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The foundation of generalized parallel connections, 2-sums, and segment-cosegment exchanges of matroids

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Abstract. We show that, under suitable hypotheses, the foundation of a generalized parallel connection of matroids is the relative tensor product of the foundations. Using this result, we show that the foundation of a 2-sum of matroids is the absolute tensor product of the foundations, and that the foundation of a matroid is invariant under segment-cosegment exchange.

Keywords. Matroids, pastures, foundations, generalized parallel connection, 2-sum, segment-cosegment exchange

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1. Introduction

Pastures are algebraic objects that generalize partial fields and hyperfields. In [BL25a], Baker and Lorscheid study the *foundation* of a matroid M, which is a pasture canonically attached to M that governs the representability of M over arbitrary pastures. In particular, the foundation F_M determines the set of projective equivalence classes of representations of M over partial fields. More precisely, for any pasture P, the set of (weak) P-representations of M, modulo a suitable equivalence relation generalizing projective equivalence, is canonically identified with the set of pasture homomorphisms from F_M to P.

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Let M_1, M_2 be matroids with ground sets E_1 and E_2 respectively. If $E_1 \cap E_2 = T$ with $M_1|T = M_2|T$ and T is a modular flat[1] in either M_1 or M_2, then one can define the *generalized parallel connection* $P_T(M_1, M_2)$ (cf. [Oxl06, p.441]) as the matroid on $E = E_1 \cup E_2$ such that F is a flat of $P_T(M_1, M_2)$ if and only if $F \cap E_i$ is a flat of M_i for $i = 1, 2$.

There are some important constructions in matroid theory which make use of the generalized parallel connection, two of the most important being:

1. If M_1 and M_2 are simple and $T = \{p\}$ is a singleton, then T is automatically a modular flat in both M_1 and M_2. In this case, we define the *2-sum of M_1 and M_2 along p*, denoted $M_1 \oplus_2 M_2$ (or $M_1 \oplus_p M_2$, if we want to emphasize the dependence on p), to be the minor $P_T(M_1, M_2)\backslash T$ of $P_T(M_1, M_2)$.

2. If T is a coindependent triangle (i.e., 3-element circuit) in a matroid M, we define the *Delta-Wye exchange of M along T*, denoted $\Delta_T(M)$, to be the minor $P_T(M, M(K_4))\backslash T$ of $P_T(M, M(K_4))$, where T is identified with a triangle in $M(K_4)$.

   More generally, if M is a matroid and $X \subseteq E(M)$ is a coindependent set such that $M|X \cong U_{2,n}$ for some $n \geqslant 2$, one defines the *segment-cosegment exchange of M along X* to be $P_X(M, \Theta_n)\backslash X$, where Θ_n is a certain matroid on $2n$ elements defined in Section 5. When $n = 3$, we have $\Theta_3 \cong M(K_4)$ and the segment-cosegment exchange of M along X coincides with $\Delta_X(M)$.

It is known that a 2-sum of matroids M_1 and M_2 is representable over a partial field P if and only if M_1 and M_2 are both representable over P [vZ09, Corollary 2.4.31]. It is also known that if M is a matroid containing a coindependent set X such that $M|X \cong U_{2,n}$ for some $n \geqslant 2$, then M is representable over a partial field P if and only if the segment-cosegment exchange of M along X is representable over P [OSV00, Corollary 3.6]. In this paper, we generalize these results in two important ways:

- We establish bijections between suitable rescaling classes of P-representations.
- We prove analogous results for representations over arbitrary pastures.

Our main theorems are as follows:

Theorem A. *Let M_1 and M_2 be matroids so that $E(M_1) \cap E(M_2) = T$ and $M_1|T = M_2|T$, and let $N = M_1|T$. Suppose that either:*

(1) T is a modular flat of both M_1 and M_2; or

(2) T is isomorphic to $U_{2,n}$ for some $n \geqslant 2$ and M_2 is isomorphic to Θ_n.

Then the foundation of $P_T(M_1, M_2)$ is isomorphic to $F_{M_1} \otimes_{F_N} F_{M_2}$.

Part (1) of Theorem A is proved in Section 3, and part (2) is proved in Section 5.

In the special case where $T = \varnothing$, we obtain the following corollary (also proved in [BLZ25]):

[1] A flat T of a matroid M is called *modular* if $r(T) + r(F) = r(T \cap F) + r(T \cup F)$ for every flat F of M, where r is the rank function of M.

Corollary B. *The foundation of a direct sum $M_1 \oplus M_2$ is isomorphic to $F_{M_1} \otimes F_{M_2}$.*

Remark. When T is a modular flat in M_2 but not necessarily in M_1, the generalized parallel connection $M = P_T(M_1, M_2)$ is still well-defined, but the identity $F_{P_T(M_1,M_2)} \cong F_{M_1} \otimes_{F_{M_1|T}} F_{M_2}$ does not necessarily hold, even when $r(T) = 2$. We give an example at the end of Section 3.

In certain situations, the foundations of $P_T(M_1, M_2)$ and $P_T(M_1, M_2)\backslash T$ turn out to be isomorphic. The two most important examples are that of 2-sums and segment-cosegment exchanges:

Theorem C. *Let M_1 and M_2 be matroids on E_1 and E_2, respectively, so that $E_1 \cap E_2 = \{p\}$ and p is not a loop or a coloop in M_1 or M_2. Then the foundation of the 2-sum $M_1 \oplus_p M_2$ is isomorphic to $F_{M_1} \otimes F_{M_2}$.*

Theorem D. *Let M be a matroid and let $X \subseteq E(M)$ be a coindependent set such that $M|X \cong U_{2,n}$ for some $n \geqslant 2$. Then the foundation of the segment-cosegment exchange of M along X is isomorphic to F_M.*

A proof of Theorem C is given in Section 4. Theorem C implies, in particular, that (under the hypotheses of Theorem C) for every partial field P there is a bijection between rescaling equivalence classes of P-representations of $M_1 \oplus_p M_2$ and pairs of rescaling equivalence classes of P-representations of M_1 and M_2. To the best of our knowledge, even this particular consequence of Theorem C is new.

Theorem D is proved in Section 5. It generalizes a result of Oxley–Semple–Vertigan [OSV00, Corollary 3.6] which says that, under the hypotheses of Theorem D, for every partial field P there is a bijection between rescaling equivalence classes of P-representations of M and rescaling equivalence classes of P-representations of the segment-cosegment exchange of M along X.

The proof of Theorem C relies on part (1) of Theorem A, and the proof of Theorem D relies on part (2) of Theorem A.

Remark. The foundation of $M' = P_T(M_1, M_2)\backslash T$ is not in general isomorphic to the foundation of $M = P_T(M_1, M_2)$, even when $E(M_1)$ and $E(M_2)$ are both modular in M. For example, if N is any non-regular matroid on E and $M_i = N \oplus e_i$ with $e_i \notin E$ for $i = 1, 2$, then $E(N)$ is a modular flat of both M_1 and M_2, so by Theorem A we have $F_M \cong F_{M_1} \otimes_{F_N} F_{M_2}$. However, $F_{M'} = F_{e_1 \oplus e_2} \cong \mathbb{F}_1^{\pm}$, whereas $F_{M_1} \otimes_{F_N} F_{M_2} \cong F_N \not\cong \mathbb{F}_1^{\pm}$.

Since the universal partial field of a matroid can be computed from its foundation (cf. [BL21, Lemma 7.48] and Section 5.1 below), Theorem D implies in particular an affirmative solution to Conjecture 3.4.4 in Stefan van Zwam's thesis [vZ09] (see Section 5.1 for a proof):

Corollary E. *Let M be a matroid and let $X \subseteq E(M)$ be a coindependent set such that $M|X \cong U_{2,n}$ for some $n \geqslant 2$, and assume that M is representable over some partial field. Then the universal partial field of the segment-cosegment exchange of M along X is isomorphic to the universal partial field of M.*

Theorem D also has the following consequence for excluded minors (which is proved in [OSV00, Theorem 1.1] in the special case where P is a partial field); for a proof, see Corollary 5.15.

Corollary F. *Let P be a pasture, and let M be an excluded minor for representability over P. Then every segment-cosegment exchange of M is also an excluded minor for representability over P.*

By applying Theorems C and D to $\mathrm{Hom}(F_M, P)$ for certain pastures P, we obtain some interesting consequences for P-representability. These consequences are already known when P is a partial field, but when $P = \mathbb{S}$ (the sign hyperfield) or $\mathbb{T}$ (the tropical hyperfield), we obtain what appear to be new results. In order to state these corollaries precisely, we recall the following definitions:

Definition.

(1) A matroid M is called *orientable* if $\mathrm{Hom}(F_M, \mathbb{S})$ is non-empty. (This is equivalent to the usual notion of orientability, cf. [BB19, Example 3.33].)

(2) A matroid M is called *rigid* if $\mathrm{Hom}(F_M, \mathbb{T})$ has just one element. (This is equivalent to the condition that the base polytope of M has no non-trivial regular matroid polytope subdivision, cf. [BL24, Proposition B.1].) Equivalently, M is rigid if and only if every homomorphism $F_M \to \mathbb{T}$ factors through the canonical inclusion $\mathbb{K} \to \mathbb{T}$, where $\mathbb{K}$ is the Krasner hyperfield.

We have the following straightforward corollaries of Theorems C and D, respectively.

Corollary G. *Let M_1 and M_2 be matroids on E_1 and E_2, respectively, so that $E_1 \cap E_2 = \{p\}$ and p is not a loop or a coloop of M_1 or M_2. Then the 2-sum $M_1 \oplus_p M_2$ is orientable (resp. rigid) if and only if M_1 and M_2 are both orientable (resp. rigid).*

Proof. Let $N = M_1 \oplus_p M_2$ and F_{M_1}, F_{M_2} and F_N be the foundations of M_1, M_2 and N, respectively. Then M_1 and M_2 are both orientable if and only if both $\mathrm{Hom}(F_{M_1}, \mathbb{S})$ and $\mathrm{Hom}(F_{M_2}, \mathbb{S})$ are non-empty. By the universal property of the tensor product in the category of pastures [BL25a, Lemma 2.7], there is a canonical bijection

$$\mathrm{Hom}(F_{M_1}, \mathbb{S}) \times \mathrm{Hom}(F_{M_2}, \mathbb{S}) \;=\; \mathrm{Hom}(F_{M_1} \otimes F_{M_2}, \mathbb{S}).$$

Moreover, by Theorem C we have $F_N \cong F_{M_1} \otimes F_{M_2}$. Thus M_1 and M_2 are both orientable if and only if

$$\mathrm{Hom}(F_{M_1}, \mathbb{S}) \times \mathrm{Hom}(F_{M_2}, \mathbb{S}) \;=\; \mathrm{Hom}(F_{M_1} \otimes F_{M_2}, \mathbb{S}) \;=\; \mathrm{Hom}(F_N, \mathbb{S})$$

is non-empty. This is, in turn, equivalent to $N = M_1 \oplus_p M_2$ being orientable.

The claim for rigid matroids follows from the same proof, replacing “orientable” by “rigid”, non-empty by singleton, and $\mathbb{S}$ by $\mathbb{T}$ throughout. □

Corollary H. *Let M be a matroid and let $X \subseteq E(M)$ be a coindependent set such that $M|X \cong U_{2,n}$ for some $n \geqslant 2$. Then the segment-cosegment exchange of M along X is orientable (resp. rigid) if and only if M is orientable (resp. rigid).*

Proof. By Theorem D, the foundation of the segment-cosegment exchange of M along X is isomorphic to the foundation of M. Since the notions of orientability and rigidity for a matroid M depend only on the foundation of M, the claim follows. □

2. Background on foundations and representations of matroids over pastures

In this section, we recall some background material from [BL25a] which will be used throughout this paper. We also discuss some preliminary facts about generalized parallel connections which we will need.

2.1. Pastures

Pastures are a generalization of the notion of field in which we still have a multiplicative abelian group G, an absorbing element 0, and an "additive structure", but we relax the requirement that the additive structure come from a binary operation.

By a *pointed monoid* we mean a multiplicatively written commutative monoid P with an element 0 that satisfies $0 \cdot a = 0$ for all $a \in P$. We denote the unit of P by 1 and write $P^\times$ for the group of invertible elements in P. We denote by $\mathrm{Sym}_3(P)$ all elements of the form $a+b+c$ in the monoid semiring $\mathbb{N}[P]$, where $a, b, c \in P$.

Definition 2.1. A *pasture* is a pointed monoid P, together with a subset N_P of $\mathrm{Sym}_3(P)$, such that $a \in P^\times$ for all nonzero $a \in P$ and for all $a, b, c, d \in P$ we have:

(P1) $a + 0 + 0 \in N_P$ if and only if $a = 0$,

(P2) if $a + b + c \in N_P$, then $ad + bd + cd$ is in N_P,

(P3) there is a unique element $\epsilon \in P^\times$ such that $1 + \epsilon + 0 \in N_P$.

We call N_P the *nullset of* P, and say that $a+b+c$ *is null*, and write symbolically $a + b + c = 0$, if $a + b + c \in N_P$. The element ϵ plays the role of an additive inverse of 1, and the relations $a + b + c = 0$ express that certain sums of elements are zero, even though the multiplicative monoid P does not carry an addition. For this reason, we will write frequently $-a$ for ϵa and $a - b$ for $a + \epsilon b$. In particular, we have $\epsilon = -1$.

A *morphism* of pastures is a multiplicative map $f : P \to P'$ of monoids such that $f(0) = 0$, $f(1) = 1$ and $f(a) + f(b) + f(c) = 0$ in P' whenever $a + b + c = 0$ in P.

2.1.1 Examples

Every field F can be considered as a pasture whose underlying monoid equals that of F and whose nullset is $N_F = \{a + b + c \mid a + b + c = 0 \text{ in } F\}$.

Other examples of interest are the following:

1. The *regular partial field* is the pointed monoid $\mathbb{F}_1^\pm = \{0, 1, -1\}$ (with the obvious multiplication) together with the nullset $N_{\mathbb{F}_1^\pm} = \{0,\ 1 - 1\}$.

2. The *Krasner hyperfield* is the pointed monoid $\mathbb{K} = \{0, 1\}$ (with the obvious multiplication) together with the nullset $N_{\mathbb{K}} = \{0,\ 1 + 1,\ 1 + 1 + 1\}$.

3. The *sign hyperfield* is the pointed monoid $\mathbb{S} = \{0, 1, -1\}$ (with the obvious multiplication) together with the nullset $N_{\mathbb{S}} = \{0,\ 1-1,\ 1+1-1,\ 1-1-1\}$.

4. The *tropical hyperfield* is the pointed monoid $\mathbb{T} = \mathbb{R}_{\geqslant 0}$ (with the obvious multiplication) together with the nullset $N_{\mathbb{T}} = \{a + b + b \mid a \leqslant b\}$.

2.1.2 Tensor products

The category of pastures contains all limits and colimits. For example, $\mathbb{F}_1^{\pm}$ is initial and $\mathbb{K}$ is terminal, i.e., for every pasture P, there are unique morphisms $\mathbb{F}_1^{\pm} \to P$ and $P \to \mathbb{K}$.

The categorical construction that is most essential to this paper is the tensor product (or push-out). Namely given pasture morphisms $\alpha_1 : P_0 \to P_1$ and $\alpha_2 : P_0 \to P_2$, there is a pasture $P_1 \otimes_{P_0} P_2$ together with morphisms $\iota_1 : P_1 \to P_1 \otimes_{P_0} P_2$ and $\iota_2 : P_2 \to P_1 \otimes_{P_0} P_2$ such that $\iota_1 \circ \alpha_1 = \iota_2 \circ \alpha_2$ that is universal in the sense that for every pair of pasture morphisms $f_1 : P_1 \to Q$ and $f_2 : P_2 \to Q$ with $f_1 \circ \alpha_1 = f_2 \circ \alpha_2$, there is a unique pasture morphism $f : P_1 \otimes_{P_0} P_2 \to Q$ such that $f_1 = f \circ \iota_1$ and $f_2 = f \circ \iota_2$. In other words, there is a canonical bijection

$$\mathrm{Hom}(P_1 \otimes_{P_0} P_2,\ Q) \longrightarrow \mathrm{Hom}(P_1,\ Q) \times_{\mathrm{Hom}(P_0, Q)} \mathrm{Hom}(P_2,\ Q)$$

that is functorial in Q. This property determines $P_1 \otimes_{P_0} P_2$, together with ι_1 and ι_2, uniquely up to unique isomorphism. For the construction of $P_1 \otimes_{P_0} P_2$, we refer the reader to [Cre21].

2.2. Representations of matroids over pastures

Let P be a pasture and let M be a matroid on the finite set E. There are various "cryptomorphic" descriptions of weak P-matroids, for example in terms of "weak P-circuits", cf. [BB19]. For the purposes of the present paper, however, it will be more convenient to define weak P-matroids in terms of modular systems of hyperplane functions, as in [BL25a, Section 2.3]. The point here is that generalized parallel connections are defined in terms of flats, so we have easier access to the hyperplanes of a generalized parallel connection than to the bases or circuits.

Definition 2.2. Let $\mathscr{H}$ be the set of hyperplanes of M.

1. Given $H \in \mathscr{H}$, we say that $f_H : E \to P$ is a *P-hyperplane function* for H if $f_H(e) = 0$ if and only if $e \in H$.

2. A triple of hyperplanes $(H_1, H_2, H_3) \in \mathscr{H}^3$ is *modular* if $F = H_1 \cap H_2 \cap H_3$ is a flat of corank 2 such that $F = H_i \cap H_j$ for all distinct $i, j \in \{1, 2, 3\}$.

3. A *modular system* of P-hyperplane functions for M is a collection of P-hyperplane functions $f_H : E \to P$, one for each $H \in \mathscr{H}$, such that whenever H_1, H_2, H_3 is a modular triple of hyperplanes in $\mathscr{H}$, the corresponding functions f_{H_i} are linearly dependent, i.e., there exist constants c_1, c_2, c_3 in P, not all zero, such that

$$c_1 f_{H_1}(e) + c_2 f_{H_2}(e) + c_3 f_{H_3}(e) = 0$$

for all $e \in E$.

Definition 2.3.
1. A P*-representation* of M is a modular system of P-hyperplane functions for M.

2. Two P-representations $\{f_H\}$ and $\{f'_H\}$ of M are *isomorphic* if there is a function $H \mapsto c_H$ from $\mathscr{H}$ to $P^\times$ such that $f'_H(e) = c_H f_H(e)$ for all $e \in E$ and $H \in \mathscr{H}$.

3. Two P-representations $\{f_H\}$ and $\{f'_H\}$ of M are *rescaling equivalent* if there are functions $H \mapsto c_H$ from $\mathscr{H}$ to $P^\times$ and $e \mapsto c_e$ from E to $P^\times$ such that $f'_H(e) = c_H c_e f_H(e)$ for all $e \in E$ and $H \in \mathscr{H}$.

