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SIDORENKO HYPERGRAPHS AND RANDOM TURÁN NUMBERS

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Abstract. Let $\mathrm{ex}(G^r_{n,p}, F)$ denote the maximum number of edges in an F-free subgraph of the random r-uniform hypergraph $G^r_{n,p}$, and let

$$s(F) := \sup\{s : \exists H,\ t_F(H) = t_{K^r_r}(H)^{s+e(F)} > 0\}.$$

Following recent work of Conlon, Lee, and Sidorenko, we prove non-trivial lower bounds on $\mathrm{ex}(G^r_{n,p}, F)$ whenever $s(F) > 0$, i.e. F is not Sidorenko. This connection between Sidorenko's conjecture and random Turán problems gives new lower bounds on $\mathrm{ex}(G^r_{n,p}, F)$ whenever $s(F) > 0$, and further allows us to establish upper bounds for $s(F)$ whenever upper bounds for $\mathrm{ex}(G^r_{n,p}, F)$ are known. As a consequence, we prove that $s(\mathrm{E}^r(K^k_{k+1})) = \frac{1}{r-k}$ where $\mathrm{E}^r(K^k_{k+1})$ is the r-expansion of K^k_{k+1}.

Keywords. Hypergraph, Sidorenko Conjecture, Random Turán Problem

Mathematics Subject Classifications. 05C65, 05C80, 05D05, 05D40

1. Introduction

In recent work, Conlon, Lee, and Sidorenko [CLS24] established a connection between two seemingly unrelated problems in extremal combinatorics: Sidorenko's conjecture and Turán problems. Building on their work, we further establish a connection between Sidorenko's conjecture and *random* Turán problems. Along the way, we obtain good estimations for how "far" certain hypergraphs F are from being Sidorenko.

Throughout this paper we consider r-uniform hypergraphs, or r-graphs for short. For an r-graph $H = (V, E)$, we use the notation $v(H) := |V|$ and $e(H) := |E|$.

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1.1. Sidorenko's conjecture

Recall that a *homomorphism* from an r-graph F to an r-graph H is a map $\phi : V(F) \to V(H)$ such that $\phi(e)$ is an edge of H whenever e is an edge of F. We let $\hom(F,H)$ denote the number of homomrophisms from F to H and define the *homomorphism density*

$$t_F(H) = \frac{\hom(F,H)}{v(H)^{v(F)}},$$

which is equivalently the probability that a random map $\phi : V(F) \to V(H)$ is a homomorphism. Sidorenko famously conjectured the following.

Conjecture 1.1 (Sidorenko's Conjecture [Sid92, Sid93]). For every bipartite graph F and every graph H,

$$t_F(H) \geqslant t_{K_2}(H)^{e(F)}. \tag{1.1}$$

This bound is asymptotically best possible by considering random graphs. A large body of literature is dedicated towards Sidorenko's conjecture [CFS10, CL17, CL21, CKLL18, CR21, FW17, Hat10, KLL16, LS11, Lov11, Sze14], but overall this problem remains very wide open. We call a graph F *Sidorenko* if it satisfies (1.1). This notion naturally extends to hypergraphs as follows.

Definition 1.2. We say that an r-graph F is *Sidorenko* if for every r-graph H we have

$$t_F(H) \geqslant t_{K_r^r}(H)^{e(F)},$$

where K_r^r is the r-graph consisting of a single edge.

This bound is also asymptotically best possible by considering random hypergraphs. Observe that if F is not r-partite, then F is not Sidorenko (due to $H = K_r^r$, for example). As such, it suffices to only consider r-partite r-graphs when discussing Sidorenko hypergraphs.

It is known that Sidorenko's conjecture does not extend to hypergraphs. In particular, Sidorenko [Sid93] showed that r-uniform loose triangles (see Definition 1.9) are not Sidorenko for $r \geqslant 3$. Because of this, there seems to have been relatively little interest in studying which r-graphs are Sidorenko. This changed very recently with the work of Conlon, Lee, and Sidorenko [CLS24] who showed the following surprising connection between non-Sidorenko hypergraphs and Turán numbers. Here we recall that the *Turán number* $\mathrm{ex}(n,F)$ of an r-graph F is the maximum number of edges that an n-vertex F-free r-graph can have.

Theorem 1.3 ([CLS24]). *If F is not Sidorenko, then there exists $c = c(F)$ such that*

$$\mathrm{ex}(n,F) = \Omega\left(n^{r - \frac{v(F)-r}{e(F)-1} + c}\right).$$

We note that this improves upon the trivial lower bound $\mathrm{ex}(n,F) = \Omega(n^{r - \frac{v(F)-r}{e(F)-1}})$ which holds for all F by a standard random deletion argument. A number of other results around non-Sidorenko hypergraphs were proven in [CLS24], such as the fact that r-partite r-graphs of odd girth are always non-Sidorenko.

1.2. The random Turán problem

We now discuss our second problem of interest: the random Turán problem. Given r-graphs G and F, we define $\mathrm{ex}(G,F)$ to be the maximum number of edges in an F-free subgraph of G. Define $G^r_{n,p}$ to be the random r-graph on n vertices obtained by including each possible edge independently and with probability p. When $r=2$ we simply write $G_{n,p}$ instead of $G^2_{n,p}$. We call $\mathrm{ex}(G^r_{n,p},F)$ the *random Turán number*, which is the maximum number of edges in an F-free subgraph of $G^r_{n,p}$. Note that when $p=1$ we have $\mathrm{ex}(G^r_{n,1},F)=\mathrm{ex}(n,F)$, so this can be viewed as a probabilistic analog of the classical Turán number.

The asymptotics of $\mathrm{ex}(G^r_{n,p},F)$ has essentially been determined whenever F is not an r-partite r-graph due to independent breakthrough work of Schacht [Sch16] and of Conlon and Gowers [CG16], and as such we will focus only on the case when F is r-partite. For $r=2$, McKinely and Spiro [MS23] recently conjectured the following, where here we define the r-density of an r-graph F with $v(F)>r$ by

$$m_r(F) := \max_{F'\subseteq F : v(F')>r} \left\{ \frac{e(F')-1}{v(F')-r} \right\},$$

and we say that F is r-*balanced* if $m_r(F)=\frac{e(F)-1}{v(F)-r}$.

