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SOME ENUMERATIVE PROPERTIES OF PARKING FUNCTIONS

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Abstract. A *parking function* is a sequence $(\pi_1, \ldots, \pi_n)$ of positive integers such that if $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ is the increasing rearrangement of $\pi_1, \ldots, \pi_n$, then $\lambda_i \leqslant i$ for $1 \leqslant i \leqslant n$. In this paper we obtain some new results on the enumeration of parking functions. We will consider the joint distribution of several sets of statistics on parking functions. The distribution of most of these individual statistics is known, but the joint distributions are new. Parking functions of length n are in bijection with labelled forests on the vertex set $[n] = \{1, 2, \ldots, n\}$ (or rooted trees on $[n]_0 = \{0, 1, \ldots, n\}$ with root 0), so our results can also be applied to labelled forests. Extensions of our techniques are discussed, including an extension to a probabilistic scenario.

Keywords. Parking function, labelled forest, generating function, recurrence, Pollak's circle argument

Mathematics Subject Classifications. 05A15, 60C05, 05A19

1. Introduction

In this paper we obtain some new results on the enumeration of parking functions and, by way of the bijection between parking functions and labelled forests, some new results on the enumeration of labelled forests. We begin with the necessary definitions. In the classical parking function scenario due to Konheim and Weiss [KW66], we have n parking spaces on a one-way street, labelled $1, 2, \ldots, n$ in consecutive order as we drive down the street. There are n cars $C_1, \ldots, C_n$. Each car C_i has a preferred spot $1 \leqslant \pi_i \leqslant n$. The cars drive down the street one at a time in the order $C_1, \ldots, C_n$. The car C_i drives immediately to spot π_i and then parks in the first available

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spot. Thus if spot π_i is empty, then C_i parks there; otherwise C_i next goes to spot $\pi_i + 1$, etc. If all cars are able to park, then the sequence $\pi = (\pi_1, \dots, \pi_n)$ is called a *parking function* of length n. It is well-known and easy to see that if $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ is the (weakly) increasing rearrangement of $\pi_1, \dots, \pi_n$, then π is a parking function if and only if $\lambda_i \leqslant i$ for $1 \leqslant i \leqslant n$. In particular, any permutation of a parking function is a parking function. Write $\mathrm{PF}(n)$ for the set of parking functions of length n.

The first significant result on parking functions, due to Pyke [Pyk59] in another context and then to Konheim and Weiss [KW66], is that the number of parking functions of length n is equal to $(n+1)^{n-1}$. A famous combinatorial proof was given by Pollak (unpublished but recounted in [FR74] and [Rio69]). It boils down to the following easily verified statement: let G denote the group of all n-tuples $(\pi_1, \dots, \pi_n) \in [n+1]^n$ with componentwise addition modulo $n+1$. Let H be the subgroup generated by $(1, 1, \dots, 1)$. Then every coset of H contains exactly one parking function. Some of our proofs will be based on generalizations of Pollak's argument. A central feature of our paper is the first systematic development of refinements of Pollak's proof technique.

Let us review some statistics on parking functions π and what was previously known about their enumeration. For a permutation $\sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ (the symmetric group on $[n]$), we define a *descent* to be an index $1 \leqslant i \leqslant n-1$ for which $\sigma_i > \sigma_{i+1}$ and a *right-to-left maximum* to be an index $1 \leqslant i \leqslant n$ for which $\sigma_i > \sigma_j$ for all $j > i$.

- Parking outcome: the permutation $\mathrm{oc}(\pi) = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ such that σ_i is the spot occupied by car C_i. Thus if $\mathrm{oc}(\pi)^{-1} = \tau_1\tau_2\cdots\tau_n$, then C_{τ_i} is the car in spot i. For example, if $\pi = (2, 2, 1, 3)$ then $\mathrm{oc}(\pi) = 2314$ and $\mathrm{oc}(\pi)^{-1} = 3124$.

- Displacement: total number of failed attempts before all cars find their parking spaces, denoted $\mathrm{dis}(\pi)$. Note that if $\pi = (\pi_1, \dots, \pi_n)$ and $\mathrm{oc}(\pi) = \sigma_1 \cdots \sigma_n$, then $\mathrm{dis}(\pi) = \sum(\sigma_i - \pi_i) = \binom{n+1}{2} - \sum \pi_i$. Thus the displacement statistic is equivalent to the "sum of elements" statistic $\sum \pi_i$. It is known [Yan15, Thm. 1.4] that if

$$P_n(q) = \sum_{\pi \in \mathrm{PF}(n)} q^{\mathrm{dis}(\pi)},$$

  then

$$\begin{aligned} P_1(q) &= 1, \\ P_{n+1}(q) &= \sum_{i=0}^{n} \binom{n}{i} (q^i + q^{i-1} + \cdots + 1) P_i(q) P_{n-i}(q). \end{aligned}$$

  From this one can derive the generating function [Yan15, Thm. 1.6]:

$$\sum_{n \geqslant 1} P_{n-1}(q)(q-1)^{n-1} \frac{x^n}{n!} = \log \sum_{n \geqslant 0} q^{\binom{n}{2}} \frac{x^n}{n!}.$$

- Unluckiness: total number of cars that fail to park at their desired spot, denoted $\mathrm{unl}(\pi)$. The complementary statistic "luckiness" (denoted $\mathrm{lucky}(\pi)$) for parking functions was studied by Gessel and Seo, who gave an explicit formula for the corresponding enumerator [GS06, Thm. 10.1]:

$$\sum_{\pi\in\mathrm{PF}(n)} q^{\mathrm{lucky}(\pi)} = q\prod_{i=1}^{n-1}(i+(n-i+1)q).$$

- Descents: total number of descents in the inverse outcome $\mathrm{oc}(\pi)^{-1}$, denoted $\mathrm{idoc}(\pi)$. For example, $\pi=(2,2,1,3)$ yields $\mathrm{oc}(\pi)^{-1}=3124$, so $\mathrm{idoc}(\pi)=1$.
- Repeats: total number of cars whose desired spot is the same as that of the previous car, denoted $\mathrm{rep}(\pi)$. Using a bijection between parking functions and Prüfer codes, and recognizing that a zero in the Prüfer code indicates a pair of consecutive like numbers in the parking function, an explicit formula for the enumerator of parking functions by repeats was established [Yan15, Cor. 1.3]:

$$\sum_{\pi\in\mathrm{PF}(n)} q^{\mathrm{rep}(\pi)} = (q+n)^{n-1}.$$

- Leading elements: total number of cars whose desired spot is the same as that of the first car, denoted $\mathrm{lel}(\pi)$. This statistic has not been considered before.
- 1's: total number of cars whose desired spot is spot 1, denoted $\mathrm{ones}(\pi)$.
- Right-to-left maxima: total number of right-to-left maxima in the inverse outcome $\mathrm{oc}(\pi)^{-1}$, denoted $\mathrm{rlm}(\pi)$. For example, $\pi=(2,2,1,3)$ yields $\mathrm{oc}(\pi)^{-1}=3124$, so $\mathrm{rlm}(\pi)=1$.
- Inversions: a pair of vertices (i,j) of a labelled rooted tree T with root 0 constitutes an inversion if $i<j$ and j lies on the unique path connecting the root vertex to i. Let $\mathcal{T}(n)$ denote the set of all rooted trees on the vertex set $[n]_0$ with root 0. We write $\mathrm{inv}(T)$ for the number of inversions of $T\in\mathcal{T}(n)$. The distribution of inv on trees $T\in\mathcal{T}(n)$ coincides with the distribution of dis on parking functions π of length n [Yan15, §1.2.2].
- Leaders (proper vertices): a leader is a vertex of a tree $T\in\mathcal{T}(n)$ which is the smallest among all the vertices of the subtree rooted at this vertex. The root vertex 0 is not counted, either as a leader or as a non-leader. We write $\mathrm{ldr}(T)$ for the number of leaders of T and $\mathrm{nld}(T)$ for the complementary number of non-leaders of T. This statistic for trees was studied by Gessel and Seo, who gave an explicit formula for the corresponding enumerator [Seo06, Cor. 8] and [GS06, Thm. 6.1]:

$$\sum_{T\in\mathcal{T}(n)} q^{\mathrm{ldr}(T)} = q\prod_{i=1}^{n-1}(i+(n-i+1)q).$$

See also Seo and Shin [SS07] for a simple bijective proof of Gessel and Seo's formula using a variation of Prüfer code. Note that ldr on trees $T\in\mathcal{T}(n)$ is equidistributed with lucky on parking functions $\pi\in\mathrm{PF}(n)$.

- Leaves: a leaf is a vertex of a tree $T \in \mathcal{T}(n)$ with no children. A single root is also considered to be a leaf. We write $\mathrm{lev}(T)$ for the number of leaves of T.
- Root degree: The number of children of the root vertex in a labelled rooted tree T with root 0, denoted $\deg_T(0)$.
- Edge descents: an edge descent of a tree $T \in \mathcal{T}(n)$ is an edge of T whose vertex nearer the root is larger than the other vertex. The number of edge descents of T is denoted $\mathrm{edes}(T)$.

Here is a summary of our notations just discussed.

$\mathrm{oc}(\pi)$:	parking outcome	$\mathrm{dis}(\pi)$:	displacement
$\mathrm{unl}(\pi)$:	unluckiness	$\mathrm{lucky}(\pi)$:	luckiness
$\mathrm{idoc}(\pi)$:	number of descents of $\mathrm{oc}(\pi)^{-1}$	$\mathrm{rep}(\pi)$:	repeats
$\mathrm{lel}(\pi)$:	leading elements	$\mathrm{ones}(\pi)$:	1's
$\mathrm{rlm}(\pi)$:	number of right-to-left maxima of $\mathrm{oc}(\pi)^{-1}$	$\mathrm{inv}(T)$:	inversions
$\mathrm{ldr}(T)$:	leaders	$\mathrm{nld}(T)$:	non-leaders
$\mathrm{lev}(T)$:	leaves	$\deg_T(0)$:	root degree
$\mathrm{edes}(T)$:	edge descents		

2. A recurrence relation for refined trees and parking functions

Define

$$P_n(x,y,z,w) := \sum_{\pi \in \mathrm{PF}(n)} x^{\mathrm{unl}(\pi)} y^{\mathrm{dis}(\pi)} z^{\mathrm{idoc}(\pi)} w^{\mathrm{rlm}(\pi)}$$

and

$$Q_n(x,y,z,w) := \sum_{T \in \mathcal{T}(n)} x^{\mathrm{nld}(T)} y^{\mathrm{inv}(T)} z^{\mathrm{lev}(T)-1} w^{\deg_T(0)}.$$

We will show that $P_n(x,y,z,w) = Q_n(x,y,z,w)$ by showing that they satisfy the same recurrence relation. By convention $P_0(x,y,z,w) = Q_0(x,y,z,w) = 1$.

Theorem 2.1.