When P is a partial field, a rescaling equivalence class of P-representations of M is the same thing as a projective equivalence class of P-representations of M in the sense of [PvZ10]. When P is a field, the equivalence between the notion of representability provided in Definition 2.3 and the usual notion of matroid representability over a field is precisely the content of "Tutte's representation theorem", cf. [Tut65, Theorem 5.1].

Remark 2.4. The notion of rescaling classes of P-representations given by Definition 2.3 is compatible with the notion of rescaling classes of P-representations given in [BL21, Section 1.4.7]. Indeed, by [BL25a, Thm. 2.16], for every modular system $\{f_H\}$ of hyperplane functions for M in P, there is a weak Grassmann–Plücker function $\Delta : E^r \to P$ representing M such that

$$\frac{f_H(e)}{f_H(e')} \;=\; \frac{\Delta(e, e_2, \ldots, e_r)}{\Delta(e', e_2, \ldots, e_r)}$$

for every $H \in \mathscr{H}$ and all $e, e', e_2, \ldots, e_r \in E$ such that $\{e_2, \ldots, e_r\}$ spans H and $\{e', e_2, \ldots, e_r\}$ is a basis of M. The weak Grassmann–Plücker function Δ is uniquely determined up to a constant $c \in P^\times$, and two modular systems of hyperplane functions $\{f_H\}$ and $\{f'_H\}$ correspond to the same weak Grassmann–Plücker function $\Delta : E^r \to P$ (up to a constant) if and only if they are isomorphic.

Two weak Grassmann–Plücker functions Δ and Δ' are rescaling equivalent if there are a constant $c \in P^\times$ and a function $e \mapsto c_e$ from $E \to P^\times$ such that

$$\Delta'(e_1, \ldots, e_r) \;=\; c \cdot c_{e_1} \cdots c_{e_r} \cdot \Delta(e_1, \ldots, e_r).$$

Consequently, we have

$$\frac{\Delta'(e, e_2, \ldots, e_r)}{\Delta'(e', e_2, \ldots, e_r)} \;=\; \frac{c_e \cdot \Delta(e, e_2, \ldots, e_r)}{c_{e'} \cdot \Delta(e', e_2, \ldots, e_r)} \;=\; \frac{c_e \cdot f_H(e)}{c_{e'} \cdot f_H(e')},$$

where $H \in \mathscr{H}$ and $e, e', e_2, \ldots, e_r \in E$ are as before. This establishes a bijection

$$\left\{ \begin{array}{c} \text{rescaling classes of weak Grassmann–} \\ \text{Plücker functions for } M \text{ in } P \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{rescaling classes of modular systems} \\ \text{of hyperplane functions for } M \text{ in } P \end{array} \right\}.$$

2.3. The universal pasture and the foundation

Let $\mathcal{X}^I_M(P)$ (resp. $\mathcal{X}^R_M(P)$) be the set of isomorphism classes (resp. rescaling equivalence classes) of P-representations of M. It is shown in [BL25a] that the functors $\mathcal{X}^I_M$ and $\mathcal{X}^R_M$ are representable by the universal pasture $\tilde{F}_M$ and the foundation F_M, respectively. This is equivalent to the fact that $\mathcal{X}^I_M(P) = \mathrm{Hom}(\tilde{F}_M, P)$ (resp. $\mathcal{X}^R_M(P) = \mathrm{Hom}(F_M, P)$) functorially in P.

In particular, in order to show that some pasture F' is isomorphic to the foundation of M, it is equivalent to show that for every morphism of pastures $P \to P'$ there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(F', P) & \xrightarrow{\cong} & \mathcal{X}^R_M(P) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(F', P') & \xrightarrow{\cong} & \mathcal{X}^R_M(P'). \end{array}$$

We will use this observation (which is a version of the famous Yoneda Lemma in category theory) frequently throughout the paper. A similar characterization holds, of course, for the universal pasture of M.

Example 2.5. As an example, we compute the foundation of a regular matroid. Since a regular matroid M has a unique rescaling class of P-representations for every P (which is given by a unimodular matrix), we conclude that $\mathrm{Hom}(F_M, P) = \mathcal{X}^R_M(P)$ is a singleton for every P. In other words, F_M has a unique morphism to any other pasture, which characterizes F_M as the intial object $F_M = \mathbb{F}_1^\pm$ of the category of pastures.

This holds, in particular, for the foundation $F_p = \mathbb{F}_1^\pm$ of the matroid $M = U^1_1$ of rank 1 with one element p.

2.3.1 Induced representations for embedded minors

Let $\mathcal{H}$ be a modular system of P-hyperplane functions for a matroid M over a pasture P, and let $A \subseteq E(M)$. For $f_H \in \mathcal{H}$ and $X \subseteq E(M)$, we write $f_H|_X$ for the restriction of the function f_H to X. Define $\mathcal{H}/A = \{f_H|_{E(M)-A} \mid A \subseteq H\}$, and define $\mathcal{H}\backslash A = \{f_H|_{E(M)-A} \mid H - A \text{ is a hyperplane of } M\backslash A\}$. The following was originally stated in terms of weak P-circuits, but we obtain the following statement via the cryptomorphism between weak P-circuits and P-hyperplane functions.

Theorem 2.6 (Theorem 3.29 of [BB19])**.** *Let M be a matroid, let P be pasture, let $\mathcal{H}$ be a modular system of P-hyperplane functions for M, and let $A \subseteq E(M)$. Then, up to multiplying functions by scalars, $\mathcal{H}/A$ and $\mathcal{H}\backslash A$ are modular systems of P-hyperplane functions for M/A and $M\backslash A$, respectively.*

An *embedded minor* of a matroid M is a minor $N = M\backslash I/J$ together with the pair (I, J), where I is a coindependent subset and J is an independent subset of $E(M)$ such that $I \cap J = \varnothing$. Given an embedded minor $N = M\backslash I/J$ and a P-representation of M over a pasture P, Theorem 2.6 gives an induced P-representation for N. In general, this representation depends on the choices of I and J, meaning that if $N = M\backslash I/J = M\backslash I'/J'$, the representation induced by (I, J) may not be rescaling equivalent to the representation induced by (I', J'). However, when N is a restriction of M (or dually, a contraction of M), the induced representation is independent of the minor embedding. Before proving this, we highlight the following corollary of Theorem 2.6, which we will use repeatedly in our proofs.

Proposition 2.7. *Let M be a matroid, let $T \subseteq E(M)$, let P be a pasture, and let $\mathcal{H}$ be a modular system of P-hyperplane functions for M. If H and K are hyperplanes of M so that $H \cap T = K \cap T$ and this set is a hyperplane of $M|T$, then the functions $f_H|_T$ and $f_K|_T$ are scalar multiples of each other.*

Given a matroid M with $T \subseteq E(M)$, we will use Proposition 2.7 to define an induced system of hyperplane functions for $M|T$ that is independent of the minor embedding of $M|T$.

Proposition 2.8. *Let M be a matroid and let T and J be disjoint subsets of $E(M)$ so that $r(T) + r(J) = r(T \cup J)$. Let P be a pasture, and let $\mathcal{H}$ be a modular system of P-hyperplane functions for M. Let $\mathscr{T}_J$ be the set of hyperplanes of M that contain J and whose restriction to T is a hyperplane of $M|T$, and let $\mathcal{H}|_T = \{f_H|_T \mid H \in \mathscr{T}_J\}$. Then, up to multiplying functions by scalars, $\mathcal{H}|_T$ is a modular system of P-hyperplane functions for $M|T$, and is independent of the choice of J.*

Proof. By Proposition 2.7 we may assume, by rescaling, that if H and K are hyperplanes in $\mathscr{T}_J$ with $H \cap T = K \cap T$, then $f_H|_T = f_K|_T$. We will first show that every hyperplane of $M|T$ has an associated function in $\mathcal{H}|_T$. Fix a basis B of M/T with $J \subseteq B$. For each hyperplane L of $M|T$, the set $L' = \mathrm{cl}_M(L \cup B)$ is a hyperplane in $\mathscr{T}_J$ with $L' \cap T = L$, so L has associated P-hyperplane function $f_{L'}|_T \in \mathcal{H}|_T$. So it suffices to show that $\mathcal{H}|_T$ is a modular system. Let (L_1, L_2, L_3) be a modular triple of hyperplanes of $M|T$, and for each $i \in [3]$ let $L'_i = \mathrm{cl}_M(L_i \cup B)$. Then (L'_1, L'_2, L'_3) is a modular triple of hyperplanes of M, so there are constants $c_1, c_2, c_3 \in P^\times$ so that $c_1 \cdot f_{L'_1}(e) + c_2 \cdot f_{L'_2}(e) + c_3 \cdot f_{L'_3}(e) = 0$ for all $e \in E(M)$. Then $c_1 \cdot f_{L_1}(e) + c_2 \cdot f_{L_2}(e) + c_3 \cdot f_{L_3}(e) = 0$ for all $e \in T$, so $\mathcal{H}|_T$ is a modular system of P-hyperplane functions for $M|T$. Since $\{f_H|_T \mid H \in \mathscr{T}_J\} = \{f_H|_T \mid H \in \mathscr{T}_\varnothing\}$ because $\mathscr{T}_J \subseteq \mathscr{T}_\varnothing$, it follows that the modular system is independent of the choice of J. □

Given a matroid M with $T \subseteq E(M)$, a pasture P, and a P-representation $\mathcal{H}$ of M, we define $\mathcal{H}|_T = \{f_H|_T \mid H \text{ is a hyperplane of } M|T\}$. Let $E(M) - T = I \sqcup J$ be a decomposition of the complement of T in M into a coindependent set I and an independent set J. Then $M|T \simeq M\backslash I/J$, which induces a morphism of foundations

$$\iota_{M|T} : F_T \simeq F_{M\backslash I/J} \longrightarrow F_M$$

where we write F_T for $F_{M|T}$.

Lemma 2.9. *The morphism $\iota_{M|T}$ does not depend on the choices of I and J.*

Proof. Two choices of decompositions $E(M) - T = I_i \sqcup J_i$ (for $i = 1, 2$) induce two morphisms $\iota_i : F_T \to F_M$, each arising from the restriction of (the rescaling classes of) a modular system of P-hyperplane functions of M to $M|T$. Since these restrictions are independent of the choices of the decomposition $E(M) - T = I_i \sqcup J_i$, this means that the induced morphism of functors $\mathrm{Hom}(F_M, -) \to \mathrm{Hom}(F_T, -)$ is independent of $E(M) - T = I_i \sqcup J_i$. By the Yoneda lemma, this means that the morphism $F_T \to F_M$ is independent of this decomposition. □

As a consequence, the tensor product $F_{M_1} \otimes_{F_T} F_{M_2}$ of the foundations of two matroids M_1 and M_2 with common restriction $M_1|T = M_2|T$ has an intrinsic meaning that does not depend on the choice of minor embeddings of $M|T$ into M_1 and M_2.

2.3.2 Cross ratios

Let Ω_M be the collection of 5-tuples $(J; e_1, e_2, e_3, e_4)$, where J is an independent subset of $E(M)$ of cardinality $r-2$ and $e_1, e_2, e_3, e_4 \in E(M)$ are elements such that Je_ie_j is a basis for all $i \in \{1,2\}$ and $j \in \{3,4\}$, writing Je_ie_j for $J \cup \{e_i, e_j\}$. This means in particular that Je_i has rank $r-1$, and thus $H_i = \mathrm{cl}(Je_i)$ is a hyperplane, and that $e_j \notin H_i$ for $i \in \{1,2\}$ and $j \in \{3,4\}$.

The identification $\mathrm{Hom}(F_M, F_M) = \mathcal{X}_M^R(F_M)$ associates with the identity map $\mathrm{id}: F_M \to F_M$ the *universal rescaling class of* M, which is the rescaling class of some F_M-representation $\{f_H \mid H \in \mathcal{H}\}$ of M. We define the *universal cross ratio of* $(J; e_1, e_2, e_3, e_4) \in \Omega_M$ as

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_J = \frac{f_{H_1}(e_3) \cdot f_{H_2}(e_4)}{f_{H_1}(e_4) \cdot f_{H_2}(e_3)},$$

where $H_i = \mathrm{cl}(Je_i)$. Since rescaling by $c = \big((c_e), (c_H)\big) \in (P^\times)^E \times (P^\times)^{\mathcal{H}}$ yields

$$\frac{(cf_{H_1})(e_3) \cdot (cf_{H_2})(e_4)}{(cf_{H_1})(e_4) \cdot (cf_{H_2})(e_3)} = \frac{c_{H_1}c_{e_3}f_{H_1}(e_3) \cdot c_{H_2}c_{e_4}f_{H_2}(e_4)}{c_{H_1}c_{e_4}f_{H_1}(e_4) \cdot c_{H_2}c_{e_3}f_{H_2}(e_3)} = \frac{f_{H_1}(e_3) \cdot f_{H_2}(e_4)}{f_{H_1}(e_4) \cdot f_{H_2}(e_3)},$$

the universal cross ratio $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ depends only on the universal rescaling class, which shows that $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is a well-defined element of F_M.

We have $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = 1$ if Je_1e_2 or Je_3e_4 is not a basis, i.e., if $H_1 = H_2$ or $\mathrm{cl}(Je_3) = \mathrm{cl}(Je_4)$. In these cases, we say that $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is *degenerate*.

A more profound result, which is a consequence to Tutte's path theorem [Tut58, Theorem 5.1], is that $F_M^\times$ is generated by -1 and all universal cross ratios [BL21, Corollary 7.11]. Similarly, Tutte's homotopy theorem [Tut58, Theorem 6.1] can be used to exhibit a complete system of relations between the cross ratios as elements of $F_M^\times$ (see [BL25a, Theorem 4.19]), but we won't need this latter result for our purposes.

2.4. Facts about generalized parallel connections

Throughout this section, let M_1, M_2 be matroids with ground sets E_1 and E_2, respectively, with $E_1 \cap E_2 = T$ such that $M_1|T = M_2|T$ and T is a modular flat in M_2.

We have the following formula for the rank of flats in $P_T(M_1, M_2)$.

Proposition 2.10 (Proposition 5.5 of [Bry75]). *If r, r_1, r_2 are the rank functions of $P_T(M_1, M_2)$, M_1, and M_2 respectively, then for any flat F of $P_T(M_1, M_2)$ we have:*

$$r(F) = r_1(F \cap E_1) + r_2(F \cap E_2) - r_1(F \cap T). \quad \text{(a)}$$

In particular,

$$r(P_T(M_1, M_2)) = r(M_1) + r(M_2) - r(M_1|T). \tag{b}$$

When T is modular in both M_1 and M_2, there is a straightforward description of the hyperplanes of $P_T(M_1, M_2)$.

Proposition 2.11 (Proposition 22 of [HN11])**.** *Assume that T is a modular flat in both M_1 and M_2. A subset $H \subseteq E_1 \cup E_2$ is a hyperplane of $P_T(M_1, M_2)$ if and only if*

(1) $H \cap E_1$ is a hyperplane of M_1 that contains T, and H contains E_2, or

(2) $H \cap E_2$ is a hyperplane of M_2 that contains T, and H contains E_1, or

(3) $H \cap E_i$ is a hyperplane of M_i for $i = 1, 2$, and $r_{M_1}(H \cap T) = r_{M_1}(T) - 1$.

Proof. Let r be the rank function of $P_T(M_1, M_2)$. First suppose that H is a hyperplane of $P_T(M_1, M_2)$. Then $H \cap E_i$ is a flat of M_i for $i = 1, 2$. Let $r(H \cap T) = r(T) - k$ where $0 \leqslant k \leqslant r(T)$. Since T is a modular flat in M_i we have

$$\begin{aligned} r(T) + r(H \cap E_i) &= r(T \cap H) + r((T \cup H) \cap E_i) \\ &= r(T) - k + r((T \cup H) \cap E_i) \\ &\leqslant r(T) - k + r(E_i), \end{aligned}$$

and it follows that $r(H \cap E_i) \leqslant r(E_i) - k$. Then we have

$$\begin{aligned} r(H) &= r(E_1) + r(E_2) - r(T) - 1 \\ &= r(H \cap E_1) + r(H \cap E_2) - r(H \cap T) \\ &= r(H \cap E_1) + r(H \cap E_2) - (r(T) - k) \\ &\leqslant (r(E_1) - k) + (r(E_2) - k) - (r(T) - k) \\ &= r(E_1) + r(E_2) - r(T) - k, \end{aligned}$$

where the first line follows from (b) and the fact that H is a hyperplane of $P_T(M_1, M_2)$, and the second follows from (a). By comparing the first and last lines, we see that $k \leqslant 1$. By comparing the first and third lines, we have

$$r(E_1) + r(E_2) - 1 = r(H \cap E_1) + r(H \cap E_2) + k. \tag{c}$$

If $k = 0$, then $r(H \cap T) = r(T)$, and since H is a flat of $P_T(M_1, M_2)$ it follows that $T \subseteq H$. By (c), there is some $j \in \{1, 2\}$ so that $r(H \cap E_j) = r(E_j) - 1$ and $r(H \cap E_{3-j}) = r(E_{3-j})$. Since $H \cap E_i$ is a flat of M_i for $i = 1, 2$ by the definition of $P_T(M_1, M_2)$, it follows that if $j = 1$ then (1) holds, and if $j = 2$ then (2) holds. If $k = 1$, then $r(H \cap T) = r(T) - 1$. By (c) and the observation that $r(H \cap E_i) \leqslant r(E_i) - k$ for $i = 1, 2$ we see that $r(H \cap E_i) = r(E_i) - 1$ for $i = 1, 2$. Since $H \cap E_i$ is a flat of M_i for $i = 1, 2$ by the definition of $P_T(M_1, M_2)$, we see that (3) holds.

Conversely, suppose that (1), (2), or (3) holds for H. Since $H \cap E_i$ is a flat of M_i for $i = 1, 2$, it follows from the definition of $P_T(M_1, M_2)$ that H is a flat of $P_T(M_1, M_2)$, so it suffices to show that $r(H) = r(P_T(M_1, M_2)) - 1$. If (1) or (2) holds, then by (a) we see that $r(H) = r(M_1)+r(M_2)-r(T)-1$, and it follows from (b) that $r(H) = r(P_T(M_1, M_2))-1$. If (3) holds, then by (a) we see that $r(H) = r(M_1) + r(M_2) - r(T) - 1$, and by (b) it follows that $r(H) = r(P_T(M_1, M_2)) - 1$. □

A similar result holds for corank-2 flats.