Conjecture 1.4 ([MS23]). If F is a graph with $\mathrm{ex}(n,F)=\Theta(n^\alpha)$ for some $\alpha\in(1,2]$, then a.a.s.

$$\mathrm{ex}(G_{n,p},F)=\begin{cases}\Theta(p^{\alpha-1}n^\alpha) & p\geqslant n^{\frac{2-\alpha-1/m_2(F)}{\alpha-1}}(\log n)^{O(1)}\\ n^{2-1/m_2(F)}(\log n)^{O(1)} & n^{\frac{2-\alpha-1/m_2(F)}{\alpha-1}}(\log n)^{O(1)}\geqslant p\geqslant n^{-1/m_2(F)},\\ (1+o(1))p\binom{n}{2} & n^{-1/m_2(F)}\gg p\gg n^{-2}.\end{cases}$$

In particular, Conjecture 1.4 predicts that for bipartite graphs, $\mathrm{ex}(G_{n,p},F)$ always has a "flat middle range" where $\mathrm{ex}(G_{n,p},F)=n^{2-1/m_2(F)}(\log n)^{O(1)}$ for the entire range

$$n^{-1/m_2(F)}\leqslant p\leqslant n^{\frac{2-\alpha-1/m_2(F)}{\alpha-1}}(\log n)^{O(1)};$$

see Figure 1.1 for an example. Conjecture 1.4 is known to hold (assuming certain conjectures regarding $\mathrm{ex}(n,F)$) for complete bipartite graphs and even cycles [MS16], and for theta graphs [MS23], with the lower bound being known to hold for powers of rooted trees [Spi24].

It is known that Conjecture 1.4 does not extend to hypergraphs. In particular, Nie, Spiro, and Verstraëte [NSV21] and Nie [Nie23] showed this does not hold for r-uniform loose triangles with $r\geqslant 3$.

1.3. Our results

The reader may notice that the situation for Sidorenko r-graphs and for random Turán problems parallel each other quite closely, in particular with regard to loose triangles serving as a counterexample to each problem. The main result of this paper (Theorem 1.5) shows that this is not

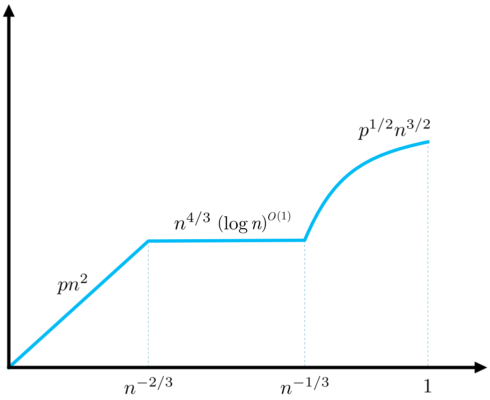

Figure 1.1: The plot of $\mathrm{ex}(G_{n,p},C_4)$ as proven by Füredi [F94]. More generally, Conjecture 1.4 predicts that $\mathrm{ex}(G_{n,p},F)$ should have a "flat middle range" for every bipartite graph F starting at $p=n^{-1/m_2(F)}$.

a coincidence: in many cases, r-graphs which are not Sidorenko have a stronger lower bound on $\mathrm{ex}(G^r_{n,p},F)$ than the generalization of Conjecture 1.4 would predict.

We prove this by extending Theorem 1.3 to give lower bounds for random Turán numbers. In addition to this, we give a quantitative statement which in several cases gives optimal bounds for $\mathrm{ex}(G^r_{n,p},F)$. To state the quantitative result, we define[1] for an r-graph F the quantity

$$s(F):=\sup\{s:\exists H,\ t_F(H)=t_{K_r^r}(H)^{s+e(F)}>0\}.$$

Note that $s(F)=0$ if and only if F is Sidorenko, and more generally, $s(F)$ measures how "far" an r-graph is from being Sidorenko.

The following is our main result.

Theorem 1.5. *If F is an r-graph with $e(F)\geqslant 2$ and $\frac{v(F)-r}{e(F)-1}<r$, then for any $p=p(n)\geqslant n^{-\frac{v(F)-r}{e(F)-1}}$, we have a.a.s.*

$$\mathrm{ex}(G^r_{n,p},F)\geqslant n^{r-\frac{v(F)-r}{e(F)-1}-o(1)}\left(pn^{\frac{v(F)-r}{e(F)-1}}\right)^{\frac{s(F)}{e(F)-1+s(F)}}.$$

Our strategy for proving Theorem 1.5 is based off [CLS24] with a more careful execution. Because of this, the bound of Theorem 1.5 will turn out to always be at least as strong as the (implicit quantitative version of) Theorem 1.3; see Corollary 2.2 and the surrounding discussion for more. In particular, the quantitative bounds of Theorem 1.5 combined with known results for random Turán numbers can be used to give effective (or even tight) bounds on $s(F)$ for a number of hypergraphs; see Corollaries 1.10 and 1.12 for more.

We emphasize that Theorem 1.5 is only non-trivial when $s(F)>0$, i.e. when F is not Sidorenko. When $s(F)>0$ and F is r-balanced, Theorem 1.5 shows that $\mathrm{ex}(G^r_{n,p},F)$ does not have a "flat middle range" as predicted in the case of graphs by Conjecture 1.4.

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[1] We emphasize that we define $s(F)$ in terms of a supremum rather than a maximum. As such, we do not require there to exist an H satisfying $t_F(H)=t_{K_r^r}(H)^{s+e(F)}$, though we note that such an H can be guaranteed if one shifts to the essentially equivalent perspective of hypergraphons.

Assuming F has no isolated vertices, then the only F not satisfying our conditions are matchings, which are known to be Sidorenko. Furthermore, the random Turán problem for matching of size two (i.e. the EKR problem in random hypergraph) has been essentially solved by Balogh, Bohman, and Mubayi [BBM09], see also [HK19a, HK19b, GHO17, BKL23]. Their results have recently been extended to matchings of any size by Frankl, Nie, and Wang [FNW24].

To get effective bounds in Theorem 1.5, one must determine how large $s(F)$ is, i.e. how "far" from Sidorenko F is; and we believe this to be a problem of independent interest.

Problem 1.6. Given an r-graph F, determine (bounds for) $s(F)$.

Theorem 1.5 allows us to translate between the random Turán problem and the Sidorenko-type problem Problem 1.6. Specifically, Theorem 1.5 shows that lower bounds on $s(F)$ give lower bounds for the random Turán problem, and conversely, upper bounds on the random Turán problem give upper bounds on $s(F)$. For example, the non-Sidorenko r-graphs established by Conlon, Lee, and Sidorenko [CLS24, Theorem 3.1] gives the following result for the random Turán problem.

