n-1} > c$. The condition $g_0 \leqslant 2c$ is already guaranteed because $g_0 \leqslant c < 2c$ (as $g \in S$ and $c > 0$). So $C^n(g) \in S$ is equivalent to $g_{n-1} > c$, as needed. Consider the case $0 < p < n$. The hypothesis in (A2′) simplifies to "$g_p \leqslant c$ and $g_{p-1} > c$." Taking $e = 0$ in Lemma 3.16(b), we see this condition is equivalent to $C^p(g) \in S$. □

After one last technical adjustment to Condition (A2), we will be ready to prove Theorem 3.10.

Proposition 3.19. *An area vector $g = (g_0, g_1, \ldots, g_{n-1})$ is k-skeletal iff these conditions hold:*

(A0) $g_0 \leqslant c$.

(A1′) $\mathrm{pos}(g) > k$.

(A2‴) For all $p \in \mathbb{Z}_{>0}$ such that $C^p(g) \in S$, $\mathrm{pos}(C^p(g)) \leqslant k$.

Proof. Assume $g \in \mathrm{AV}_n$ satisfies (A0) and (A1′), so $g \in S$. Certainly, if g satisfies (A2‴), then g satisfies (A2″). To prove the converse, assume g fails (A2‴), meaning there exists $p^* > 0$ with $C^{p^*}(g) \in S$ and $\mathrm{pos}(C^{p^*}(g)) > k$. We prove g fails (A2″) by finding $p \in \{1, 2, \ldots, n\}$ with $C^p(g) \in S$ and $\mathrm{pos}(C^p(g)) > k$.

Case 1: $p^* \leqslant n$. Then we take $p = p^*$.

Case 2: $p^* = en$ for some $e > 1$. Here, $C^{p^*}(g) = (g_0 - ec, g_1 - ec, \ldots, g_{n-1} - ec)$. Take $p = n$, so $C^p(g) = (g_0 - c, g_1 - c, \ldots, g_{n-1} - c)$. We obtain $C^p(g)$ from $C^{p^*}(g)$ by increasing each entry by $(e-1)c > 0$. Since $\mathrm{pos}(C^{p^*}(g)) > k$, the last $k+1$ entries of $C^{p^*}(g)$ are positive. So the last $k+1$ entries of $C^p(g)$ are positive, and $\mathrm{pos}(C^p(g)) > k$. Since $C^{p^*}(g)$ is in S, Lemma 3.16(a) gives $g_0 \leqslant (e+1)c$ and $g_{n-1} > ec$. Since $g_0 \leqslant c$ also, we deduce $g_0 \leqslant 2c$ and $g_{n-1} > c$. Lemma 3.16(a) now shows $C^p(g) \in S$, as needed.

Case 3: $p^* = ne + r$ for some $e \geqslant 1$ and $0 < r < n$. By assumption,

$$C^{p^*}(g) = (g_r - ec, \ldots, g_{n-1} - ec, g_0 - (e+1)c, \ldots, g_{r-1} - (e+1)c)$$

is in S and its last $k+1$ entries are positive. Applying C^{-ne} to $C^{p^*}(g)$ increases all entries by ec, leading to the vector

$$C^r(g) = (g_r, \ldots, g_{n-1}, g_0 - c, \ldots, g_{r-1} - c).$$

This vector also has its last $k+1$ entries positive, but $C^r(g)$ need not be in S because $g_r > c$ could occur. Let ℓ be as large as possible such that the first ℓ entries of $C^r(g)$ strictly exceed c. We have $0 \leqslant \ell \leqslant n-r$ since the $(n-r+1)$th entry is $g_0 - c$, and $g_0 - c \leqslant 0 < c$. Let $p = r + \ell$, which is in $\{1, 2, \ldots, n\}$. We compute

$$C^p(g) = C^\ell(C^r(g)) = (g_p, \ldots, g_{n-1}, g_0 - c, \ldots, g_{r-1} - c, g_r - c, \ldots, g_{p-1} - c).$$

The ℓ values that cycle to the right end when we pass from $C^r(g)$ to $C^p(g)$ start larger than c and get decreased by c, so the cycled values are still strictly positive. So $\mathrm{pos}(C^p(g)) \geqslant \mathrm{pos}(C^r(g)) > k$. To finish, we show $C^p(g) \in S$. If $r < p < n$, then $C^p(g) \in S$ because $g_p \leqslant c$ (by choice of ℓ) and $g_{p-1} - c > 0$ (due to g_{p-1} being cycled to the right end). If $p = r$, then $\ell = 0$, so $g_r \leqslant c$ and $C^p(g) = C^r(g)$ is in S. If $p = n$, then $C^p(g) \in S$ because $g_{n-1} - c > 0$ (due to g_{n-1} being cycled to the right end) and $g_0 - c < g_0 \leqslant c$. □

Proof of Theorem 3.10. Fix $k \in \{0, 1, \ldots, n-1\}$ and an equivalence class T of $\sim$. Every $g \in T$ belongs to S and therefore satisfies $g_0 \leqslant c$, as required by (A0). Since applying C decreases $|g|$, g satisfies (A1′) and (A2‴) iff g is the least object in T (relative to $\succeq$) such that $\mathrm{pos}(g) > k$. Because T is finite (Lemma 3.13) and contains a Dyck object g^+ with $\mathrm{pos}(g^+) = n > k$ (Lemma 3.14), there exists a unique least $g \in T$ with $\mathrm{pos}(g) > k$. This g is the unique k-skeletal area vector in T. □

4. Skeletal functions

This section can be read independently of Section 2, except for §4.5 which contains the proof of Theorem 2.3.

4.1. Representing functions as labeled paths

We continue to assume $\mathcal{G}$ is an additive subgroup of $\mathbb{R}$ and n is a fixed positive integer. Let $[n] = \{1, 2, \ldots, n\}$, and let $F_n = F_n(\mathcal{G})$ be the set of all functions $f : [n] \to \mathcal{G}$. For each such function f, we define the (unlabeled) *path of* f to be $\pi(f) = \{(x_i, i) : 0 \leqslant i < n\}$, where $x_0, x_1, \ldots, x_{n-1}$ is the list of function values $f(1), f(2), \ldots, f(n)$ sorted into weakly increasing order. The *labeled path of* f is the ordered pair $(\pi(f), w)$, where $w = (w_0, w_1, \ldots, w_{n-1})$ is the unique rearrangement of $1, 2, \ldots, n$ such that $f(w_i) = x_i$ for all i, and whenever $x_i = x_{i+1}$, we have $w_i < w_{i+1}$. We call w the *label sequence* for f. Informally, we draw the labeled path for f as follows. Put a north step labeled a on the line $x = b$ whenever $f(a) = b$. Arrange these north steps at different heights so that the x-coordinates weakly increase as we move up the figure, producing a path of height n. For north steps on the same line $x = b$, make their labels increase reading from bottom to top. This process defines a bijection from $F_n(\mathcal{G})$ to the set of labeled paths where all x-coordinates belong to $\mathcal{G}$. The inverse bijection maps a labeled path to the function f such that $f(j)$ is the x-coordinate of the north step with label j.

Example 4.1. Let $\mathcal{G} = \mathbb{Z}$, $n = 12$, $c = 6$, and $m = 2$. Consider the function f with values shown in Table 4.1. Figure 4.1 shows the labeled path for f. The area vector for the unlabeled path is

$$g = (5, 7, 9, 11, 10, 10, 6, 8, 6, 8, 10, 11).$$

The label sequence of f is $w = (4, 7, 8, 11, 1, 3, 2, 5, 6, 9, 12, 10)$.