Proposition 2.12. *Assume that T is a modular flat in both M_1 and M_2. A subset $F \subseteq E_1 \cup E_2$ is a corank-2 flat of $P_T(M_1, M_2)$ if and only if*

(1) $T \subseteq F$ and there is some $i \in \{1, 2\}$ so that $E_i \subseteq F$ and $F \cap E_{3-i}$ is a corank-2 flat of M_{3-i}, or

(2) $T \subseteq F$ and $F \cap E_i$ is a hyperplane of M_i for $i = 1, 2$, or

(3) $r_{M_1}(F \cap T) = r_{M_1}(T) - 1$, and there is some $i \in \{1, 2\}$ so that $F \cap E_i$ is a hyperplane of M_i and $F \cap E_{3-i}$ is a corank-2 flat of M_{3-i}, or

(4) $r_{M_1}(F \cap T) = r_{M_1}(T) - 2$, and $F \cap E_i$ is a corank-2 flat of M_i for $i = 1, 2$.

Proof. Let r be the rank function of $P_T(M_1, M_2)$. First suppose that F is a corank-2 flat of $P_T(M_1, M_2)$. Then $F \cap E_i$ is a flat of M_i for $i = 1, 2$. Let $r(F \cap T) = r(T) - k$ where $0 \leqslant k \leqslant r(T)$. As in the proof of Proposition 2.11, we know that $r(F \cap E_i) \leqslant r(E_i) - k$ for $i = 1, 2$. Then we have

$$\begin{aligned} r(F) &= r(E_1) + r(E_2) - r(T) - 2 \\ &= r(F \cap E_1) + r(F \cap E_2) - r(F \cap T) \\ &= r(F \cap E_1) + r(F \cap E_2) - (r(T) - k) \\ &\leqslant (r(E_1) - k) + (r(E_2) - k) - (r(T) - k) \\ &= r(E_1) + r(E_2) - r(T) - k, \end{aligned}$$

where the first line follows from (b) and the fact that F is a corank-2 flat of $P_T(M_1, M_2)$, and the second follows from (a). By comparing the first and last lines, we see that $k \leqslant 2$. By comparing the first and third lines, we have

$$r(E_1) + r(E_2) - 2 = r(F \cap E_1) + r(F \cap E_2) + k. \tag{d}$$

If $k = 0$, then $T \subseteq F$. By (d), either there is some $j \in \{1, 2\}$ so that $r(F \cap E_j) = r(E_j) - 1$ and $r(F \cap E_{3-j}) = r(E_{3-j})$ and (1) holds because $F \cap E_i$ is a flat for $i = 1, 2$, or $r(F \cap E_j) = r(E_j) - 1$ for $j = 1, 2$ and (2) holds. If $k = 1$, then $r(F \cap T) = r(T) - 1$. By (d) and the observation that $r(F \cap E_i) \leqslant r(E_i) - k$ for $i = 1, 2$ we see that (3) holds. If $k = 2$, then $r(F \cap T) = r(T) - 2$. By (d) and the observation that $r(F \cap E_i) \leqslant r(E_i) - k$ for $i = 1, 2$ we see that (4) holds.

Conversely, suppose that (1), (2), (3), or (4) holds for F. Since $F \cap E_i$ is a flat of M_i for $i = 1, 2$, it follows from the definition of $P_T(M_1, M_2)$ that F is a flat of $P_T(M_1, M_2)$. In each case it follows directly from (a) that $r(F) = r(M_1) + r(M_2) - r(T) - 2$, and by (b) it follows that F is a corank-2 flat of $P_T(M_1, M_2)$. □

We will also need analogous results when $r(T) = 2$ and T is not assumed to be modular in M_1. We replace T with X here, because we will apply this result in the case that $M_2 = \Theta_n$.

Proposition 2.13. *Let M_1, M_2 be matroids with ground sets E_1 and E_2, respectively, with $E_1 \cap E_2 = X$ such that $M_1|X = M_2|X$ and X is a modular flat in M_2. Assume furthermore that $M_2|X \cong U_{2,n}$ for some $n \geqslant 2$. A subset $H \subseteq E_1 \cup E_2$ is a hyperplane of $P_X(M_1, M_2)$ if and only if*

(1) $E_1 \subseteq H$ and $H \cap E_2$ is a hyperplane of M_2 that contains X, or

(2) $E_2 \subseteq H$ and $H \cap E_1$ is a hyperplane of M_1 that contains X, or

(3) $H \cap E_i$ is a hyperplane of M_i for $i = 1, 2$ and $|H \cap X| = 1$, or

(4) $H \cap E_1$ is a hyperplane of M_1 that is disjoint from X, and $H \cap E_2$ is a corank-2 flat of M_2 that is disjoint from X.

Proof. Let r be the rank function of $P_X(M_1, M_2)$. First suppose that H is a hyperplane of $P_X(M_1, M_2)$. Then $H \cap E_i$ is a flat of M_i for $i = 1, 2$. Let $r(H \cap X) = r(X) - k$ where $0 \leqslant k \leqslant 2$. Then we have

$$\begin{aligned} r(H) &= r(E_1) + r(E_2) - r(X) - 1 \\ &= r(H \cap E_1) + r(H \cap E_2) - r(H \cap X) \\ &= r(H \cap E_1) + r(H \cap E_2) - (r(X) - k), \end{aligned}$$

where the first line follows from (b) and the fact that H is a hyperplane of $P_X(M_1, M_2)$, and the second follows from (a). It follows that

$$r(E_1) + r(E_2) - 1 = r(H \cap E_1) + r(H \cap E_2) + k. \tag{e}$$

If $k = 0$ then $X \subseteq H$ because H is a flat, and it follows from (e) that (1) or (2) holds. If $k = 1$ then $|H \cap X| = 1$ because $M_i|X$ is simple, and $r(H \cap E_i) < r(E_i)$ for $i = 1, 2$ because H does not contain X. Then it follows from (e) that (3) holds. Finally, if $k = 2$ then $X \cap H = \varnothing$. Since X is modular flat in M_2, we know that

$$r(X) + r(H \cap E_2) = r(X \cup (H \cap E_2)),$$

and since $r(X) = 2$ and $r(X \cup (H \cap E_2)) \leqslant r(E_2)$ it follows that $r(H \cap E_2) \leqslant r(E_2) - 2$. Since $X \cap H = \varnothing$ we know that $r(H \cap E_1) \leqslant r(E_1) - 1$, and now (e) implies that (4) holds.

Conversely, suppose that (1), (2), (3), or (4) holds for H. Since $H \cap E_i$ is a flat of M_i for $i = 1, 2$, it follows from the definition of $P_T(M_1, M_2)$ that H is a flat of $P_T(M_1, M_2)$, so it suffices to show that $r(H) = r(P_T(M_1, M_2)) - 1$. In each case it follows directly from (a) that $r(H) = r(M_1) + r(M_2) - r(T) - 1$, and then (b) implies that $r(H) = r(P_T(M_1, M_2)) - 1$. □

A similar result holds for corank-2 flats.

Proposition 2.14. *With hypotheses as in Proposition 2.13, a subset $F \subseteq E_1 \cup E_2$ is a corank-2 flat of $P_X(M_1, M_2)$ if and only if*

(1) $E_1 \subseteq F$ and $F \cap E_2$ is a corank-2 flat of M_2 that contains X,

(2) $E_2 \subseteq F$ and $F \cap E_1$ is a corank-2 flat of M_1 that contains X,

(3) For each $i = 1, 2$, $F \cap E_i$ is a hyperplane of M_i that contains X,

(4) $|F \cap X| = 1$, $F \cap E_1$ is a hyperplane of M_1, and $F \cap E_2$ is a corank-2 flat of M_2, or

(5) $|F \cap X| = 1$, $F \cap E_1$ is a corank-2 flat of M_1, and $F \cap E_2$ is a hyperplane of M_2,

(6) $F \cap X = \varnothing$, $F \cap E_1$ is a hyperplane of M_1, and $F \cap E_2$ is a corank-3 flat of M_2, or

(7) $F \cap X = \varnothing$, and $F \cap E_i$ is a corank-2 flat of M_i for $i = 1, 2$.

Proof. Let r be the rank function of $P_X(M_1, M_2)$. First suppose that F is a corank-2 flat of $P_T(M_1, M_2)$. Then $F \cap E_i$ is a flat of M_i for $i = 1, 2$. Let $r(F \cap X) = r(X) - k$ where $0 \leqslant k \leqslant 2$. Then we have

$$\begin{aligned} r(F) &= r(E_1) + r(E_2) - r(X) - 2 \\ &= r(F \cap E_1) + r(F \cap E_2) - r(F \cap X) \\ &= r(F \cap E_1) + r(F \cap E_2) - (r(X) - k), \end{aligned}$$

where the first line follows from (b) and the fact that F is a corank-2 flat of $P_X(M_1, M_2)$, and the second follows from (a). It follows that

$$r(E_1) + r(E_2) - 2 = r(F \cap E_1) + r(F \cap E_2) + k. \tag{f}$$

If $k = 0$ then $X \subseteq F$ because F is a flat, and (f) implies that (1), (2), or (3) holds. If $k = 1$ then $|F \cap X| = 1$ because $M_i|X$ is simple, and $r(F \cap E_i) < r(M_i)$ for $i = 1, 2$ because F does not contain X. Then (f) implies that (4) or (5) holds. Finally, if $k = 2$ then $X \cap F = \varnothing$. Since X is modular flat in M_2, we know that

$$r(X) + r(F \cap E_2) = r(X \cup (F \cap E_2)),$$

and since $r(X) = 2$ and $r(X \cup (F \cap E_2)) \leqslant r(E_2)$ it follows that $r(F \cap E_2) \leqslant r(E_2) - 2$. Since $X \cap F = \varnothing$ we know that $r(F \cap E_1) \leqslant r(E_1) - 1$, and now (e) implies that (6) or (7) holds.

Conversely, suppose that one of (1)– (7) holds for F. Since $F \cap E_i$ is a flat of M_i for $i = 1, 2$, it follows from the definition of $P_T(M_1, M_2)$ that F is a flat of $P_T(M_1, M_2)$, so it suffices to show that $r(F) = r(P_T(M_1, M_2)) - 2$. In each case it follows directly from (a) that $r(F) = r(M_1) + r(M_2) - r(T) - 2$, and then (b) implies that $r(F) = r(P_T(M_1, M_2)) - 2$. □

We will also need to understand interactions between hyperplanes of a matroid. Given a matroid M, a *linear subclass* is a set $\mathscr{H}$ of hyperplanes of M so that if $H, H' \in \mathscr{H}$ and (H, H') is a modular pair, then every hyperplane containing $H \cap H'$ is also in $\mathscr{H}$. The canonical example of a linear subclass is the set of hyperplanes containing a fixed flat. The following proposition will be useful for inductive arguments involving hyperplanes that avoid a fixed linear subclass.

Proposition 2.15. *Let M be a matroid and let $\mathscr{H}$ be a linear subclass of M. If H and K are distinct hyperplanes of M with $H, K \notin \mathscr{H}$, then there is a hyperplane L of M so that $L \notin \mathscr{H}$, the pair (H, L) is modular, $(H \cap K) \subseteq L$, and $r(L \cap K) > r(H \cap K)$.*

Proof. Let F be a corank-2 flat of M with $(H \cap K) \subseteq F \subseteq H$. Let $\mathscr{F}$ be the set of hyperplanes of M that contain F and some element of $K - F$. If $|\mathscr{F}| = 1$, then $\mathrm{cl}_M(F \cup e) = \mathrm{cl}_M(F \cup e')$ for all $e, e' \in K - F$. Then $K \subseteq \mathrm{cl}_M(F \cup e)$, so (H, K) is a modular pair and the claim holds with $L = K$. So we may assume that $|\mathscr{F}| \geqslant 2$. If $\mathscr{F} \subseteq \mathscr{H}$, then $H \in \mathscr{H}$ because $\mathscr{H}$ is a linear subclass and all of the hyperplanes in $\mathscr{F}$ contain the corank-2 flat F. So there is some $L \in \mathscr{F} - \mathscr{H}$. Since $(H \cap K) \subseteq F \subseteq L$ we see that (H, L) is a modular pair and $(H \cap K) \subseteq L$. Since L contains an element in $K - F$ we see that $r(L \cap K) > r(H \cap K)$, and the statement holds. □

3. The foundation of a generalized parallel connection

The following theorem implies Theorem A (1), and also proves the analogous result for universal pastures. Recall from Lemma 2.9 that the restriction of a matroid M to a subset T induces a (well-defined) morphism $F_{M|T} \to F_M$ of foundations. We will write $\tilde{F}_T$ and F_T for $\tilde{F}_{M|T}$ and $F_{M|T}$, respectively.

Theorem 3.1. *Let M_1 and M_2 be matroids with ground sets E_1 and E_2, respectively, with $E_1 \cap E_2 = T$ so that $M_1|T = M_2|T$ and T is a modular flat of both M_1 and M_2, and let $M = P_T(M_1, M_2)$. Then $\tilde{F}_M \cong \tilde{F}_{M_1} \otimes_{\tilde{F}_T} \tilde{F}_{M_2}$ and $F_M \cong F_{M_1} \otimes_{F_T} F_{M_2}$.*

Proof. Let P be a pasture. Let $\mathcal{X}^I(M_1, M_2, T, P)$ (resp. $\mathcal{X}^R(M_1, M_2, T, P)$) be the subset of $\mathcal{X}^I_{M_1}(P) \times \mathcal{X}^I_{M_2}(P)$ (resp. $\mathcal{X}^R_{M_1}(P) \times \mathcal{X}^R_{M_2}(P)$) for which the induced representations of $M_1|T$ and $M_2|T$ are in the same isomorphism class (resp. rescaling equivalence class). We will define a map Φ from $\mathcal{X}^I(M_1, M_2, T, P)$ to $\mathcal{X}^I_M(P)$ and a map Ψ from $\mathcal{X}^I_M(P)$ to $\mathcal{X}^I(M_1, M_2, T, P)$. Then we will show that these maps are well-defined and inverse to each other. It will be clear from the definition of the resulting bijection that it is functorial in P. Therefore, by the universal property of the tensor product, we will obtain an isomorphism $\tilde{F}_M \cong \tilde{F}_{M_1} \otimes_{\tilde{F}_T} \tilde{F}_{M_2}$. Passing to rescaling classes instead of isomorphism classes shows that $F_M \cong F_{M_1} \otimes_{F_T} F_{M_2}$ as well.

For Φ, if we have a modular system $\mathcal{H}$ of P-hyperplane functions of M, then $\mathcal{H}|_{E_1}$ and $\mathcal{H}|_{E_2}$ are modular systems of hyperplane functions for M_1 and M_2, respectively, whose induced representations of $M|T$ are clearly isomorphic. For Ψ, let $\mathcal{H}_i$ be a modular system of P-hyperplane functions of M_i for $i = 1, 2$ so that $\mathcal{H}_1|_T$ and $\mathcal{H}_2|_T$ are isomorphic. By Propositions 2.7 and 2.11, we may assume, by scaling functions in $\mathcal{H}_1$ and $\mathcal{H}_2$, that if $f, f' \in \mathcal{H}_1 \cup \mathcal{H}_2$ have the same support in T, then $f(e) = f'(e)$ for all $e \in T$. For each hyperplane H of M we define a function f_H by declaring that if $H \cap E_i$ is a hyperplane for some $i = 1, 2$, then $f_H(e) = f_{H \cap E_i}(e)$ for all $e \in E_i$. Let $\mathcal{H}$ be the set of all f_H for hyperplanes H of M. By Proposition 2.11, the complements of the supports of the functions in $\mathcal{H}$ forms the set of hyperplanes of M. Clearly Φ and Ψ are inverse to each other because restricting the functions in $\mathcal{H}$ to E_i for $i = 1, 2$ results in the systems $\mathcal{H}_1$ and $\mathcal{H}_2$. So it remains to show that $\mathcal{H}$ is in fact a modular system.

Let F be a corank-2 flat of M and let (H, H', H'') be a modular triple of hyperplanes of M such that $H \cap H' \cap H'' = F$. We will show that $f_H, f_{H'}, f_{H''}$ are linearly dependent. There are four different cases to consider, stemming from the four cases for F in Proposition 2.12.

Case 1: Suppose $T \subseteq F$ and there is some $i \in \{1, 2\}$ so that $E_i \subseteq F$ and $F \cap E_{3-i}$ is a corank-2 flat of M_{3-i}. We may assume that $i = 1$. Then $(H \cap E_2, H' \cap E_2, H'' \cap E_2)$ is a modular triple of hyperplanes of M_2, and since $f_{H\cap E_2}, f_{H'\cap E_2}, f_{H'',\cap E_2}$ are linearly dependent in $\mathcal{H}_2$ it follows that $f_H, f_{H'}, f_{H''}$ are linearly dependent in $\mathcal{H}$.

Case 2: Suppose $T \subseteq F$ and $F \cap E_i$ is a hyperplane of M_i for $i = 1, 2$. By Proposition 2.11, the only hyperplanes of M containing F are $F \cup E_1$ and $F \cup E_2$, so there is no modular triple of hyperplanes that all contain F.

Case 3: Suppose $r_{M_1}(F \cap T) = r_{M_1}(T) - 1$, and there is some $i \in \{1, 2\}$ so that $F \cap E_i$ is a hyperplane of M_i and $F \cap E_{3-i}$ is a corank-2 flat of M_{3-i}. We may assume that $i = 1$. By Proposition 2.11 we see that $(H \cap E_2, H' \cap E_2, H'' \cap E_2)$ is a modular triple of hyperplanes of M_2, so there are constants c, c', c'' so that

$$c \cdot f_{H\cap E_2}(e) + c' \cdot f_{H'\cap E_2}(e) + c'' \cdot f_{H''\cap E_2}(e) = 0$$

for all $e \in E_2$. If none of H, H', H'' contains E_1, then $c + c' + c'' = 0$ because $H \cap E_2$, $H' \cap E_2$, $H'' \cap E_2$ all have the same restriction to T. Similarly, if $E_1 \subseteq H''$, then $c + c' = 0$. In either case it follows that $c \cdot f_H(e) + c' \cdot f_{H'}(e) + c'' \cdot f_{H''}(e) = 0$ for all $e \in E_1 \cup E_2$.