(a) $P_n(x,y,z,w)$ *satisfies the following recurrence relation:*

$$\begin{aligned} P_n(x,y,z,w) &= (1+xy+\cdots+xy^{n-1})wP_{n-1}(x,y,z,1) \\ &\quad + \sum_{i=0}^{n-2} \binom{n-1}{i} (1+xy+\cdots+xy^{i}) zwP_i(x,y,z,1)P_{n-i-1}(x,y,z,w). \end{aligned}$$

(b) $Q_n(x,y,z,w)$ *satisfies the following recurrence relation:*

$$\begin{aligned} Q_n(x,y,z,w) &= (1+xy+\cdots+xy^{n-1})wQ_{n-1}(x,y,z,1) \\ &\quad + \sum_{i=0}^{n-2} \binom{n-1}{i} (1+xy+\cdots+xy^{i}) zwQ_i(x,y,z,1)Q_{n-i-1}(x,y,z,w). \end{aligned}$$

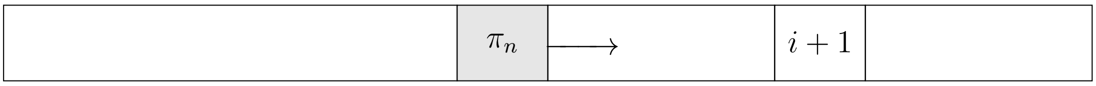

Figure 2.1: Parking function case.

Proof of (a). Cars $1, \dots, n-1$ have all parked along the street before the nth car enters, leaving only one open spot for the nth car to park. We denote this spot by $i+1$, where $i = 0, \dots, n-1$. Since a car cannot jump over an empty spot if its favorite spot is taken, the parking protocol implies that $(\pi_1, \dots, \pi_{n-1})$ may be decomposed into two parking functions $\alpha \in \mathrm{PF}(i)$ and $\beta \in \mathrm{PF}(n-1-i)$, where β is formed by subtracting $i+1$ from the relevant entries in π, and α and β do not interact with each other. The binomial coefficient $\binom{n-1}{i}$ chooses an i-element subset of $[n-1]$ to constitute the index set of α.

See Figure 2.1. This open spot $i+1$ could be either the same as the preference of the last car π_n, in which case the car parks directly without contributing to displacement or unluckiness. That is where the 1 in $1 + xy + \dots + xy^i$ comes from for $i = 0, \dots, n-1$. Or, $i+1$ could be larger than π_n, in which case the car travels forward to park, contributing to the increase of both displacement and unluckiness. That is why we have $xy + xy^2 + \dots + xy^i$. In more detail, we have:

$$\mathrm{dis}(\pi) = \mathrm{dis}(\alpha) + \mathrm{dis}(\beta) + (i + 1 - \pi_n),$$

$$\mathrm{unl}(\pi) = \mathrm{unl}(\alpha) + \mathrm{unl}(\beta) + (\text{either } 1 \text{ or } 0),$$

where the last term in the sum is 1 if $\pi_n < i+1$ and 0 if $\pi_n = i+1$. The z factor comes into play when the last car does not end up parking at the last spot; whether its preference is equal to the parking outcome does not matter. Lastly, for the w factor, we note that P_i corresponds to the total number of rlm cars that are parked to the left of the last car, and P_{n-i-1} corresponds to the total number of rlm cars that are parked to the right of the last car. Now the last car must be an rlm, so after it enters, the total number of rlm cars that are parked to the left of it does not matter anymore, and the rlm cars of π now consists of the last car and the rlm cars that are parked to the right of it. ☐

Proof of (b). Consider a tree T with $n+1$ vertices $0, 1, \dots, n$ rooted at 0. If in the unique chain joining 0 to n we remove the edge from 0, which we denote by $0r$, one obtains (a) a tree T' with $n-i$ vertices of which one is 0, and (b) a tree T'' with $i+1$ vertices of which one is r, where $0 \leqslant i \leqslant n-1$. See Figure 2.2. Recall that n is automatically a vertex of T'' by construction. The binomial coefficient $\binom{n-1}{i}$ chooses i vertices out of $[n]_0 \setminus \{0, n\}$ to form T'' and the remaining vertices of T will constitute T'. Changing the vertex labels of the descendants of 0 in T' and of r in T'' to their relative order among all the descendants of the tree will not affect the quadruple statistics (non-leaders, inversions, leaves -1, degree of root vertex) contribution from these vertices. T' is rooted at 0 and so the quadruple statistics contribution from T' is $Q_{n-i-1}(x, y, z, w)$. T'' is rooted at r not 0, but we could further change the label of vertex r to 0 (hence the contribution of $Q_i(x, y, z, 1)$) and then add back the contribution from vertex r.

The additional factor $1 + xy + \dots + xy^i$ indicates the double statistics (non-leaders, inversions) contribution from the relative order of r in tree T''. When r is the smallest vertex in T'', it is a leader and does not contribute to inversions. In all other cases, r is a non-leader and depending on

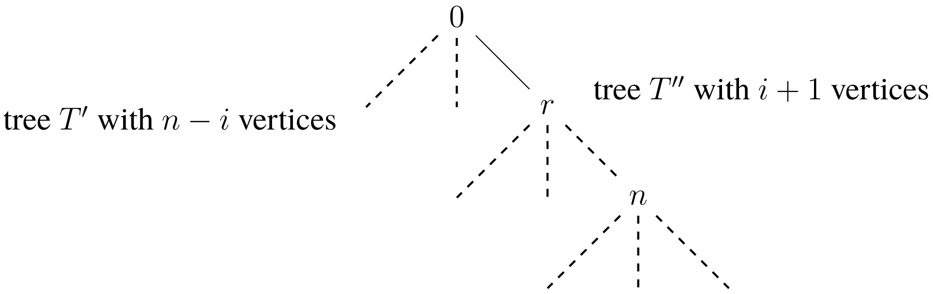

Figure 2.2: Tree case.

how many descendants of r are less than r, the inversion contribution from r would be different, ranging from y to y^i as there are i descendants of r in total. Next we consider the effect of merging the two trees T' and T'' on the single statistic $(\mathrm{leaves}-1)$. If $i=n-1$, we are merely affixing 0 to r as the new root of the tree and the $(\mathrm{leaves}-1)$ statistic stays the same. Otherwise there is an extra factor of z because $\mathrm{lev}(T)-1=\mathrm{lev}(T')+\mathrm{lev}(T'')-1=1+(\mathrm{lev}(T')-1)+(\mathrm{lev}(T'')-1)$. Lastly, for the w factor, we note that Q_{n-i-1} corresponds to the degree of vertex 0 in tree T' and Q_i corresponds to the degree of vertex r in tree T''. For the entire tree T, the degree of vertex 0 is always the degree of vertex 0 in tree T' plus 1, the degree of vertex r in tree T'' no longer matters.

Note that the above "break apart" argument in both the parking function case and the tree case could be repeated, which corresponds to the iterative process described by this recurrence formula. ☐

Under the classical parking protocol, when a car's desired spot is taken, it always drives forward to look for an available parking spot. Let us incorporate a probabilistic scenario in the parking protocol. Fix $p\in[0,1]$ and consider a coin which flips to heads with probability p and tails with probability $1-p$. Our probabilistic parking protocol proceeds as follows: if a car arrives at its preferred spot and finds it unoccupied it parks there. If instead the spot is occupied, then the driver tosses the biased coin. If the coin lands on heads, with probability p, the driver continues moving forward in the street. However, if the coin lands on tails, with probability $1-p$, the car moves backward and tries to find an unoccupied parking spot. When $p=1$, the probabilistic parking model reduces to the classical parking model. See [DHH$^+$23] for more details of this setup. We will see later that there is a deep implication of this probabilistic scenario on the parking protocol; it is not merely an artificial twist that is being added.

An alternative way to interpret this probabilistic setup is that every car arrives at the one-way street with a preferred spot and a preferred movement direction, and the movement directions of different cars are independent. If its preferred spot is not taken, the car directly parks. If on the other hand, the preferred spot is taken, the car follows its preferred movement direction trying to find a spot to park. We denote the forward direction by $\uparrow$ and the backward direction by $\downarrow$. Each car chooses the $\uparrow$ direction with probability p and the $\downarrow$ direction with probability $1-p$. To fully represent the parking preferences of all n cars, we use a pair $(\pi,\delta)\in[n]^n\times\{\uparrow,\downarrow\}^n$ where the first component denotes the preferred spaces and the second component denotes the preferred directions.

Not surprisingly, adding a probabilistic scenario to the parking protocol has some major consequences. In the case of classical parking functions, a preference vector $\pi \in [n]^n$ is either deterministically a parking function or not, and we can talk about the set of parking functions $\mathrm{PF}(n)$ and its cardinality $|\mathrm{PF}(n)|$. Contrarily, under the probabilistic parking protocol, whether the preference vector π constitutes a valid parking function depends on the movement vector δ, which happens with different probabilities. We denote by $\mathbb{P}((\pi,\delta) \in \mathrm{PF}(n))$ the probability that n cars with the preference vector π park and δ is the movement vector. By the law of conditional probability,

$$\mathbb{P}((\pi,\delta) \in \mathrm{PF}(n)) = \mathbb{P}((\pi,\delta) \in \mathrm{PF}(n)|\delta)\mathbb{P}(\delta).$$

Here the second factor on the right is the combined probability of preferred movement directions, that is, an independent product of p's and $(1-p)$'s, whereas the first factor on the right is deterministically either 1 or 0 as whether a car parks or not is deterministic given the preferred movement direction.

For example, consider parking preference $\pi = (2,2,1) \in [3]^3$. In the classical parking model, π is deterministically a parking function. Under the probabilistic scenario, parking or not however is no longer deterministic as we need to take into account the movement directions. Car 1 directly parks at its desired spot 2, so either $\uparrow$ or $\downarrow$ works. Now car 2 enters, its preference spot is taken by car 1, so car 2 would either move forward (indicated by $\uparrow$) with probability p and park at spot 3 or move backward (indicated by $\downarrow$) with probability $1-p$ and park at spot 1. Next car 3 enters. In the first situation, it will directly park at its desired spot 1, so either $\uparrow$ or $\downarrow$ works. While in the second situation, it will move forward (indicated by $\uparrow$) with probability p to park at spot 3. Altogether,

$$\begin{aligned}
\mathbb{P}(((2,2,1),(\uparrow,\uparrow,\uparrow)) \in \mathrm{PF}(n)) &= p^3,\\
\mathbb{P}(((2,2,1),(\uparrow,\uparrow,\downarrow)) \in \mathrm{PF}(n)) &= p^2(1-p),\\
\mathbb{P}(((2,2,1),(\uparrow,\downarrow,\uparrow)) \in \mathrm{PF}(n)) &= p^2(1-p),\\
\mathbb{P}(((2,2,1),(\downarrow,\uparrow,\uparrow)) \in \mathrm{PF}(n)) &= p^2(1-p),\\
\mathbb{P}(((2,2,1),(\downarrow,\uparrow,\downarrow)) \in \mathrm{PF}(n)) &= p(1-p)^2,\\
\mathbb{P}(((2,2,1),(\downarrow,\downarrow,\uparrow)) \in \mathrm{PF}(n)) &= p(1-p)^2.
\end{aligned}$$

As explained above, these are the only movement directions δ that will make it possible for the preference vector $\pi = (2,2,1)$ to park, i.e., $(\pi,\delta) \in \mathrm{PF}(n)$, and the probability of $\mathbb{P}((\pi,\delta) \in \mathrm{PF}(n))$ coincides with the probability that δ is chosen as $\mathbb{P}((\pi,\delta) \in \mathrm{PF}(n)|\delta) = 1$.