Table 4.1: Function analyzed in Example 4.1 and illustrated in Figure 4.1.

a	1	2	3	4	5	6	7	8	9	10	11	12
$f(a)$	4	12	6	1	12	16	1	1	16	17	1	16

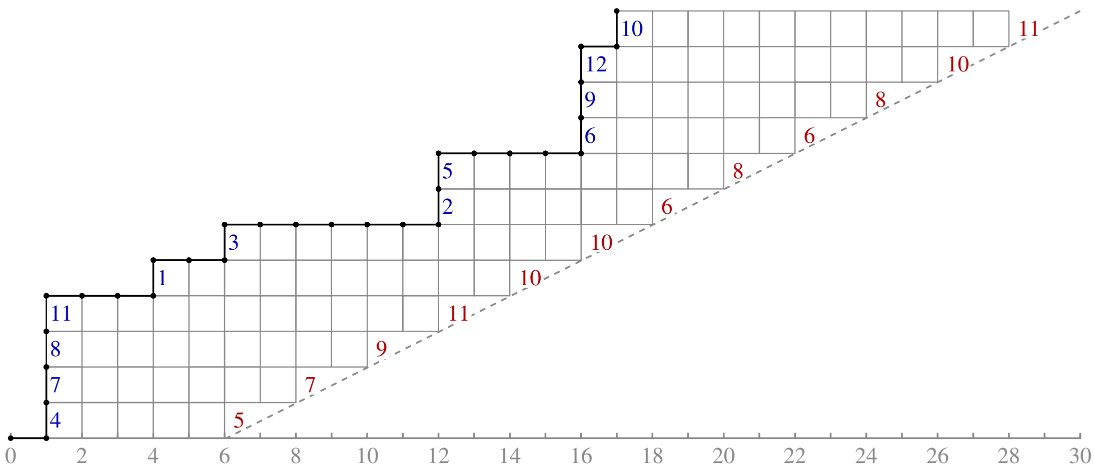

Figure 4.1: Labeled path for the function in Example 4.1. Labels are placed in blue to the right of each north step. Entries of the g-vector are listed in each row in red along the line $x = 2y + 6$.

For any path $\pi = \{(x_i, i)\} \in P_n$, a *run of north steps* in π is a maximal interval $\{i, i + 1, \ldots, j\}$ of consecutive indices such that $x_i = x_{i+1} = \cdots = x_j$. The *length* of this run is $j - i + 1$, which is the number of north steps of π lying on the line $x = x_i$. Let $\text{run}(\pi)$ be the multiset consisting of the lengths of all runs of north steps in π. Suppose π has area vector $G(\pi) = (g_0, g_1, \ldots, g_{n-1})$. By definition of G, $x_i = x_{i+1}$ iff $g_{i+1} = g_i + m$. A *run* in an area vector g is a maximal subsequence of consecutive entries in which each successive entry exceeds the previous one by m. Let $\text{run}(g)$ be the multiset of run lengths in g. For all $\pi \in P_n$, we have $\text{run}(\pi) = \text{run}(G(\pi))$.

An unlabeled path $\pi \in P_n$ may have the form $\pi(f)$ for several different functions $f \in F_n$. The run structure of π determines how many such f there are, as follows.

Proposition 4.2. *Suppose $\pi \in P_n$ has $\mathrm{run}(\pi) = [r_1, r_2, \ldots, r_a]$. The number of $f \in F_n$ with $\pi(f) = \pi$ is the multinomial coefficient $\binom{n}{r_1, r_2, \ldots, r_a}$.*

Proof. Choose distinct $z_1, z_2, \ldots, z_a$ such that π has r_i north steps with x-coordinate z_i. A function $f \in F_n$ has $\pi(f) = \pi$ iff the word $f(1), f(2), \ldots, f(n)$ is a rearrangement of r_1 copies of z_1, r_2 copies of z_2, and so on. The number of such rearrangements is $\binom{n}{r_1, r_2, \ldots, r_a}$. □

Example 4.3. The path π shown in Figure 4.1 has run multiset $\mathrm{run}(\pi) = [4, 1, 1, 2, 3, 1]$. This path is $\pi(f)$ for $\binom{12}{4,1,1,2,3,1} = 1663200$ choices of $f \in F_n$. The area vector $g = G(\pi)$ has the same run multiset as π. We write $g = (\underline{5, 7, 9, 11}, \underline{10}, \underline{10}, \underline{6, 8}, \underline{6, 8, 10}, \underline{11})$, where entries in the same run are underlined.

4.2. Main result for skeletal functions

A function $f \in F_n$ is called k-*skeletal* (for parameters c and m) iff the unlabeled path $\pi(f)$ is k-skeletal. Similarly, a labeled path (π, w) is k-skeletal iff π is k-skeletal. Let SKF_k be the set of k-skeletal functions in F_n with parameters c and m and values in $\mathcal{G}$.

Theorem 4.4. *(a) For all $k, k' \in \{0, 1, \ldots, n-1\}$, there is a canonical bijection from SKF_k to $\mathrm{SKF}_{k'}$. (b) When $\mathcal{G} = \mathbb{Z}$, for all $k \in \{0, 1, \ldots, n-1\}$, we have $|\,\mathrm{SKF}_k\,| = |\,\mathrm{SKF}_{n-1}\,| = c(mn+c)^{n-1}$.*

The equality $|\,\mathrm{SKF}_{n-1}\,| = c(mn+c)^{n-1}$ in (b) is a classical result (compare to the proof of Corollary 3.3). Specifically, SKF_{n-1} is the set of *trapezoidal parking functions*, which are functions f whose path $\pi(f)$ stays in the trapezoidal region

$$\{(x, y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant n, 0 \leqslant x \leqslant my + c - 1\}.$$

The stated formula for the number of such functions can be found, for example, in [SW18, Theorem 1.2].

4.3. Run structure of equivalence classes

The proof of Theorem 4.4 uses the set S and equivalence relation $\sim$ from Section 3.3.

Lemma 4.5. *Let T be an equivalence class of $\sim$ in S with unique Dyck representative g^+. For every $h \in T$, $\mathrm{run}(h) = \mathrm{run}(g^+)$.*

Proof. Let $T = \{g^+ \succ g^1 \succ g^2 \succ \cdots\}$ be the finite sequence of all area vectors in T ordered by decreasing area. By induction, it suffices to show that if g, h are two consecutive objects in this sequence, then $\mathrm{run}(g) = \mathrm{run}(h)$.

Let $g = (g_0, g_1, \ldots, g_{n-1}) \in T$. To reach h from g, we must apply C one or more times. Choose the largest s so that the first s entries of g are $\leqslant c$; we have $s \geqslant 1$ since $g_0 \leqslant c$. When C cycles these entries to the end, they each become $\leqslant 0$. So $C^1(g), \ldots, C^s(g)$ are not in S. If $s = n$, then applying C additional times never leads to an object in S, contradicting the fact that h follows g in the sequence. So $0 < s < n$. Next, choose the largest t so that the t entries of g scanning forward from g_s are $> c$. We must have $t > 0$ because $s < n$. Since the first entry of any vector in S must be $\leqslant c$, we must cycle all t of these entries to the end to reach the next object in S. Thus, $h = C^{s+t}(g)$ where $0 < s + t \leqslant n$. In the case $s + t = n$, $h = (g_0 - c, g_1 - c, \ldots, g_{n-1} - c)$, which certainly has the same run multiset as g. In the case $s + t < n$,

$$h = (g_{s+t}, \ldots, g_{n-1}; g_0 - c, \ldots, g_{s+t-1} - c). \tag{4.1}$$

The key point is that $g_{s+t-1} > c$ but $g_{s+t} \leqslant c$, so g_{s+t} cannot be $g_{s+t-1} + m$. So $(g_0, \ldots, g_{s+t-1})$ is a union of certain runs of g, and the run lengths in this part are unaffected when we subtract c from every entry. Similarly, $(g_{s+t}, \ldots, g_{n-1})$ contributes the same run lengths to g and to h, once we notice (using the last sentence of the proof of Lemma 3.12) that g_{n-1} and $g_0 - c$ cannot belong to the same run in h. So $\mathrm{run}(h) = \mathrm{run}(g)$. □

Example 4.6. In Example 3.15, all vectors g^+, g', g, g'' in the equivalence class have run multiset $[5, 1, 3, 3, 1, 1]$.

Example 4.7. Continuing Example 4.1, the path in Figure 4.1 has Dyck area vector

$$g = g^+ = (\underline{5, 7, 9, 11}, \underline{10}, \underline{10}, \underline{6, 8}, \underline{6, 8, 10}, \underline{11}) \in S.$$

To reach the next element of S equivalent to g, we must cycle the initial entries less than or equal to c (namely the first entry 5) and continue to cycle everything after that exceeding c (namely 7, 9, 11, 10 and 10). This leads to

$$g' = (\underline{6, 8}, \underline{6, 8, 10}, \underline{11}, \underline{-1, 1, 3, 5}, \underline{4}, \underline{4}).$$

Repeating this process, we reach two more area vectors in the equivalence class:

$$g'' = (\underline{6, 8, 10}, \underline{11}, \underline{-1, 1, 3, 5}, \underline{4}, \underline{4}, \underline{0, 2});$$

$$g''' = (\underline{-1, 1, 3, 5}, \underline{4}, \underline{4}, \underline{0, 2}, \underline{0, 2, 4}, \underline{5}).$$

We have $\mathrm{pos}(g) = 12$, $\mathrm{pos}(g') = 5$, $\mathrm{pos}(g'') = 1$, $\mathrm{pos}(g''') = 3$, and all these vectors have run multiset $[4, 1, 1, 2, 3, 1]$. So g is k-skeletal for $5 \leqslant k < 12$, g' is k-skeletal for $3 \leqslant k < 5$, g'' is not k-skeletal for any k, and g''' is k-skeletal for $0 \leqslant k < 3$. Note that exactly one of the elements $\{g, g', g'', g'''\}$ is k-skeletal for each k in the range $0 \leqslant k \leqslant 11$, as assured by Theorem 4.4(a).