Corollary 1.7. *Let F be an r-partite r-graph of odd girth. Then there exists an $\epsilon > 0$ such that if $p \geqslant n^{-\frac{v(F)-r}{e(F)-1}}$, then a.a.s.*

$$\mathrm{ex}(G^r_{n,p}, F) \geqslant n^{r-\frac{v(F)-r}{e(F)-1}-o(1)}\left(pn^{\frac{v(F)-r}{e(F)-1}}\right)^{\epsilon}.$$

On the other hand, the simplest case of $p = 1$ in Theorem 1.5 gives the following general bound on $s(F)$.

Corollary 1.8. *If F is an r-graph with $\mathrm{ex}(n, F) = O(n^{\alpha})$ and $\alpha < r$, then*

$$s(F) \leqslant \frac{v(F) - \alpha}{r - \alpha} - e(F).$$

While Corollary 1.8 can sometimes provide tight bounds (see Corollary 1.12), in general more effective bounds are obtained by considering smaller values of p. To state these improved bounds, we make the following definitions which will be used throughout the paper.

Definition 1.9. Given a k-graph F, we define the r-expansion $\mathrm{E}^r(F)$ to be the r-graph obtained by enlarging each k-edge of F with a set of $r - k$ vertices of degree one. When F is the graph cycle C_ℓ, we write $C^r_\ell := \mathrm{E}^r(C_\ell)$ and refer to this as the r-uniform *loose cycle* of length ℓ.

Lastly, we define an r-graph T to be a *tight r-tree* if its edges can be ordered as $e_1, \dots, e_t$ so that

$$\forall i \geqslant 2\ \exists v \in e_i \textit{ and } 1 \leqslant s \leqslant i-1 \textit{ such that } v \notin \cup_{j=1}^{i-1} e_j \textit{ and } e_i - v \subset e_s.$$

The following bounds for $s(F)$ turn out to follow from Theorem 1.5 together with known bounds for random Turán numbers [MY23, Nie23, Nie24]; see the end of Section 2 for further details on the derivation.

Corollary 1.10.

(a) *For $r > k \geqslant 2$, we have*

$$s(\mathrm{E}^r(K_{k+1}^k)) \leqslant \frac{1}{r-k}.$$

(b) *For $\ell \geqslant 1$ and $r \geqslant 3$, we have*

$$s(C_{2\ell+1}^r) \leqslant \frac{2\ell-1}{r-2}.$$

(c) *For $r \geqslant k \geqslant 2$ and any tight k-tree T, $s(\mathrm{E}^r(T)) = 0$. That is, expansions of tight trees are Sidorenko.*

(d) *For $\ell \geqslant 1$ and $r \geqslant 3$ we have $s(C_{2\ell}^r) = 0$. That is, loose even cycles are Sidorenko.*

In addition to these immediate consequences of Theorem 1.5, we prove two new results related to $s(F)$ and expansions. First, we show that expansions of F which contain K_{k+1}^k are not Sidorenko.

Theorem 1.11. *If F is a k-graph which contains K_{k+1}^k as a subgraph, then for all $r > k$ we have*

$$s(\mathrm{E}^r(F)) \geqslant \frac{1}{r-k}.$$

In particular, $\mathrm{E}^r(F)$ is not Sidorenko.

The proof of Theorem 1.11 makes use of a construction of Gowers and Janzer [GJ21], which generalized the seminal construction of Ruzsa and Szemerédi [RS78]. We note that the quantitative lower bound of Theorem 1.11 combined with Theorem 1.5 gives optimal lower bounds for the random Turán number $\mathrm{ex}(G_{n,p}^r, \mathrm{E}^r(K_{k+1}^k))$; see [Nie23, NSV21]. Moreover, this result together with Corollary 1.10(a) gives the following.

Corollary 1.12. *For $r > k \geqslant 2$, we have*

$$s(\mathrm{E}^r(K_{k+1}^k)) = \frac{1}{r-k}.$$

Corollary 1.12, in other words, says that the minimum $t_{\mathrm{E}^r(K_{k+1}^k)}(H)$ among all r-graph H with edge density δ is $\delta^{k+1+\frac{1}{r-k}\pm o(1)}$. For $\mathrm{E}^3(K_3) = C_3^3$ this result was proved previously by Fox, Sah, Sawhney, Stoner, and Zhao [FSS$^+$20, Theorem 1.2]. We note that the $r = k+1$ case of this corollary gives an example where the general upper bound of Corollary 1.8 is tight.

Finally, we establish an upper bound on $s(\mathrm{E}^r(F))$ in terms of $s(F)$.

Theorem 1.13. *If F is a k-graph with $s(F) < \infty$, then for all $r \geqslant k$ we have*

$$s(\mathrm{E}^r(F)) \leqslant \frac{v(F)-k}{v(F)-k+(r-k)(s(F)+e(F)-1)} \cdot s(F).$$

For example, one can check that knowing $s(\mathrm{E}^{k+1}(K_{k+1}^{k})) \leqslant 1$ from Corollary 1.10 together with Theorem 1.13 implies $s(\mathrm{E}^{r}(K_{k+1}^{k})) \leqslant \frac{1}{r-k}$ for all $r > k$, and similarly one recovers our upper bound for $C_{2\ell+1}^{r}$ assuming the upper bound for $r = 3$. We also obtain the following nice corollary by taking $s(F) = 0$.

Corollary 1.14. *If F is a Sidorenko k-graph, then its expansions $\mathrm{E}^{r}(F)$ are Sidorenko for all $r \geqslant k$.*

2. Proof of Theorem 1.5 and its corollaries

Our proof of Theorem 1.5 is based off the proof of Theorem 1.3 from [CLS24] which relies on the tensor product trick. Given two r-graphs G, H, we define the *tensor product* $G \otimes H$ to be r-graph on $V(G) \times V(H)$ where $((x_1, y_1), \ldots, (x_r, y_r)) \in E(G \otimes H)$ if and only if $(x_1, \ldots, x_r) \in E(G)$ and $(y_1, \ldots, y_r) \in E(H)$. For N a positive integer, we define the N-fold tensor product $H^{\otimes N}$ inductively by setting $H^{\otimes 1} = H$ and $H^{\otimes N} = H \otimes H^{\otimes (N-1)}$. The key property we need regarding tensor products is the fact that for any r-graphs F, H and $N \geqslant 1$, we have

$$t_F(H^{\otimes N}) = t_F(H)^N,$$

which is straightforward to verify.

By incorporating the tensor product trick from [CLS24] together with random homomorphisms, we can show that r-graphs G with few copies of F have large F-free subgraphs; by copies of F we mean subgraphs of G that are isomorphic to F. To this end, we let $\mathcal{N}_F(G)$ denote the number of copies of F in G and recall that $\mathrm{ex}(G, F)$ is the maximum number of edges in an F-free subgraph of G.