Note that the parking statistics unl, dis, idoc, rlm depend on both π and δ and eventually on $\mathrm{oc}(\pi)$. Suppressing the δ dependence in these parking statistics for ease of notation, let

$$\sum_{(\pi,\delta)} x^{\mathrm{unl}(\pi)} y^{\mathrm{dis}(\pi)} z^{\mathrm{idoc}(\pi)} w^{\mathrm{rlm}(\pi)}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}(n)) := P'_n(x,y,z,w),$$

where for the p dependence in $P'_n(x,y,z,w)$ is also suppressed. We point out that the only randomness in the above definition is in the choice of δ: $P'_n(x,y,z,w)$ is not the expected value of a statistic from a randomly selected (π,δ), but instead is the sum over all π of the expected

Figure 2.3: Parking function case under probabilistic scenario.

value of the statistic when δ is randomly selected. We continue with our example to illustrate this compact definition further. For the preference vector $\pi = (2, 2, 1)$, the parking outcome $\mathrm{oc}(\pi)$ is either 231 or 213. The outcome 231 happens when δ is given by $(\uparrow, \uparrow, \uparrow)$, $(\uparrow, \uparrow, \downarrow)$, $(\downarrow, \uparrow, \uparrow)$, or $(\downarrow, \uparrow, \downarrow)$, with combined probability p, while the outcome 213 happens when δ is given by $(\uparrow, \downarrow, \uparrow)$ or $(\downarrow, \downarrow, \uparrow)$, with combined probability $(1-p)p$. For $\mathrm{oc}(\pi) = 231$, we have $\mathrm{unl}(\pi) = 1$, $\mathrm{dis}(\pi) = 1$, $\mathrm{idoc}(\pi) = 1$, and $\mathrm{rlm}(\pi) = 2$. For $\mathrm{oc}(\pi) = 213$, we have $\mathrm{unl}(\pi) = 2$, $\mathrm{dis}(\pi) = 3$, $\mathrm{idoc}(\pi) = 1$, and $\mathrm{rlm}(\pi) = 1$. The contribution of $\pi = (2, 2, 1)$ to $P'_n(x, y, z, w)$ is thus $pxyzw^2 + (1-p)px^2y^3zw$.

Theorem 2.1'. *$P'_n(x, y, z, w)$ satisfies the following recurrence relation:*

$$\begin{aligned}P'_n(x,y,z,w) &= (1 + p(xy + \cdots + xy^{n-1}))wP'_{n-1}(x,y,z,1)\\ &\quad + \sum_{i=0}^{n-2} \binom{n-1}{i} (1 + p(xy + \cdots + xy^i) + (1-p)(xy + \cdots + xy^{n-i-1}))\cdot\\ &\qquad zwP'_i(x,y,z,1)P'_{n-i-1}(x,y,z,w). \qquad (2.1)\end{aligned}$$

Proof. The proof is almost identical to the proof of Theorem 2.1. But note that under the probabilistic scenario, when a car's desired spot is taken, it could either move forward with probability p or backward with probability $1-p$, there are thus three possible scenarios for the preference of the last car π_n, as compared with the open spot, which we denote by $i+1$. Either $\pi_n = i+1$ (giving the contribution of 1), or $\pi_n < i+1$ (giving the contribution of $p(xy + \cdots + xy^i)$), or $\pi_n > i+1$ (giving the contribution of $(1-p)(xy + \cdots + xy^{n-i-1})$). See Figure 2.3. ☐

Set $z = 1$ and $w = 1$ in (2.1). For simplicity we write $P'_n(x, y, 1, 1)$ by $P'_n(x, y)$. Interestingly, the p dependence disappears. Similar phenomenon was observed in Theorem 4 of Tian and Treviño [TTn22] and Theorem 1 in Durmić et al. [DHH+23].

Corollary 2.2. *$P'_n(x, y)$ is independent of p and satisfies the following recurrence relation:*

$$P'_n(x,y) = \sum_{i=0}^{n-1} \binom{n-1}{i} (1 + xy + \cdots + xy^i) P'_i(x,y) P'_{n-i-1}(x,y).$$

Proof. By convention $P'_0(x, y) = 1$. From Theorem 2.1', for $n \geqslant 1$,

$$P'_n(x,y) = \sum_{i=0}^{n-1} \binom{n-1}{i} (1+p(xy+\cdots+xy^i)+(1-p)(xy+\cdots+xy^{n-i-1}))P'_i(x,y)P'_{n-i-1}(x,y).$$

We note that the index set $\{0, \dots, n-1\}$ for i may be partitioned into

$$\{0, n-1\}, \{1, n-2\}, \dots, \left\{\frac{n-3}{2}, \frac{n+1}{2}\right\}, \left\{\frac{n-1}{2}\right\} \text{ if } n \text{ is odd,}$$

and

$$\{0, n-1\}, \{1, n-2\}, \dots, \left\{\frac{n-2}{2}, \frac{n}{2}\right\} \text{ if } n \text{ is even.}$$

We group the summands based on the above partition:

$$\binom{n-1}{i}(1+p(xy+\dots+xy^i)+(1-p)(xy+\dots+xy^{n-i-1}))P_i'(x,y)P_{n-i-1}'(x,y)+$$
$$\binom{n-1}{n-i-1}(1+p(xy+\dots+xy^{n-i-1})+(1-p)(xy+\dots+xy^i))P_{n-i-1}'(x,y)P_i'(x,y)$$
$$=\binom{n-1}{i}\left((1+xy+\dots+xy^i)+(1+xy+\dots+xy^{n-i-1})\right)P_i'(x,y)P_{n-i-1}'(x,y).$$

When n is odd, there is one extra term:

$$\binom{n-1}{\frac{n-1}{2}}(1+p(xy+\dots+xy^{\frac{n-1}{2}})+(1-p)(xy+\dots+xy^{\frac{n-1}{2}}))P_{\frac{n-1}{2}}'(x,y)P_{\frac{n-1}{2}}'(x,y)$$
$$=\binom{n-1}{\frac{n-1}{2}}(1+xy+\dots+xy^{\frac{n-1}{2}})P_{\frac{n-1}{2}}'(x,y)P_{\frac{n-1}{2}}'(x,y).$$

Our proof is complete. □

Remark 2.3. We verified the cancellation of p dependence in Corollary 2.2 via direct computation, but there is a deeper reason behind this disappearance of p which we will later elaborate upon.

Take $m \leqslant n$. We use a bijection [KY23] between parking functions $\mathrm{PF}(m,n)$ with m cars and n spots and spanning forests $\mathcal{F}(n+1, n-m+1)$ with $n+1$ vertices and $n-m+1$ distinct trees having specified roots: for a parking function $\pi \in \mathrm{PF}(m,n)$, there are $n-m$ parking spots that are never attempted by any car. Let $k_i(\pi)$ for $i = 1, \dots, n-m$ represent these spots, so that $0 := k_0 < k_1 < \dots < k_{n-m} < k_{n-m+1} := n+1$. This separates π into $n-m+1$ disjoint noninteracting segments, with each segment a classical parking function of length $k_i - k_{i-1} - 1$ after translation. Each of these classical parking functions corresponds to a rooted tree in the forest $F \in \mathcal{F}(n+1, n-m+1)$. See Figure 2.4 for a parking function π and its associated forest, constructed using an extension of Knuth's bijection between classical parking functions and rooted trees [Knu98, Section 6.4]. For more details of the generalized construction between parking functions and forests, see [KY23, Section 2.4].

We now extend statistics that were introduced in Section 1 for classical parking functions $\mathrm{PF}(n)$ of length n and rooted trees $\mathcal{T}(n)$ on vertex set $[n]_0$ to parking functions $\mathrm{PF}(m,n)$ and rooted forests $\mathcal{F}(n+1, n-m+1)$. These extended statistics are explained in detail below.

For a parking function $\pi \in \mathrm{PF}(m,n)$, as explained earlier, the inverse parking outcome $\mathrm{oc}(\pi)^{-1}$ is in general not a permutation but broken up into disjoint noninteracting segments separated by empty spots. For $\pi = (3,1,9,1,10,7,3,11,10)$ as in Figure 2.4, $\mathrm{oc}(\pi)^{-1} = 2417\square\square6\square3589$, where the number in spot i denotes the car number that parks in spot i and $\square$ denotes an empty spot that is not parked in by any car. Alternatively, π is comprised of disjoint segments, each of which is a classical parking function after translation:

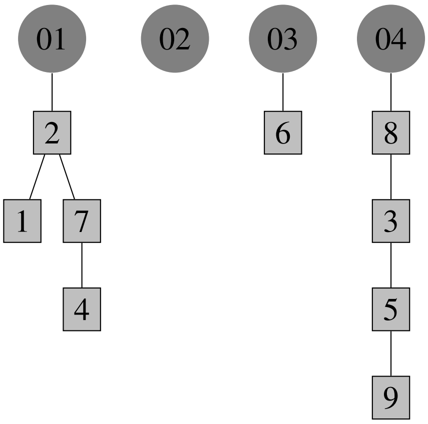

Figure 2.4: Rooted spanning forest $F \in \mathcal{F}(13,4)$ associated with parking function $\pi = (3,1,9,1,10,7,3,11,10) \in \mathrm{PF}(9,12)$.

$\pi_1 = (3,1,1,3)$, $\pi_2 = ()$ (the empty parking function, as there are two consecutive empty spots in $\mathrm{oc}(\pi)^{-1}$), $\pi_3 = (7) \to (1)$, and $\pi_4 = (9,10,11,10) \to (1,2,3,2)$. We then define the statistics $\mathrm{unl}(\pi)$, $\mathrm{dis}(\pi)$, $\mathrm{idoc}(\pi)$, $\mathrm{rlm}(\pi)$ for the parking function π as the sum of the respective statistics in $\pi_1, \pi_2, \pi_3, \pi_4$. We have

$$\mathrm{unl}(\pi) = \mathrm{unl}(\pi_1) + \mathrm{unl}(\pi_2) + \mathrm{unl}(\pi_3) + \mathrm{unl}(\pi_4) = 2+0+0+1 = 3,$$

$$\begin{aligned}\mathrm{dis}(\pi) &= \mathrm{dis}(\pi_1) + \mathrm{dis}(\pi_2) + \mathrm{dis}(\pi_3) + \mathrm{dis}(\pi_4) = 2+0+0+2 = 4,\\ \mathrm{idoc}(\pi) &= \mathrm{idoc}(\pi_1) + \mathrm{idoc}(\pi_2) + \mathrm{idoc}(\pi_3) + \mathrm{idoc}(\pi_4) = 1+0+0+0 = 1,\\ \mathrm{rlm}(\pi) &= \mathrm{rlm}(\pi_1) + \mathrm{rlm}(\pi_2) + \mathrm{rlm}(\pi_3) + \mathrm{rlm}(\pi_4) = 1+0+1+1 = 3.\end{aligned}$$

Note that $\mathrm{unl}(\pi)$ and $\mathrm{dis}(\pi)$ may be directly defined on π following the description given in Section 1. Also note that the concept of descent and right-to-left maximum used in the definition of idoc and rlm is essentially built on the relative order between the indices and not the index value itself. For example, $\mathrm{oc}(\pi_1)^{-1} = 2314$ has the same relative order as 2417 (the first segment of $\mathrm{oc}(\pi)^{-1}$) and $\mathrm{oc}(\pi_4)^{-1} = 1234$ has the same relative order as 3589 (the last segment of $\mathrm{oc}(\pi)^{-1}$), so we may equivalently define $\mathrm{idoc}(\pi)$ and $\mathrm{rlm}(\pi)$ directly on $\mathrm{oc}(\pi)^{-1}$.