4.4. Proof of Theorem 4.4.

Fix $k, k' \in \{0, 1, \ldots, n-1\}$ and $f \in \mathrm{SKF}_k = \mathrm{SKF}_k(\mathcal{G}; c, m)$. Let g be the area vector of $\pi(f)$, and let g' be the unique k'-skeletal area vector with $g' \sim g$ (Theorem 3.2). We can go from g to g' by stepping up or down through the equivalence class of g using powers of C^{-1} or C, as described in the proof of Lemma 4.5. Each step cycles the area vector in a way that preserves the run multiset. So we can cycle the labels of the north steps in the same way without violating the rules for labeled paths. For example, suppose we start with $g = (g_0, g_1, \ldots, g_{n-1})$ and label sequence $w = (w_0, w_1, \ldots, w_{n-1})$. If the next step replaces g by the h shown in (4.1), then we replace the label sequence w by $(w_{s+t}, \ldots, w_{n-1}; w_0, \ldots, w_{s+t-1})$. We eventually reach g' and a new label sequence w', which corresponds to some function $f' \in \mathrm{SKF}_{k'}$. The map sending f to f' is the required bijection from SKF_k to $\mathrm{SKF}_{k'}$.

Example 4.8. Let f be the function shown in Table 4.1. The area vector of $\pi(f)$ is the vector g from Example 4.7, which is 7-skeletal, so $f \in \mathrm{SKF}_7$. Let us find the image of f under the bijection from SKF_7 to SKF_4. The label sequence of f is $w = (\underline{4,7,8,11}, \underline{1}, \underline{3}, \underline{2,5}, \underline{6,9,12}, \underline{10})$. When g cycles to g', the label sequence cycles to $w' = (\underline{2,5}, \underline{6,9,12}, \underline{10}, \underline{4,7,8,11}, \underline{1}, \underline{3})$. The labeled path encoded by g' and w' is shown in Figure 4.2. The corresponding 4-skeletal function f' has the values shown in Table 4.2.

Table 4.2: Function analyzed in Example 4.8 and illustrated in Figure 4.2.

a	1	2	3	4	5	6	7	8	9	10	11	12
$f'(a)$	22	0	24	19	0	4	19	19	4	5	19	4

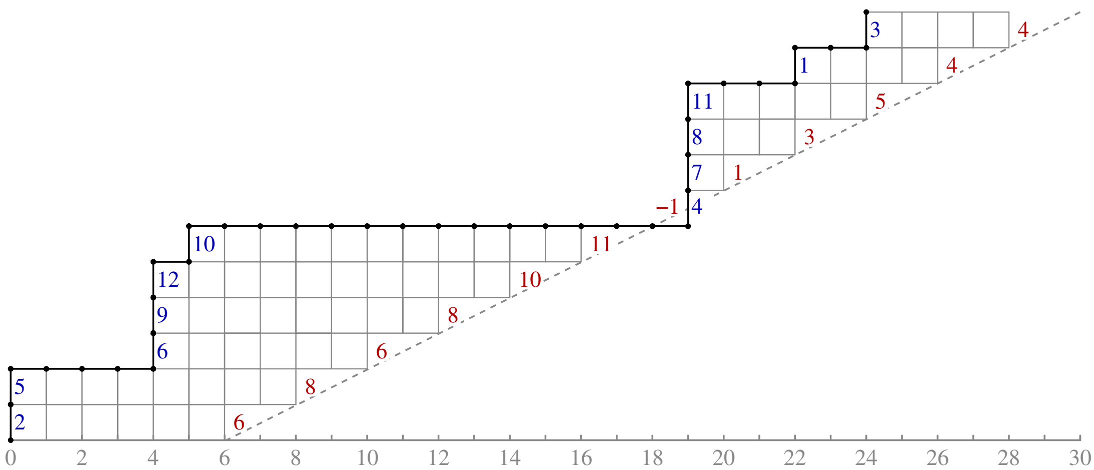

Figure 4.2: Labeled path for the function f' in Example 4.7.

Remark 4.9. The symmetric group $\mathfrak{S}_n$ acts on each set SKF_k by permuting the inputs: $\sigma \cdot f = f \circ \sigma^{-1}$ for $\sigma \in \mathfrak{S}_n$ and $f \in \mathrm{SKF}_k$. To see that $\sigma \cdot f$ does belong to SKF_k, note that f and $\sigma \cdot f$ have the same multiset of output values. So the (unlabeled) path of $\sigma \cdot f$ equals the path of f, which is k-skeletal by assumption on f.

4.5. Proof of Theorem 2.3.

Theorem 2.3 is an immediate consequence of Theorem 4.4, once we show that k-skeletal chip configurations (for given parameters $\mathcal{G}, n, c, m$) are the same thing as k-skeletal functions. Let $D : [n] \to \mathcal{G}$ be a chip configuration with area vector $g = (g_0, \ldots, g_{n-1})$. On one hand, D is a k-skeletal function iff g is a k-skeletal area vector iff g satisfies the conditions in Proposition 3.17. On the other hand, D is a k-skeletal chip configuration iff g satisfies the conditions in Theorem 2.10. The two sets of conditions are easily seen to be equivalent. This completes the proof.

We finish this section by elaborating on how the operators C and C^{-1} furnish the explicit bijections of Theorem 2.3. The reader may compare this approach to [AdDL16, Sec. 2, Sec.7]. In the special case of $c = m = 1$ and $\mathcal{G} = \mathbb{Z}$, our operator C is closely related to their ψ and C^{-1} to their φ.

Fix k between 0 and $n-1$, and let SKC_k^+ be the set of chip configurations $D : [n] \to \mathcal{G}$ that satisfy Conditions (C0) and (C1). Define a map $U : \mathrm{SKC}_k^+ \to \mathrm{SKC}_k^+$ as follows. For all $D \in \mathrm{SKC}_k$, let $U(D) = D$. For $D \in \mathrm{SKC}_k^+ \setminus \mathrm{SKC}_k$, D must fail Condition (C2). By the negation of Condition (C2′) in Lemma 2.7, there exists $p \in [n]$ such that borrowing at the poorest p vertices of D produces another configuration D' satisfying Conditions (C0) and (C1). Choose the minimal such p, and let $U(D) = D'$ be the result of applying this borrow move to D.

Next define a "projection operator" $U^* : \mathrm{SKC}_k^+ \to \mathrm{SKC}_k$ as follows. Starting at $D \in \mathrm{SKC}_k^+$, apply U zero or more times until reaching a configuration in SKC_k, and call this configuration $U^*(D)$. An argument like that in Lemma 3.13 shows that a k-skeletal configuration must be reached in finitely many steps. U^* is surjective since every k-skeletal configuration maps to itself. For $k' > k$, Condition (C1) for k' implies Condition (C1) for k; in particular, $\mathrm{SKC}_{k'} \subseteq \mathrm{SKC}_k^+$. Based on theorems proven so far, we can now see that the restriction of U^* to $\mathrm{SKC}_{k'}$ must be a *bijection* from $\mathrm{SKC}_{k'}$ to SKC_k. The reason is that each application of U corresponds to applying some power C^p to the labeled lattice path representing D. We have seen that each C-equivalence class of labeled paths contains exactly one k-skeletal configuration and exactly one k'-skeletal configuration. Thus, the restricted U^* is one-to-one and onto.

5. q-Analogues of skeletal paths and skeletal functions

Throughout this section, we take $\mathcal{G} = \mathbb{Z}$ and fix positive integer parameters n, m, c. A *statistic* on area vectors is any function $\mathrm{stat} : \mathrm{AV}_n \to \mathbb{Z}_{\geqslant 0}$. A *$q$-analogue of* $|\,\mathrm{SKV}_k\,|$ is the polynomial generating function $\mathrm{SKV}_k(q; \mathrm{stat}) = \sum_{g \in \mathrm{SKV}_k} q^{\mathrm{stat}(g)}$.