Lemma 2.1. *If F is an r-graph such that there exists an r-graph H with $t_{K_r^r}(H) = \alpha$ and $t_F(H) = \alpha^{s+e(F)}$, then for all r-graphs G and integers $N \geqslant 1$ we have*

$$\mathrm{ex}(G, F) \geqslant \alpha^N e(G) - \alpha^{(s+e(F))N} \mathcal{N}_F(G).$$

Proof. Let $\phi : V(G) \to V(H^{\otimes N})$ be chosen uniformly at random, and let $G' \subseteq G$ be the subgraph consisting of all hyperedges $e \in G$ which are mapped bijectively onto an edge of $H^{\otimes N}$. By linearity of expectation, we have

$$\mathbb{E}[e(G')] = t_{K_r^r}(H^{\otimes N}) \cdot e(G) = \alpha^N \cdot e(G),$$

and

$$\mathbb{E}[\mathcal{N}_F(G')] = t_F(H^{\otimes N}) \cdot \mathcal{N}_F(G) = \alpha^{(s+e(F))N} \cdot \mathcal{N}_F(G).$$

Thus if we define $G'' \subseteq G'$ by deleting an edge from each copy of F in G', then G'' is F-free and satisfies

$$\mathbb{E}[e(G'')] \geqslant \alpha^N e(G) - \alpha^{(s+e(F))N} \mathcal{N}_F(G),$$

and hence there must exist an F-free subgraph of G with at least this many edges, proving the result. □

With this we can prove our main result.

Proof of Theorem 1.5. Recall that we wish to prove if F is an r-graph with $e(F) \geqslant 2$ and $\frac{v(F)-r}{e(F)-1} < r$, then for any $p = p(n) \geqslant n^{-\frac{v(F)-r}{e(F)-1}}$, we have a.a.s.

$$\mathrm{ex}(G_{n,p}^r, F) \geqslant n^{r-\frac{v(F)-r}{e(F)-1}-o(1)}\left(pn^{\frac{v(F)-r}{e(F)-1}}\right)^{\frac{s(F)}{e(F)-1+s(F)}}.$$

If $s(F) = 0$ then this result follows by a standard random deletion argument, so from now on we assume $s(F) > 0$.

Consider any $0 < \epsilon \leqslant s(F)$ (which exists by assumption $s(F) > 0$). By definition of $s(F)$, there exists a non-empty r-graph H with $t_{K_r^r}(H) = \alpha > 0$ and $t_F(H) = \alpha^{s+e(F)}$ with $0 \leqslant s(F) - \epsilon \leqslant s \leqslant s(F)$. By Lemma 2.1 we find

$$\mathrm{ex}(G_{n,p}^r, F) \geqslant \alpha^N e(G_{n,p}^r) - \alpha^{(s+e(F))N}\mathcal{N}_F(G_{n,p}^r), \tag{2.1}$$

so it suffices to choose an N such that this is sufficiently large a.a.s.

Given p and any function $\delta(n) = o(1)$, let $N \geqslant 1$ be the smallest integer such that

$$q := \delta(n)n^{-\frac{v(F)-r}{e(F)-1+s}}p^{-\frac{e(F)-1}{e(F)-1+s}} \geqslant \alpha^N.$$

Note that such an integer exists since $0 < \alpha < 1$. Also note that $\alpha^N \leqslant q \leqslant \alpha^{N-1}$ by the minimality of N.

Let A denote the event that $e(G_{n,p}^r) \geqslant \frac{1}{2r!}pn^r$. Because $e(G_{n,p}^r)$ is a binomial random variable and $pn^r \geqslant n^{r-\frac{v(F)-r}{e(F)-1}} \to \infty$ by hypothesis, the Chernoff bound implies that the event A holds a.a.s.

Let B denote the event that $\mathcal{N}_F(G_{n,p}^r) \leqslant \delta(n)^{-1/2}p^{e(F)}n^{v(F)}$. Since $\mathbb{E}[\mathcal{N}_F(G_{n,p}^r)] \leqslant p^{e(F)}n^{v(F)}$, it follows by Markov's inequality and $\delta(n) = o(1)$ that B holds a.a.s.

Because $A \cap B$ hold a.a.s., we find that a.a.s. the bound in (2.1) is at least

$$\begin{aligned}
&\frac{1}{2r!}\alpha^N pn^r - \delta(n)^{-1/2}\alpha^{(s+e(F))N}p^{v(F)}n^{e(F)}\\
&\qquad\geqslant \frac{1}{2r!}\alpha^N pn^r(1 - 2r!\delta(n)^{-1/2}q^{s+e(F)-1}p^{e(F)-1}n^{v(F)-r})\\
&\qquad= \frac{1}{2r!}\alpha^N pn^r(1 - 2r!\delta(n)^{s+e(F)-3/2}).
\end{aligned}$$

Note that $s + e(F) - 3/2 > 0$ since $s \geqslant 0$ and $e(F) \geqslant 2$. Thus for n sufficiently large the quantity above is at least

$$\begin{aligned}
\frac{1}{4r!}\alpha^N pn^r &\geqslant \frac{\alpha}{4r!}\delta(n)n^{r-\frac{v(F)-r}{e(F)-1}}\left(pn^{\frac{v(F)-r}{e(F)-1}}\right)^{\frac{s}{e(F)-1+s}}\\
&\geqslant \frac{\alpha}{4r!}\delta(n)n^{-\frac{\epsilon}{e(F)-1+s(F)}}\cdot n^{r-\frac{v(F)-r}{e(F)-1}}\left(pn^{\frac{v(F)-r}{e(F)-1}}\right)^{\frac{s(F)}{e(F)-1+s(F)}},
\end{aligned}$$

with this last step used $s \geqslant s(F) - \epsilon$. As $\epsilon > 0$ was arbitrary and $\delta(n)$ tends to 0 arbitrarily slowly, we conclude the desired result. ☐

As an aside, the bound of Theorem 1.5 continues to hold in expectation even if $\frac{v(F)-r}{e(F)-1} \geqslant r$. However, in this case we can not say $G^r_{n,p}$ has any edges a.a.s., and hence no non-trivial lower bound for $\mathrm{ex}(G^r_{n,p}, F)$ can hold a.a.s.

Focusing on the $p = 1$ case, Lemma 2.1 quickly gives the following.