Case 4: Suppose $r_{M_1}(F \cap T) = r_{M_1}(T) - 2$ and $F \cap E_i$ is a corank-2 flat of M_i for $i = 1, 2$. If outcome (1) or (2) of Proposition 2.11 holds for H, then by (a) we see that $r_M(H) \geqslant r_M(F) + 2$, a contradiction. So outcome (3) of Proposition 2.11 holds for H, H', and H'', and since $F \cap E_i$ is a corank-2 flat of M_i for $i = 1, 2$, it follows that $(H \cap E_i, H' \cap E_i, H'' \cap E_i)$ is a modular triple of hyperplanes of M_i for $i = 1, 2$. Then there are constants c, c, c'' so that

$$c \cdot f_{H\cap E_1}(e) + c' \cdot f_{H'\cap E_1}(e) + c'' \cdot f_{H''\cap E_1}(e) = 0$$

for all $e \in E_1$, and constants d, d', d'' so that

$$d \cdot f_{H\cap E_2}(e) + d' \cdot f_{H'\cap E_2}(e) + d'' \cdot f_{H''\cap E_2}(e) = 0$$

for all $e \in E_2$. Since outcome (3) of Proposition 2.11 holds for H, H', and H'', we know that $r_M(H \cap T) = r_M(H' \cap T) = r_M(H'' \cap T) = r_M(T) - 1$. Since F and H do not agree on T, there is an element $t \in (H \cap T) - F$ so that $\mathrm{cl}_M(F \cup t) = H$. Then $t \notin H' \cup H''$, or else $H = H'$ or $H = H''$. By setting $e = t$, the first equation shows that $\frac{c}{c'} = -\frac{f_{H'\cap E_1}(t)}{f_{H\cap E_1}(t)}$, and the second equation shows that $\frac{d}{d'} = -\frac{f_{H'\cap E_2}(t)}{f_{H\cap E_2}(t)}$. It follows that $\frac{c}{c'} = \frac{d}{d'}$. Repeating this argument with an element $t' \in (H' \cap T) - (H \cup H'')$ and an element $t'' \in (H'' \cap T) - (H \cup H')$ shows that (c, c', c'') is a scalar multiple of (d, d', d''), and it follows that $c \cdot f_H(e) + c' \cdot f_{H'}(e) + c'' \cdot f_{H''}(e) = 0$ for all $e \in E_1 \cup E_2$.

The four cases combine to show that $\mathcal{H}$ is a modular system of P-hyperplane functions for M, as desired. So we have defined maps from $\mathcal{X}^I(M_1, M_2, T, P)$ to $\mathcal{X}^I_M(P)$ and vice versa that are inverse to each other and functorial in P, which shows that $\tilde{F}_M \cong \tilde{F}_{M_1} \otimes_{\tilde{F}_T} \tilde{F}_{M_2}$. Since these maps induce maps from $\mathcal{X}^R(M_1, M_2, T, P)$ to $\mathcal{X}^R_M(P)$ and vice versa that are also inverse to each other and functorial in P, it follows that $F_M \cong F_{M_1} \otimes_{F_T} F_{M_2}$ as well. □

Remark 3.2. When T is only a modular flat in M_2, the generalized parallel connection $P_T(M_1, M_2)$ is still well-defined. However, the identity $F_{P_T(M_1,M_2)} \cong F_{M_1} \otimes_{F_T} F_{M_2}$ does not always hold in this more general setting, even when $r(T) = 2$. For example, let M_1 and M_2 be the rank-3 matroids spanned by the two planes of the matroid shown in Figure 3.1, and let T be the line spanned by the intersection of these two planes. Then T is a modular flat of M_2, so $M = P_T(M_1, M_2)$ is well-defined. However, one can check, using the Macaulay2 package developed by Chen and Zhang (cf. [CZ23])[2], that $F_M \ncong F_{M_1} \otimes_{F_T} F_{M_2}$. Specifically, $F_{M_1} \otimes_{F_T} F_{M_2}$ has 30 hexagons (in the sense of [BL25a, Figure 4.1]) while F_M has 31 hexagons.

We briefly explain how this extra hexagon in F_M arises from the fact that T is not a modular flat of M_1. Let $H = E(M_1) - T$ and let $\{a, b, c, d\} = E(M_2) - T$. Then $H \cup a$, $H \cup b$, $H \cup c$, and $H \cup d$ are all hyperplanes of M that are not of the form described in Proposition 2.11. Moreover, $(H \cup a, H \cup b, H \cup c, H \cup d)$ is a modular quadruple of hyperplanes of M, which corresponds to a hexagon of F_M (see [BL25a, Definitions 3.3 and 3.4]). The pasture obtained from F_M by deleting this hexagon is isomorphic to $F_{M_1} \otimes_{F_T} F_{M_2}$ (as verified via Macaulay2), so the discrepancy between F_M and $F_{M_1} \otimes_{F_T} F_{M_2}$ arises directly from the fact that T is not a modular flat of M_1.

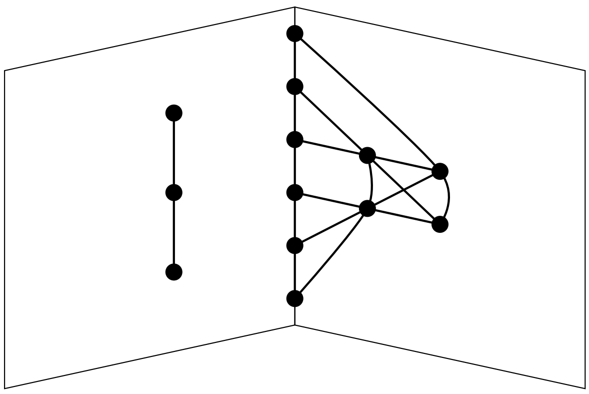

Figure 3.1: A generalized parallel connection for which the foundation of $P_T(M_1, M_2)$ is not isomorphic to $F_{M_1} \otimes_{F_T} F_{M_2}$.

[2]The software described in [CZ23] is now available through the standard distribution of Macaulay2 as the package "foundations.m2".

4. The foundation of a 2-sum

In this section, we study the special case in which $T = \{p\}$ is a singleton that is not a loop or a coloop in either M_1 or M_2. In this case, the *2-sum* of M_1 and M_2 with basepoint p is the matroid with ground set $(E(M_1) \cup E(M_2)) - p$ and set of circuits

$$\mathcal{C}(M_1 \backslash p) \cup \mathcal{C}(M_2 \backslash p) \cup \{(C_1 \cup C_2) - p \mid p \in C_1 \in \mathcal{C}(M_1) \text{ and } p \in C_2 \in \mathcal{C}(M_2)\},$$

where $\mathcal{C}(N)$ denotes the set of circuits of the matroid N. The 2-sum of M_1 and M_2 with basepoint p is denoted by $M_1 \oplus_2 M_2$ or $M_1 \oplus_p M_2$. When $\{p\}$ is a flat of M_1 or M_2, we can also define $M_1 \oplus_p M_2$ to be $P_p(M_1, M_2) \backslash p$, where $P_p(M_1, M_2)$ is the parallel connection of M_1 and M_2 along p [Oxl06, Proposition 7.1.20].

We seek to prove Theorem C, which states that $F_{M_1 \oplus_p M_2} \cong F_{M_1} \otimes F_{M_2}$, where we use that $F_p = \mathbb{F}_1^{\pm}$, as noted in Example 2.5. We know from Theorem 3.1 that $F_{P_p(M_1,M_2)} \cong F_{M_1} \otimes F_{M_2}$, so it suffices to show that $F_{M_1 \oplus_p M_2} \cong F_{P_p(M_1,M_2)}$. We first show that the sets of hyperplanes of $M_1 \oplus_p M_2$ and $P_p(M_1, M_2)$ are closely related.

Lemma 4.1. *Let M_1 and M_2 be matroids on E_1 and E_2, respectively, so that $E_1 \cap E_2 = \{p\}$ where p is not a loop or a coloop of M_1 or M_2, and $\{p\}$ is a flat of M_1 or M_2. Let $M = P_p(M_1, M_2)$ and $M' = M_1 \oplus_p M_2$, and let $\mathscr{H}$ and $\mathscr{H}'$ be the sets of hyperplanes of M and M' respectively. Then*

(1) $\mathscr{H}' = \{H - p \mid H \in \mathcal{H}\}$,

(2) if (H_1, H_2, H_3) is a modular triple of hyperplanes of M, then $(H_1 - p, H_2 - p, H_3 - p)$ is a modular triple of hyperplanes of M', and

(3) conversely, if (H'_1, H'_2, H'_3) is a modular triple of hyperplanes of M', then

$$(\mathrm{cl}_M(H'_1), \mathrm{cl}_M(H'_2), \mathrm{cl}_M(H'_3))$$

is a modular triple of hyperplanes of M.

Proof. We first prove (1). Since M is obtained from M' by deleting p, it follows that $\mathscr{H}' \subseteq \{H - p \mid H \in \mathscr{H}\}$, so we need only show that the reverse containment holds as well. If $p \notin H$ then clearly $H - p \in \mathscr{H}'$. If $p \in H$ then $E_i \subseteq H$ for some $i \in \{1, 2\}$ by Proposition 2.11. Since p is not a coloop of M_i, it follows that H and $H - p$ have the same rank in M, and so $H - p \in \mathscr{H}'$.

We next prove (2). Suppose (H_1, H_2, H_3) is a modular triple of hyperplanes of M. Let $L = H_1 \cap H_2 \cap H_3$. It suffices to show that if $p \in L$, then $r_M(L - p) = r_M(L)$. If $p \in L$, then by Proposition 2.11, each of H_1, H_2, and H_3 contains E_1 or E_2. If $E_1 \in H_1$ and $E_2 \in H_2$ then (H_1, H_2, H_3) is not a modular triple, so without loss of generality we may assume $E_1 \subseteq L$. Since p is not a coloop of M_1, it follows that $r_M(L - p) = r_M(L)$, as desired.

Finally, we prove (3). Suppose (H'_1, H'_2, H'_3) is a modular triple of hyperplanes of M', and let $L' = H'_1 \cap H'_2 \cap H'_3$. Then

$$r(M) - 2 = r(M') - 2 = r_{M'}(L') = r_M(\mathrm{cl}_M(L')) = r_M(\mathrm{cl}_M(H'_1) \cap \mathrm{cl}_M(H'_2) \cap \mathrm{cl}_M(H'_3)),$$

which shows that $(\mathrm{cl}_M(H'_1), \mathrm{cl}_M(H'_2), \mathrm{cl}_M(H'_3))$ is a modular triple of hyperplanes of M. $\square$

The following is a restatement of Theorem C.

Theorem 4.2. *Let M_1 and M_2 be matroids on E_1 and E_2, respectively, so that $E_1 \cap E_2 = \{p\}$ and p is not a loop or a coloop of M_1 or M_2. Then $F_{M_1 \oplus_p M_2} \cong F_{M_1} \otimes F_{M_2}$.*

Proof. We will write $\mathrm{si}(M)$ for the simplification of a matroid M. We will reduce to the case in which M_1 and M_2 are simple. First suppose that p is a coloop of $\mathrm{si}(M_i)$ for some $i \in \{1, 2\}$; we may assume that $i = 1$. Since p is not a coloop of M_1, it is in a nontrivial parallel class of M_1. By [Oxl06, Proposition 7.1.15 (v)] and [BL25a, Corollary 4.10] we may assume that this parallel class is $\{p, p'\}$ for some $p' \in E_1$. Then p' is a coloop of $M\backslash p$, so $\mathcal{C}(M_1\backslash p) = \mathcal{C}(M_1\backslash\{p, p'\})$, and $\{p, p'\}$ is the unique circuit of M_1 that contains p. Let $\hat{M}_2$ be the matroid obtained from M_2 by adding p' in parallel to p and then deleting p. Then $E(\hat{M}_2) = (E_2 - p) \cup p'$, and clearly $\hat{M}_2 \cong M_2$ and $\mathcal{C}(M_2\backslash p) = \mathcal{C}(\hat{M}_2\backslash p')$. Since $\{p, p'\}$ is the unique circuit of M_1 that contains p, we see that

$$\{(C_1 \cup C_2) - p \mid p \in C_1 \in \mathcal{C}(M_1) \text{ and } p \in C_2 \in \mathcal{C}(M_2)\}$$

is equal to $\{C_2 \in \mathcal{C}(\hat{M}_2) \mid p' \in C_2\}$, because the only choice for C_1 is $\{p, p'\}$. Since $\mathcal{C}(M_1\backslash p) = \mathcal{C}(M_1\backslash\{p, p'\})$ and $\mathcal{C}(M_2\backslash p) = \mathcal{C}(\hat{M}_2\backslash p)$, it follows that

$$\begin{aligned}\mathcal{C}(M_1 \oplus_p M_2) &= \mathcal{C}(M_1\backslash\{p, p'\}) \cup \mathcal{C}(\hat{M}_2\backslash p) \cup \{C_2 \in \mathcal{C}(\hat{M}_2) \mid p' \in C_2\} \\ &= \mathcal{C}(M_1\backslash\{p, p'\}) \cup \mathcal{C}(\hat{M}_2) \\ &= \mathcal{C}(M_1\backslash\{p, p'\} \oplus \hat{M}_2),\end{aligned}$$

where the last equality is due to [Oxl06, 4.2.12]. Therefore $M_1 \oplus_p M_2 \cong M_1\backslash\{p, p'\} \oplus \hat{M}_2$. Since $F_{M_1\backslash\{p,p'\}} \cong F_{M_1}$ by [BL25a, Corollary 4.10] and $F_{\hat{M}_2} \cong F_{M_2}$ because $\hat{M}_2 \cong M_2$, it follows from Corollary B that $F_{M_1 \oplus_p M_2} \cong F_{M_1} \otimes F_{M_2}$. So we may assume that p is not a coloop of $\mathrm{si}(M_1)$ or $\mathrm{si}(M_2)$. Then it follows from [Oxl06, Proposition 7.1.15 (v)] that $\mathrm{si}(M_1 \oplus_p M_1) = \mathrm{si}(M_1) \oplus_p \mathrm{si}(M_2)$, so by [BL25a, Corollary 4.10] we may assume that M_1 and M_2 are simple.

Let $E = E_1 \cup E_2$, let $E' = E - p$, and let $E_i' = E_i - p$ for $i = 1, 2$. Let P be a pasture. Given functions $f_i \colon E_i' \to P$ for $i = 1, 2$, we define $f_1 * f_2$ to be the function from E' to P so that $(f_1 * f_2)(e) = f_i(e)$ when $e \in E_i'$. Using modular systems of hyperplane functions, we first define a map Φ from $\mathcal{X}_M^R(P)$ to $\mathcal{X}_{M'}^R(P)$ and a map Ψ from $\mathcal{X}_{M'}^R(P)$ to $\mathcal{X}_M^R(P)$. Then we will show that these two maps are well-defined and inverse to each other. The maps will be functorial in P by construction, and so we will obtain an isomorphism $F_{M_1 \oplus_p M_2} \cong F_{M_1} \otimes F_{M_2}$.

Let $\mathcal{H}$ be a modular system of P-hyperplane functions of M. We define $\Phi(\mathcal{H}) = \mathcal{H}|_{E'}$. Now let $\mathcal{H}'$ be a modular system of P-hyperplane functions of M'. We define Ψ by extending the functions in $\mathcal{H}'$ to p. If f_H is in $\mathcal{H}'$ and H contains E_1' or E_2', then $H \cup p$ is a hyperplane of M, so we define $f_{H \cup p}(p) = 0$. Otherwise, H is also a hyperplane of M by Lemma 4.1, and we will extend f_H to p with the help of a fixed hyperplane H_0 of M' that does not contain E_1' or E_2'. (To see that H_0 exists, for each $i \in \{1, 2\}$, let H_i be a hyperplane of M_i that does not contain p. Then $H_1 \cup H_2$ is a hyperplane of M by Proposition 2.11, and therefore $H_1 \cup H_2$ is also a hyperplane of M' by Lemma 4.1 (1). Let $H_0 = H_1 \cup H_2$.) Our definition of Ψ will rely on the following observations, which we will use freely throughout the remainder of the proof:

- If H and K are hyperplanes of M' so that $H \cap E_i = K \cap E_i$ for some $i \in \{1,2\}$ and $f_H, f_K \in \mathcal{H}'$, then $f_H|_{E_i}$ and $f_K|_{E_i}$ are scalar multiples of each other.
- If H is a hyperplane of M' that does not contain E_1' or E_2', then $(H \cap E_1') \cup (H_0 \cap E_2')$ is a hyperplane of M'.

The first follows from Proposition 2.7, and the second follows from Proposition 2.11 and Lemma 4.1. From these two observations, if H is a hyperplane of M' that does not contain E_1' or E_2', then $f_{(H\cap E_1')\cup(H_0\cap E_2')}|_{E_1'}$ is a scalar multiple of $f_H|_{E_1'}$ and $f_{(H\cap E_1')\cup(H_0\cap E_2')}|_{E_2'}$ is a scalar multiple of $f_{H_0}|_{E_2'}$, and it follows that there is a unique $c \in P^\times$ such that $f_{(H\cap E_1')\cup(H_0\cap E_2')}$ is a scalar multiple of $f_H|_{E_1'} * (c \cdot f_{H_0}|_{E_2'})$. We define $f_H(p) = c$, which completes the definition of $\Psi(\mathcal{H}')$. Before proving that $\Psi(\mathcal{H}')$ is a modular system of P-hyperplane functions for M', we will show that this definition is symmetric in E_1' and E_2'. To do so, we first prove the following technical claim.

Claim 4.3. *Let K and L be hyperplanes of M' so that neither contains E_1' or E_2' and $K \cap E_2' = L \cap E_2'$. Let K' and L' be hyperplanes of M' so that $K' \cap E_1' = K \cap E_1'$ and $L' \cap E_1' = L \cap E_1'$, and $K' \cap E_2' = L' \cap E_2'$. Let g_K and g_L be scalar multiples of $f_K, f_L \in \mathcal{H}'$, respectively, so that $g_K|_{E_2'} = g_L|_{E_2'}$. Then, for any scalar multiples $g_{K'}$ and $g_{L'}$ of $f_{K'}, f_{L'} \in \mathcal{H}'$, respectively, with $g_{K'}|_{E_1'} = g_K|_{E_1'}$ and $g_{L'}|_{E_1'} = g_L|_{E_1'}$, we have $g_{K'}|_{E_2'} = g_{L'}|_{E_2'}$.*

Proof. Fix L, and suppose that the claim is false for L. Choose K so that the claim is false for L and K, and $r_{M'}(K \cap L)$ is maximal with this property. Since $K \cap E_2' = L \cap E_2'$, this is equivalent to the maximality of $r_{M'}(K \cap L \cap E_1')$. Assume we are given K', L', $g_{K'}$, and $g_{L'}$. If $K \cap E_1' = L \cap E_1'$, then $K = L = K' = L'$ and the result holds. So $K \cap E_1' \neq L \cap E_1'$. It follows from Lemma 4.1 that K and L are also hyperplanes of M. Let $\mathscr{P}$ be the linear subclass of hyperplanes of M that contain p. By Proposition 2.15 with $(H, K, \mathscr{H}) = (K, L, \mathscr{P})$, there is a hyperplane H of M (possibly $H = L$) so that (K, H) is a modular pair, $p \notin H$, $r_M(H \cap L) > r_M(K \cap L)$, and $(K \cap L) \subseteq H$. Since $p \notin H$, Lemma 4.1 implies that H is also a hyperplane of M'. Then since $p \notin H$ and $K \cap L$ contains $L \cap E_2'$ which is a hyperplane of M_2, we see that $H \cap E_2' = L \cap E_2' = K \cap E_2'$.