We prove the following q-analogue of Theorem 3.2 and Corollary 3.3.

Theorem 5.1. *Suppose $F : \mathbb{R} \to \mathbb{R}$ is a function satisfying $F(z) = F(c - z)$ for all $z \in \mathbb{R}$, and* $\mathrm{stat} : \mathrm{AV}_n \to \mathbb{Z}_{\geqslant 0}$ *satisfies*

$$\mathrm{stat}(g_0, g_1, \ldots, g_{n-1}) = \sum_{0 \leqslant i < j < n} F(g_i - g_j). \tag{5.1}$$

Then for all $k \in \{0, 1, \ldots, n-1\}$, $\mathrm{SKV}_k(q; \mathrm{stat}) = \mathrm{SKV}_{n-1}(q; \mathrm{stat})$. *This is a q-analogue of the m-ballot number $\frac{c}{(m+1)n+c}\binom{(m+1)n+c}{n}$ that is independent of k.*

Proof. Let $g = (g_0, g_1, \dots, g_{n-1})$ be any area vector. Since $C(g) = (g_1, \dots, g_{n-1}, g_0 - c)$, we have

$$\begin{aligned}\operatorname{stat}(C(g)) &= \sum_{1\leqslant i<j<n} F(g_i - g_j) + \sum_{0<j<n} F(g_j - (g_0 - c)) \\ &= \sum_{1\leqslant i<j<n} F(g_i - g_j) + \sum_{0<j<n} F(c - (g_0 - g_j)) \\ &= \sum_{1\leqslant i<j<n} F(g_i - g_j) + \sum_{0<j<n} F(g_0 - g_j) = \sum_{0\leqslant i<j<n} F(g_i - g_j) = \operatorname{stat}(g).\end{aligned}$$

Therefore, any area vector h reachable from g by applying powers of C has $\operatorname{stat}(h) = \operatorname{stat}(g)$. Recall (from Theorem 3.10 and the following statement) that there is a bijection $\mathrm{SKV}_k \to \mathrm{SKV}_{n-1}$ sending $g \in \mathrm{SKV}_k$ to the unique Dyck area vector $h \in \mathrm{SKV}_{n-1}$ with $g \sim h$. Using this bijection, we compute

$$\mathrm{SKV}_k(q; \operatorname{stat}) = \sum_{g\in \mathrm{SKV}_k} q^{\operatorname{stat}(g)} = \sum_{h\in \mathrm{SKV}_{n-1}} q^{\operatorname{stat}(h)} = \mathrm{SKV}_{n-1}(q; \operatorname{stat}). \qquad \square$$

Some statistics that arise in the theory of q, t-Catalan numbers have the form (5.1). Consider first the case $c = m = 1$. Jim Haglund's celebrated combinatorial formula [Hag04] for the q, t-Catalan numbers can be written $\mathrm{Cat}_n(q, t) = \sum_{g\in \mathrm{SKV}_{n-1}} q^{\operatorname{area}(g)} t^{\operatorname{bounce}(g)}$, where $\operatorname{area}(g_0, g_1, \dots, g_{n-1}) = g_0 + g_1 + \cdots + g_{n-1} - n$ and bounce is a certain statistic defined on the Dyck path with area vector g. We subtract n in the area formula since we compute area vectors relative to the reference line $x = y + 1$, rather than $x = y$. For instance, under our conventions, the path NENENE in Figure 3.3 has area vector $(1, 1, 1)$ and area statistic 0. Under the classical conventions in [Hag08], the area vector of NENENE would be $(0, 0, 0)$.

In a personal communication to Haglund in 2000, Mark Haiman conjectured a companion formula $\mathrm{Cat}_n(q, t) = \sum_{g\in \mathrm{SKV}_{n-1}} q^{\operatorname{dinv}(g)} t^{\operatorname{area}(g)}$, where $\operatorname{dinv}(g_0, g_1, \dots, g_{n-1})$ is the number of pairs $i < j$ with $g_i - g_j \in \{0, 1\}$. There is an explicit bijection proving the equivalence of Haglund's and Haiman's formulas [Hag08, Thm. 3.15]. Haiman's dinv statistic has the form (5.1) if we take F to be the characteristic function of $\{0, 1\}$, namely $F(0) = F(1) = 1$ and $F(z) = 0$ for all other real z. The bijection just cited shows that dinv and area are equidistributed on Dyck vectors. Hence, Theorem 5.1 shows that for every k,

$$\sum_{g\in \mathrm{SKV}_k} q^{\operatorname{dinv}(g)} = \sum_{h\in \mathrm{SKV}_{n-1}} q^{\operatorname{area}(h)} = \mathrm{Cat}_n(q, 1),$$

which is the q-analogue of the Catalan numbers first studied by Fürlinger and Hofbauer [FH85].

For parameters $m = 1$ and $c \geqslant 1$, we can define $\operatorname{dinv}^{(c)}(g)$ to be the number of pairs $i < j$ with $0 \leqslant g_i - g_j \leqslant c$. In this variation, $F(z) = 1$ for $0 \leqslant z \leqslant c$ and $F(z) = 0$ otherwise. For parameters $c = 1$ and $m \geqslant 1$, we can use the function $F(z) = \max(0, m + 1/2 - |z - 1/2|)$. On integer inputs, $F(z) = m + 1 - z$ for $1 \leqslant z \leqslant m$, $F(z) = m + z$ for $-m \leqslant z \leqslant 0$, and $F(z) = 0$ otherwise. So this F recovers the "slope-m dinv statistic" used to define higher-order q, t-Catalan numbers [Loe05, Sec. 2.1]. For general $c, m \in \mathbb{Z}_{>0}$, we can take F to be the piecewise linear function that goes from $(-m, 0)$ to $(0, m)$ to (c, m) to $(c+m, 0)$ and is 0 outside this range. Or we may use $F(z) = \max(0, m + c/2 - |z - c/2|)$.

The q, t-analogue $\mathrm{SKV}_k(q, t) = \sum_{g \in \mathrm{SKV}_k} q^{\mathrm{dinv}(g)} t^{\mathrm{area}(g)}$ is not independent of k. It would be interesting to compare the combinatorial properties of these polynomials to q, t-Catalan polynomials and their generalizations. Alternatively, perhaps there is a k-skeletal variation of the area statistic on SKV_k which, when paired with dinv, does give the q, t-Catalan polynomial for every k. The q, t-Catalan numbers satisfy the joint symmetry property $\mathrm{Cat}_n(q, t) = \mathrm{Cat}_n(t, q)$. This joint symmetry is not true for $k < n - 1$, as seen in the next example.

Example 5.2. By computing dinv for the paths shown in Figure 3.3 from left to right, we find

$$\begin{aligned}\mathrm{SKV}_0(q) = \mathrm{SKV}_1(q) = \mathrm{SKV}_2(q) &= q^3 + q^1 + q^2 + q^1 + q^0 \\ &= 1 + 2q + q^2 + q^3 = \mathrm{Cat}_3(q, 1) = \mathrm{Cat}_3(1, q).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{SKV}_0(q, t) &= q^3 t^0 + q^1 t^{-2} + q^2 t^{-1} + q^1 t^{-1} + q^0 t^{-3};\\ \mathrm{SKV}_1(q, t) &= q^3 t^0 + q^1 t^1 + q^2 t^1 + q^1 t^{-1} + q^0 t^0;\\ \mathrm{SKV}_2(q, t) &= q^3 t^0 + q^1 t^1 + q^2 t^1 + q^1 t^2 + q^0 t^3 = \mathrm{Cat}_3(q, t) = \mathrm{Cat}_3(t, q).\end{aligned}$$

For general n, the duality property (cf. Figure 3.2) implies that $\mathrm{SKV}_0(q, t^{-1}) = \mathrm{SKV}_{n-1}(q, t) = \mathrm{Cat}_n(q, t)$. Indeed, using the classical version of area vectors, rotating the Dyck path replaces the classical area vector $(g_0, g_1, \ldots, g_{n-1})$ by $(-g_{n-1}, \ldots, -g_1, -g_0)$. This preserves dinv and negates the area statistic. But for $0 < k < n-1$, there are no obvious joint symmetry properties of $\mathrm{SKV}_k(q, t)$.

The q, t-Catalan numbers (defined using area and bounce) were given chip-firing interpretations in [DL13]. A result similar to Theorem 5.1 holds for q-analogues of k-skeletal functions.

