On a question of Gowers on clique differences

Ryan Alweiss*[1]

[1]*Department of Pure Mathematics and Mathematical Statistics and Trinity College, University of Cambridge, U.K.*
ra699@cam.ac.uk

Submitted: Dec 28, 2020; Accepted: Jun 16, 2025; Published: Dec 20, 2025

Abstract. We solve a question of Gowers from 2009 on clique differences in chains, thus ruling out any Sperner-type proof of the polynomial density Hales–Jewett Theorem for alphabets of size 2.

Keywords. Hales–Jewett, Polynomial Hales–Jewett, Ramsey Theory

Mathematics Subject Classifications. 05D10

1. Introduction

The polynomial density Hales–Jewett question is a central open problem in ergodic Ramsey theory, and is a common generalization of the Bergelson–Leibman theorem [BL96] and the density Hales–Jewett theorem [FK91]. One special case was proposed by Gowers as a potential polymath project [Gow09].

Conjecture 1.1 (Gowers 2009, [Gow09])**.** For every $\delta > 0$, there exists n such that if A is any collection of at least $\delta 2^{\binom{n}{2}}$ graphs with vertex set $\{1, \cdots, n\}$, then A contains two distinct graphs G and H so that $G \subset H$ and $H \setminus G$ is a clique.

In the same blog post, Gowers asks the following question in an attempt to prove Conjecture 1.1.

Question 1.2 (Gowers, 2009)**.** Is it true that for every $\delta > 0$ there exists a sequence of distinct graphs $G_1 \subset G_2 \subset \cdots \subset G_r$ such that for every subset $A \subset \{1, 2, \ldots, r\}$ of size at least δr there exists $i < j$ such that $i, j \in A$ and $G_j \setminus G_i$ is a clique?

If the answer to the above question is positive, then it seems plausible that, by uniformly covering the set of all graphs by sequences of the above form, Conjecture 1.1 would follow from a positive answer to Question 1.2; see [Gow09] for more details.

*Supported by a Trinity College Cambridge Junior Research Fellowship

Given a sequence of distinct graphs $G_1 \subset \cdots \subset G_r$, we form the graph G on them by connecting G_i, G_j for $i < j$ when $G_j \setminus G_i$ is a clique. In this note we settle Question 1.2 in the negative, and we show that G always has an independent set of size at least $\frac{r}{20}$.

Theorem 1.3. *For every sequence of distinct graphs $G_1 \subset G_2 \subset \cdots \subset G_r$, there exists $A \subset \{1, 2, \ldots, r\}$ of size at least $r/20$ so that for all $i, j \in A$, $G_j \setminus G_i$ is not a clique.*

2. Proof of Theorem 1.3

Lemma 2.1. *Given $a < b < c < d$, if $G_c \setminus G_a$ and $G_d \setminus G_b$ are cliques, then $G_c \setminus G_b$ is also a clique.*

Proof. $G_c \setminus G_a$ and $G_d \setminus G_b$ are cliques, so their nonempty edge intersection $G_c \setminus G_b$ is also a clique. □

Lemma 2.2. *For any y, it cannot be true that all of G_y, G_{y+1}, G_{y+2} have at least three right-neighbors and three left-neighbors in G.*

Proof. Assume this was the case. Then there is some $z \geqslant y + 3$ so that $G_z \setminus G_y$ is a clique. Also there is some $x < y$ so that $G_{y+1} \setminus G_x$ is a clique. By Lemma 2.1 on $(x, y, y + 1, z)$, then $G_{y+1} \setminus G_y$ is a clique. Similarly, $G_{y+2} \setminus G_{y+1}$ is a clique. Also there is some $u < y$ so that $G_{y+2} \setminus G_u$ is a clique, and then by Lemma 2.1 on $(u, y, y+2, z)$, $G_{y+2} \setminus G_y$ is a clique, but it is the edge-disjoint union of the cliques $G_{y+2} \setminus G_{y+1}$ and $G_{y+1} \setminus G_y$, which is a contradiction. □

Call a graph G_i *good* if it has either at most two right-neighbors in G or at most two left-neighbors in G. For any $G_1 \subset G_2 \subset \cdots \subset G_r$, at least $\frac{r-2}{3}$ are good by Lemma 2.2 because we cannot have three bad in a row. So without loss of generality at least $\frac{r-2}{6}$ of the graphs have at most two right-neighbors. This means we can go through these graphs from left to right and greedily select an independent set of size at least $\frac{r-2}{18} \geqslant \frac{r}{20}$. We are done.

Following the initial proof, Noga Alon communicated an improvement of the constant in Theorem 1.3.

Lemma 2.3. *There are no $G_{3i-2}, G_{3i-1}, G_{3i}$ so that:*

- *There is some $j > 3i - 1$ so that $G_j \setminus G_{3i-2}$ is a clique.*
- *There is some $k > 3i - 1$ and some $\ell < 3i - 1$ so that $G_k \setminus G_{3i-1}$ and $G_{3i-1} \setminus G_\ell$ are cliques.*
- *There is some $m < 3i - 1$ so that $G_{3i} \setminus G_m$ is a clique.*

Proof. Assume this were the case. Then if $\ell = 3i-2$, $G_{3i-1} \setminus G_{3i-2}$ is a clique. Else if $\ell < 3i-2$, then by Lemma 2.1 on $(\ell, 3i-2, 3i-1, j)$, $G_{3i-1} \setminus G_{3i-2}$ is a clique anyway. Similarly, $G_{3i} \setminus G_{3i-1}$ is also a clique. Finally if $j = 3i$ or $m = 3i - 2$ then $G_{3i} \setminus G_{3i-2}$ is a clique. Else if $j > 3i$ and $m < 3i - 2$ then by Lemma 2.1 on $(m, 3i - 2, 3i, j)$, $G_{3i} \setminus G_{3i-2}$ is a clique anyway. This contradicts Lemma 2.2. □

For each $i \geqslant 1$, either G_{3i-2} violates the first bulleted condition, G_{3i-1} the second, or G_{3i} the third. Select one of them correspondingly. Direct all edges from left to right. Now on this induced subgraph, every vertex G_{3i-2} has outdegree 0, every vertex G_{3i-1} has either indegree 0 (if there does not exist ℓ) or outdegree 0 (if there does not exist k), and every vertex G_{3i} has indegree 0. So every vertex has indegree or outdegree 0, and thus there is an independent set of size at least $\lceil \lfloor r/3 \rfloor /2 \rceil \geqslant \frac{r-2}{6}$.

Acknowledgements

Thanks to Noga Alon, Timothy Gowers, and an anonymous reviewer for helpful comments.

References

[BL96] Vitaly Bergelson and Alex Leibman. Polynomial extensions of van der Waerden's Theorem and Szemerédi's Theorem. *J. Amer. Math. Soc.*, 90:725–753, 1996. doi:10.1090/S0894-0347-96-00194-4.

[FK91] Hillel Furstenberg and Yitzhak Katznelson. A density version of the Hales–Jewett Theorem. *J. Anal. Math*, 57:64–119, 1991. doi:10.1007/BF03041066.

[Gow09] William Timothy Gowers. The first unknown case of polynomial DHJ [online]. 11 2009. URL: https://gowers.wordpress.com/2009/11/14/the-first-unknown-case-of-polynomial-dhj/.