Likewise, for a forest $F \in \mathcal{F}(n+1, n-m+1)$, the statistics $\mathrm{nld}(F)$, $\mathrm{inv}(F)$, $\mathrm{lev}(F)$, $\deg_F(0)$ are all defined to be the sum of the respective statistics in its component trees. For example, $\deg_F(0)$ is the total number of children of all the $n-m+1$ root vertices in the forest. Let T_1, T_2, T_3, T_4 respectively be the component trees from left to right in the forest F in Figure 2.4, we have

$$\mathrm{nld}(F) = \mathrm{nld}(T_1) + \mathrm{nld}(T_2) + \mathrm{nld}(T_3) + \mathrm{nld}(T_4) = 2+0+0+1 = 3,$$

$$\begin{aligned}\mathrm{inv}(F) &= \mathrm{inv}(T_1) + \mathrm{inv}(T_2) + \mathrm{inv}(T_3) + \mathrm{inv}(T_4) = 2+0+0+2 = 4,\\ \mathrm{lev}(F) &= \mathrm{lev}(T_1) + \mathrm{lev}(T_2) + \mathrm{lev}(T_3) + \mathrm{lev}(T_4) = 2+1+1+1 = 5,\\ \deg_F(0) &= \deg_{T_1}(0) + \deg_{T_2}(0) + \deg_{T_3}(0) + \deg_{T_4}(0) = 1+0+1+1 = 3.\end{aligned}$$

From Theorem 2.1, we may let

$$\sum_{\pi\in \mathrm{PF}(m,n)} x^{\mathrm{unl}(\pi)}y^{\mathrm{dis}(\pi)}z^{\mathrm{idoc}(\pi)}w^{\mathrm{rlm}(\pi)} = \sum_{F\in\mathcal{F}(n+1,n+1-m)} x^{\mathrm{nld}(F)}y^{\mathrm{inv}(F)}z^{\mathrm{lev}(F)-(n-m+1)}w^{\deg_F(0)}$$
$$:=P_{m,n}(x,y,z,w).$$

Similarly, under the probabilistic scenario, let

$$\sum_{(\pi,\delta)} x^{\mathrm{unl}(\pi)}y^{\mathrm{dis}(\pi)}z^{\mathrm{idoc}(\pi)}w^{\mathrm{rlm}(\pi)}\,\mathbb{P}((\pi,\delta)\in \mathrm{PF}(m,n)) := P'_{m,n}(x,y,z,w),$$

where $\mathbb{P}((\pi,\delta)\in \mathrm{PF}(m,n))$ denotes the probability that m cars with the preference vector π park on a street with n spots and δ is the movement vector.

The following propositions are immediate following the above explanations, where we take $\boldsymbol{s} = (k_1-k_0-1,\dots,k_{n-m+1}-k_{n-m}-1)$.

Proposition 2.4.

$$P_{m,n}(x,y,z,w) = \sum_{\boldsymbol{s}\in C(m)} \binom{m}{\boldsymbol{s}} \prod_{i=1}^{n-m+1} P_{s_i}(x,y,z,w),$$

where $C(m)$ *consists of (weak) compositions of* m *of length* $n-m+1$*:* $\boldsymbol{s}=(s_1,\dots,s_{n-m+1})\models m$ *with* $\sum_{i=1}^{n-m+1} s_i = m$*.*

Proposition 2.4′.

$$P'_{m,n}(x,y,z,w) = \sum_{\boldsymbol{s}\in C(m)} \binom{m}{\boldsymbol{s}} \prod_{i=1}^{n-m+1} P'_{s_i}(x,y,z,w),$$

where $C(m)$ *consists of (weak) compositions of* m *of length* $n-m+1$*:* $\boldsymbol{s}=(s_1,\dots,s_{n-m+1})\models m$ *with* $\sum_{i=1}^{n-m+1} s_i = m$*.*

3. Circular Symmetry: Part I

In this section we derive various results for valid parking preferences (π,δ), using a combination of probabilistic and combinatorial tools. We will follow Pollak's ingenious circle argument [FR74] for the street parking model and assign n cars on a circle with $n+1$ spots. Those car assignments where spot $n+1$ is left empty after circular rotation give valid parking functions. A forward (or backward) movement of a car on a street could be interpreted as a clockwise (or counterclockwise) movement of a car on a circle. We will write our results for the parking situation under the probabilistic scenario, where a car moves forward with probability p and backward with probability $1-p$ when its desired spot is taken. But note that when $p=1$ we recover the classical deterministic parking model and so readers who are less familiar with probability may interpret all results deterministically.

Unlike Pollak's original argument where the parking statistics are studied after all cars have parked, we will investigate the individual parking statistics for each car the moment it is parked on the circle. It turns out that this "seemingly small step ahead" will provide a lot more useful information about a range of parking statistics, such as "unluckiness" and "displacement", that we studied earlier. We note that $\mathrm{unl}(\pi)$ and $\mathrm{dis}(\pi)$ are sums of individual parking statistics $\mathrm{unl}(\pi_i)$ and $\mathrm{dis}(\pi_i)$, and more importantly only the parking preferences of the first i cars matter when studying the corresponding statistics of the ith car that enters. We also note that the mean of the parking statistics are invariant under circular rotation and the direction that the car chooses to move when its preferred spot is occupied. This is the deeper reason behind the disappearance of the probabilistic parameter p in the generating function $P'_n(x, y)$ in Corollary 2.2.

Based on the above crucial observations, we first present some probabilistic results for the parking statistics unluckiness and displacement.

Proposition 3.1. *Consider a street with n spots. We park m cars on the street under the probabilistic parking protocol, where $m \leqslant n$. Take $0 \leqslant i \leqslant m-1$. The probability that the displacement of the $(i+1)$th car is k, where $0 \leqslant k \leqslant i$, is given by*

$$\begin{aligned} P_k &:= \mathbb{P}\left(\textit{displacement of } (i+1)\textit{th car is } k \mid (\pi,\delta) \in \mathrm{PF}(m,n)\right) \\ &= \frac{n-i}{(n+1)^i}\left[p\sum_{j=k+1}^{i+1}\binom{i}{j-1} j^{j-2}(n-j+1)^{i-j}\right. \\ &\qquad\qquad \left. +(1-p)\sum_{j=n-i}^{n-k}\binom{i}{n-j}(n-j+1)^{n-j-1} j^{i-n+j-1}\right] \\ &= \frac{n-i}{(n+1)^i}\sum_{s=k}^{i}\binom{i}{s}(s+1)^{s-1}(n-s)^{i-s-1}. \end{aligned}$$

The mean of the displacement of parking the $(i+1)$th car is $D_i = \sum_{k=0}^{i} kP_k$, and the mean of the displacement of parking a random car is $\frac{1}{m}\sum_{i=0}^{m-1} D_i$.

Proof. As observed earlier, no matter how many cars are to be parked in total, only the parking preferences of the first $i+1$ cars have an effect on the displacement of the $(i+1)$th car that enters. The minimum possible displacement of the $(i+1)$th car is 0 and the maximum possible displacement of the $(i+1)$th car is i, which happens when the first i entering cars occupy an entire block and the desired spot of the $(i+1)$th car is at one end of the block.

We assign i cars on a circle with $n+1$ spots. This leaves open $n-i+1$ spots. One of these would become the street exit in the street parking model, and any of the other $n-i$ available spots may be parked in by the $(i+1)$th car. Adapting circular symmetry to the street parking situation and denoting the available spot that is later parked in by the $(i+1)$th car by j, we only need to consider two extremal cases in the street parking model: (a) spot j is the leftmost open spot after the first i cars have parked and the $(i+1)$th car travels forward to park, and (b) spot j is the rightmost open spot after the first i cars have parked and the $(i+1)$th car travels backward to park. For case (a), there are $j-1$ cars and $j-1$ spots to the left of spot j and $i-j+1$ cars and $n-j$ spots to the right of spot j. The fact that the forward-moving displacement of the $(i+1)$th car is k implies that $j \geqslant k+1$. We must also impose that $j \leqslant i+1$ to ensure that $i-j+1 \geqslant 0$. Similarly, for case (b), there are $n-j$ cars and $n-j$ spots to the right of spot j and $i-n+j$ cars and $j-1$ spots to the left of spot j. The backward-moving displacement of the $(i+1)$th car is k implies that $j \leqslant n-k$. We must also impose that $j \geqslant n-i$ to ensure that $i-n+j \geqslant 0$.

From a slight generalization of Corollary 2.2, the sum of probabilities $\mathbb{P}((\pi,\delta) \in \mathrm{PF}(m,n))$ over all preference pairs $(\pi,\delta) \in [n]^m \times \{\uparrow,\downarrow\}^m$ is independent of the forward probability p and assumes the same formula as in the classical setting:

$$\sum_{(\pi,\delta)} \mathbb{P}((\pi,\delta) \in \mathrm{PF}(m,n)) = (n-m+1)(n+1)^{m-1}.$$

Therefore taking into account all parking preference and movement choices, the sum of probabilities of the first $i+1$ cars successfully parking is $(n-(i+1)+1)(n+1)^{(i+1)-1} = (n-i)(n+1)^i$. Now the sum of probabilities of case (a) happening is

$$p \sum_{j=k+1}^{i+1} \binom{i}{j-1} j^{j-2}(n-i)(n-j+1)^{i-j},$$

where the binomial coefficient $\binom{i}{j-1}$ chooses $j-1$ cars out of i cars to assign to the left of spot j, and the remaining $i-j+1$ cars are assigned to the right of spot j. Similarly, the sum of probabilities of case (b) happening is

$$(1-p) \sum_{j=n-i}^{n-k} \binom{i}{n-j} (n-j+1)^{n-j-1}(n-i)j^{i-n+j-1},$$

where the binomial coefficient $\binom{i}{n-j}$ chooses $n-j$ cars out of i cars to assign to the right of spot j and the remaining $i-n+j$ cars are assigned to the left of spot j. Recalling that there are $n-i$ available spots after the first i cars have parked, we have

$$\begin{aligned}
&\mathbb{P}\left(\text{displacement of } (i+1)\text{th car is } k \mid (\pi,\delta) \in \mathrm{PF}(m,n)\right) \\
&= \frac{(n-i)^2}{(n-i)(n+1)^i}\left[p \sum_{j=k+1}^{i+1} \binom{i}{j-1} j^{j-2}(n-j+1)^{i-j}\right. \\
&\qquad\qquad \left. +(1-p) \sum_{j=n-i}^{n-k} \binom{i}{n-j} (n-j+1)^{n-j-1} j^{i-n+j-1}\right] \\
&= \frac{n-i}{(n+1)^i}\left[p \sum_{s=k}^{i} \binom{i}{s} (s+1)^{s-1}(n-s)^{i-s-1}\right. \\
&\qquad\qquad \left. +(1-p) \sum_{s=k}^{i} \binom{i}{s} (s+1)^{s-1}(n-s)^{i-s-1}\right] \\
&= \frac{n-i}{(n+1)^i} \sum_{s=k}^{i} \binom{i}{s} (s+1)^{s-1}(n-s)^{i-s-1}.
\end{aligned}$$

The rest is immediate. □

From Proposition 3.1, we see that the displacement statistic is rather correlated: the displacement of a car depends on the parking preferences of all the cars that enter before it. The next corollary shows that there is more independence structure with the unluckiness statistic.