Theorem 5.3. *Extend C to act on pairs consisting of an area vector g and a label sequence w by writing*

$$C((g_0, g_1, \ldots, g_{n-1}), (w_0, w_1, \ldots, w_{n-1})) = ((g_1, \ldots, g_{n-1}, g_0 - c), (w_1, \ldots, w_{n-1}, w_0)).$$

If stat *is a statistic on labeled paths that satisfies* $\mathrm{stat}(C(g, w)) = \mathrm{stat}(g, w)$ *for all inputs* (g, w), *then* $\sum q^{\mathrm{stat}(g,w)}$ *(summed over all pairs* (g, w) *encoding functions in* SKF_k*) is a q-analogue of* $|\,\mathrm{SKF}_k\,|$ *that is independent of k.*

This theorem follows from the bijective proof of Theorem 4.4 (Section 4.4).

For example, let $m = c = 1$ and define $\mathrm{dinv}(g, w)$ to be the number of pairs $i < j$ with $g_i - g_j = 0$ and $w_i < w_j$, or $g_i - g_j = 1$ and $w_i > w_j$ (see [Hag08, Chapter 5]). It is easy to check that dinv is preserved by C. The q, t-analogue of classical parking functions studied in q, t-Catalan theory is $\sum_{(g,w)} q^{\mathrm{dinv}(g)} t^{\mathrm{area}(g,w)}$ where we sum over labeled Dyck paths of height n. Setting $t = 1$ and summing over labeled paths for k-skeletal functions gives a q-analogue that is independent of k. Here too, perhaps there are variations of area (depending on k) that pair with dinv to give the same q, t-polynomial for every k.

6. First-return recursion and bijection

Corollary 3.3 provides an enumeration of the number of k-skeletal paths when $\mathcal{G} = \mathbb{Z}$. Here we give an alternative proof when $c = 1$ using a generalization of the first-return recursion for the Catalan numbers.

For any $n \geqslant 0$ we identify elements of $P_n(\mathbb{Z})$ with the classical lattice paths starting at the origin. We write $\mathcal{D}_n^m$ for the set of *augmented m-Dyck paths of height n*, that is, the subset of paths in $P_n(\mathbb{Z})$ that remain weakly above the line $x = my$ except for their last step, which we require to be an east step ending at $(mn+1, n)$. The cardinality of $\mathcal{D}_n^m$ is well known to be given by the Fuss–Catalan number $C_n^m = \frac{1}{mn+1}\binom{mn+n}{n}$ (see [Duc00, vF95]). We often identify a path in $\mathcal{D}_n^m$ with a certain string of n north steps and $mn + 1$ east steps, which may be translated to a starting point other than the origin in some situations.

Proposition 6.1 (See [Cat87, LF03, Ted11]). *For $n, m > 0$, any $\pi \in \mathcal{D}_n^m$ has a unique decomposition of the form*

$$N\pi_1\pi_2 \cdots \pi_{m+1} \tag{6.1}$$

where each $\pi_j \in \mathcal{D}_{p_j}^m$ for some $p_j \geqslant 0$ and $p_1 + \cdots + p_{m+1} = n - 1$. Furthermore, if $G(\pi) = (g_0, g_1, \ldots, g_{n-1})$, then for any i with $1 \leqslant i \leqslant p_1 + \cdots + p_j$, $g_i \geqslant m - j + 2$.

Proof. Let $\pi \in \mathcal{D}_n^m$. We note that π begins on the line $x = my$ and ends on the line $x = my+1$. Consider any point (x_0, y_0) on π lying on the line $x = my - b$ for some $b \in \mathbb{Z}$. If the next step in π is a north step, then the next point on π lies on the line $x = my - b - m$; if an east step, then the next point on π lies on the line $x = my - b + 1$.

Since π stays weakly above $x = my$ except at the end, it must begin with a north step. So the first point on the path after the starting point of $(0, 0)$ is $(0, 1)$, which is on the line $x = my - m$. As π ends at the point $(mn + 1, n)$, which is on the line $x = my + 1$, it must visit each of the lines $x = my - i$ at least once, for each i in the range $m \geqslant i \geqslant -1$. Furthermore, its initial visit to $x = my - j$ must occur before any visit to $x = my - i$ whenever $m \geqslant j > i \geqslant -1$. Letting N denote a north step and E an east step, it follows that π has a unique factorization of the form $\mathrm{N}\pi_1\pi_2 \cdots \pi_{m+1}$ where for each j with $1 \leqslant j \leqslant m + 1$, $\mathrm{N}\pi_1\pi_2 \cdots \pi_j$ is the shortest initial segment of π ending on the line $x = my - m + j$. By construction, each $\pi_j \in \mathcal{D}_{p_j}^m$ for some $p_j \geqslant 0$, as otherwise there would be a shorter initial segment ending on the line $x = my - m + j$. Since π ends at $(mn+1, n)$, it follows that $p_1 + \cdots + p_{m+1} = n - 1$.

The claimed inequalities satisfied by the area vector follow from the fact that the north steps in $\mathrm{N}\pi_1 \cdots \pi_j$ (except the first one) all start weakly left of $x = my - m + j - 1$. So the area vector entries for this part of the path satisfy

$$g_i = mi + 1 - x_i \geqslant mi + 1 - (mi - m + j - 1) = m - j + 2. \qquad \square$$

Corollary 6.2. *We have* $C_n^m = \sum_{\substack{i_1+\cdots+i_{m+1}=n-1 \\ i_j \geqslant 0 \text{ for } 1 \leqslant j \leqslant m+1}} C_{i_1}^m C_{i_2}^m \cdots C_{i_{m+1}}^m.$

Let $\mathcal{D}_n^{m,k}$ denote the set of k-skeletal paths for parameters m and n (again augmenting each path with one east step at the end), and let $C_n^{m,k} = |\mathcal{D}_n^{m,k}|$.

Theorem 6.3. *For all $m, n \geqslant 1$ and $0 \leqslant k \leqslant n-1$, $C_n^m = C_n^{m,k}$ via an explicit bijection.*

Proof. We consider m fixed and prove the theorem by induction on n. For each k, m, and n, we define a map $\varphi_n^{m,k} : \mathcal{D}_n^m \to \mathcal{D}_n^{m,k}$ that we prove to be a bijection. To avoid clutter, we will frequently suppress the dependencies of $\varphi_n^{m,k}$ on m, n and k and simply write φ.

Given a path $\pi \in \mathcal{D}_n^m$, find the unique decomposition (as in Proposition 6.1)

$$\pi = \mathrm{N}\pi_1\pi_2\cdots\pi_{m+1} \tag{6.2}$$

such that each $\pi_i \in \mathcal{D}_{p_i}^m$ for some $p_i \geqslant 0$. Choose the maximum $s \in \{1, 2, \ldots, m+2\}$ such that $p_1 + \cdots + p_{s-1} \leqslant k$ (such s exists since the sum on the left side is zero for $s = 1$). Write $\overleftarrow{\sigma}$ for the stepwise reversal of a path σ. We define φ by

$$\varphi(\pi) = \begin{cases} \pi, & n = k+1, \\ \overleftarrow{\pi_{s+1}\cdots\pi_{m+1}}\mathrm{N}\pi_1\cdots\pi_{s-1}\pi_s, & n > k+1 \geqslant p_s, \\ \overleftarrow{\pi_{s+1}\cdots\pi_{m+1}}\mathrm{N}\pi_1\cdots\pi_{s-1}\varphi_{p_s}^{m,k}(\pi_s), & n > k+1 < p_s. \end{cases} \tag{6.3}$$

Since $p_1 + \cdots + p_{m+1} = n-1$ and $0 \leqslant k \leqslant n-1$, we have $s = m+2$ if and only if $k+1 = n$. Hence, in the definition of φ above, and in the discussion below, we have $s \leqslant m+1$ whenever $n > k+1$.

For any n, when $k = n-1$ it is routine to check that m-Dyck paths coincide with k-skeletal paths, so the identity map gives the required bijection. This case covers the base case $n = 1$ of the induction. From now on, fix $n > 1$ and assume $\varphi_p^{m,k}$ is a bijection for all $p < n$.

In the rest of this proof, define the *level* of a point $(x, y) \in \mathbb{Z}^2$ to be $\mathrm{lvl}(x, y) = my - x$, which is the signed horizontal distance from (x, y) to the line $x = my$. We rephrase Conditions (P0), (P1), and (P2) for k-skeletal paths (see Section 3.1) in terms of levels. Condition (P0) is automatically satisfied since all lattice paths considered here start at the origin. Since $c = 1$ here, Conditions (P1) and (P2) may be rewritten as:

(P1′) The last $k+1$ north steps of the path start at levels $\geqslant 0$.