Let g_H be the scalar multiple of f_H so that $g_H|_{E_2'} = g_K|_{E_2'} = g_L|_{E_2'}$. Define H' to be the hyperplane of M' with $H' \cap E_1' = H \cap E_1'$ and $H' \cap E_2' = K' \cap E_2' = L' \cap E_2'$. Let $g_{H'}$ be the scalar multiple of $f_{H'}$ so that $g_{H'}|_{E_1'} = g_H|_{E_1'}$. Since $r_{M'}(H \cap L) > r_{M'}(K \cap L)$, by the maximality of $r_{M'}(K \cap L)$ we know that the claim is true for H and L, and so $g_{H'}|_{E_2'} = g_{L'}|_{E_2'}$. We will complete the proof by showing that $g_{K'}|_{E_2'} = g_{H'}|_{E_2}$. Let $X_1 = K \cap H \cap E_1'$, so X_1 is a corank-2 flat of M_1'. Let $X = [\mathrm{cl}_{M_1}(X \cup p) \cup E_2] - p$. By Proposition 2.11 and Lemma 4.1, X is a hyperplane of M'. Moreover, (K, H, X) is a modular triple of hyperplanes of M', so there are constants c, c' so that

$$g_K(e) + c \cdot g_H(e) + c' \cdot f_X(e) = 0$$

for all $e \in E'$. Since $g_K|_{E_2'} = g_H|_{E_2'}$ and $f_X(e) = 0$ for all $e \in E_2'$, we see that $c = -1$, and so

$$g_K(e) - g_H(e) + c' \cdot f_X(e) = 0$$

for all $e \in E'$.

Next, note that (K', H', X) is also a modular triple of M', because $K' \cap H' \cap X$ is the union of $K \cap H \cap E_1'$ and $K' \cap E_2$, which is a corank-2 flat of M'. So there are constants d, d' so that

$$g_{K'}(e) + d \cdot g_{H'}(e) + d' \cdot f_X(e) = 0$$

for all $e \in E'$. Let $a \in (H - (K \cup X)) \cap E_1'$, and note that $a \in (H' - (K' \cup X)) \cap E_1'$ because $H|E_1' = H'|E_1'$ and $K|E_1' = K'|E_1'$. By plugging in a to both equations, we see that $g_K(a) + c' \cdot f_X(a) = 0$ and $g_{K'}(a) + d' \cdot f_X(a) = 0$. Since $g_K(a) = g_{K'}(a)$ because $g_K|_{E_1'} = g_{K'}|_{E_1'}$, it follows that $c' = d'$.

Now let $b \in (K - (H \cup X)) \cap E_1'$, and note that $b \in (K' - (H' \cup X)) \cap E_1'$ because $H|E_1' = H'|E_1'$ and $K|E_1' = K'|E_1'$. By plugging in b to both equations we see that $-g_H(b) + c' \cdot f_X(b) = 0$ and $d \cdot g_{H'}(b) + d' \cdot f_X(b) = 0$. Since $c' = d'$ and $g_{H'}(b) = g_H(b)$ because $g_H|_{E_1'} = g_{H'}|_{E_1'}$, it follows that $d = -1$. Since $d = -1$, for any $e \in E_2' - (H' \cup K')$ we have $g_{K'}(e) - g_{H'}(e) = 0$, and so $g_{K'}|_{E_2'} = g_{H'}|_{E_2'}$, as desired. □

We have the following corollary, which is the only application of Claim 4.3 that we will need. It shows that the map Ψ from $\mathcal{H}'$ to $\mathcal{H}$ does not depend on whether we restrict H_0 to E_1' or to E_2'.

Claim 4.4. *Let H be a hyperplane of M' that contains neither E_1' nor E_2'. If $f_H|_{E_1'} * (c \cdot f_{H_0}|_{E_2'})$ is in $\mathcal{H}'$ for some scalar c, then a scalar multiple of $(c \cdot f_{H_0}|_{E_1'}) * f_H|_{E_2'}$ is also in $\mathcal{H}'$.*

Proof. Let K be the hyperplane of M' with $K \cap E_1' = H \cap E_1'$ and $K \cap E_2' = H_0 \cap E_2'$. Note that $f_K = f_H|_{E_1'} * (c \cdot f_{H_0}|_{E_2'})$ by assumption. Let $L = H_0$ and $K' = H$, and let L' be the hyperplane of M' with $L' \cap E_1' = H_0' \cap E_1'$ and $L' \cap E_2' = H \cap E_2'$. Note that $f_K|_{E_1'} = f_{K'}|_{E_1'}$ because $K' = H$. Let $g_K = f_K$ and $g_L = c \cdot f_L$; then $g_K|_{E_2'} = g_L|_{E_2'} = c \cdot f_{H_0}|_{E_2'}$. Let $g_{K'}$ and $g_{L'}$ be scalar multiples of $f_{K'}, f_{L'} \in \mathcal{H}'$, respectively, so that $g_{K'}|_{E_1'} = g_K|_{E_1'}$ and $g_{L'}|_{E_1'} = g_L|_{E_1'}$. Then

$$g_{K'}|_{E_1'} = g_K|_{E_1'} = f_K|_{E_1'} = f_{K'}|_{E_1'},$$

and since $g_{K'}$ is a scalar multiple of $f_{K'}$ it follows that $g_{K'} = f_{K'}$. By applying Claim 4.3, we know that $g_{K'}|_{E_2'} = g_{L'}|_{E_2'}$. Then

$$g_{L'}|_{E_1'} = g_L|_{E_1'} = c \cdot f_L|_{E_1'} = c \cdot f_{H_0}|_{E_1'}$$

and

$$g_{L'}|_{E_2'} = g_{K'}|_{E_2'} = f_{K'}|_{E_2'} = f_H|_{E_2'},$$

and so $g_{L'} = (c \cdot f_{H_0}|_{E_1'}) * f_H|_{E_2'}$ and the claim holds. □

Next, we will show that $\Psi(\mathcal{H}')$ is a modular system of P-hyperplane functions of M. Let F be a corank-2 flat of M, and let (H_1, H_2, H_3) be a modular triple of hyperplanes of M so that $H_1 \cap H_2 \cap H_3 = F$. By Lemma 4.1, $(H_1 - p, H_2 - p, H_3 - p)$ is a modular triple of hyperplanes of M', so there are constants c_1, c_2, c_3 so that

$$c_1 \cdot f_{H_1}(e) + c_2 \cdot f_{H_2}(e) + c_3 \cdot f_{H_3}(e) = 0$$

for all $e \in E'$. We need only show that this also holds for $e = p$. We consider two cases.

Case 1: Suppose $p \in F$. Then $f_{H_i}(p) = 0$ for $i = 1, 2, 3$.

Case 2: Suppose $p \notin F$. Then outcome (3) of Proposition 2.12 holds for F, so there is some $i \in \{1, 2\}$ so that $F \cap E_i$ is a hyperplane of M_i and $F \cap E_{3-i}$ is a corank-2 flat of M_{3-i}. We consider two subcases.

First suppose that $p \notin H_1 \cup H_2 \cup H_3$. Then H_1, H_2, and H_3 all have the same restriction to E'_i, and so $f_{H_1}|_{E'_i}$, $f_{H_2}|_{E'_i}$, and $f_{H_3}|_{E'_i}$ are scalar multiples of each other. If $i = 1$, then H_1, H_2, and H_3 agree on E'_1, so $(f_{H_1}(p), f_{H_2}(p), f_{H_3}(p))$ is a scalar multiple of $(f_{H_1}(e), f_{H_2}(e), f_{H_3}(e))$ for any $e \in E'_1 - F$. Hence $c_1 \cdot f_{H_1}(p) + c_2 \cdot f_{H_2}(p) + c_3 \cdot f_{H_3}(p) = 0$. If $i = 2$, then H_1, H_2, and H_3 agree on E'_2, and it follows from Claim 4.4 that $(f_{H_1}(p), f_{H_2}(p), f_{H_3}(p))$ is a scalar multiple of $(f_{H_1}(e), f_{H_2}(e), f_{H_3}(e))$ for any $e \in E'_2 - F$. Again, $c_1 \cdot f_{H_1}(p) + c_2 \cdot f_{H_2}(p) + c_3 \cdot f_{H_3}(p) = 0$.

In the second subcase, suppose that $p \in H_j$ for some $j \in \{1, 2, 3\}$. We may assume that $j = 1$. Then H_1 contains E_i, so $f_{H_1}|_{E_i} = 0$ and we have $f_{H_3}|_{E'_i} = -\frac{c_2}{c_3} \cdot f_{H_2}|_{E'_i}$. First suppose that $i = 1$. Then by the definition of $f_{H_2}(p)$, a multiple of $f_{H_2}|_{E'_1} * (f_{H_2}(p) \cdot f_{H_0}|_{E'_2})$ is in $\mathcal{H}'$. Similarly, a multiple of $f_{H_3}|_{E'_1} * (f_{H_3}(p) \cdot f_{H_0}|_{E'_2})$ is in $\mathcal{H}'$. Since $f_{H_3}|_{E'_1} = -\frac{c_2}{c_3} \cdot f_{H_2}|_{E'_1}$, a multiple of $-\frac{c_2}{c_3} \cdot f_{H_2}|_{E'_1} * (f_{H_3}(p) \cdot f_{H_0}|_{E'_2})$ is in $\mathcal{H}'$, and by scaling we see that a multiple of $f_{H_2}|_{E'_1} * (-\frac{c_3}{c_2} \cdot f_{H_3}(p) \cdot f_{H_0}|_{E'_2})$ is in $\mathcal{H}'$. Therefore $f_{H_2}(p) = -\frac{c_3}{c_2} \cdot f_{H_3}(p)$, so $f_{H_3}(p) = -\frac{c_2}{c_3} \cdot f_{H_2}(p)$, and when $e = p$ we have

$$0 + c_2 \cdot f_{H_2}(p) + c_3 \cdot \left(-\frac{c_2}{c_3} \cdot f_{H_2}(p) \right) = 0,$$

as desired. If $i = 2$, then Claim 4.4 allows us to use an identical argument, which we briefly describe. First, by the definition of $f_{H_2}(p)$ and Claim 4.4, a multiple of $(f_{H_2}(p) \cdot f_{H_0}|_{E'_1}) * f_{H_2}|_{E'_2}$ is in $\mathcal{H}'$. Similarly, a multiple of $(f_{H_3}(p) \cdot f_{H_0}|_{E'_1}) * f_{H_3}|_{E'_2}$ is in $\mathcal{H}'$. Since $f_{H_3}|_{E'_2} = -\frac{c_2}{c_3} \cdot f_{H_2}|_{E'_2}$, a multiple of $(f_{H_3}(p) \cdot f_{H_0}|_{E'_1}) * (-\frac{c_2}{c_3} \cdot f_{H_2}|_{E'_2})$ is in $\mathcal{H}'$. Once again, it follows that $f_{H_3}(p) = -\frac{c_2}{c_3} \cdot f_{H_2}(p)$, and so $c_1 \cdot f_{H_1}(p) + c_2 \cdot f_{H_2}(p) + c_3 \cdot f_{H_3}(p) = 0$, as desired. It follows from Cases 1 and 2 that $\Psi(\mathcal{H}')$ is a modular system of hyperplane functions, as claimed.

Next we will show that Φ and Ψ are inverses of one another. It is clear that $\Phi \circ \Psi$ is the identity map regardless of the choice of H_0. In the case of $\Psi \circ \Phi$, let H_0 be the hyperplane that we fixed. Note that H_0 is also a hyperplane of M. Let $f_{H_0} \in \mathcal{H}$, and let $f_H \in \mathcal{H}$ for an arbitrary hyperplane H of M. Let $\overline{f_H}$ be the function in $\Psi \circ \Phi(\mathcal{H})$ such that $\overline{f_H}(e) = f_H(e)$ for all $e \in E'$. If $p \in H$ then $\overline{f_H} = f_H$. If $p \notin H$, then let $K = (H \cap E_1) \cup (H_0 \cap E_2)$; by Proposition 2.11, we know that K is a hyperplane of M. Since $K \cap E_2 = H_0 \cap E_2$ we may assume, by scaling $f_K \in \mathcal{H}$, that $f_K|_{E_2} = f_{H_0}|_{E_2}$. In particular, $f_K(p) = f_{H_0}(p)$. Since $K \cap E_1 = H \cap E_1$, we know that $f_K|_{E_1}$ is a scalar multiple of $f_H|_{E_1}$, and in particular we have $f_K|_{E_1} = \frac{f_K(p)}{f_H(p)} \cdot f_H|_{E_1} = \frac{f_{H_0}(p)}{f_H(p)} \cdot f_H|_{E_1}$. Then $f_K = \left(\frac{f_{H_0}(p)}{f_H(p)} \cdot f_H|_{E_1} \right) * f_{H_0}|_{E_2}$. So, by definition, $\overline{f_H}(p) = \frac{1}{f_{H_0}(p)} \cdot f_H(p)$. The constant $\frac{1}{f_{H_0}(p)}$ only depends on the hyperplane H_0, so $\mathcal{H}$ and $\Psi \circ \Phi(\mathcal{H})$ are in the same rescaling class. □

5. The foundation of a segment-cosegment exchange

In this section we show that if M is a matroid and $X \subseteq E(M)$ is a coindependent set such that $M|X \cong U_{2,n}$ for some $n \geqslant 2$, then the segment-cosegment exchange of M along X has the same foundation as M. We first recall the relevant definitions, which first appeared in [OSV00].

For each integer $n \geqslant 2$, the matroid Θ_n has ground set $X \sqcup Y$ where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, and the following bases:

- Y,
- $(Y - y_i) \cup x_j$ for distinct $i, j \in [n]$, and
- $(Y - Y') \cup X'$ where $Y' \subseteq Y$ and $X' \subseteq X$ and $|Y'| = |X'| = 2$.

The set X is a modular flat of Θ_n and $\Theta_n|X \cong U_{2,n}$. Therefore, if M is any matroid with $M|X \cong U_{2,n}$, the generalized parallel connection $P_X(M, \Theta_n)$ is well-defined.

The matroid $P_X(M, \Theta_n)\backslash X$, often denoted $\Delta_X(M)$, is called the *segment-cosegment exchange* of M along X. When $n = 2$, $\{x_i, y_i\}$ is a series pair of $P_X(M, \Theta_2)$ for $i = 1, 2$, so $P_X(M, \Theta_2)\backslash X \cong M$. When $n = 3$ we have $\Theta_3 \cong M(K_4)$ (the cycle matroid of the graph K_4), and $P_X(M, \Theta_3)\backslash X$ is also called the *Delta-Wye exchange* of M along X [AO91].

We next state some properties of Θ_n. There are three different types of hyperplanes of Θ_n, depending on the size of their intersection with X. This is straightforward to prove using the above description of the bases of Θ_n.

Proposition 5.1. *If H is a hyperplane of Θ_n, then either*

(1) $H = (Y - y_i) \cup x_i$ for some $i \in [n]$, or

(2) $H = (Y - \{y_i, y_j\}) \cup x_k$ for distinct $i, j, k \in [n]$, or

(3) $H = (X \cup Y) - \{y_i, y_j, y_k\}$ for distinct $i, j, k \in [n]$.

Using the previous proposition, it is straightforward to show that there are four types of corank-2 flats of Θ_n. Note that outcomes (1) and (2) only occur when $n \geqslant 4$.

Proposition 5.2. *If F is a corank-2 flat of Θ_n, then either*

(1) $F = (X \cup Y) - \{y_i, y_j, y_k, y_l\}$ for distinct $i, j, k, l \in [n]$, or

(2) $F = (Y - \{y_i, y_j, y_k\}) \cup x_l$ for distinct $i, j, k, l \in [n]$, or

(3) $F = (Y - \{y_i, y_j, y_k\}) \cup x_i$ for distinct $i, j, k \in [n]$, or

(4) $F = Y - \{y_i, y_j\}$ for distinct $i, j \in [n]$.

We next turn our attention to representations of $U_{2,n}$, and prove two properties that hold for any modular system of hyperplane functions of $U_{2,n}$.

Proposition 5.3. *Let P be a pasture, and let $\mathcal{H}$ be a modular system of P-hyperplane functions for $U_{2,n}$ on the ground set $X = \{x_1, x_2, \dots, x_n\}$. Then*

(1) $f_{x_i}(x_j) = -f_{x_j}(x_i)$ *for all distinct $i, j, \in [n]$, and*

(2) for all $1 \leqslant i < j < k \leqslant n$ we have

$$f_{x_j}(x_k) \cdot f_{x_i}(e) + f_{x_k}(x_i) \cdot f_{x_j}(e) + f_{x_i}(x_j) \cdot f_{x_k}(e) = 0$$

for all $e \in X$.

Proof. It follows from [BL25a, Theorem 2.16] that the function $\Delta\colon X^2 \to P$ defined by $\Delta(x_i x_j) = f_{x_i}(x_j)$ is a (weak) Grassmann–Plücker function, which implies that (1) and (2) hold. □

Finally, we need a general lemma about rescaling a modular system of hyperplane functions along a triangle.

Lemma 5.4. *Let M be a matroid, let $T = \{x, y, z\}$ be a triangle of M, and let P be a pasture. Let $\mathcal{H}$ be a modular system of P-hyperplane functions for M. Then there is a modular system $\mathcal{H}'$ of P-hyperplane functions for M that is rescaling equivalent to $\mathcal{H}$ and has the following properties:*

(1) If H is a hyperplane of M so that $|H \cap T| = 1$, then $f_H \in \mathcal{H}'$ has values 0, 1, and -1 on T.

(2) If H is a hyperplane of M disjoint from T, then $f_H \in \mathcal{H}'$ satisfies $f_H(x)+f_H(y)+f_H(z)=0$.

Proof. Let B be a basis of M/T, and let $L = \mathrm{cl}_M(B)$. Let H_x, H_y, and H_z be $\mathrm{cl}(L \cup x)$, $\mathrm{cl}(L \cup y)$, and $\mathrm{cl}(L \cup z)$, respectively. Note that (H_x, H_y, H_z) is a modular triple of hyperplanes of M. By scaling functions in $\mathcal{H}$, we may assume that if H is a hyperplane and $H \cap T = \{x\}$, then $f_H(y) = 1$. Similarly, we may assume that if $H \cap T = \{y\}$ then $f_H(z) = 1$, and if $H \cap T = \{z\}$ then $f_H(x) = 1$. Now, scale $\mathcal{H}$ by $\frac{-1}{f_{H_x}(z)}$ at z and by $\frac{-1}{f_{H_y}(x)}$ at x, and let $\mathcal{H}'$ be the resulting system of P-hyperplane functions for M. Note that $f_{H_x}(z) = -1$ and $f_{H_y}(x) = -1$, as desired.