Corollary 3.2.

$$\mathbb{P}((i+1)\textit{th car is unlucky} \mid (\pi,\delta) \in \mathrm{PF}(m,n)) = \frac{i}{n+1}.$$

Proof. Note that a car is lucky if and only if its displacement is zero. Using the notation of Proposition 3.1,

$$\mathbb{P}((i+1)\text{th car is lucky}) = P_0 = \frac{n-i}{(n+1)^i}\sum_{s=0}^{i}\binom{i}{s}(s+1)^{s-1}(n-s)^{i-s-1} = 1-\frac{i}{n+1},$$

where the last equality comes from Abel's extension of the binomial theorem. □

Remark 3.3. We will later give a direct proof of Corollary 3.2 in the more general case of (r,k)-parking functions. See Theorem 3.4.

The circular symmetry argument we have been employing is of wide applicability and is not only restricted to classical parking functions. It produces many new results and renders various known results as corollaries. We now present some of them.

Let $\boldsymbol{u} = (u_1,\dots,u_m)$ be a weakly increasing sequence of positive integers. A *$\boldsymbol{u}$-parking function* is a sequence $(\pi_1,\dots,\pi_m)$ of positive integers whose increasing rearrangement $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m$ satisfies $\lambda_i \leqslant u_i$. Fix positive integers r,k and set $\boldsymbol{u} = (k, r+k, 2r+k, \dots, (m-1)r+k)$. A $\boldsymbol{u}$-parking function of this form is referred to in [SW18] as an *(r,k)-parking function* of length m. There is a similar interpretation for such $\boldsymbol{u}$-parking functions in terms of the classical parking scenario: one wishes to park m cars on a street with u_m spots, all spots begin unoccupied and the ith occupied spot after parking has label no larger than u_i for $1 \leqslant i \leqslant m$. Under the probabilistic scenario (p not necessarily equal to 1), following notation in Section 2, denote the probability that δ is the movement vector and the ith spot parked in by the m cars read from left to right is at most u_i for $1 \leqslant i \leqslant m$ under the preference vector π by $\mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k))$. By the law of conditional probability,

$$\mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k)) = \mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k) | \delta)\mathbb{P}(\delta),$$

where the first factor on the right is deterministically either 1 or 0 and the second factor on the right is the combined probability of preferred movement directions, that is, an independent product of p's and $(1-p)$'s. When $p=1$, $\mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k))$ has probability either 0 or 1, and we denote the deterministic set of valid parking functions also by $\mathrm{PF}_m(r,k)$.

For example, take $k=r=m=2$ so $\boldsymbol{u}=(2,4)$. We have

$$\begin{aligned}\mathbb{P}(((3,3),(\uparrow,\downarrow)) \in \mathrm{PF}_2(2,2)) &= \mathbb{P}(((3,3),(\uparrow,\downarrow)) \in \mathrm{PF}_2(2,2)|(\uparrow,\downarrow))\mathbb{P}((\uparrow,\downarrow)) \\ &= 1\cdot p(1-p) = p(1-p),\end{aligned}$$

$$\begin{aligned}\mathbb{P}(((3,3),(\downarrow,\downarrow)) \in \mathrm{PF}_2(2,2)) &= \mathbb{P}(((3,3),(\downarrow,\downarrow)) \in \mathrm{PF}_2(2,2)|(\downarrow,\downarrow))\mathbb{P}((\downarrow,\downarrow)) \\ &= 1\cdot(1-p)^2 = (1-p)^2.\end{aligned}$$

Here car 1 takes the empty spot 3 upon entering the street, so either ↑ or ↓ works. Now car 2 enters and finds its desired spot 3 taken by car 1, prompting it to make a choice. If car 2 drives forward to park at spot 4 (indicated by ↑), which happens with probability p, then the parking spots occupied by the two cars are $(3,4) \not\leqslant (2,4)$. But if car 2 drives backward to park at spot 2 (indicated by ↓), which happens with probability $1-p$, then the parking spots occupied by the two cars are $(2,3) \leqslant (2,4)$.

Theorem 3.4.

$$\sum_{(\pi,\delta)} x^{\mathrm{unl}(\pi)} y^{\mathrm{rep}(\pi)}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k)) = k\prod_{i=1}^{m-1}(xy+(i-1)x+k+mr-i). \tag{3.1}$$

Proof. We think deterministically and take the forward probability $p=1$. But as will be clear in the proof, with minor adaptation our argument works for a generic p. We utilize a generalization of Pollak's original circle argument [FR74] for (r,k)-parking functions introduced in [Sta97]. To generate an (r,k)-parking function of length m uniformly at random, we proceed as follows:

1. Pick an element $\pi \in (\mathbb{Z}/(k+mr)\mathbb{Z})^m$, where the equivalence class representatives are taken in $1,\dots,k+mr$.
2. For $i \in \{0,\dots,k+mr-1\}$, record i if $\pi + i(1,\dots,1)$ (modulo $k+mr$) is an (r,k)-parking function, where $(1,\dots,1)$ is a vector of length m. There should be exactly k such i's.
3. Pick one i from (2) uniformly at random. Then $\pi+i(1,\dots,1)$ is an (r,k)-parking function of length m taken uniformly at random.

The main takeaway from this procedure is that a random (r,k)-parking function could be generated by assigning m cars independently on a circle of length $k+mr$ and then applying circular rotation. Since circular rotation does not affect unluckiness nor repeats, and whether the $(i+1)$th car is unlucky or repeats the behavior of the ith car is independent of the particular preferences of all the previous i cars that have already parked, the generating function factors completely.

The first car is always lucky. For $1 \leqslant i \leqslant m-1$, note that if the $(i+1)$th car prefers any of the i spots that are taken by the previous i cars, then it must be unlucky. In particular, if the $(i+1)$th car repeats the preference of the ith car, then it must be unlucky. We have

$$\mathbb{P}((i+1)\text{th car repeats the preference of } i\text{th car}) = \frac{1}{k+mr},$$

$$\mathbb{P}((i+1)\text{th car is unlucky but does not repeat the preference of } i\text{th car}) = \frac{i-1}{k+mr},$$

$$\mathbb{P}((i+1)\text{th car is lucky}) = \frac{k+mr-i}{k+mr}.$$

Recall that

$$\sum_{(\pi,\delta)} \mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k)) = k(k+mr)^{m-1}.$$

This implies that

$$\begin{aligned}
&\sum_{(\pi,\delta)} x^{\mathrm{unl}(\pi)} y^{\mathrm{rep}(\pi)}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k)) \\
=&k(k+mr)^{m-1} \prod_{i=1}^{m-1} \left(xy\frac{1}{k+mr} + x\frac{i-1}{k+mr} + \frac{k+mr-i}{k+mr} \right) \\
=&k \prod_{i=1}^{m-1} (xy + (i-1)x + k + mr - i).
\end{aligned}$$

□

Corollary 3.5. *Taking $k = n - m + 1$ and $r = y = 1$ in (3.1), we have*

$$\sum_{(\pi,\delta)} x^{\mathrm{unl}(\pi)}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}(m,n)) = (n-m+1) \prod_{i=1}^{m-1} (ix + (n-i+1)).$$

This agrees with the corresponding (deterministic) formula in Gessel and Seo [GS06, Theorem 10.1, Corollary 10.2].

Corollary 3.6. *Taking $k = r = x = 1$ in (3.1), we have*

$$\sum_{(\pi,\delta)} y^{\mathrm{rep}(\pi)}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}(m)) = (y+m)^{m-1}.$$

This agrees with the corresponding (deterministic) formula in Yan [Yan15, Corollary 1.3].

A similar approach as in the proof of Theorem 3.4 also yields the following result.

Theorem 3.7.

$$\sum_{(\pi,\delta)} x^{\mathrm{unl}(\pi)} y^{\mathrm{lel}(\pi)}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k)) = ky \prod_{i=1}^{m-1} (xy + (i-1)x + k + mr - i).$$

Proof. The proof is almost identical to the proof of Theorem 3.4. We note that for $1 \leqslant i \leqslant m-1$, if the desired spot of the $(i+1)$th car coincides with that of the first car, then it must be unlucky. □

Proposition 3.8.

$$\sum_{(\pi,\delta)} x^{\mathrm{unl}(\pi)} y^{\#\{i:\ \pi_i = \pi_2\}}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k)) = ky \prod_{i=1}^{m-1} (xy + (i-1)x + k + mr - i).$$

Proof. The above result will not come as a surprise once we note that an instance of unluckiness occurs in the first two cars exactly when $\pi_1 = \pi_2$. Alternatively, we may switch the preferences of car 1 and car 2, which has no effect on the unluckiness index. $\square$

Remark 3.9. For $s \geqslant 3$, in general

$$\sum_{(\pi,\delta)} x^{\mathrm{unl}(\pi)} y^{\#\{i\colon\, \pi_i=\pi_s\}}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k)) \neq ky \prod_{i=1}^{m-1} (xy + (i-1)x + k + mr - i).$$

The first counterexample occurs when $m = 3$.

Remark 3.10. It is however true that for $s \geqslant 1$,

$$\sum_{(\pi,\delta)} y^{\#\{i\colon\, \pi_i=\pi_s\}}\, \mathbb{P}((\pi,\delta) \in \mathrm{PF}_m(r,k)) = ky \prod_{i=1}^{m-1} (y + (k + mr - 1)).$$

Using standard probability tools, some asymptotic analysis of the above parking statistics readily follows.

Proposition 3.11. *Take m large. Take $r \geqslant 1$ any integer and $k = cm + r$ for some $c \geqslant 0$. Consider the parking preference π chosen uniformly at random. Let $U(\pi)$ be the number of unlucky cars and $R(\pi)$ be the number of repeats in π reading from left to right. Then for fixed $j = 0, 1, \ldots$,*

$$\mathbb{P}\left(R(\pi) = j \mid (\pi,\delta) \in \mathrm{PF}_m(r,k)\right) \sim \frac{\left(\frac{1}{c+r}\right)^j e^{-\frac{1}{c+r}}}{j!},$$

and for fixed x, $-\infty < x < \infty$,

$$\mathbb{P}\left(\frac{U(\pi) - \frac{1}{2(c+r)}m}{\sqrt{\frac{3(c+r)-2}{6(c+r)^2}m}} \leqslant x \mid (\pi,\delta) \in \mathrm{PF}_m(r,k)\right) \sim \Phi(x),$$

where $\Phi(x)$ is the standard normal distribution function.

Proof. From the proof of Theorem 3.4, the probability generating function of $\boldsymbol{X} := (U(\pi), R(\pi))$ may be decomposed into $\boldsymbol{X} = \sum_{i=1}^{m-1} \boldsymbol{X}_i$, with $\boldsymbol{X}_i = (U_i(\pi), R_i(\pi))$ independent, U_i and R_i both Bernoulli, and

$$\mathbb{E}(U_i) = \frac{i}{k+mr}, \qquad \mathbb{E}(R_i) = \frac{1}{k+mr},$$

$$\mathbb{V}\mathrm{ar}(U_i) = \frac{i}{k+mr}\left(1 - \frac{i}{k+mr}\right), \qquad \mathbb{V}\mathrm{ar}(R_i) = \frac{1}{k+mr}(1 - \frac{1}{k+mr}).$$

We compute

$$\sum_{i=1}^{m-1} \mathbb{E}(U_i) \sim \frac{1}{2(c+r)}m, \qquad \sum_{i=1}^{m-1} \mathbb{E}(R_i) \sim \frac{1}{c+r},$$

$$\sum_{i=1}^{m-1} \mathbb{V}\mathrm{ar}(U_i) \sim \frac{3(c+r)-2}{6(c+r)^2}m, \qquad \sum_{i=1}^{m-1} \mathbb{V}\mathrm{ar}(R_i) \sim \frac{1}{c+r}.$$

We recognize that $R(\pi)$ may be approximated by Poisson$(\frac{1}{c+r})$ while $U(\pi)$ may be approximated by normal$(0,1)$ after standardization. ☐

Corollary 3.12. *Taking $c = 0$ and $r = 1$ (and thus $k = 1$) in Proposition 3.11, we have*

$$\mathbb{P}\left(R(\pi) = j \mid (\pi,\delta) \in \mathrm{PF}(m)\right) \sim \frac{1}{ej!},$$

$$\mathbb{P}\left(\frac{U(\pi) - \frac{m}{2}}{\sqrt{\frac{m}{6}}} \leqslant x \mid (\pi,\delta) \in \mathrm{PF}(m)\right) \sim \Phi(x).$$

This agrees with the corresponding formula in Diaconis and Hicks [DH17, Theorem 4, Theorem 6].