Corollary 2.2. *If F is an r-graph such that there exists a non-empty r-graph H with $t_F(H) = t_{K^r_r}(H)^{s+e(F)}$, then*

$$\mathrm{ex}(n, F) = \Omega\left(n^{r-\frac{v(F)-r}{e(F)-1}+\frac{(v(F)-r)s}{(e(F)-1)(s+e(F)-1)}}\right).$$

Proof. Take $G = K^r_n$ in Lemma 2.1, which means $e(G) \geqslant \frac{1}{2r!}n^r$ and $\mathcal{N}_F(T) \leqslant n^{v(F)}$, so for $\alpha = t_{K^r_r}(H)$ and any $N \geqslant 1$ we have

$$\mathrm{ex}(n, F) = \mathrm{ex}(G, F) \geqslant \frac{1}{2r!}\alpha^N n^r \left(1 - 2r!\alpha^{(s+e(F)-1)N} n^{v(F)-r}\right).$$

We conclude the result by taking N such that α^N is a sufficiently small constant times

$$n^{-\frac{v(F)-r}{s+e(F)-1}} = n^{(v(F)-r)\left(\frac{-1}{e(F)-1}+\frac{s}{(e(F)-1)(s+e(F)-1)}\right)}. \qquad \square$$

As a point of comparison, a more careful analysis of the proof giving Theorem 1.3 yields the following quantitative bound.

Theorem 2.3 (Quantitative Theorem 1.3)**.** *If F is an r-graph such that there exists a non-empty r-graph H with $t_F(H) = t_{K^r_r}(H)^{s+e(F)}$ and $t_{K^r_r}(H) = v(H)^{-\delta}$, then*

$$\mathrm{ex}(n, F) = \Omega\left(n^{r-\frac{v(F)-r}{e(F)-1}+\frac{\delta s}{e(F)-1}}\right).$$

It is not difficult to show that $\delta \leqslant \frac{v(F)-r}{s+e(F)-1}$ in Theorem 2.3 (see Lemma 4.1 for a formal proof). If δ obtains this maximum possible value then Theorem 2.3 matches Corollary 2.2; otherwise Corollary 2.2 does strictly better.

We now sketch how Theorem 1.5 together with known results for random Turán bounds implies Corollaries 1.8 and 1.10.

Proof of Corollary 1.8. Recall that we wish to show that if F is an r-graph with $\mathrm{ex}(n, F) = O(n^\alpha)$, then

$$s(F) \leqslant \frac{v(F) - \alpha}{r - \alpha} - e(F).$$

Let s be such that there exists a non-empty r-graph H with $t_F(H) = t_{K^r_r}(H)^{s+e(F)}$. By Corollary 2.2, we have

$$\Omega(n^{r-\frac{v(F)-r}{s+e(F)-1}}) = \mathrm{ex}(n, F) = O(n^\alpha).$$

This implies $r - \frac{v(F)-r}{s+e(F)-1} \leqslant \alpha$, and rearranging gives $s \leqslant \frac{v(F)-\alpha}{r-\alpha} - e(F)$. As $s(F)$ is the supremum over all such s, we conclude the result. $\square$

Proof of Corollary 1.10. Throughout we implicitly utilize the fact that every F we consider is r-balanced and hence $m_r(F) = \frac{v(F)-r}{e(F)-1}$.

We first show (a): for $r > k \geqslant 2$ that $s(\mathrm{E}^r(K_{k+1}^k)) \leqslant \frac{1}{r-k}$. By [Nie23, Theorem 1.4], we have for $p = n^{-\frac{1}{m_r(\mathrm{E}^r(K_{k+1}^k))}+c} = n^{-r+k-\frac{1}{k}+c}$ with sufficiently small $c = c(k,r) > 0$ that a.a.s.

$$\mathrm{ex}(G_{n,p}^r, \mathrm{E}^r(K_{k+1}^k)) = p^{\frac{1}{(r-k)k+1}} n^{k+o(1)},$$

hence by Theorem 1.5

$$\frac{s(\mathrm{E}^r(K_{k+1}^k))}{k + s(\mathrm{E}^r(K_{k+1}^k))} \leqslant \frac{1}{(r-k)k+1},$$

which implies the desired upper bound.

For (b), by [Nie23, Theorem 1.8], we have for $p = n^{-\frac{1}{m_r(C_{2\ell+1}^r)}+c} = n^{-r+1+\frac{1}{2\ell}+c}$ with sufficiently small $c = c(\ell, r) > 0$ that a.a.s.

$$\mathrm{ex}(G_{n,p}^r, C_{2\ell+1}^r) \leqslant p^{\frac{2\ell-1}{2\ell(r-1)-1}} n^{2+o(1)}, 4$$

hence by Theorem 1.5

$$\frac{s(C_{2\ell+1}^r)}{2\ell + s(C_{2\ell+1}^r)} \leqslant \frac{2\ell-1}{2\ell(r-1)-1},$$

which implies

$$s(C_{2\ell+1}^r) \leqslant \frac{2\ell-1}{r-2}.$$

For (c), by [Nie23, Theorem 1.5], we have $\mathrm{ex}(G_{n,p}^r, \mathrm{E}^r(T)) = n^{k-1+o(1)}$ a.a.s. for $p = n^{-\frac{1}{m_r(\mathrm{E}^r(T))}+c} = n^{-r+k-1+c}$ with sufficiently small $c = c(k,r) > 0$, which implies the bound.

For (d), it was proven in [MY23, Nie24] that $\mathrm{ex}(G_{n,p}^r, C_{2\ell}^r) = n^{1+\frac{1}{2\ell-1}+o(1)}$ a.a.s. for $p = n^{-\frac{1}{m_r(C_{2\ell}^r)}+c} = n^{-r+1+\frac{1}{2\ell-1}+c}$ with $c = c(\ell, r) > 0$, from which the bound follows. ☐

3. Proof of Theorem 1.11

To prove our lower bounds on $s(\mathrm{E}^r(F))$ when F contains K_{k+1}^k, we need the following construction of Gowers and Janzer [GJ21], which is a generalization of the seminal construction of Ruzsa and Szemerédi [RS78].

Theorem 3.1 ([GJ21, RS78]). *For $r > k \geqslant 2$ and $n \geqslant 1$, there exists a graph $G_{n,r,k}$ on n vertices with the following two properties:*

(i) It has $n^k e^{-O(\sqrt{\log n})}$ subgraphs isomorphic to K_r;

(ii) For any t with $k < t \leqslant r$ and any subgraph G_1 isomorphic to K_k, if there exist a subgraph G_2 isomorphic to K_t and a subgraph G_3 isomorphic to K_r such that $G_1 \subseteq G_2$ and $G_1 \subseteq G_3$, then $G_2 \subseteq G_3$.