(P2′) There do not exist $k+1$ consecutive rows such that the north steps in those rows all start at levels > 0.

Suppose π_i (one of the factors in (6.2)) appears as a subword of some path and starts in that path at level ℓ. Then all steps of π_i start at levels $\geqslant \ell$, and the final east step of π_i ends at level $\ell - 1$. Next suppose $\overleftarrow{\pi_i}$ appears as a subword of some path and starts in that path at level ℓ. Then the first step of $\overleftarrow{\pi_i}$ goes east to level $\ell - 1$, all subsequent steps of $\overleftarrow{\pi_i}$ start at levels $< \ell$, and $\overleftarrow{\pi_i}$ ends at level $\ell - 1$.

Using the above observations, consider the levels reached by various subpaths of $\varphi(\pi)$ when $n > k+1$. First consider the shared prefix $\overleftarrow{\pi_{s+1}\cdots\pi_{m+1}}\mathrm{N}$. If $s = m+1$ then $\overleftarrow{\pi_{s+1}\cdots\pi_{m+1}}$ is the empty word. If $s < m+1$, $\overleftarrow{\pi_{s+1}\cdots\pi_{m+1}}$ is the concatenation of $m+1-s$ factors $\overleftarrow{\pi_{m+1}}, \ldots, \overleftarrow{\pi_{s+1}}$, it starts at level 0 with an east step, it has all subsequent steps starting at negative levels, and it ends at level $s - (m+1) < 0$. In either case, the terminal north step of $\overleftarrow{\pi_{s+1}\cdots\pi_{m+1}}\mathrm{N}$ starts at level $s - (m+1) \leqslant 0$ and ends at level $s - 1 \geqslant 0$; this north step is the first step in $\varphi(\pi)$ ending at a nonnegative level.

We now consider the subpath $\pi_1 \cdots \pi_{s-1}$. If $s - 1 = 0$ then it is empty. Otherwise, by the previous two paragraphs we know it has all its steps starting at positive levels and its final step ending at level 0. Finally, we consider the subpath of $\varphi(\pi)$ following $\pi_1 \cdots \pi_{s-1}$. In the case of $k+1 \geqslant p_s$, the final subpath π_s has all its steps starting at levels $\geqslant 0$, and ends at level -1. In the case of $k+1 < p_s$, the final subpath $\varphi^{m,k}_{p_s}(\pi_s)$ of $\varphi(\pi)$ starts at level 0, ends at level -1, and satisfies Conditions (P1′) and (P2′) by the induction hypothesis.

We now prove that $\varphi(\pi)$ satisfies Condition (P1′) when $n > k+1$. We know that $p_1 + \cdots + p_s \geqslant k+1$, by definition of s. In the case of $k+1 \geqslant p_s$, the last $k+1$ north steps in $\varphi(\pi)$ all appear in the suffix $\pi_1\pi_2 \cdots \pi_s$. By the preceding analysis, all these north steps start at levels $\geqslant 0$, as needed. In the case of $k+1 < p_s$, the last $k+1$ north steps in $\varphi(\pi)$ all appear in the suffix $\varphi^{m,k}_{p_s}(\pi_s)$. By induction hypothesis, these north steps all start at levels $\geqslant 0$.

We now prove that $\varphi(\pi)$ satisfies Condition (P2′) when $n > k+1$. To get a contradiction, assume there are $k+1$ rows in $\varphi(\pi)$ where the north steps in those rows all start at levels > 0. The last factor of $\varphi(\pi)$, namely π_s when $k+1 \geqslant p_s$ or $\varphi^{m,k}_{p_s}(\pi_s)$ when $k+1 < p_s$, either has no north steps at all or has first north step starting at level 0. The north steps violating Condition (P2′) must either all occur after this north step or all occur before it. We rule out the first possibility as follows. When $k+1 \geqslant p_s$ there are not enough north steps in π_s (following the first north step) to cause a violation. When $k+1 < p_s$ we reach the same conclusion by invoking the induction hypothesis to see that $\varphi^{m,k}_{p_s}(\pi_s)$ has no violation. Next we rule out the possibility of a violation earlier in $\varphi(\pi)$. By the level analysis above, the $k+1$ violating north steps must all occur in the subword $\pi_1\pi_2 \cdots \pi_{s-1}$ of $\varphi(\pi)$. But, regardless of how $k+1$ compares to p_s, $p_1 + \cdots + p_{s-1} < k+1$ by definition of s, so there are not enough available north steps in this region to cause a violation.

So far, we have proved that $\varphi^{m,k}_n$ maps $\mathcal{D}^m_n$ into $\mathcal{D}^{m,k}_n$. It remains to show that $\varphi^{m,k}_n$ is a bijection. We define an inverse $\psi = \psi^{m,k}_n : \mathcal{D}^{m,k}_n \to \mathcal{D}^m_n$ recursively as follows. Decompose $\omega \in \mathcal{D}^{m,k}_n$ as $\omega = \omega_1 \mathrm{N} \omega_2 \omega_3$ where:

- $\omega_1\mathrm{N}$ is the shortest initial segment of ω ending at a nonnegative level;
- $\omega_1\mathrm{N}\omega_2$ is the shortest initial segment ending at level zero.

Note that ω_1 is either empty or begins with an east step and that ω_2 may be empty. Let p be the number of north steps in ω_3. We define ψ using three cases.

$$\psi(\omega) = \begin{cases} \omega, & n = k+1, \\ \mathrm{N}\omega_2\omega_3\overleftarrow{\omega_1}, & n > k+1 \geqslant p, \\ \mathrm{N}\omega_2\psi^{m,k}_p(\omega_3)\overleftarrow{\omega_1}, & n > k+1 < p. \end{cases}$$

By a level analysis similar to what appears above, one may show that: ω_3 belongs to $\mathcal{D}^{m,k}_p$ when $n > k+1 < p$; ψ does map $\mathcal{D}^{m,k}_n$ into $\mathcal{D}^m_n$; and ψ is the two-sided inverse of φ. We omit these details. □

Example 6.4. Consider $k = 1$ and

$$\pi = \mathrm{N}\,\overbrace{\mathrm{NEEE}}^{\pi_1}\,\overbrace{\mathrm{E}}^{\pi_2}\,\overbrace{\mathrm{NNENEEEENNEEEENEEEEEE}}^{\pi_3} \in \mathcal{D}^2_8,$$

as illustrated in Figure 6.1(a). Since $s = 3$, we find that $\varphi(\pi) = \mathrm{N}\pi_1\pi_2\varphi(\pi_3) = \mathrm{NNEEEE}\varphi(\pi_3)$. We now decompose π_3 as

$$\pi_3 = \mathrm{N}\,\overbrace{\mathrm{NENEEENNEEEENEEEE}}^{\pi'_1}\,\overbrace{\mathrm{E}}^{\pi'_2}\,\overbrace{\mathrm{E}}^{\pi'_3},$$

from which it follows that $s' = 1$ and that

$$\varphi(\pi_3) = \overleftarrow{\pi'_2\pi'_3}\mathrm{N}\varphi(\pi'_1) = \mathrm{EEN}\varphi(\pi'_1).$$

Then we decompose π'_1 as

$$\pi'_1 = \mathrm{N}\,\overbrace{\mathrm{E}}^{\pi''_1}\,\overbrace{\mathrm{NEEE}}^{\pi''_2}\,\overbrace{\mathrm{NNEEENEEEE}}^{\pi''_3},$$

so $s'' = 3$ and $\varphi(\pi'_1) = \mathrm{N}\pi''_1\pi''_2\varphi(\pi''_3) = \mathrm{NENEEE}\varphi(\pi''_3)$. We now decompose π''_3 as

$$\pi''_3 = \mathrm{N}\,\overbrace{\mathrm{NEEE}}^{\pi'''_1}\,\overbrace{\mathrm{NEEE}}^{\pi'''_2}\,\overbrace{\mathrm{E}}^{\pi'''_3},$$

from which it follows that $s''' = 2$ and $\varphi(\pi''_3) = \overleftarrow{\pi'''_3}\mathrm{N}\pi'''_1\varphi(\pi'''_2) = \mathrm{ENNEEE}\varphi(\pi'''_2)$. Finally, when we decompose π'''_2 we find that $s'''' = 4$, so $\varphi(\pi'''_2) = \pi'''_2 = \mathrm{NEEE}$. Combining our results, we conclude that

$$\begin{aligned}
\varphi(\pi) &= \mathrm{NNEEEE}\varphi(\pi_3)\\
&= \mathrm{NNEEEE\,EEN}\varphi(\pi'_1)\\
&= \mathrm{NNEEEE\,EEN\,NENEEE}\,\varphi(\pi''_3)\\
&= \mathrm{NNEEEE\,EEN\,NENEEE\,ENNEEE}\varphi(\pi'''_2)\\
&= \mathrm{NNEEEE\,EEN\,NENEEE\,ENNEEE\,NEEE},
\end{aligned}$$

as illustrated in Figure 6.1(b).