Proposition 3.13. *Take m large. Take $1 \leqslant s \leqslant m$ any integer. Take $r \geqslant 1$ any integer and $k = cm + r$ for some $c \geqslant 0$. Consider the parking preference π chosen uniformly at random. Let $L_s(\pi)$ be the number of cars with the same preference as car s. Then for fixed $j = 0, 1, \dots,$*

$$\mathbb{P}\left(L_s(\pi) = 1 + j \mid (\pi,\delta) \in \mathrm{PF}_m(r,k)\right) \sim \frac{\left(\frac{1}{c+r}\right)^j e^{-\frac{1}{c+r}}}{j!}.$$

Proof. This follows from Theorem 3.7 (see also Remark 3.10). In the same way as the proof of Theorem 3.7 is almost identical to the proof of Theorem 3.4, the proof of Proposition 3.13 is also almost identical to the proof of Proposition 3.11. ☐

4. Circular Symmetry: Part II

We have been focusing on the cars' perspective of parking functions until now. In this section we also take into consideration the spots' perspective. In the classical parking situation where there are n cars parking on a street with n spots, the unluckiness statistic could be interpreted in two ways. Either it is the total number of cars that fail to park at their desired spot, or it is the total number of spots occupied by a car for which that spot is not the car's first preference. In particular, spot 1 is always lucky. We note that this alternative interpretation for "unluckiness" fails for more general parking situations.

There is also a difference between cars and spots. Cars are actively moving to find available spots; when a car's first preference is taken, the car moves forward or backward and an extra probabilistic scenario may be added to the parking model indicating the direction the car moves. Contrarily, spots are passively waiting for cars to park in them and a probabilistic scenario associated with the movement direction is irrelevant. All statistics involving spots are thus only valid for the deterministic parking model, where the forward probability $p = 1$ (or equivalently $p = 0$). As a result, instead of investigating the probabilistic questions involving $(\pi,\delta) \in \mathrm{PF}(n)$, we study the deterministic scenario when $\pi \in \mathrm{PF}(n)$.

We note some curious features of the pair of statistics (leading elements, 1's). While the leading elements statistic is invariant under circular rotation, it does not satisfy permutation symmetry as permuting the entries might change the first element. On the other hand, though the 1's statistic is invariant under permuting all the entries, it does not exhibit circular rotation invariance. Indeed, only 1 out of $n+1$ rotations of an assignment of n cars on a circle with $n+1$ spots gives a valid parking function.

We establish a recurrence relation for the pair of statistics (unluckiness, leading elements) and (unluckiness, 1's). Write

$$P_n(x,y)=\sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)}y^{\mathrm{lel}(\pi)}$$

and

$$Q_n(x,y)=\sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)}y^{\mathrm{ones}(\pi)}.$$

Note that $P_n(x,y)$ is a special (deterministic) case of Theorem 3.7 with $k=r=1$. As in the proof of Theorem 2.1, we focus on the last car. The parking protocol implies that $(\pi_1,\dots,\pi_{n-1})$ may be decomposed into two parking functions $\alpha\in\mathrm{PF}(i)$ and $\beta\in\mathrm{PF}(n-1-i)$, where $0\leqslant i\leqslant n-1$, β is formed by subtracting $i+1$ from the relevant entries in π, and α and β do not interact with each other. This open spot $i+1$ could be either the same as the preference of the last car π_n, in which case the car parks directly; or $i+1$ could be bigger than π_n, in which case the car travels forward to park. Also, depending on whether $\pi_1<i+1$ or $\pi_1>i+1$ (note that it is impossible for $\pi_1=i+1$), the count on $\mathrm{lel}(\pi)$ would be different, as j has the possibility of equalling π_1 in the former case but no possibility of equalling π_1 in the latter case.

$$\begin{aligned}P_n(x,y)=\sum_{i=0}^{n-1}\Bigg(\binom{n-2}{i-1}&(xy+(i-1)x+1)P_i(x,y)P_{n-i-1}(x,1)\\&+\binom{n-2}{i}(ix+1)P_i(x,1)P_{n-i-1}(x,y)\Bigg).\end{aligned}$$

$$Q_n(x,y)=\sum_{i=1}^{n-1}\binom{n-1}{i}(xy+(i-1)x+1)Q_i(x,y)Q_{n-i-1}(x,1)+yQ_{n-1}(x,1).$$

The recurrence formulas for $P_n(x,y)$ and $Q_n(x,y)$ look very different, yet we will show that $P_n(x,y)=Q_n(x,y)$ in Theorem 4.2. We first present a direct combinatorial argument for counting the number of parking functions $\pi\in\mathrm{PF}(n)$ with $\pi_1=1$. This will shed light on the structure of parking functions and will be useful in the proof of Theorem 4.2.

Lemma 4.1. *We have*

$$\#\{\pi\in\mathrm{PF}(n):\pi_1=1\}=2(n+1)^{n-2}.$$

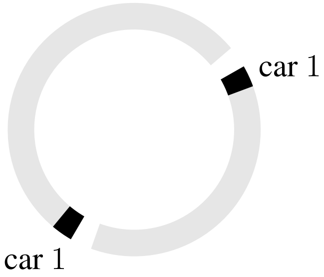

Figure 4.1: Possible positions of car 1 on the circle.

Proof. We assign cars $2, \ldots, n$ independently on a circle of length $n+1$, of which there are $(n+1)^{n-1}$ possibilities. This leaves two empty spots on the circle. Now in order for car 1's preference to be recorded as 1 after circular rotation, car 1 has to choose the adjacent spot (clockwise) to either of the empty spots, so there are only two possibilities. See Figure 4.1. Note that a valid parking function is produced when spot $n+1$ is left unoccupied after circular rotation, and one $n+1$ scalar factor goes away. ☐

Theorem 4.2.

$$\sum_{\pi \in \mathrm{PF}(n)} x^{\mathrm{lel}(\pi)} y^{\mathrm{ones}(\pi)} z^{\mathrm{unl}(\pi)}$$
$$= xy \left((n-1) \prod_{i=1}^{n-2} (xz + yz + (i-1)z + n - i) + (xyz+1) \prod_{i=1}^{n-2} (xyz + iz + n - i) \right). \tag{4.1}$$

Proof. We classify into two situations: $\pi_1 \neq 1$ and $\pi_1 = 1$.

Case A: $\pi_1 \neq 1$. As usual, we assign n cars independently on a circle of length $n+1$ and then apply circular rotation. Recall that for classical parking functions, spot 1 is always taken by some car. Since $\pi_1 \neq 1$, there are two special spots on the circle, one corresponding to π_1 and the other corresponding to 1 (both pre-rotation). These two special spots must correspond to lucky cars. We could think that the first car that picks spot 1 enters the street right after the leading car that prefers spot π_1 and ahead of all other cars, and this will have no effect on either the unluckiness, or the leading elements statistic, or the 1's statistic. When a new car comes in and picks the special spot which will rotate to spot π_1, it must be unlucky. This situation adds 1 to the leading elements statistic. Similarly, when the new car picks the special spot which will rotate to spot 1 upon entering, it must also be unlucky. This situation adds 1 to the 1's statistic. Lastly, if the new car picks any of the already taken spots, it must be unlucky as well. Suppose i spots (including the two special spots corresponding to π_1 and 1) have been taken, where $2 \leqslant i \leqslant n-1$, then with the entrance of the new car, we have

$$\mathbb{P}(\text{new car picks spot } 1) = \frac{1}{n+1},$$

$$\mathbb{P}(\text{new car picks spot } \pi_1) = \frac{1}{n+1},$$

$$\mathbb{P}(\text{new car picks an unavailable spot that is neither spot 1 nor spot } \pi_1) = \frac{i-2}{n+1},$$

$$\mathbb{P}(\text{new car picks an available spot}) = \frac{n-i+1}{n+1}.$$

From Lemma 4.1, the total number of parking functions $\pi \in \mathrm{PF}(n)$ where $\pi_1 \neq 1$ is given by

$$(n+1)^{n-1} - 2(n+1)^{n-2} = (n+1)^{n-2}(n-1).$$

This implies that

$$\begin{aligned}
&\sum_{\pi\in\mathrm{PF}(n)\colon\, \pi_1\neq 1} x^{\mathrm{lel}(\pi)} y^{\mathrm{ones}(\pi)} z^{\mathrm{unl}(\pi)} \\
=&(n+1)^{n-2}(n-1)xy\prod_{i=2}^{n-1}\left(xz\frac{1}{n+1} + yz\frac{1}{n+1} + z\frac{i-2}{n+1} + \frac{n-i+1}{n+1}\right) \\
=&xy(n-1)\prod_{i=1}^{n-2}(xz+yz+(i-1)z+n-i).
\end{aligned}$$

Case B: $\pi_1 = 1$. Since $\pi_1 = 1$, there is only one special spot on the circle, corresponding to 1 pre-rotation. This special spot must correspond to a lucky car. But from the proof of Lemma 4.1, we see that there is some other car that is also special. It either parks in the adjacent-to-empty spot that is picked by car 1 (thus repeating the preference of car 1) or it parks in the other adjacent-to-empty spot (thus lucky). The first situation contributes xyz to the generating function, while the second situation contributes 1 to the generating function. Let us now focus on the other $n-2$ cars. Suppose i spots (including the special spot corresponding to 1 and the other special car) have been taken, where $2 \leqslant i \leqslant n-1$, then with the entrance of the new car, we have

$$\mathbb{P}(\text{new car picks spot } 1) = \frac{1}{n+1},$$

$$\mathbb{P}(\text{new car picks an unavailable spot that is not spot } 1) = \frac{i-1}{n+1},$$

$$\mathbb{P}(\text{new car picks an available spot}) = \frac{n-i+1}{n+1}.$$

Again from the proof of Lemma 4.1, the total number of parking preferences for these non-special $n-2$ cars is given by $(n+1)^{n-2}$. This implies that

$$\begin{aligned}&\sum_{\pi\in\mathrm{PF}(n)\colon\,\pi_1=1} x^{\mathrm{lel}(\pi)}y^{\mathrm{ones}(\pi)}z^{\mathrm{unl}(\pi)}\\ =&(n+1)^{n-2}xy(xyz+1)\prod_{i=2}^{n-1}\left(xyz\frac{1}{n+1}+z\frac{i-1}{n+1}+\frac{n-i+1}{n+1}\right)\\ =&xy(xyz+1)\prod_{i=1}^{n-2}(xyz+iz+n-i).\end{aligned}$$ ☐

Corollary 4.3. *Taking $x=1$ and $z=1$ in (4.1), we have*

$$\sum_{\pi\in\mathrm{PF}(n)} y^{\mathrm{ones}(\pi)} = y(y+n)^{n-1}. \tag{4.2}$$

This agrees with the corresponding formula in Yan [Yan15, Corollary 1.16]. The coefficients count forests of rooted trees with a specified number of components (trees). We note that (4.2) is also the characteristic polynomial (after replacing y by $-y$ and multiplying by $(-1)^n$) of both the Shi arrangement and the Ish arrangement. See Stanley [Sta07, Theorem 5.16] and Armstrong and Rhoades [AR12, Theorem 3.2].