Property (ii) is indeed slightly stronger than the original statement of Theorem 1.2 in [GJ21] which states "every K_k is contained in at most one K_r". This strengthened property is observed by Nie in [Nie23]. In fact, property (ii) is inherently implied by the proof of Lemma 3.1 in [GJ21]. To see this, we roughly explain the idea of constructing $G_{n,r,k}$:

1. Let $S_{k-1} \subseteq \mathbb{R}^k$ be the $(k-1)$-dimension unit sphere. Pick r points $p_1, \ldots, p_r \in S_{k-1}$ in general position.
2. Randomly pick r point sets $V_1, \ldots, V_r \subseteq S_{d-1} \subseteq \mathbb{R}^d$, each of size n/r, where d is some suitable integer depending on n. By deleting a small portion of clustering points, we can make sure that all these points are reasonably well separeated. These points form the vertex set of $G_{n,r,k}$
3. For any $1 \leqslant i < j \leqslant r$ and points $v_i \in V_i, v_j \in V_j$, they form an edge in $G_{n,r,k}$ if and only if $|\langle v_i, v_j\rangle - \langle p_i, p_j\rangle| < \epsilon$, where ϵ is some resonably small constant.

By design, any copy of K_r in $G_{n,r,k}$ must be close to a configuration with angles determined by the points $p_1, \ldots, p_r$. Note that once we fix k points $v_1 \in V_1, \ldots, v_k \in V_k$, then for all $k+1 \leqslant i \leqslant r$, v_i is constrained to lie in a set with small diameter. Since all points are well separated, there are at most one choice for each v_i. This property guarantees property *(ii)* in Theorem 3.1—See [GJ21] for detailed computations.

Consider an r-graph $H_{n,r,k}$ on $V(G_{n,r,k})$ whose edges are the vertex sets of copies of K_r in $G_{n,r,k}$. The following properties of $H_{n,r,k}$ is proved in [Nie23]. We include a proof for completeness.

Proposition 3.2 (Proposition 5.4, [Nie23])**.** *For $r > k \geqslant 2$ and $n \geqslant 1$, $H_{n,r,k}$ has the following properties:*

(i) $e(H_{n,r,k}) \geqslant n^k e^{(-O(\sqrt{\log n}))}$*;*

(ii) Any two edges intersect in at most $k-1$ vertices;

(iii) $H_{n,r,k}$ does not contain any subgraph isomorphic to $\mathrm{E}^r(K^k_{k+1})$.

Proof. Property (i) follows immediately. Now assume, for contradiction, that there exist two edges in the hypergraph that share at least k vertices. This would imply the presence of a K_k subgraph lying in the intersection of two distinct K_r-subgraphs within $G_{n,r,k}$, violating condition (ii) of $G_{n,r,k}$.

Likewise, suppose that $H_{n,r,k}$ contains a subhypergraph isomorphic to $\mathrm{E}^r(K^k_{k+1})$. Then, in the underlying graph $G_{n,r,k}$, this would correspond to a K_k subgraph that is a common part of a K_{k+1} and a K_r which does not include the K_{k+1}. This again contradicts property (ii) of $G_{n,r,k}$. □

Now we are ready to prove our main result for this section.

Proof of Theorem 1.11. Recall that we wish to prove that if F is a k-graph which contains K_{k+1}^k as a subgraph, then for all $r > k$ we have

$$s(\mathrm{E}^r(F)) \geqslant \frac{1}{r-k}.$$

That is, for any $\epsilon > 0$ we want to find an r-graph H such that

$$t_{\mathrm{E}^r(F)}(H) \leqslant t_{K_r^r}(H)^{e(F)+\frac{1}{r-k}-\epsilon}.$$

Let $H = H_{n,r,k}$ with n to be chosen sufficiently large in terms of ϵ. The crucial observation is the following.

Claim 3.3. *If $\phi : V(\mathrm{E}^r(K_{k+1}^k)) \to V(H)$ is a homomorphism, then $\phi(e) = \phi(f)$ for all $e, f \in \mathrm{E}^r(K_{k+1}^k)$.*

Proof. Denote the edges of $\mathrm{E}^r(K_{k+1}^k)$ by $e_1, \ldots, e_{k+1}$, where each edge e_i is defined as

$$e_i = \{v_1, \ldots, v_{k+1}, w_{i,1}, \ldots, w_{i,r-k}\} \setminus \{v_i\}.$$

Assume, without loss of generality, that $\phi(e_1) \neq \phi(e_2)$ for the sake of contradiction. By Proposition 3.2(ii), it follows that

$$|\phi(e_1) \cap \phi(e_2)| \leqslant k - 1.$$

However, observe that the image of the intersection satisfies

$$\phi(e_1 \cap e_2) \subseteq \phi(e_1) \cap \phi(e_2),$$

and since $|e_1 \cap e_2| = k - 1$, the size of the left-hand side is $k - 1$, forcing equality:

$$\phi(e_1 \cap e_2) = \phi(e_1) \cap \phi(e_2).$$

In particular, this implies that $\phi(v_1) \notin \phi(e_1)$.

Observe that v_1 belongs to every edge e_i with $i > 1$, and thus $\phi(v_1) \in \phi(e_i)$ for each $i > 1$. Consequently, since $\phi(v_1) \notin \phi(e_1)$ as established earlier, it follows that $\phi(e_1) \neq \phi(e_i)$ for all $i > 1$. Therefore, we have

$$|\phi(e_1) \cap \phi(e_i)| = k - 1 \quad \text{for all } i > 1.$$

By the symmetry of the construction, it follows that

$$|\phi(e_i) \cap \phi(e_j)| = k - 1 \quad \text{for all } i \neq j.$$

Hence, the set of images $\phi(e_1), \ldots, \phi(e_{k+1})$ forms a subhypergraph in $H_{m,r,k}$ isomorphic to $\mathrm{E}^r(K_{k+1}^k)$, which contradicts Proposition 3.2(iii). □

This allows us to prove the following.

Claim 3.4. *If $x \in V(F)$ is contained in a K_{k+1}^k, then for any map $\phi : V(F)\setminus\{x\} \to V(H)$, there are at most $O(1)$ homomorphisms $\phi' : V(\mathrm{E}^r(F)) \to V(H)$ such that the restriction $\phi'|_{V(F)\setminus\{x\}}$ equals ϕ.*

Proof. Let $x_1, \ldots, x_k$ be the other vertices of the K_{k+1}^k containing x. Because $\mathrm{E}^r(F)$ contains an edge e containing $\{x_1, \ldots, x_k\}$, if the set $X = \{\phi(x_1), \ldots, \phi(x_k)\}$ either has size less than k or is not contained in an edge of H, then no homomorphism restricts to ϕ, so we may assume this is not the case. By Proposition 3.2(ii), there exists a unique edge $h \in H$ containing X, and any homomorphism ϕ' which restricts to ϕ must map e to h. By Claim 3.3, we must have $\phi'(x) \in h$.