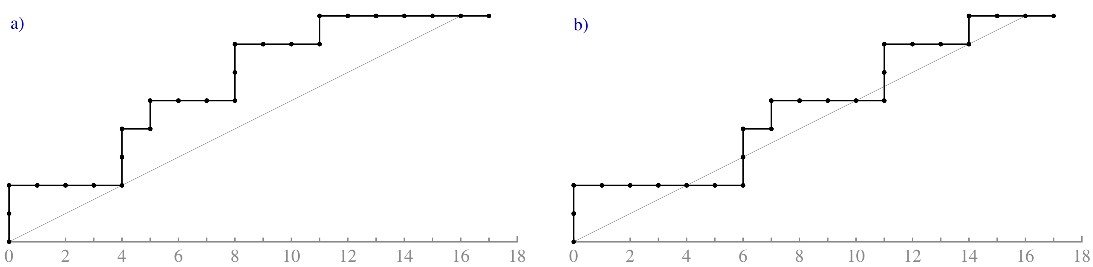

Figure 6.1: (a) The path π from Example 6.4 and (b) its image under φ. Note that we are not showing the reference line $x = my + c$, but rather the standard diagonal $x = my$.

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References

[AB10] Arash Asadi and Spencer Backman. Chip-firing and Riemann–Roch theory for directed graphs. 2010. arXiv:1012.0287.

[ABKS14] Yang An, Matthew Baker, Greg Kuperberg, and Farbod Shokrieh. Canonical representatives for divisor classes on tropical curves and the Matrix–Tree Theorem. In *Forum of Mathematics, Sigma*, volume 2, page e24. Cambridge University Press, 2014. doi:10.1017/fms.2014.25.

[AdDL16] Jean-Christophe Aval, Michele d'Adderio, Mark Dukes, and Yvan Le Borgne. Two operators on sandpile configurations, the sandpile model on the complete bipartite graph, and a cyclic lemma. *Advances in Applied Mathematics*, 73:59–98, 2016. doi:10.1016/j.aam.2015.09.018.

[ALW16] Drew Armstrong, Nicholas A. Loehr, and Gregory S. Warrington. Rational parking functions and Catalan numbers. *Annals of Combinatorics*, 20:21–58, 2016. doi:10.1007/s00026-015-0293-6.

[AM10] Omid Amini and Madhusudan Manjunath. Riemann-Roch for sub-lattices of the root lattice A_n. *The Electronic Journal of Combinatorics*, 17:R124–R124, 2010. doi:10.37236/396.

[ARW13] Drew Armstrong, Brendon Rhoades, and Nathan Williams. Rational Catalan combinatorics: The associahedron. In *DMTCS Proceedings vol. AS, 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013)*, pages 933–944. Discrete Mathematics & Theoretical Computer Science, Episciences. org, 2013. doi:10.46298/dmtcs.2355.

[Bac12] Spencer Backman. A bijection between the recurrent configurations of a hereditary chip-firing model and spanning trees. 2012. arXiv:1207.6175.

[BCT10] Brian Benson, Deeparnab Chakrabarty, and Prasad Tetali. G-parking functions, acyclic orientations and spanning trees. *Discrete Mathematics*, 310(8):1340–1353, 2010. doi:10.1016/j.disc.2010.01.002.

[BHM$^+$23] Jonah Blasiak, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. A shuffle theorem for paths under any line. *Forum Math. Pi*, 11:Paper No. e5, 38, 2023. doi:10.1017/fmp.2023.4.

[Big99] N. Biggs. Chip-firing and the critical group of a graph. *J. of Alg. Comb.*, 9:25–45, 1999. doi:10.1023/A:1018611014097.

[BKO+24] Zach Benton, Jane Kwak, SuHo Oh, Mateo Torres, and Mckinley Xie. On z-superstable and critical configurations of chip firing pairs. 2024. arXiv:2412.02679.

[BLHN97] Roland Bacher, Pierre de La Harpe, and Tatiana Nagnibeda. The lattice of integral flows and the lattice of integral cuts on a finite graph. *Bulletin de la société mathématique de France*, 125(2):167–198, 1997. doi:10.24033/bsmf.2303.

[BLS91] Anders Björner, László Lovász, and Peter W. Shor. Chip-firing games on graphs. *European Journal of Combinatorics*, 12(4):283–291, 1991. doi:10.1016/S0195-6698(13)80111-4.

[BLW] Spencer Backman, Nicholas A. Loehr, and Gregory S. Warrington. Quantized rational chip firing. In preparation.

[BM23] François Bergeron and Mikhail Mazin. Combinatorics of triangular partitions. *Enumer. Comb. Appl.*, 3(1):Paper No. S2R1, 20, 2023. doi:10.54550/eca2023v3s1r1.

[BN07] Matthew Baker and Serguei Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007. doi:10.1016/j.aim.2007.04.012.

[BS13] Matthew Baker and Farbod Shokrieh. Chip-firing games, potential theory on graphs, and spanning trees. *Journal of Combinatorial Theory, Series A*, 120(1):164–182, 2013. doi:10.1016/j.jcta.2012.07.011.

[BTW87] Per Bak, Chao Tang, and Kurt Wiesenfeld. Self-organized criticality: An explanation of the $1/f$ noise. *Physical Review Letters*, 59(4):381–384, Jul 1987. doi:10.1103/PhysRevLett.59.381.

[Cat87] M. E. Catalan. Sur les nombres de Segner. *Rend. Circ. Mat. Palermo*, 1:190–201, 1887. doi:10.1007/BF03020089.

[Cay89] Arthur Cayley. A theorem on trees. *Quarterly Journal of Mathematics*, 23:376–378, 1889. doi:10.1017/CBO9780511703799.010.

[CL03] Robert Cori and Yvan Le Borgne. The sand-pile model and Tutte polynomials. *Advances in Applied Mathematics*, 30(1-2):44–52, 2003. doi:10.1016/S0196-8858(02)00524-9.

[CL13] Robert Cori and Yvan Le Borgne. The Riemann-Roch theorem for graphs and the rank in complete graphs. 2013. arXiv:1308.5325.

[CLB16] Robert Cori and Yvan Le Borgne. On computation of Baker and Norine's rank on complete graphs. *Electron. J. Combin.*, 23(1):Paper 1.31, 47, 2016. doi:10.37236/4350.

[CP18] Scott Corry and David Perkinson. *Divisors and sandpiles*, volume 114. American Mathematical Soc., 2018. doi:10.1090/mbk/114.

[CPS12] Sergio Caracciolo, Guglielmo Paoletti, and Andrea Sportiello. Multiple and inverse topplings in the abelian sandpile model. *The European Physical Journal Special Topics*, 212(1):23–44, 2012. doi:10.1140/epjst/e2012-01652-9.

[CPS16] Robert Cori, Pasquale Petrullo, and Domenico Senato. Hall–Littlewood symmetric functions via the chip-firing game. *European Journal of Combinatorics*, 58:225–237, 2016. doi:10.1016/j.ejc.2016.05.008.

[CRS02] Robert Cori, Dominique Rossin, and Bruno Salvy. Polynomial ideals for sandpiles and their Gröbner bases. *Theoretical Computer Science*, 276(1):1–15, 2002. doi:10.1016/S0304-3975(00)00397-2.

[DDI$^+$24] Michele D'Adderio, Mark Dukes, Alessandro Iraci, Alexander Lazar, Yvan Le Borgne, and Anna Vanden Wyngaerd. Shuffle theorems and sandpiles. 2024. arXiv:2401.06488.

[Dha90] Deepak Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64:1613–1616, 1990. doi:10.1103/PhysRevLett.64.1613.

[DK21] Anton Dochtermann and Westin King. Trees, parking functions, and standard monomials of skeleton ideals. *Australasian Journal of Combinatorics*, 81(1):126–151, 2021.