We first show that $f_{H_z}(y) = -1$. Since (H_x, H_y, H_z) is a modular triple, there are constants c', c'' so that

$$f_{H_x}(e) + c' \cdot f_{H_y}(e) + c'' \cdot f_{H_z}(e) = 0$$

for all $e \in E(M)$. Setting $e = z$ shows that $c' = 1$, and setting $e = x$ shows that $c'' = 1$. Then setting $e = y$ shows that $f_{H_z}(y) = -1$, as desired.

Now we prove (1). We present the argument only for hyperplanes H with $H \cap T = \{x\}$, but the argument is very similar when $H \cap T \in \{y, z\}$. Suppose there is a hyperplane H of M with $H \cap T = \{x\}$ so that $f_H(z) \neq -1$, and let $r(H \cap H_x)$ be maximal with these properties. Let $\mathscr{T}$ be the linear subclass of hyperplanes of M that contain T. By Proposition 2.15 with $(H, K, \mathscr{H}) = (H, H_x, \mathscr{T})$, there is a hyperplane H' (possibly H_x) so that (H, H') is a modular pair, H' contains $H \cap H_x$ but not T, and $r(H' \cap H_x) > r(H \cap H_x)$. Since H' contains $H \cap H_x$ but not T we see that $H' \cap T = \{x\}$. By the maximality of $r_M(H \cap H_x)$, it follows that $f_{H'}(z) = -1$. Let $F = H \cap H'$, and let $H'' = \mathrm{cl}(F \cup T)$. Then (H, H', H'') is a modular triple because F is a corank-2 flat of M, so there are constants c, c'' so that

$$c \cdot f_H(e) + f_{H'}(e) + c'' \cdot f_{H''}(e) = 0$$

for all $e \in E(M)$. Setting $e = y$ shows that $c = -1$, and then setting $e = z$ shows that $f_H(z) = -1$, a contradiction. This establishes (1).

We now prove (2). Let H be a hyperplane of M which is disjoint from T. Let F be a corank-2 flat of M contained in H, and let $H_x = \mathrm{cl}(F \cup x)$, $H_y = \mathrm{cl}(F \cup y)$. Then (H, H_x, H_y) is a modular triple, so there are constants c and c' so that

$$f_H(e) + c \cdot f_{H_x}(e) + c' \cdot f_{H_y}(e) = 0$$

for all $e \in E(M)$. By setting $e = x$, we see that $c' = -\frac{f_H(x)}{f_{H_y}(x)}$, and by setting $e = y$, we see that $c = -\frac{f_H(y)}{f_{H_x}(y)}$. Setting $e = z$ then gives

$$f_H(z) - \frac{f_H(y)}{f_{H_x}(y)} \cdot f_{H_x}(z) - \frac{f_H(x)}{f_{H_y}(x)} \cdot f_{H_y}(z) = 0,$$

and since $\frac{f_{H_x}(z)}{f_{H_x}(y)} = \frac{f_{H_y}(z)}{f_{H_y}(x)} = -1$ by (1), this simplifies to $f_H(z) + f_H(y) + f_H(x) = 0$. □

We now prove that forming the generalized parallel connection with Θ_n preserves foundations. Note that we do not require X to be coindependent; that is only necessary for the subsequent argument in which we delete X.

Theorem 5.5. *Let M_1 be a matroid, let $X \subseteq E(M_1)$ so that $M_1|X \cong U_{2,n}$ for some $n \geqslant 2$, and let $M = P_X(M_1, \Theta_n)$. Then $F_M \cong F_{M_1}$.*

Proof. When $n = 2$ we know that the cosimplification of M is isomorphic to M_1 because $\{x_i, y_i\}$ is a series pair of M for $i = 1, 2$. So by [BL25a, Corollary 4.10], we may assume that $n \geqslant 3$. Let E_1 be the ground set of M_1, and let $E_2 = X \cup Y$ be the ground set of Θ_n with $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Let $E = E_1 \cup E_2$.

Let P be a pasture. Given a modular system $\mathcal{H}$ of P-hyperplane functions for M, we define a modular system $\mathcal{H}_1$ of P-hyperplane functions for M_1 by restriction to E_1, so $\mathcal{H}_1 = \mathcal{H}|_{E_1}$. Conversely, let $\mathcal{H}_1$ be a modular system of P-hyperplane functions for M_1. Note that $\mathcal{H}_1$ induces a modular system $\mathcal{H}_1|_X$ of P-hyperplane functions of $U_{2,n}$ by restriction to X; we write f_{x_i} for the function in $\mathcal{H}_1|_X$ corresponding to the hyperplane x_i of $M_1|X$. By Proposition 2.7 we may assume, by rescaling the functions in $\mathcal{H}_1$, that for all distinct $i, j \in [n]$, if H_1 is a hyperplane of M_1 with $H_1 \cap X = \{x_i\}$, then $f_{H_1}(x_j) = f_{x_i}(x_j)$. We will define a modular system $\mathcal{H}$ of P-hyperplane functions for M so that $\mathcal{H}|_{E_1} = \mathcal{H}_1$, up to rescaling equivalence.

For each hyperplane H of M, we will define the corresponding function $f_H \in \mathcal{H}$ by separately considering the five different possibilities for the type of H. These five possibilities arise by applying Propositions 2.13, 5.1, and 5.2; note that we split outcome (3) of Proposition 2.13 into two separate cases depending on the form of the hyperplane of Θ_n:

(1) If $H = E_1 \cup (Y - \{y_i, y_j, y_k\})$ for distinct $i, j, k \in [n]$ with $i < j < k$, define

- $f_H(y_i) = f_{x_j}(x_k)$,
- $f_H(y_j) = f_{x_k}(x_i)$, and
- $f_H(y_k) = f_{x_i}(x_j)$.

(2) If $H = H_1 \cup E_2$, where H_1 is a hyperplane of M_1 that contains X, define $f_H(e) = f_{H_1}(e)$ for all $e \in E$.

(3) If $H = H_1 \cup ((Y - y_i) \cup x_i)$ for $i \in [n]$, where H_1 is a hyperplane of M_1 with $H_1 \cap X = \{x_i\}$, define

- $f_H(e) = f_{H_1}(e)$ for all $e \in E_1$ (in particular, $f_H(x_j) = f_{x_i}(x_j)$ for all distinct $i, j \in [n]$), and
- $f_H(y_i) = 1$.

(3') If $H = H_1 \cup ((Y - \{y_i, y_j\}) \cup x_k)$ for distinct $i, j, k \in [n]$ with $i < j$, where H_1 is a hyperplane of M_1 with $H_1 \cap X = \{x_k\}$, define

- $f_H(e) = f_{H_1}(e)$ for all $e \in E_1$ (in particular, $f_H(x_l) = f_{x_k}(x_l)$ for all $l \notin \{i, j, k\}$),
- $f_H(y_i) = \frac{-f_{x_j}(x_k)}{f_{x_i}(x_j)}$, and
- $f_H(y_j) = \frac{f_{x_i}(x_k)}{f_{x_i}(x_j)}$.

(4) If $H = H_1 \cup (Y - \{y_i, y_j\})$ for distinct $i, j \in [n]$, where H_1 is a hyperplane of M_1 disjoint from X, define

- $f_H(e) = f_{H_1}(e)$ for all $e \in E_1$,
- $f_H(y_i) = \frac{f_{H_1}(x_j)}{f_{x_i}(x_j)}$, and
- $f_H(y_i) = \frac{f_{H_1}(x_i)}{f_{x_j}(x_i)}$.

We now have a well-defined map from $\mathcal{H}_1$ to a set $\mathcal{H}$ of hyperplane functions for M. Clearly $\mathcal{H}|_{E_1} = \mathcal{H}_1$, so it suffices to show that $\mathcal{H}$ is a modular system.

Let F be a corank-2 flat of M, and let (H, H', H'') be a modular triple of hyperplanes of M with $H \cap H' \cap H'' = F$. By Proposition 2.14, there are seven possibilities for F, which we consider separately. (Some cases only occur when $n \geqslant 4$ or $n \geqslant 5$.) We split outcome (4) of Proposition 2.14 into two cases depending on the form of the hyperplane of Θ_n. Also, each hyperplane or corank-2 flat of Θ_n is associated with a given subset of $[n]$; we will explicitly choose this subset without loss of generality to improve readability. We also choose (H, H', H'') up to permutation.

Case 1: $F = E - \{y_1, y_2, y_3, y_4\}$. Then $(H, H', H'') = (F \cup y_1, F \cup y_2, F \cup y_3)$. We will show that

$$[f_{x_1}(x_4)] \cdot f_H(e) - [f_{x_2}(x_4)] \cdot f_{H'}(e) + [f_{x_3}(x_4)] \cdot f_{H''}(e) = 0$$

for all $e \in E$. Without loss of generality, this only needs to be checked for $e = y_1$ and $e = y_4$. When $e = y_1$, by applying (1) we have

$$-[f_{x_2}(x_4)] \cdot f_{x_3}(x_4) + [f_{x_3}(x_4)] \cdot f_{x_2}(x_4) = 0.$$

When $e = y_4$, using (1) we have

$$[f_{x_1}(x_4)] \cdot f_{x_2}(x_3) - [f_{x_2}(x_4)] \cdot f_{x_1}(x_3) + [f_{x_3}(x_4)] \cdot f_{x_1}(x_2),$$

which is equal to 0 by Proposition 5.3.

Case 2: $F = F_1 \cup E_2$, where F_1 is a corank-2 flat of M_1 that contains X. Then there is a modular triple (H_1, H_1', H_1'') of hyperplanes of M_1 so that $(H, H', H'') = (H_1 \cup E_2, H_1' \cup E_2, H_1'' \cup E_2)$. So there are constants c, c', c'' such that

$$c \cdot f_{H_1}(e) + c' \cdot f_{H_1'}(e) + c'' \cdot f_{H_1''}(e) = 0$$

for all $e \in E_1$, and it follows from (2) that

$$c \cdot f_H(e) + c' \cdot f_{H'}(e) + c'' \cdot f_{H''}(e) = 0$$

for all $e \in E$.

Case 3: $F = H_1 \cup (Y - \{y_1, y_2, y_3\})$, where H_1 is a hyperplane of M_1 that contains X. Then there is no modular triple of hyperplanes containing F, because the only hyperplanes containing F are $F \cup E_1$ and $F \cup E_2$.

Case 4: $F = H_1 \cup ((Y - \{y_1, y_2, y_3\}) \cup x_4)$ where H_1 is a hyperplane of M_1 with $H_1 \cap X = \{x_4\}$. There are two subcases. In the first subcase, $(H, H', H'') = (F \cup y_1, F \cup y_2, F \cup y_3)$. We will show that

$$[f_{x_1}(x_4) \cdot f_{x_2}(x_3)] \cdot f_H(e) + [f_{x_4}(x_2) \cdot f_{x_1}(x_3)] \cdot f_{H'}(e) + [f_{x_3}(x_4) \cdot f_{x_1}(x_2)] \cdot f_{H''}(e) = 0 \quad \text{(a)}$$

for all $e \in E$. When $e \in E_1$, this follows from Proposition 5.3 and the fact that $f_H(e) = f_{H'}(e) = f_{H''}(e)$ by (3'). When $e = y_1$, using (3'), the left-hand side of (a) becomes

$$[f_{x_4}(x_2) \cdot f_{x_1}(x_3)] \cdot \frac{-f_{x_3}(x_4)}{f_{x_1}(x_3)} + [f_{x_3}(x_4) \cdot f_{x_1}(x_2)] \cdot \frac{-f_{x_2}(x_4)}{f_{x_1}(x_2)},$$

which is equal to 0 by Proposition 5.3.

In the second subcase, $(H, H', H'') = (F \cup y_1, F \cup y_2, F \cup E_1)$. We will show that

$$[f_{x_3}(x_1) \cdot f_{x_2}(x_3)] \cdot f_H(e) + [f_{x_2}(x_3) \cdot f_{x_1}(x_3)] \cdot f_{H'}(e) + [f_{x_3}(x_4)] \cdot f_{H''}(e) = 0$$

for all $e \in E$. When $e \in E_1$ this follows from the fact that $f_H(e) = f_{H'}(e)$ by (3'). When $e = y_1$, using (1) and (3'), we have

$$[f_{x_3}(x_1) \cdot f_{x_2}(x_3)] \cdot \frac{-f_{x_3}(x_4)}{f_{x_1}(x_3)} + [f_{x_3}(x_4)] \cdot f_{x_2}(x_3) = 0,$$

and when $e = y_3$, using (1) and (3'), the left-hand side of (a) becomes

$$[f_{x_3}(x_1) \cdot f_{x_2}(x_3)] \cdot \frac{f_{x_2}(x_4)}{f_{x_2}(x_3)} + [f_{x_2}(x_3) \cdot f_{x_1}(x_3)] \cdot \frac{f_{x_1}(x_4)}{f_{x_1}(x_3)} + [f_{x_3}(x_4)] \cdot f_{x_1}(x_2),$$

which is equal to 0 by Proposition 5.3.

Case 4': $F = H_1 \cup ((Y - \{y_1, y_2, y_3\}) \cup x_1)$, where H_1 is a hyperplane of M_1 with $H_1 \cap X = \{x_1\}$. Then $(H, H', H'') = (F \cup y_1, F \cup \{y_2, y_3\}, F \cup E_1)$. We will show that

$$[f_{x_2}(x_3)] \cdot f_H(e) + [f_{x_3}(x_2)] \cdot f_{H'}(e) + f_{H''}(e) = 0$$

for all $e \in E$. When $e \in E_1$, this follows from the fact that $f_H(e) = f_{H'}(e) = f_{H_1}(e)$ by (3) and (3'). When $e = y_1$, using (1) and (3), we have

$$[f_{x_3}(x_2)] \cdot 1 + f_{x_2}(x_3) = 0.$$

When $e = y_3$, using (1) and (3'), we have

$$[f_{x_2}(x_3)] \cdot \frac{f_{x_2}(x_1)}{f_{x_2}(x_3)} + f_{x_1}(x_2) = 0.$$

Case 5: $F = F_1 \cup H_2$, where F_1 is a corank-2 flat of M_1, H_2 is a hyperplane of Θ_n, and $F_1 \cap X = H_2 \cap X = \{x_1\}$. Then $(H \cap E_1, H' \cap E_1, H'' \cap E_1)$ is a modular triple of hyperplanes of M_1, so there are constants c, c', c'' so that

$$c \cdot f_{H \cap E_1}(e) + c' \cdot f_{H' \cap E_1}(e) + c'' \cdot f_{H'' \cap E_1}(e) = 0 \tag{b}$$

for all $e \in E_1$. By (2), (3), and (3'), this implies that $c \cdot f_H(e) + c' \cdot f_{H'}(e) + c'' \cdot f_{H''}(e) = 0$ for all $e \in E_1$, so we only need to show that this also holds for all y_i. At most one of H, H', H'' contains E_2; we may assume that H and H' do not contain E_2. We consider two cases depending on whether or not $E_2 \subseteq H''$. First suppose that H'' does not contain E_2. Then $x_2 \notin H \cup H' \cup H''$. Since $H \cap X = H' \cap X = H'' \cap X = \{x_1\}$ we know that $f_{H \cap E_1}(x_i) = f_{H' \cap E_1}(x_i) = f_{H' \cap E_1}(x_i) = f_{x_1}(x_i)$ for all $i \in [n]$ due to the scaling assumption on $\mathcal{H}_1$. Then plugging in $e = x_2$ to (b) shows that $c + c' + c'' = 0$. Since H, H', H'' all have the same restriction to E_2 (namely H_2), either f_H, $f_{H'}$, $f_{H''}$ are all defined using (3) or they are all defined using (3'), and it follows from (3) or (3') that $f_H(y_i) = f_{H'}(y_i) = f_{H''}(y_i)$ for all $i \in [n]$. Therefore $c \cdot f_H(y_i) + c' \cdot f_{H'}(y_i) + c'' \cdot f_{H''}(y_i) = 0$ for all $i \in [n]$.

In the second case, suppose that $E_2 \subseteq H''$. Since $H \cap X = H' \cap X = \{x_1\}$ we know that $f_{H \cap E_1}(x_i) = f_{H' \cap E_1}(x_i) = f_{x_1}(x_i)$ for all $i \in [n]$ due to the scaling assumption on $\mathcal{H}_1$. Then plugging in $e = x_2$ to (b) shows that $c + c' = 0$, because $x_2 \in H''$. Since H and H' have the same restriction to E_2 (namely H_2), either f_H and $f_{H'}$ are both defined using (3) or they are both defined using (3'), and it follows from (3) or (3') that $f_H(y_i) = f_{H'}(y_i)$ for all $i \in [n]$. Since $f_{H''}(y_i) = 0$ for all $i \in [n]$ and $c + c' = 0$, we see that $c \cdot f_H(y_i) + c' \cdot f_{H'}(y_i) + c'' \cdot f_{H''}(y_i) = 0$ for all $i \in [n]$.

Case 6: $F = H_1 \cup (Y - \{y_1, y_2, y_3\})$, where H_1 is a hyperplane of M_1 disjoint from X. Lemma 5.4 (1) implies that by scaling $\mathcal{H}_1$ at the triangle $\{x_1, x_2, x_3\}$, we may assume that if H_0 is a hyperplane of M_1 with $|H_0 \cap \{x_1, x_2, x_3\}| = 1$ then f_{H_0} takes values 0, 1, and -1 on $\{x_1, x_2, x_3\}$. It follows from Lemma 5.4 (2) that $f_{H_1}(x_1) + f_{H_1}(x_2) + f_{H_1}(x_3) = 0$. We may further assume, by rescaling functions, that $f_{x_1}(x_2) = 1$.

We now consider two subcases. In the first subcase, $(H, H', H'') = (F \cup y_1, F \cup y_2, F \cup y_3)$. We will show that

$$[f_{H_1}(x_1) \cdot f_{x_2}(x_3)] \cdot f_H(e) + [f_{H_1}(x_2) \cdot f_{x_3}(x_1)] \cdot f_H(e) + [f_{H_1}(x_3) \cdot f_{x_1}(x_2)] \cdot f_H(e) = 0 \quad \text{(c)}$$

for all $e \in E$.

When $e \in E_1$, we know that $f_H(e) = f_{H'}(e) = f_{H''}(e) = f_{H_1}(e)$ by (4). Since $f_{x_1}(x_2) = 1$, we know that $f_{x_1}(x_3) = -1$, and so by Proposition 5.3 we have $f_{x_3}(x_1) = 1$. Similarly, $f_{x_2}(x_3) = 1$, and then (c) holds because $f_{H_1}(x_1) + f_{H_1}(x_2) + f_{H_1}(x_3) = 0$ by Lemma 5.4 (2).