Let $y \in h$ and define $\phi_y : V(F) \to V(H)$ by having $\phi_y(x) = y$ and $\phi_y(z) = \phi(z)$ for all other z. By the observation above, any ϕ' which restricts to ϕ must restrict to ϕ_y for one of the at most $O(1)$ choices $y \in h$. We claim that for any $y \in h$ there are at most $O(1)$ homomorphism ϕ' which restricts to ϕ_y, from which the result will follow.

Indeed, consider any vertex $z \in V(\mathrm{E}^r(F)) \setminus V(F)$, which by definition of the expansion means there is an edge $\{z_1, \ldots, z_k\} \in E(F)$ such that $\{z, z_1, \ldots, z_k\}$ is contained in an edge e' of $\mathrm{E}^r(F)$. If the set $Z = \{\phi_y(z_1), \ldots, \phi_y(z_k)\}$ has size less than k or is not contained in an edge of H, then no homomorphism restricts to ϕ_y, so we may assume this is not the case. By Proposition 3.2(ii), there exists a unique edge $h' \in H$ containing Z, and any homomorphism ϕ' which restricts to ϕ_y must map e' to h'. In conclusion, for any $z \in V(\mathrm{E}^r(F)) \setminus V(F)$ there are at most $O(1)$ vertices z can map to in a homomorphism ϕ' which restricts to ϕ_y. Thus there are at most $O(1)$ homomorphisms ϕ' which restrict to ϕ_y, proving the claim. □

Observe that the number of maps $\phi : V(F) \setminus \{x\} \to V(H)$ is at most $n^{v(F)-1}$, and hence the claim above implies

$$t_{\mathrm{E}^r(F)}(H) \leqslant n^{v(F)-1-v(\mathrm{E}^r(F))} = n^{-1-(r-k)e(F)}.$$

On the other hand,

$$t_{K_r^r}(H)^{e(F)+\frac{1}{r-k}-\epsilon} = n^{(k-r-o(1))(e(F)+\frac{1}{r-k}-\epsilon)} = n^{-1-(r-k)e(F)+(r-k)\epsilon-o(1)},$$

and for n sufficiently large in terms of ϵ this is greater than the bound for $t_{\mathrm{E}^r(F)}(H)$ obtained above, proving the result. □

Before going on, we note that one can easily adapt the proof above to give stronger quantitative bounds on $s(F)$ in certain cases. For example, if there exists a subset $V \subseteq V(F)$ such that for every $x \in V$ there exist vertices $x_1, \ldots, x_k \in V(F) \setminus V$ forming a K_{k+1}^k with x, then one can prove

$$s(\mathrm{E}^r(F)) \geqslant \frac{|V|}{r-k}.$$

4. Proof of Theorem 1.13

Here we establish an upper bound on $s(\mathrm{E}^r(F))$ in terms of F. For this the following will be useful.

Lemma 4.1. *If F' is an r-partite r-graph and H is an r-graph such that $t_{F'}(H) \leqslant t_{K_r^r}(H)^{s'+e(F')}$ for some s', then $t_{K_r^r}(H) \geqslant v(H)^{-\frac{v(F')-r}{s'+e(F')-1}}$.*

Proof. Let H satisfy $t_{F'}(H) \leqslant t_{K_r^r}(H)^{s'+e(F')}$ and let δ be such that $t_{K_r^r}(H) = v(H)^{-\delta}$. This means H has $(r!)^{-1}v(H)^{r-\delta}$ edges, and hence $\hom(F', H) \geqslant v(H)^{r-\delta}$ (since F' mapping onto a single edge of H is always a homomrphism by assumption of F' being r-partite). Hence

$$v(H)^{r-\delta-v(F')} \leqslant t_{F'}(H) \leqslant v(H)^{-\delta(s'+e(F'))},$$

and rearranging shows $\delta \leqslant \frac{v(F')-r}{s'+e(F')-1}$, proving the result. □

For our proof, it will be convenient to work with weighted r-graphs W where the weight of an r-set $\{x_1, \ldots, x_r\}$ is denoted $W(x_1, \ldots, x_r)$. We let $\mathrm{Hom}(F, W)$ denote the set of all maps $\phi : V(F) \to V(W)$ which are injective on $e \in E(F)$. We define the weight $w(\phi)$ of $\phi \in \mathrm{Hom}(F, W)$ to be $\prod_{e \in E(F)} W(\phi(e))$ and we define $\hom(F, W) = \sum_{\phi \in \mathrm{Hom}(F,W)} w(\phi)$. With this we can define the notion of homomorphism densities $t_F(W)$ exactly as before, and it is not difficult to show that if $s(F) = s$ then $t_F(W) \geqslant t_{K_r^r}(W)^{s+e(F)}$ for all weighted r-graphs W.

Proof of Theorem 1.13. Recall that we wish to show that if F is a k-graph with $s(F) < \infty$, then for all $r \geqslant k$ we have

$$s(\mathrm{E}^r(F)) \leqslant s' := \frac{v(F) - k}{v(F) - k + (r - k)(s(F) + e(F) - 1)} \cdot s(F).$$

Assume for contradiction that there exists an n-vertex r-graph H such that $t_{\mathrm{E}^r(F)}(H) < t_{K_r^r}(H)^{s'+e(F)}$. Since $s(F) < \infty$ by assumption, F must be k-partite and hence $\mathrm{E}^r(F)$ must be r-partite. Thus by Lemma 4.1 we must have $t_{K_r^r}(H) \geqslant n^{-\frac{v(F)+(r-k)e(F)-r}{s'+e(F)-1}}$, or equivalently

$$n \geqslant t_{K_r^r}(H)^{-\frac{s'+e(F)-1}{v(F)+(r-k)e(F)-r}}. \tag{4.1}$$

We define an auxiliary weighted k-graph W on $V(H)$ such that for any k-set X we have $W(X) = (r-k)!\deg_H(X)$, i.e. $(r-k)!$ times the number of edges of H containing X. By definition of $s(F)$ we have

$$\begin{aligned} t_F(W) &= \frac{\sum_{\phi \in \mathrm{Hom}(F,W)} \prod_{e \in E(F)} (r-k)!\deg_H(\phi(e))}{n^{v(F)}} \\ &\geqslant t_{K_k^k}(W)^{s(F)+e(F)} = \left(\frac{k!}{n^k} \sum_{X \in W} (r-k)!\deg_H(X)\right)^{s(F)+e(F)} = \left(\frac{r!e(H)}{n^k}\right)^{s(F)+e(F)}. \end{aligned} \tag{4.2}$$