Remark 4.4. Using standard probability tools, some asymptotic analysis for the 1's parking statistic was done in Diaconis and Hicks [DH17, Theorem 7].

Setting $z=1$ in our "master formula" (4.1), we may use the multinomial theorem to get a simple explicit formula for the coefficients. We will present a direct combinatorial proof for the formula. The following structural result about multisets will be useful in our proof.

Lemma 4.5. *Suppose for some $k\geqslant 1$, elements $1,2,\dots,k$ all appear in a parking function $\pi\in\mathrm{PF}(n)$. Fix the elements in π that are equal to $1,2,\dots,k$ (denote the total number of such elements by s) and rotate the remaining elements of π in the cycle $(k{+}1,k{+}2,\dots,n,n{+}1)$. If $s<n$, then exactly $s-k+1$ of these are parking functions.*

Proof. We create an extra parking spot $n+1$ and let the cars go around in a circle if necessary. By a slight generalization of Pollak's original argument [FR74], we note that when there are l cars parking on a circle with m spots, exactly $m-l$ of the rotations of a fixed parking preference will leave spot $n+1$ unoccupied and are thus valid parking functions. Now spots $1,2,\dots,k$ are all taken and the parking situation is equivalent to parking $n-s$ cars on a circle with $n+1-k$ spots, so $(n+1-k)-(n-s)=s-k+1$ rotations will give valid parking functions. ☐

Proposition 4.6. *We have for $s \neq t$,*

$$\#\{\pi \in \mathrm{PF}(n)\colon\ \mathrm{lel}(\pi) = s+1 \textit{ and } \mathrm{ones}(\pi) = t+1\} = \binom{n-2}{s, t, n-s-t-2}(n-1)^{n-s-t-1},$$

and for $s = t$,

$$\begin{aligned}&\#\{\pi \in \mathrm{PF}(n)\colon\ \mathrm{lel}(\pi) = s+1 \textit{ and } \mathrm{ones}(\pi) = s+1\}\\ &\qquad = \binom{n-2}{s, s, n-2s-2}(n-1)^{n-2s-1} + \binom{n-2}{s}n^{n-s-2} + \binom{n-2}{s-1}n^{n-s-1}.\end{aligned}$$

Proof. We use a variant of Pollak's circle argument [FR74]. We classify into two situations: $\pi_1 \neq 1$ and $\pi_1 = 1$.

Case A: $\pi_1 \neq 1$. Car 1's parking preference is already fixed to be π_1, but we have to choose the parking preferences for the other $n-1$ cars. This contributes a multinomial coefficient to the list of preferences π. Note that parking functions are invariant under the action of the symmetric group $\mathfrak{S}_n$ permuting the n cars. We create an extra parking spot $n+1$ and let the cars go around in a circle if necessary. Since cars with the preference 1 will never back up to spot $n+1$, we could think that all the cars that prefer spot 1 enter the street right after the leading car that prefers spot π_1 and ahead of all other cars. We now focus attention on these trailing cars, and note that s of these cars prefer spot π_1 (which has $n-1$ possible values since $\pi_1 \neq 1$), and each of the remaining $n-s-t-2$ cars could choose any spot from $\{1, \dots, n+1\} \setminus \{1, \pi_1\}$. The parking functions among these are exactly those for which spot $n+1$ remains unoccupied. Since there are $n-t-2$ cars in total and $n-1$ spots, this is a fraction $(t+1)/(n-1)$ of these functions, as explained in Lemma 4.5. Thus we obtain

$$\frac{t+1}{n-1}(n-1)(n-1)^{n-s-t-2} = (t+1)(n-1)^{n-s-t-2}$$

valid preferences from these later cars. Putting all this together,

$$\begin{aligned}&\#\{\pi \in \mathrm{PF}(n)\colon \pi_1 \neq 1,\ \mathrm{lel}(\pi) = s+1 \text{ and } \mathrm{ones}(\pi) = t+1\}\\ &=\binom{n-1}{s, t+1, n-s-t-2}(t+1)(n-1)^{n-s-t-2} = \binom{n-2}{s, t, n-s-t-2}(n-1)^{n-s-t-1}.\end{aligned}$$

Case B: $\pi_1 = 1$. The argument works similarly but is easier. Car 1's parking preference is fixed to be 1 so $s = t$. We choose the parking preferences for the other $n-1$ cars. This contributes a multinomial (actually, binomial in this case) coefficient to the list of preferences π. Again we could think that all the cars that prefer spot 1 enter the street before all the other cars. Focusing attention on these trailing cars, we note that each of the $n-s-1$ cars could choose any spot from $\{2, \dots, n+1\}$. The parking functions among these are exactly those for which spot $n+1$ remains unoccupied. Since there are $n-s-1$ cars in total and n spots, this is a fraction $(s+1)/n$ of these functions, as explained in Lemma 4.5. Thus we obtain

$$\frac{s+1}{n}n^{n-s-1} = (s+1)n^{n-s-2}$$

valid preferences from these later cars. Putting all this together,

$$\begin{aligned}&\#\{\pi \in \mathrm{PF}(n)\colon \pi_1 = 1,\ \mathrm{lel}(\pi) = s+1 \text{ and } \mathrm{ones}(\pi) = s+1\}\\ =&\binom{n-1}{s}(s+1)n^{n-s-2} = \binom{n-2}{s}n^{n-s-2} + \binom{n-2}{s-1}n^{n-s-1}.\end{aligned}$$

Finally, we note that the $s \neq t$ term could only come from Case A, whereas the $s = t$ term could come from either Case A or Case B. □

From Theorems 3.4 and 4.2, we recognize that the number of classical parking functions of length n with k repeats coincides with the number of parking functions with $k+1$ elements equal to the leading element and also coincides with the number of parking functions where there are $k+1$ 1's appearing in the parking preference sequence. See Table A for the case where $n = 3$. Here the first two columns are connected by the correspondence between labelled trees and their Prüfer code. The number of 0's in the Prüfer code plus 1 is the same as the number of children of the root vertex 0. The bijection between the first and third columns is due to Pollak [Rio69]. The number of 0's in the Prüfer code is the same as the number of repeats in the parking function. The fourth column arises from the Prüfer code via circular symmetry. We add a 0 in front of the Prüfer code and interpret 0 as spot $n+1$ on the circle. Then there is exactly one rotation of the circle that gives a valid parking function. The number of 0's in the Prüfer code plus 1 is the same as the number of times the leading element appears in the parking function. The fifth column, which goes back to Foata and Riordan [FR74], is generated from the labelled tree using a breadth first search as described in Yan [Yan15, Section 1.2.3]. The number of children of the root vertex 0 is the same as the number of 1's in the parking preference sequence.

We end this section with some observations. Our first observation concerns corresponding statistics between parking functions and trees. We note that even though we have the generating function equality for individual parking statistics,

$$\sum_{\pi\in\mathrm{PF}(n)} y^{\mathrm{lel}(\pi)} = \sum_{\pi\in\mathrm{PF}(n)} y^{\mathrm{ones}(\pi)} = \sum_{T\in\mathcal{T}(n)} y^{\deg_T(0)} = y(y+n)^{n-1}$$

and

$$\sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)} = \sum_{T\in\mathcal{T}(n)} x^{\mathrm{nld}(T)} = \prod_{i=1}^{n-1}(ix + (n-i+1)),$$

the equality nevertheless fails for pairs:

$$\sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)}y^{\mathrm{lel}(\pi)} = \sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)}y^{\mathrm{ones}(\pi)} \neq \sum_{T\in\mathcal{T}(n)} x^{\mathrm{nld}(T)}y^{\deg_T(0)}.$$

Indeed, from Theorem 4.2, for parking functions

$$\sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)}y^{\mathrm{lel}(\pi)} = \sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)}y^{\mathrm{ones}(\pi)} = y\prod_{i=1}^{n-1}(xy + (i-1)x + (n-i+1)),$$

while for trees, the counterpart formula was derived in Gessel and Seo [GS06, Theorem 6.1], which gives

$$\sum_{T\in\mathcal{T}(n)} x^{\mathrm{nld}(T)} y^{\deg_T(0)} = y\prod_{i=1}^{n-1}(ix+(n-i+y)).$$

The first counterexample is located when $n = 2$. (See Hou [Hou16] for some similar but contrasting generating functions for statistics in trees.)

Similarly, we also note that even though

$$\sum_{\pi\in\mathrm{PF}(n)} y^{\mathrm{dis}(\pi)} = \sum_{T\in\mathcal{T}(n)} y^{\mathrm{inv}(T)} \quad \text{and} \quad \sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)} = \sum_{T\in\mathcal{T}(n)} x^{\mathrm{edes}(T)},$$

we have

$$\sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)} y^{\mathrm{dis}(\pi)} \neq \sum_{T\in\mathcal{T}(n)} x^{\mathrm{edes}(T)} y^{\mathrm{inv}(T)}.$$

The first counterexample is located when $n = 4$.

Our second observation examines statistics of parking functions alone. Borrowing notation $\mathrm{lel}(\pi)$ from "leading elements", we write $\mathrm{nlel}(\pi)$ for the total number of cars whose desired spot is the same as that of the second car. From Theorem 3.7, Proposition 3.8 and the previous observation, we have

$$\sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)} y^{\mathrm{lel}(\pi)} = \sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)} y^{\mathrm{nlel}(\pi)} = \sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{unl}(\pi)} y^{\mathrm{ones}(\pi)}.$$

Comparing with Theorem 4.2, we will show that the generating functions for the triple

$$(\mathrm{nlel}(\pi), \mathrm{ones}(\pi), \mathrm{unl}(\pi)) \quad \text{and} \quad (\mathrm{lel}(\pi), \mathrm{ones}(\pi), \mathrm{unl}(\pi))$$

are the same, but the generating function for the triple $(\mathrm{lel}(\pi), \mathrm{nlel}(\pi), \mathrm{unl}(\pi))$ is different.

Theorem 4.7.

$$\begin{aligned}&\sum_{\pi\in\mathrm{PF}(n)} x^{\mathrm{nlel}(\pi)} y^{\mathrm{ones}(\pi)} z^{\mathrm{unl}(\pi)}\\ &= xy\left((n-1)\prod_{i=1}^{n-2}(xz+yz+(i-1)z+n-i)+(xyz+1)\prod_{i=1}^{n-2}(xyz+iz+n-i)\right).\end{aligned}$$

Proof. As in the proof of Proposition 3.8, we switch the preferences of car 1 and car 2, which has no effect on the unluckiness or the 1's statistic. The result readily follows from Theorem 4.2. ☐

We now present a direct combinatorial argument for counting the number of parking functions $\pi \in \mathrm{PF}(n)$ with $\pi_1 = \pi_2$ (actually, in the more general case $\pi_1 = \cdots = \pi_k$ for some k). This will shed light on the structure of parking functions and will be useful in the proof of Theorem 4.9. Note that the parking statistics in Lemma 4.8 and Theorem 4.9 involve only car preferences and not spots, so the statements could be interpreted probabilistically as in Section 3.