When $e = y_1$, using (4), the equation (c) reduces to

$$[f_{H_1}(x_2) \cdot f_{x_3}(x_1)] \cdot \frac{f_{H_1}(x_3)}{f_{x_1}(x_3)} + [f_{H_1}(x_3) \cdot f_{x_1}(x_2)] \cdot \frac{f_{H_1}(x_2)}{f_{x_1}(x_2)} = 0.$$

In the second subcase, $(H, H', H'') = (F \cup y_1, F \cup y_2, F \cup E_1)$. It is similarly straightforward to check that

$$[f_{x_1}(x_3) \cdot f_{x_2}(x_3)] \cdot f_H(e) + [f_{x_3}(x_1) \cdot f_{x_2}(x_3)] \cdot f_{H'}(e) + [f_{H_1}(x_3)] \cdot f_{H''}(e) = 0 \quad \text{(d)}$$

for all $e \in E$. When $e \in E_1$, this follows from the fact that $f_H(e) = f_{H'}(e) = f_{H_1}(e)$ by (4). When $e = y_1$, applying (1) and (4) gives

$$[f_{x_3}(x_1) \cdot f_{x_2}(x_3)] \cdot \frac{f_{H_1}(x_3)}{f_{x_1}(x_3)} + [f_{H_1}(x_3)] \cdot f_{x_2}(x_3) = 0,$$

and when $e = y_3$, applying (4) shows that the left-hand side of (d) is equal to

$$[f_{x_1}(x_3) \cdot f_{x_2}(x_3)] \cdot \frac{f_{H_1}(x_2)}{f_{x_3}(x_2)} + [f_{x_3}(x_1) \cdot f_{x_2}(x_3)] \cdot \frac{f_{H_1}(x_1)}{f_{x_3}(x_1)} + [f_{H_1}(x_3)] \cdot f_{x_1}(x_2).$$

This is equal to 0 because, as described in the previous subcase, $f_{x_1}(x_2) = f_{x_3}(x_1) = f_{x_2}(x_3) = 1$ and $f_{H_1}(x_1) + f_{H_1}(x_2) + f_{H_1}(x_3) = 0$.

Case 7: $F = F_1 \cup (Y - \{y_1, y_2\})$, where F_1 is a corank-2 flat of M_1 disjoint from X. We first prove:

Claim 5.6. *Let (H_i, H_j, H_k) be a modular triple of hyperplanes of M_1 so that $H_i \cap X = \{x_i\}$, $H_j \cap X = \{x_j\}$, and $H_k \cap X = \{x_k\}$. Then*

$$[f_{x_j}(x_k)] \cdot f_{H_i}(e) - [f_{x_i}(x_k)] \cdot f_{H_j}(e) + [f_{x_i}(x_j)] \cdot f_{H_k}(e) = 0$$

for all $e \in E_1$.

Proof. We may assume that $(i,j,k)=(1,2,3)$. There are constants c_1,c_2,c_3 so that

$$c_1\cdot f_{H_1}(e)+c_2\cdot f_{H_2}(e)+c_3\cdot f_{H_3}(e)=0$$

for all $e\in E_1$. By plugging in $e=x_1,x_2,x_3$ and using the assumption that $H_l\cap X=\{x_l\}$ implies $f_{H_l}(x_m)=f_{x_l}(x_m)$ for all $l,m\in[n]$, we see that

$$(c_1,c_2,c_3)=(f_{x_2}(x_3),f_{x_3}(x_1),f_{x_1}(x_2))$$

up to multiplication by a scalar. This proves the claim. □

We now consider three subcases.

In the first subcase, $(H,H',H'')=(F\cup\{x_1,y_2\},F\cup\{x_2,y_1\},F\cup x_3)$. We will show that

$$[f_{x_2}(x_3)]\cdot f_H(e)-[f_{x_1}(x_3)]\cdot f_{H'}(e)+[f_{x_1}(x_2)]\cdot f_{H''}(e)=0$$

for all $e\in E$. When $e\in E_1$, this holds by Claim 5.6 with $(i,j,k)=(1,2,3)$ and $(H_i,H_j,H_k)=(H,H',H'')$. When $e=y_1$, using (3) and (4) we have

$$[f_{x_2}(x_3)]\cdot 1+[f_{x_1}(x_2)]\cdot\frac{-f_{x_2}(x_3)}{f_{x_1}(x_2)}=0.$$

In the second subcase, $(H,H',H'')=(F\cup\{x_1,y_2\},F\cup x_3,F\cup x_4)$. We will show that

$$[f_{x_3}(x_4)]\cdot f_H(e)-[f_{x_1}(x_4)]\cdot f_{H'}(e)+[f_{x_1}(x_3)]\cdot f_{H''}(e)=0 \tag{e}$$

for all $e\in E$. When $e\in E_1$, this holds by Claim 5.6 with $(i,j,k)=(1,3,4)$ and $(H_i,H_j,H_k)=(H,H',H'')$. When $e=y_2$, applying (3') shows that

$$-[f_{x_1}(x_4)]\cdot\frac{f_{x_1}(x_3)}{f_{x_1}(x_2)}+[f_{x_1}(x_3)]\cdot\frac{f_{x_1}(x_4)}{f_{x_1}(x_2)}=0.$$

When $e=y_1$, by applying (3) and (3'), the left-hand side of (e) becomes

$$[f_{x_3}(x_4)]\cdot 1-[f_{x_1}(x_4)]\cdot\frac{-f_{x_2}(x_3)}{f_{x_1}(x_2)}+[f_{x_1}(x_3)]\cdot\frac{-f_{x_2}(x_4)}{f_{x_1}(x_2)},$$

which is equal to 0 by Proposition 5.3.

In the third subcase, $(H,H',H'')=(F\cup x_3,F\cup x_4,F\cup x_5)$. We will show that

$$[f_{x_4}(x_5)]\cdot f_H(e)-[f_{x_3}(x_5)]\cdot f_{H'}(e)+[f_{x_3}(x_4)]\cdot f_{H''}(e)=0 \tag{g}$$

for all $e\in E$. When $e\in E_1$ this holds by Claim 5.6 with $(i,j,k)=(3,4,5)$ and $(H_i,H_j,H_k)=(H,H',H'')$. When $e=y_2$, using (3), the left-hand side of (g) becomes

$$[f_{x_4}(x_5)]\cdot\frac{f_{x_1}(x_3)}{f_{x_1}(x_2)}-[f_{x_3}(x_5)]\cdot\frac{f_{x_1}(x_4)}{f_{x_1}(x_2)}+[f_{x_3}(x_4)]\cdot\frac{f_{x_1}(x_5)}{f_{x_1}(x_2)},$$

which is equal to 0 by Proposition 5.3.

These seven cases combine to show that $\mathcal{H}$ is in fact a modular system of P-hyperplane functions for M. So, for any pasture P, we have defined a map from $\mathcal{X}^R_{M_1}(P)$ to $\mathcal{X}^R_M(P)$. The inverse of the map from $\mathcal{X}^R_M(P)$ to $\mathcal{X}^R_{M_1}(P)$ is the natural map defined by restriction to E_1, which is clearly functorial in P. This implies that M_1 and M have isomorphic foundations. □

This has the following corollary in the special case that $M_1\cong U_{2,n}$.

Corollary 5.7. *For all $n \geqslant 2$, the matroids $U_{2,n}$ and Θ_n have isomorphic foundations.*

We next delete X from $P_X(M, \Theta_n)$ and show that this preserves the foundation when X is coindependent in M. We will use the following lemma.

Lemma 5.8. *If P is a finitely generated pasture and $f : P \to P$ is a homomorphism which restricts to a surjection $P^\times \to P^\times$ of multiplicative groups, then f is an isomorphism.*

Proof. A surjective homomorphism from a finitely generated abelian group to itself is necessarily an isomorphism, cf. [Mat80, Proof of Lemma 29.2]. So f is a bijection on underlying sets, and by construction $f(N_P) \subseteq N_P$. It suffices to prove that the map from P to P which sends $x \in P$ to $f^{-1}(x) \in P$ is a homomorphism.

Let $g : P \to P'$ be the homomorphism of pastures induced by the inverse map $f^{-1} : P \to P$, i.e., P' has the same underlying set as P, and we define the null set of P' to consist of all formal sums of the form $\sum a_i y_i$ such that $\sum a_i f^{-1}(y_i) \in N_P$. Then $g \circ f : P \to P'$ is the identity map on underlying sets, and therefore $N_P \subseteq N_{P'}$. For the reverse containment, suppose $\sum a_i y_i \in N_{P'}$. By definition, there exist $x_i \in P$ such that $f(x_i) = y_i$ and $\sum a_i x_i \in N_P$. Since $f : P \to P$ is a homomorphism, we must have $\sum a_i f(x_i) \in N_P$, which means that $N_{P'} \subseteq N_P$. □

We next describe the homomorphism to which we will apply Lemma 5.8. It will be defined using cross ratios; see Section 2.3.2 for the relevant definitions. Let N be a matroid with a coindependent set X. If $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is a cross ratio of $N \backslash X$, then $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is also a cross ratio of N. It follows from [BL25a, Proposition 4.9] that the function $\psi_{N\backslash X}$ from $F^\times_{N\backslash X}$ to $F^\times_N$ that maps $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ to $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is a homomorphism. We next show that in the special case that $N = P_X(M_1, \Theta_n)$ for some matroid M_1, this homomorphism is surjective.

Lemma 5.9. *Let M_1 be a matroid and let $X \subseteq E(M_1)$ be a coindependent set such that $M_1|X \cong U_{2,n}$ for some $n \geqslant 2$. Let $M = P_X(M_1, \Theta_n)$, let $M' = M\backslash X$, and let $\psi_{M\backslash X}$ be the homomorphism from $F^\times_{M'}$ to $F^\times_M$ that maps $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ to $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$. Then $\psi_{M\backslash X}$ is surjective.*

Proof. Let E, E_1, and $E_2 = X \cup Y$ be the ground sets of M, M_1, and Θ_n, respectively. When $n = 2$, we know that the cosimplification of M is isomorphic to M_1 because $\{x_i, y_i\}$ is a series pair of M for $i = 1, 2$. So, by [BL25a, Corollary 4.10], we may assume that $n \geqslant 3$. The following claim will allow us to show that two given cross ratios of M are equal.

Claim 5.10. *Let $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ be a cross ratio of M.*

(1) $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_2 & e_1 \\ e_4 & e_3 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_3 & e_4 \\ e_1 & e_2 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_4 & e_3 \\ e_2 & e_1 \end{smallmatrix}\right]_J$.

(2) If $\mathrm{cl}(J) = \mathrm{cl}(J')$, *then* $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_{J'}$.

(3) If $\mathrm{cl}(J \cup e_4) = \mathrm{cl}(J \cup e_4')$, *then* $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4' \end{smallmatrix}\right]_J$.

(4) If $(Ie_5; e_1, e_2, e_3, e_4)$, $(Ie_3; e_1, e_2, e_4, e_5)$, *and* $(Ie_4; e_1, e_2, e_5, e_3)$ *are all in* Ω_M, *then*

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}_{Ie_5} \cdot \begin{bmatrix} e_1 & e_2 \\ e_4 & e_5 \end{bmatrix}_{Ie_3} \cdot \begin{bmatrix} e_1 & e_2 \\ e_5 & e_3 \end{bmatrix}_{Ie_4} = 1.$$

Proof. Parts (1) and (4) are relations (Rσ) and (R4), respectively, of [BL25a, Theorem 4.21], and parts (2) and (3) are implied by [BL25a, Corollary 3.7]. □

Fix a cross ratio $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ of M, and let $F = \mathrm{cl}(J)$. We will show that $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is in the image of $\psi_{M\backslash X}$. By Proposition 2.14, there are seven possibilities for F, which we consider separately. In Cases 1–6 we will show that $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is the image of a cross ratio of M', and in Case 7 we will show that $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is the image of a product of cross ratios of M'. Each hyperplane or corank-2 flat of Θ_n is associated with a given subset of $[n]$; we will choose this subset explicitly without loss of generality to improve readability.

Case 1: $F = E - \{y_1, y_2, y_3, y_4\}$. Then $e_1, e_2, e_3, e_4 \notin X$ and X is spanned in M by $F - X$ because X is coindependent in M_1. Let J' be a basis of $F - X$. Then $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_{J'} = \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ by Claim 5.10 (2) and $J' \cup \{e_1, e_2, e_3, e_4\}$ is disjoint from X, so $\psi_{M\backslash X}$ maps $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_{J'}$ to $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$.

Case 2: $F = F_1 \cup E_2$, where F_1 is a corank-2 flat of M_1 that contains X. Then $e_1, e_2, e_3, e_4 \notin X$ and X is spanned in M by $F - X$ because $Y \subseteq F - X$. Let J' be a basis of $F - X$. Then $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_{J'} = \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ by Claim 5.10 (2) and $J' \cup \{e_1, e_2, e_3, e_4\}$ is disjoint from X, so $\psi_{M\backslash X}$ maps $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_{J'}$ to $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$.

Case 3: $F = H_1 \cup (Y - \{y_1, y_2, y_3\})$, where H_1 is a hyperplane of M_1 that contains X. Then M/J has at most two parallel classes (namely, $E_1 - F$ and $E_2 - F$), so $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is degenerate and therefore $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = 1$ in $F_M^\times$.

Case 4: $F = H_1 \cup ((Y - \{y_1, y_2, y_3\}) \cup x_i)$ for some $i \in \{1, 4\}$, where H_1 is a hyperplane of M_1 with $H_1 \cap X = \{x_i\}$. We separately consider the cases $i = 1$ and $i = 4$.

If $i = 1$, then y_2 and y_3 are parallel in M/J because $(Y - y_1) \cup x_1$ is a hyperplane of Θ_n. So M/J has at most three parallel classes: $E_1 - F$, $\{y_1\}$, and $\{y_2, y_3\}$. Therefore $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is degenerate, so $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = 1$ in $F_M^\times$.

Suppose $i = 4$. Then M/J has at most four parallel classes: $E_1 - F$, $\{y_1\}$, $\{y_2\}$, and $\{y_3\}$. We may assume that $\{e_1, e_2, e_3, e_4\}$ contains one element from each of these parallel classes, or else $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is degenerate. By swapping rows and columns of $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_{J'}$, we may assume that $e_4 \in E_1 - F$. Since X is coindependent in M_1, there is some $a \in E_1 - (H_1 \cup X)$. Then $\mathrm{cl}(J \cup e_4) = \mathrm{cl}(J \cup a)$ because e_4 and a are parallel in M/J, so by Claim 5.10 (3) we may assume that $e_4 = a$ and $\{e_1, e_2, e_3\} = \{y_1, y_2, y_3\}$. Up to re-indexing, we may assume that $e_3 = y_3$, so $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_1 & e_2 \\ y_3 & a \end{smallmatrix}\right]_J$. Let $I = J - y_4$. By Claim 5.10 (4) we have

$$\begin{bmatrix} e_1 & e_2 \\ y_3 & a \end{bmatrix}_{Iy_4} \cdot \begin{bmatrix} e_1 & e_2 \\ a & y_4 \end{bmatrix}_{Iy_3} \cdot \begin{bmatrix} e_1 & e_2 \\ y_4 & y_3 \end{bmatrix}_{Ia} = 1.$$

Since $(Y - y_4) \cup x_4$ is a hyperplane of Θ_n we see that y_1 and y_2 are parallel in $M/(I \cup y_3)$. Then $M/(I \cup y_3)$ has at most three parallel classes (namely $E_1 - \mathrm{cl}(I \cup y_3)$, $\{y_1, y_2\}$, and $\{y_4\}$), so $\left[\begin{smallmatrix} e_1 & e_2 \\ a & y_4 \end{smallmatrix}\right]_{Iy_3}$ is degenerate. Since $\mathrm{cl}(I \cup a) = E - \{y_1, y_2, y_3, y_4\}$, the cross ratio $\left[\begin{smallmatrix} e_1 & e_2 \\ y_4 & y_3 \end{smallmatrix}\right]_{Ia}$ is the image under $\psi_{M\backslash X}$ of a cross ratio of M', as proved in Case 1. Since $\psi_{M\backslash X}$ is a homomorphism, $\left[\begin{smallmatrix} e_1 & e_2 \\ y_3 & a \end{smallmatrix}\right]_{Iy_4}$ is the image under $\psi_{M\backslash X}$ of the inverse of $\left[\begin{smallmatrix} e_1 & e_2 \\ y_4 & y_3 \end{smallmatrix}\right]_{Ia}$ in $F_{M'}^\times$.

Case 5: $F = F_1 \cup H_2$, where F_1 is a corank-2 flat of M_1, H_2 is a hyperplane of Θ_n, and $F_1 \cap X = H_2 \cap X = \{x_1\}$. We consider two subcases, depending on the form of H_2.

First, suppose that $H_2 = (Y - y_1) \cup x_1$. Then $F - x_1$ contains a basis J' of F because $Y - y_1$ spans x_1 in Θ_n and therefore in M as well. Suppose that $e_i \in X$ for some $i \in [4]$. Since X is contained in a parallel class of M/J', this choice of i is unique, or else $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is degenerate. By Claim 5.10 (1) we may assume that $i = 4$. Then e_4 and y_1 are parallel in M/J', so $\mathrm{cl}(J' \cup e_4) = \mathrm{cl}(J' \cup y_1)$ and therefore $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_{J'} = \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & y_1 \end{smallmatrix}\right]_{J'}$ by Claim 5.10 (3). Since $J' \cup \{e_1, e_2, e_3, y_1\}$ is disjoint from X, we see that $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & y_1 \end{smallmatrix}\right]_{J'}$ is a cross ratio of M' whose image under $\psi_{M\backslash X}$ is $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_{J'}$.