By definition of expansions, every homomorphism $\phi : V(\mathrm{E}^r(F)) \to V(H)$ can be formed by first choosing a homomorphism $\phi' : V(F) \to V(W)$, and then for each $e' \in F$ with $e \in \mathrm{E}^r(F)$ the edge containing e', one chooses some edge $h \in E(H)$ containing the k-set $\phi(e')$ together with a bijection from $e \setminus e'$ to $h \setminus \phi(e)$. Thus we have

$$\hom(\mathrm{E}^r(F), H) = \sum_{\phi \in \mathrm{Hom}(F,W)} \prod_{e \in E(F)} (r-k)\,!\deg_H(\phi(e)).$$

Hence,

$$\begin{aligned} t_{\mathrm{E}^r(F)}(H) &= \frac{\sum_{\phi \in \mathrm{Hom}(F,W)} \prod_{e \in E(F)} (r-k)\,!\deg_H(\phi(e))}{n^{v(F)+(r-k)e(F)}} \\ &= \frac{t_F(W)}{n^{(r-k)e(F)}} \\ &\geqslant \frac{\left(r\,!e(H)n^{-k}\right)^{s(F)+e(F)}}{n^{(r-k)e(F)}} \\ &= t_{K_r^r}(H)^{s(F)+e(F)} \cdot n^{(r-k)s(F)} \\ &\geqslant t_{K_r^r}(H)^{s(F)+e(F)} \cdot t_{K_r^r}(H)^{-\frac{(r-k)s(F)(s'+e(F)-1)}{v(F)+(r-k)e(F)-r}}, \end{aligned}$$

where the first inequality used (4.2) and the second used (4.1). One can verify that this final quantity equals $t_{K_r^r}(H)^{s'+e(F)}$, a contradiction to our choice of H. We conclude the result. □

As an aside, it is tempting to generalize the statement of Theorem 1.13 to hold even at $s(F) = \infty$. Indeed, "taking the limit" in Theorem 1.13 suggests that when $s(F) = \infty$ we should have

$$s(\mathrm{E}^r(F)) \leqslant \frac{v(F)-k}{r-k}.$$

This does hold whenever $\mathrm{E}^r(F)$ satisfies $\mathrm{ex}(n, \mathrm{E}^r(F)) = O(n^k)$ by our general upper bound Corollary 1.8 since

$$\frac{v(\mathrm{E}^r(F))-k}{r-k} - e(\mathrm{E}^r(F)) = \frac{v(F)-k}{r-k}.$$

However, such a result does not hold in general. For example, it certainly fails if $\mathrm{E}^r(F)$ is not r-partite, such as when considering $\mathrm{E}^3(K_t)$ with $t > 3$.

5. Concluding remarks

There are many questions left to explore regarding (non-)Sidorenko hypergraphs which we break into three broad categories.

Sidorenko Expansions. In Corollary 1.14 we showed that expansions of Sidorenko hypergraphs are Sidorenko. This motivates the following conjecture.

Conjecture 5.1. For every bipartite graph F, there exists an $r \geqslant 2$ such that $\mathrm{E}^r(F)$ is Sidorenko.

Note that Sidorenko's conjecture predicts this holds with $r = 2$ for all F. Moreover, Corollary 1.14 suggests it may be easier to prove Conjecture 5.1 for larger values of r (since if it holds for some r_0, then it holds for all $r \geqslant r_0$). As such, Conjecture 5.1 can be viewed as a (potentially) weaker version of Sidorenko's conjecture, and it would be particularly interesting if one could verify it for some $r \geqslant 2$ independent of F.

Another question asks whether the converse of Corollary 1.14 holds.

Question 5.2. Is it true that F is Sidorenko if and only if all of its expansions $\mathrm{E}^r(F)$ are Sidorenko?

Note that if Question 5.2 has an affirmative answer, then Conjecture 5.1 would be equivalent to Sidorenko's conjecture. The simplest case that we do not know how to answer is the following.

Question 5.3. If F is a non-bipartite graph, are all of its expansions $\mathrm{E}^r(F)$ not Sidorenko?

We note that [CLS24, Theorem 3.1] shows this holds if F has odd girth, but beyond this we know nothing.

k-linear Hypergraphs. In Theorem 1.11 we proved that expansions of k-graphs containing K_{k+1}^k are not Sidorenko. We conjecture that the following stronger result holds.

Conjecture 5.4. If F is an r-graph such that $|e \cap f| < k$ for any distinct $e, f \in F$ and such that F contains an expansion $\mathrm{E}^r(K_{k+1}^k)$ as a subgraph, then F is not Sidorenko.

The case $k = 2$ was proven in [CLS24], but as they note, their construction does not seem to effectively generalize to higher uniformities. We offer an alternative proof of the $k = 2$ case in the appendix of the arXiv version of this paper to serve as another potential source of inspiration towards proving Conjecture 5.4.

Bounds for $s(F)$. Theorem 1.5 motivates the problem of determining $s(F)$ for non-Sidorenko hypergraphs, especially those for which the random Turán number is unknown. One outstanding case is that of loose odd cycles.

Problem 5.5. Determine $s(C_{2\ell+1}^r)$.

In Corollary 1.10 we showed $s(C_{2\ell+1}^r) \leqslant \frac{2\ell-1}{r-2}$, and by considering $H = K_r^r$ it is possible to prove that $s(C_{2\ell+1}^r)$ is at least roughly $r^{-2\ell-1}$. We believe the upper bound is closer to the truth, but we do not think this is tight. Our best guess (though we would not go so far as to make it a conjecture) is that

$$s(C_{2\ell+1}^r) = \frac{1}{(r-1)\ell - 1}.$$

Indeed, the lower bound $s(C_{2\ell+1}^r) \geqslant \frac{1}{(r-1)\ell-1}$ would follow if there existed n-vertex r-graphs of girth $2\ell+2$ with $n^{1+1/\ell-o(1)}$ edges, which is the densest such an r-graph can be [CCGJ18]. Such r-graphs are only known to exist when $\ell = 1$ due to Ruzsa–Szemerédi type constructions, and it is difficult for us to imagine a construction that would give a better lower bound for $s(C_{2\ell+1}^r)$ than this. We also note that the general upper bound $s(C_{2\ell+1}^r) \leqslant \frac{1}{(r-1)\ell-1}$ would follow from the $r = 3$ result by using Theorem 1.13.

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