[DL13] Mark Dukes and Yvan Le Borgne. Parallelogram polyominoes, the sandpile model on a complete bipartite graph, and a q, t-Narayana polynomial. *Journal of Combinatorial Theory, Series A*, 120(4):816–842, 2013. doi:10.1016/j.jcta.2013.01.004.

[Doc17] Anton Dochtermann. One-skeleta of G-parking function ideals: resolutions and standard monomials. 2017. arXiv:1708.04712.

[Duc00] Philippe Duchon. On the enumeration and generation of generalized Dyck words. *Discrete Mathematics*, 225(1):121–135, 2000. FPSAC'98. doi:10.1016/S0012-365X(00)00150-3.

[ECAM18] Erik Carlsson and Anton Mellit. A proof of the Shuffle Conjecture. *Journal of the American Mathematical Society*, 31(3):pp. 661–697, 2018. doi:10.1090/jams/893.

[FH85] J. Fürlinger and J. Hofbauer. q-Catalan numbers. *Journal of Combinatorial Theory, Series A*, 40(2):248–264, 1985. doi:10.1016/0097-3165(85)90089-5.

[Gab93] Andrei Gabrielov. Abelian avalanches and Tutte polynomials. *Phys. A*, 195(1-2):253–274, 1993. doi:10.1016/0378-4371(93)90267-8.

[GH16] Petar Gaydarov and Sam Hopkins. Parking functions and tree inversions revisited. *Advances in Applied Mathematics*, 80:151–179, 2016. doi:10.1016/j.aam.2016.06.010.

[GK15] Johnny Guzmán and Caroline Klivans. Chip-firing and energy minimization on M-matrices. *J. Combin. Theory Ser. A*, 132:14–31, 2015. doi:10.1016/j.jcta.2014.12.002.

[GM14] E. Gorsky and M. Mazin. Compactified Jacobians and q, t-Catalan numbers, II. *J. Alg. Comb.*, 39:153–186, 2014. doi:10.1007/s10801-013-0443-z.

[Hag04] J. Haglund. A combinatorial model for the Macdonald polynomials. *Proceedings of the National Academy of Sciences*, 101(46):16127–16131, 2004. doi:10.1073/pnas.0405567101.

[Hag08] J. Haglund. *The q,t-Catalan Numbers and the Space of Diagonal Harmonics: With an Appendix on the Combinatorics of Macdonald Polynomials*. University Lecture Series. American Mathematical Soc., 2008.

[HHL+05] James Haglund, Mark D. Haiman, Nicholas A. Loehr, Jeffrey B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. *Duke Mathematical Journal*, 126(2):195–232, 2005. doi:10.1215/S0012-7094-04-12621-1.

[Kli18] Caroline Klivans. *The Mathematics of Chip-firing*. Chapman & Hall, 2018. doi:10.1201/9781315206899.

[KLR22] Chanchal Kumar, Gargi Lather, and Amit Roy. Standard monomials of 1-skeleton ideals of graphs and generalized signless Laplacians. *Linear Algebra and its Applications*, 637:24–48, 2022. doi:10.1016/j.laa.2021.12.003.

[KLS21] Chanchal Kumar, Gargi Lather, and Sonica. Skeleton ideals of certain graphs, standard monomials and spherical parking functions. *The Electronic Journal of Combinatorics*, 28(1):P1–53, 2021. doi:10.37236/9874.

[Kre80] G. Kreweras. Une famille de polynômes ayant plusieurs propriétés énumeratives. *Periodica Mathematica Hungarica*, 11(4):309–320, 1980. doi:10.1007/BF02107572.

[KW66] Alan G. Konheim and Benjamin Weiss. An occupancy discipline and applications. *SIAM Journal on Applied Mathematics*, 14(6):1266–1274, 1966. doi:10.1137/0114101.

[KY08] Dimitrije Kostić and Catherine H Yan. Multiparking functions, graph searching, and the Tutte polynomial. *Advances in Applied Mathematics*, 40(1):73–97, 2008. doi:10.1016/j.aam.2007.03.001.

[LF03] Peter Larcombe and David French. The Catalan number k-fold self-convolution identity: the original formulation. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 46:191–204, 01 2003.

[Loe03] Nicholas A. Loehr. Trapezoidal lattice paths and multivariate analogues. *Advances in Applied Mathematics*, 31(4):597–629, 2003. doi:10.1016/S0196-8858(03)00028-9.

[Loe05] Nicholas A. Loehr. Conjectured statistics for the higher q,t-Catalan sequences. *Electron. J. Comb.*, 12, 2005. doi:10.37236/1906.

[Loe17] N. Loehr. *Combinatorics*. CRC Press, second edition, 2017. doi:10.1201/9781315153360.

[Lor89] Dino J. Lorenzini. Arithmetical graphs. *Math. Ann.*, 285(3):481–501, 1989. doi:10.1007/BF01455069.

[Lor91] Dino J. Lorenzini. A finite group attached to the Laplacian of a graph. *Discrete Math.*, 91(3):277–282, 1991. doi:10.1016/0012-365X(90)90236-B.

[LP09] L. Levine and Y. Peres. Strong spherical asymptotics for rotor-router aggregation and the divisible sandpile. *Potential Anal.*, 30:1–27, 2009. doi:10.1007/s11118-008-9104-6.

[Mos72] KM Mosesian. Strongly basable graphs. In *Akad. Nauk. Armian. SSR Dokl*, volume 54, pages 134–138, 1972.

[MS13] Madhusudan Manjunath and Bernd Sturmfels. Monomials, binomials and Riemann–Roch. *Journal of Algebraic Combinatorics*, 37:737–756, 2013. doi:10.1007/s10801-012-0386-9.

[MZ08] G. Mikhalkin and I. Zharkov. *Tropical curves, their Jacobians and theta functions*, pages 203–230. Amer. Math. Soc., Providence, RI, 2008. doi:10.1090/conm/465/09104.

[PPW13] David Perkinson, Jacob Perlman, and John Wilmes. Primer for the algebraic geometry of sandpiles. In *Tropical and non-Archimedean geometry*, volume 605 of *Contemp. Math.*, pages 211–256. Amer. Math. Soc., Providence, RI, 2013. doi:10.1090/conm/605/12117.

[Pre86] Oliver Pretzel. On reorienting graphs by pushing down maximal vertices. *Order*, 3:135–153, 1986. doi:10.1007/BF00390104.

[Pro02] James Propp. Lattice structure for orientations of graphs. 2002. arXiv:math/0209005.

[PS04] Alexander Postnikov and Boris Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans. Amer. Math. Soc.*, 356(8):3109–3142, 2004. doi:10.1090/S0002-9947-04-03547-0.

[PYY17] David Perkinson, Qiaoyu Yang, and Kuai Yu. G-parking functions and tree inversions. *Combinatorica*, 37:269–282, 2017. doi:10.1007/s00493-015-3191-y.

[Ray70] Michel Raynaud. Spécialisation du foncteur de Picard. *Publications Mathématiques de l'IHÉS*, 38:27–76, 1970. URL: https://www.numdam.org/item/PMIHES_1970__38__27_0/.

[Rio69] John Riordan. Ballots and trees. *Journal of Combinatorial Theory*, 6(4):408–411, 1969. doi:10.1016/S0021-9800(69)80039-6.

[Roy20] Amit Roy. Standard monomials of 1-skeleton ideals of multigraphs. 2020. arXiv:2010.14474.

[SF99] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999. doi:10.1017/CBO9780511609589.

[Sta13] Richard P. Stanley. Catalan Addendum, 2013. Version of May 25, 2013; accessed Aug 09, 2024. URL: https://math.mit.edu/~rstan/ec/catadd.pdf.

[SW18] Richard P. Stanley and Yinghui Wang. Some aspects of (r, k)-parking functions. *Journal of Combinatorial Theory, Series A*, 159:54–78, 2018. doi:10.1016/j.jcta.2018.05.003.

[Ted11] Steven J. Tedford. Combinatorial interpretations of convolutions of the Catalan numbers. *Integers*, 11(1):35–45, 2011. doi:10.1515/INTEG.2011.003.

[The25] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 10.6)*, 2025. https://www.sagemath.org.

[vF95] N. von Fuss. Solutio quaestionis quot modis polygonum n laterum in polygona m laterum per diagonales resolvi queat. *Nova acta Academiae scientiarum imperialis petropolitanae*, IX:243–251, 1795.

[Yue17] Chi Ho Yuen. Geometric bijections between spanning trees and break divisors. *J. Combin. Theory Ser. A*, 152:159–189, 2017. doi:10.1016/j.jcta.2017.06.004.