In the second subcase, suppose that $H_2 = (Y - \{y_2, y_3\}) \cup x_1$. Since $E_2 - H_2$ is contained in a parallel class of M/J, at most one of e_1, e_2, e_3, e_4 is in E_2 or else $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is degenerate. Suppose that $e_i \in E_2$ for some $i \in [4]$. By Claim 5.10 we may assume that $i = 4$. Then e_4 and y_2 are parallel in M/J, so $\mathrm{cl}(J \cup e_4) = \mathrm{cl}(J \cup y_2)$ and therefore $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & y_2 \end{smallmatrix}\right]_J$ by Claim 5.10 (3). Note that $\{e_1, e_2, e_3\} \subseteq E_1 - X$. Let $I = J - y_1$. By Claim 5.10 (4) we have

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & y_2 \end{bmatrix}_{Iy_1} \cdot \begin{bmatrix} e_1 & e_2 \\ y_2 & y_1 \end{bmatrix}_{Ie_3} \cdot \begin{bmatrix} e_1 & e_2 \\ y_1 & e_3 \end{bmatrix}_{Iy_2} = 1.$$

Since $(Y - y_1) \cup x_1$ is a hyperplane of Θ_n we see that y_2 and y_3 are parallel in $M/(I \cup e_3)$. Then $M/(I \cup e_3)$ has at most three parallel classes (namely $E_1 - \mathrm{cl}(I \cup e_3)$, $\{y_1\}$, and $\{y_2, y_3\}$), so $\left[\begin{smallmatrix} e_1 & e_2 \\ y_2 & y_1 \end{smallmatrix}\right]_{Ie_3}$ is degenerate. Since $\mathrm{cl}(I \cup y_2)$ is a corank-2 flat of M_1 consisting of F_1 and the hyperplane $(Y - y_1) \cup x_1$ of Θ_n, we know from the first subcase of Case 5 that $\left[\begin{smallmatrix} e_1 & e_2 \\ y_1 & e_3 \end{smallmatrix}\right]_{Iy_2}$ is the image under $\psi_{M\backslash X}$ of a cross ratio of M'. Since $\psi_{M\backslash X}$ is a homomorphism, $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & y_2 \end{smallmatrix}\right]_{Iy_1}$ is the image under $\psi_{M\backslash X}$ of the inverse of $\left[\begin{smallmatrix} e_1 & e_2 \\ y_1 & e_3 \end{smallmatrix}\right]_{Iy_2}$ in $F_{M'}^{\times}$.

Case 6: $F = H_1 \cup (Y - \{y_1, y_2, y_3\})$, where H_1 is a hyperplane of M_1 disjoint from X. Then M/J has at most four nontrivial parallel classes: $E_1 - H_1$, $\{y_1\}$, $\{y_2\}$, and $\{y_3\}$. We may assume that $\{e_1, e_2, e_3, e_4\}$ contains one element from each of these parallel classes, or else $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is degenerate. By Claim 5.10 we may assume that $e_4 \in E_1 - H_1$ and $\{e_1, e_2, e_3\} = \{y_1, y_2, y_3\}$. Let $a \in E_1 - (H_1 \cup X)$; such an element exists because X is coindependent in M_1. Then $\mathrm{cl}(J \cup e_4) = \mathrm{cl}(J \cup a)$ because e_4 and a are parallel in M/J, so $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & a \end{smallmatrix}\right]_J$ by Claim 5.10 (3). Since $J \cup \{e_1, e_2, e_3, a\}$ is disjoint from X we see that $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & a \end{smallmatrix}\right]_J$ is a cross ratio of M' whose image under $\psi_{M\backslash X}$ is $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$.

Case 7: $F = F_1 \cup (Y - \{y_1, y_2\})$, where F_1 is a corank-2 flat of M_1 disjoint from X. Note that $\{x_1, y_2\}$ and $\{x_2, y_1\}$ are parallel pairs in M/J. Let $k = |\{e_1, e_2, e_3, e_4\} \cap X|$. We will proceed by induction on k to show that every cross ratio $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ with $\mathrm{cl}(J) = F$ is in the image of $\psi_{M\backslash X}$. If $k = 0$, then $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$ is a cross ratio of M' whose image under $\psi_{M\backslash X}$ is $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$. So we may assume that $k \geqslant 1$, so $e_i = x_j$ for some $i \in [4]$ and $j \in [n]$. By Claim 5.10 we may assume that $i = 4$. If $j = 1$, then since $\mathrm{cl}(J \cup x_1) = \mathrm{cl}(J \cup y_2)$ because $\{x_1, y_2\}$ is a parallel pair of M/J, by Claim 5.10 (3) we see that $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J = \left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & y_2 \end{smallmatrix}\right]_J$. By induction, $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & y_2 \end{smallmatrix}\right]_J$ is in the image of $\psi_{M\backslash X}$, and therefore so is $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & e_4 \end{smallmatrix}\right]_J$. So we may assume that $j \neq 1$, and by similar reasoning, that $j \neq 2$. Without loss of generality, we may assume that $j = 3$, so $e_4 = x_3$.

Let $I = J - y_3$. By Claim 5.10 (4) we have

$$\begin{bmatrix} e_1 & e_2 \\ e_3 & x_3 \end{bmatrix}_{Iy_3} \cdot \begin{bmatrix} e_1 & e_2 \\ x_3 & y_3 \end{bmatrix}_{Ie_3} \cdot \begin{bmatrix} e_1 & e_2 \\ y_3 & e_3 \end{bmatrix}_{Ix_3} = 1.$$

Since $\mathrm{cl}(I \cup x_3)$ is a corank-2 flat of the form considered in Case 4, we know that $\left[\begin{smallmatrix} e_1 & e_2 \\ y_3 & e_3 \end{smallmatrix}\right]_{Ix_3}$ is in the image of $\psi_{M\backslash X}$. We next show that $\left[\begin{smallmatrix} e_1 & e_2 \\ x_3 & y_3 \end{smallmatrix}\right]_{Ie_3}$ is in the image of $\psi_{M\backslash X}$ by considering three possibilities for e_3. If $\mathrm{cl}(I \cup e_3) \cap X \neq \varnothing$ (in particular, if $e_3 \in X$), then $\mathrm{cl}(I \cup e_3)$ is a corank-2 flat of the form considered in Case 4, so $\left[\begin{smallmatrix} e_1 & e_2 \\ x_3 & y_3 \end{smallmatrix}\right]_{Ie_3}$ is in the image of $\psi_{M\backslash X}$. If $e_3 \in \{y_1, y_2\}$, then without loss of generality we may assume that $e_3 = y_1$. Then y_2 and y_3 are parallel in M/J because of the hyperplane $(Y - y_3) \cup x_3$ of Θ_n. So $\mathrm{cl}(I \cup e_3 \cup x_3) = \mathrm{cl}(I \cup e_3 \cup y_2)$, so Claim 5.10 (3) implies that $\left[\begin{smallmatrix} e_1 & e_2 \\ x_3 & y_3 \end{smallmatrix}\right]_{Ie_3} = \left[\begin{smallmatrix} e_1 & e_2 \\ y_2 & y_3 \end{smallmatrix}\right]_{Ie_3}$. By induction, $\left[\begin{smallmatrix} e_1 & e_2 \\ y_2 & y_3 \end{smallmatrix}\right]_{Ie_3}$ is in the image of $\psi_{M\backslash X}$, and therefore so is $\left[\begin{smallmatrix} e_1 & e_2 \\ x_3 & y_3 \end{smallmatrix}\right]_{Ie_3}$. Finally, if $e_3 \in E_1 - X$ and $\mathrm{cl}(I \cup e_3)$ is disjoint from X, then $\mathrm{cl}(I \cup e_3)$ is a corank-2 flat of the form considered in Case 6 and is therefore in the image of $\psi_{M\backslash X}$. Therefore, since $\psi_{M\backslash X}$ is a homomorphism, $\left[\begin{smallmatrix} e_1 & e_2 \\ e_3 & x_3 \end{smallmatrix}\right]_{Iy_3}$ is the image under $\psi_{M\backslash X}$ of the product of the inverses of $\left[\begin{smallmatrix} e_1 & e_2 \\ x_3 & y_3 \end{smallmatrix}\right]_{Ie_3}$ and $\left[\begin{smallmatrix} e_1 & e_2 \\ y_3 & e_3 \end{smallmatrix}\right]_{Ix_3}$ in $F_{M'}^{\times}$. □

We can now prove the main result of this section.

Theorem 5.11. *Let M be a matroid and let $X \subseteq E(M)$ be a coindependent set so that $M|X \cong U_{2,n}$. Then the foundation of the segment-cosegment exchange of M along X is isomorphic to the foundation of M.*

Proof. Following [Oxl06], for a matroid N with $X \subseteq E(N)$ so that X is coindependent and $N|X \cong U_{2,n}$, we write $\Delta_X(N)$ for $P_X(N, \Theta_n)\backslash X$, the segment-cosegment exchange of N along X. (We do not follow the convention from [Oxl06] of relabeling Y with X in $\Delta_X(N)$ via the natural isomorphism from Θ_n to Θ_n^* that swaps x_i and y_i for each $i \in [n]$.) Dually, if $N^*|Y \cong U_{2,n}$ then we write $\nabla_Y(N)$ for $(\Delta_Y(N^*))^*$, the cosegment-segment exchange of N along Y.

Let $P = P_X(M, \Theta_n)$ and let $M' = P_X(M, \Theta_n)\backslash X$. By [Oxl06, Lemma 11.5.6] we know that $((M')^*|Y) \cong U_{2,n}$, so let $P' = P_Y((M')^*, \Theta_n^*)$. By [Oxl06, Proposition 11.5.11 (i)] we know that $\nabla_Y(\Delta_X(M)) = M$. Taking the dual of both sides, we see that $\Delta_Y((M')^*) = M^*$, so $P'\backslash Y = M^*$. It follows from Theorem 5.5 that we have isomorphisms $F_M \to F_P$ and $F_{(M')^*} \to F_{P'}$, and since $M^* = P'\backslash Y$ and $M' = P\backslash X$ it follows from [BL25a, Proposition 4.9] that we have homomorphisms $F_{M^*} \to F_{P'}$ and $F_{M'} \to F_P$. Hence, we have the following diagram of homomorphisms of pastures:

$$F_M \overset{\cong}{\to} F_{M^*} \to F_{P'} \overset{\cong}{\to} F_{(M')^*} \overset{\cong}{\to} F_{M'} \to F_P \overset{\cong}{\to} F_M. \tag{a}$$

Here, the maps $F_M \to F_{M^*}$ and $F_{(M')^*} \to F_{M'}$ are the natural isomorphisms given by [BL25a, Proposition 4.8], and the maps $F_{P'} \to F_{(M')^*}$ and $F_P \to F_M$ are the inverses of the isomorphisms $F_M \to F_P$ and $F_{(M')^*} \to F_{P'}$.

By Lemma 5.9, the homomorphisms $F_{M^*} \to F_{P'}$ and $F_{M'} \to F_P$ restrict to surjective homomorphisms of multiplicative groups. It follows that the composition of the maps in (a) induces a surjection of multiplicative groups. By Lemma 5.8, we conclude that the composite map is an isomorphism, which means that all the intermediate maps must be isomorphisms as well. In particular, $F_{M'} \cong F_P$. On the other hand, we know from Theorem 5.5 that $F_P \cong F_M$, and thus $F_{M'} \cong F_M$ as desired. □

We have the following corollary in the case that $n = 3$.

Theorem 5.12. *Let M be a matroid and let $T \subseteq E(M)$ be a coindependent triangle. Then the foundation of the Delta-Wye exchange of M along T is isomorphic to the foundation of M.*

Remark 5.13. Note that if we replace the foundation by the universal pasture in the statement of Theorem 5.11, the result remains true. This follows formally from Corollary 7.14 and Remark 7.15 of [BL21] upon noting that there is a bijection between connected components of M and connected components of the segment-cosegment exchange of M along X; see Lemma 5.14 below for a straightforward proof of this fact.

Lemma 5.14. *If M is a matroid with $X \subseteq E(M)$ so that X is coindependent and $M|X \cong U_{2,n}$ for some $n \geqslant 2$, then there is a bijection between the connected components of M and the connected components of the segment-cosegment exchange $P_X(M, \Theta_n)\backslash X$.*

Proof. If $n = 2$, then M and $P_X(M, \Theta_n)\backslash X$ are isomorphic because $\{x_i, y_i\}$ is a series pair for $i = 1, 2$, so we may assume that $n \geqslant 3$. If M is connected, then $P_X(M, \Theta_n)\backslash X$ is connected by [Oxl06, pg. 456, Ex. 6] and the result follows, so we may assume that M is disconnected. Since $n \geqslant 3$ we know that $M|X$ is connected, and therefore X is contained some component of M. So $M = M_1 \oplus M_2$ where M_1 is connected and $X \subseteq E(M_1)$ (and M_2 may or may not be connected).

We will first show that $P_X(M, \Theta_n) = P_X(M_1, \Theta_n) \oplus M_2$. Let E, E_1, and E_2 be the ground sets of M, M_1, and M_2, respectively. For a matroid N we write $\mathcal{F}(N)$ for the set of flats of N. Then

$$\begin{aligned}
\mathcal{F}(P_X(M, \Theta_n)) &= \{F \subseteq E \cup Y \mid F \cap E \in \mathcal{F}(M) \text{ and } F \cap (X \cup Y) \in \mathcal{F}(\Theta_n)\} \\
&= \{F \subseteq E \cup Y \mid F \cap E_i \in \mathcal{F}(M_i) \text{ for } i = 1, 2 \text{ and } F \cap (X \cup Y) \in \mathcal{F}(\Theta_n)\} \\
&= \{F \subseteq E \cup Y \mid F \cap (E_1 \cup X \cup Y) \in \mathcal{F}(P_X(M_1, \Theta_n)) \text{ and } F \cap E_2 \in \mathcal{F}(M_2)\} \\
&= \mathcal{F}(P_X(M_1, \Theta_n) \oplus M_2).
\end{aligned}$$

Here, the first and third lines follow from the definition of generalized parallel connection, and the second and fourth lines follow from the characterization of flats of a direct sum [Oxl06, Proposition 4.2.16]. Therefore $P_X(M, \Theta_n) = P_X(M_1, \Theta_n) \oplus M_2$, and it follows from [Oxl06, Proposition 4.2.19] that $P_X(M, \Theta_n)\backslash X = (P_X(M_1, \Theta_n)\backslash X) \oplus M_2$. Since $P_X(M_1, \Theta_n)\backslash X$ is connected by [Oxl06, pg. 456, Ex. 6], it follows that the components of $P_X(M, \Theta_n)\backslash X$ are precisely $(E_1 - X) \cup Y$ and the components of M_2. This gives a bijection between the components of M and the components of $P_X(M, \Theta_n)\backslash X$ in which E_1 maps to $(E_1 - X) \cup Y$ and every other component of M maps to itself. □

We turn to the proof of Corollary F from the Introduction, whose statement we now recall:

Corollary 5.15. *Let P be a pasture, and let M be an excluded minor for representability over P. Then every segment-cosegment exchange of M is also an excluded minor for representability over P.*

Proof. Let M be an excluded minor for P-representability, so M is not P-representable, but every proper minor of M is P-representable. In particular, it follows from [BL25a, Lemma 4.10] that M is simple and cosimple. Let $M|X \cong U_{2,n}$ for some $n \geqslant 2$ so that X is coindependent in M, and let M' be the segment-cosegment exchange of M on X. It follows from Theorem 5.11 that M' is not P-representable, so it suffices to show that every proper minor of M' is P-representable. If $n = 2$, then $M' \cong M$ and the result holds, so we may assume that $n \geqslant 3$. Let $e \in E(M')$. We consider two cases. First suppose that $e = y_i$ for some $i \in [n]$. By [OSV00, Lemma 2.13] we know that M'/y_i is isomorphic to the segment-cosegment exchange of $M\backslash x_i$ along $X - x_i$. Since $M\backslash x_i$ is P-representable, it follows from Theorem 5.11 that M'/y_i is also P-representable. In $M'\backslash y_i$, the set $Y - y_i$ is contained in a series class because $M'|Y \cong U_{2,n}$. By [BL25a, Lemma 4.10], the cosimplification of $M'\backslash y_i$ has foundation isomorphic to the foundation of $M'\backslash y_i$. Since the cosimplification of $M'\backslash y_i$ is a minor of M'/y_j for some $j \neq i$, it follows that $M'\backslash y_i$ is P-representable.

Next suppose that $e \notin Y$. Then $M'\backslash e = P_X(M\backslash e, \Theta_n)\backslash X$ by [Oxl06, Proposition 11.4.14 (iv)], and since $M\backslash e$ is P-representable it follows from Theorem 5.11 that $M'\backslash e$ is P-representable. It remains to show that M'/e is P-representable. If e is not spanned by X in M, then by [OSV00, Lemma 2.16] we know that M'/e is isomorphic to the segment-cosegment exchange of M/e along X, and it follows from Theorem 5.11 that M'/e is P-representable. So we may assume that e is spanned by X in M. Then $M|(X \cup e) \cong U_{2,n+1}$ because M is simple, so $U_{2,n+1}$ is P-representable, and therefore $U_{n-1,n+1}$ is P-representable by [BL25a, Proposition 4.8]. By [OSV00, Lemma 2.15] we know that M'/e is isomorphic to the 2-sum of $M/e\backslash(X - x_i)$ and a copy of $U_{n-1,n+1}$ for some $i \in [n]$. Since both of these matroids are P-representable, it follows from Theorem C that M'/e is P-representable. □

5.1. Application to a conjecture by Pendavingh and van Zwam

In this final section, we turn to the proof of Corollary E. As preparation, we recall that the universal partial field $\mathbb{P}_M$ of a representable matroid M is determined by its foundation F_M.

According to [BL25b, Lemma 2.14], for every pasture P that maps to some partial field F, there is a universal map $\pi_P : P \to \Pi P$ to a partial field ΠP such every other map $f : P \to F$ to a partial field F factors uniquely through π_P.

The partial field ΠP is defined as follows: let I be the ideal of the group ring $\mathbb{Z}[P^\times]$ which is generated by all terms $a + b + c$ that appear in the null set N_P. Then ΠP is the partial field $(P^\times, \mathbb{Z}[P^\times]/I)$; as a pasture, it can be described as

$$\Pi P \;=\; P /\!\!/ \langle a + b + c \mid a + b + c \in I \rangle.$$

The pasture morphism $\pi_P : P \to \Pi P$ is the quotient map. Note that since P maps to some partial field, I is a proper ideal of $\mathbb{Z}[P^\times]$ and thus ΠP is indeed a partial field (since $1 \neq 0$).

If $P = F_M$ is the foundation of a representable matroid M, its universal partial field is $\mathbb{P}_M = \Pi F_M$. This follows at once from a comparison of the universal properties of ΠF_M

and $\mathbb{P}_M$: either of these partial fields represents the functor that associates with a partial field F the set of rescaling classes of M over F.

Corollary 5.16. *Let M be a matroid, let $X \subseteq E(M)$ so that X is coindependent and $M|X \cong U_{2,n}$ for some $n \geqslant 2$, and assume that M is representable over some partial field. Then the universal partial field of the segment-cosegment exchange of M along X is isomorphic to the universal partial field of M.*

Proof. Let M' be the segment-cosegment exchange of M along X. Let F_M and $F_{M'}$ be the foundations of M and M', respectively. By Theorem 5.11, $F_{M'} \simeq F_M$, which implies

$$\mathbb{P}_{M'} \;=\; \Pi F_{M'} \;\simeq\; \Pi F_M \;=\; \mathbb{P}_M,$$

since the functor Π preserves isomorphisms. □

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