Lemma 4.8. *Let* $k \geqslant 1$.

$$\#\{\pi \in \mathrm{PF}(n) \colon \pi_1 = \pi_2 = \cdots = \pi_k\} = (n+1)^{n-k}.$$

Proof. We apply circular symmetry. Add an additional spot $n+1$ and arrange the spaces in a circle. We select a common spot for the first k cars, and it can be done in $n+1$ ways. Then for the remaining $n-k$ cars, there are $(n+1)^{n-k}$ possible preference sequences. A valid parking function is produced when spot $n+1$ is left unoccupied after circular rotation, and one $n+1$ scalar factor goes away. □

Theorem 4.9.

$$\sum_{\pi \in \mathrm{PF}(n)} x^{\mathrm{lel}(\pi)} y^{\mathrm{nlel}(\pi)} z^{\mathrm{unl}(\pi)}$$
$$= xy \left(n \prod_{i=1}^{n-2} (xz + yz + (i-1)z + n - i) + xyz \prod_{i=1}^{n-2} (xyz + iz + n - i) \right). \tag{4.3}$$

Proof. We proceed as in the proof of Theorem 4.2. We classify into two situations: $\pi_1 \neq \pi_2$ and $\pi_1 = \pi_2$.

Case A: $\pi_1 \neq \pi_2$. As usual, we assign n cars independently on a circle of length $n+1$ and then apply circular rotation. Since $\pi_1 \neq \pi_2$, there are two special spots on the circle, one corresponding to π_1 and the other corresponding to π_2 (both pre-rotation). These two special spots must correspond to lucky cars. When a new car comes in and picks the special spot which will rotate to spot π_1, it must be unlucky. Similarly, when the new car picks the special spot which will rotate to spot π_2 upon entering, it must also be unlucky. Lastly, if the new car picks any of the already taken spots, it must be unlucky as well. Suppose i spots (including the two special spots corresponding to π_1 and π_2) have been taken, where $2 \leqslant i \leqslant n-1$, then with the entrance of the new car, we have

$$\mathbb{P}(\text{new car picks spot } \pi_1) = \frac{1}{n+1},$$

$$\mathbb{P}(\text{new car picks spot } \pi_2) = \frac{1}{n+1},$$

$$\mathbb{P}(\text{new car picks an unavailable spot that is neither spot } \pi_1 \text{ nor spot } \pi_2) = \frac{i-2}{n+1},$$

$$\mathbb{P}(\text{new car picks an available spot}) = \frac{n-i+1}{n+1}.$$

From Lemma 4.8, the total number of parking functions $\pi \in \mathrm{PF}(n)$ where $\pi_1 \neq \pi_2$ is given by

$$(n+1)^{n-1} - (n+1)^{n-2} = n(n+1)^{n-2}.$$

This implies that

$$\begin{aligned}
&\sum_{\pi\in\mathrm{PF}(n)\colon\, \pi_1\neq\pi_2} x^{\mathrm{lel}(\pi)}y^{\mathrm{nlel}(\pi)}z^{\mathrm{unl}(\pi)}\\
=&n(n+1)^{n-2}xy\prod_{i=2}^{n-1}\left(xz\frac{1}{n+1}+yz\frac{1}{n+1}+z\frac{i-2}{n+1}+\frac{n-i+1}{n+1}\right)\\
=&nxy\prod_{i=1}^{n-2}(xz+yz+(i-1)z+n-i).
\end{aligned}$$

Case B: $\pi_1=\pi_2$. Since $\pi_1=\pi_2$, π_2 parks in an adjacent spot (clockwise) to π_1. Thus car 1 is lucky and car 2 is unlucky. Let us turn our attention to the remaining $n-2$ cars. Suppose i spots (including the two special spots corresponding to π_1 and π_1+1) have been taken, where $2\leqslant i\leqslant n-1$, then with the entrance of the new car, we have

$$\mathbb{P}(\text{new car picks spot }\pi_1)=\frac{1}{n+1},$$

$$\mathbb{P}(\text{new car picks an unavailable spot that is not spot }\pi_1)=\frac{i-1}{n+1},$$

$$\mathbb{P}(\text{new car picks an available spot})=\frac{n-i+1}{n+1}.$$

Again from the proof of Lemma 4.8, the total number of parking preferences for these remaining $n-2$ cars is given by $(n+1)^{n-2}$. This implies that

$$\begin{aligned}
&\sum_{\pi\in\mathrm{PF}(n)\colon\, \pi_1=\pi_2} x^{\mathrm{lel}(\pi)}y^{\mathrm{nlel}(\pi)}z^{\mathrm{unl}(\pi)}\\
=&(n+1)^{n-2}xy(xyz)\prod_{i=2}^{n-1}\left(xyz\frac{1}{n+1}+z\frac{i-1}{n+1}+\frac{n-i+1}{n+1}\right)\\
=&xy(xyz)\prod_{i=1}^{n-2}(xyz+iz+n-i).
\end{aligned}$$

□

Setting $z=1$ in our "contrast formula" (4.3), as with Proposition 4.6, a direct combinatorial proof is straightforward.

Proposition 4.10. *We have for $s\neq t$,*

$$\#\{\pi\in\mathrm{PF}(n)\colon\ \mathrm{lel}(\pi)=s+1 \textit{ and } \mathrm{nlel}(\pi)=t+1\}=\binom{n-2}{s,t,n-s-t-2}n(n-1)^{n-s-t-2},$$

and for $s=t$,

$$\begin{aligned}
&\#\{\pi\in\mathrm{PF}(n)\colon\ \mathrm{lel}(\pi)=s+1 \textit{ and } \mathrm{nlel}(\pi)=s+1\}\\
&\qquad=\binom{n-2}{s,s,n-2s-2}n(n-1)^{n-2s-2}+\binom{n-2}{s-1}n^{n-s-1}.
\end{aligned}$$

Proof. We apply Pollak's circle argument [FR74] and classify into two situations: $\pi_1 \neq \pi_2$ and $\pi_1 = \pi_2$. The argument is analogous but easier than the proof of Proposition 4.6.

Case A: $\pi_1 \neq \pi_2$. Create an extra parking spot $n+1$ and let the cars go around in a circle if necessary. We select two spots and assign them respectively to car 1 and car 2, of which there are $(n+1)n$ possibilities. Next we choose the parking preferences for the other $n-2$ cars. Now s of these cars have the same preference as car 1 and t of these cars have the same preference as car 2, and this contributes a multinomial coefficient to the list of preferences π. Lastly, the remaining $n-s-t-2$ cars could choose any of the $n-1$ spots not preferred by car 1 and car 2, and there are $(n-1)^{n-s-t-2}$ possible preference sequences. A valid parking function is produced when spot $n+1$ is left unoccupied after circular rotation, and one $n+1$ scalar factor goes away. Putting all this together,

$$\#\{\pi \in \mathrm{PF}(n) \colon \pi_1 \neq \pi_2,\ \mathrm{lel}(\pi) = s+1 \text{ and } \mathrm{nlel}(\pi) = t+1\} = \binom{n-2}{s, t, n-s-t-2} n(n-1)^{n-s-t-2}.$$

Case B: $\pi_1 = \pi_2$. As in Case A, we assign a common spot on the circle to car 1 and car 2, of which there are $n+1$ possibilities. Next we choose the parking preferences for the other $n-2$ cars. Now $s-1$ of these cars have the same preference as car 1, and this contributes a multinomial (actually, binomial in this case) coefficient to the list of preferences π. Lastly, the remaining $n-s-1$ cars could choose any of the n spots not preferred by car 1, and there are n^{n-s-1} possible preference sequences. A valid parking function is produced when spot $n+1$ is left unoccupied after circular rotation, and one $n+1$ scalar factor goes away. Putting all this together,

$$\#\{\pi \in \mathrm{PF}(n) \colon \pi_1 = \pi_2,\ \mathrm{lel}(\pi) = s+1 \text{ and } \mathrm{nlel}(\pi) = s+1\} = \binom{n-2}{s-1} n^{n-s-1}.$$

Finally, we note that the $s \neq t$ term could only come from Case A, whereas the $s = t$ term could come from either Case A or Case B. $\square$

5. Miscellaneous results

As mentioned earlier, the techniques we have been using (recurrence, circular symmetry, ...) in this paper are not restricted to classical parking functions. We have already seen some applicability of our method to (r,k)-parking functions in Section 3. In this section we will further demonstrate the power of our systematic approach in the study of parking functions through some selected results.

5.1. Prime parking functions

A classical parking function $\pi = (\pi_1, \ldots, \pi_n)$ is said to be *prime* if for all $1 \leqslant j \leqslant n-1$, at least $j+1$ cars want to park in the first j places. (Equivalently, if we remove some term of π equal to 1, then we still have a parking function.) Denote the set of prime parking functions of length n by $\mathrm{PPF}(n)$. As with classical parking functions, we could also study prime parking functions via circular rotation [Sta24, pp. 141-142]. We note however that the circular symmetry argument for prime parking functions arises somewhat differently from the corresponding argument for classical parking functions, and only the deterministic parking protocol works for this prime parking model.

Theorem 5.1. *We have*

$$\sum_{\pi \in \mathrm{PPF}(n)} x^{\mathrm{unl}(\pi)} y^{\mathrm{rep}(\pi)} = \prod_{i=1}^{n-1} (xy + (i-1)x + n - 1 - i). \tag{5.1}$$

Proof. The proof is similar to the proof of Theorem 3.4. We utilize the circular symmetry argument of Kalikow [Sta24] and interpret in terms of probability.

1. Pick an element $\pi \in (\mathbb{Z}/(n-1)\mathbb{Z})^n$, where the equivalence class representatives are taken in $1, \ldots, n-1$.
2. For $i \in \{0, \ldots, n-2\}$, record i if $\pi + i(1, \ldots, 1)$ (modulo $n-1$) is a prime parking function, where $(1, \ldots, 1)$ is a vector of length n. There is exactly one such i, which we denote by i_0. Then $\pi + i_0(1, \ldots, 1)$ is a prime parking function of length n taken uniformly at random.

The main takeaway from this procedure is that a random prime parking function could be generated by assigning n cars independently on a circle of length $n-1$ and then applying circular rotation. As in the proof of Theorem 3.4, the generating function factors completely.

The first car is always lucky. For $1 \leqslant i \leqslant n-1$, note that if the $(i+1)$th car prefers any of the i spots that are taken by the previous i cars, then it must be unlucky. In particular, if the $(i+1)$th car repeats the preference of the ith car, then it must be unlucky. The last car is always unlucky. We have

$$\mathbb{P}((i+1)\text{th car repeats the preference of } i\text{th car}) = \frac{1}{n-1},$$

$$\mathbb{P}((i+1)\text{th car is unlucky but does not repeat the preference of } i\text{th car}) = \frac{i-1}{n-1},$$

$$\mathbb{P}((i+1)\text{th car is lucky}) = \frac{n-1-i}{n-1}.$$

Since the total number of prime parking functions of length n is $(n-1)^{n-1}$, this implies that

$$\begin{aligned} \sum_{\pi \in \mathrm{PPF}(n)} x^{\mathrm{unl}(\pi)} y^{\mathrm{rep}(\pi)} =& (n-1)^{n-1} \prod_{i=1}^{n-1} \left(xy \frac{1}{n-1} + x \frac{i-1}{n-1} + \frac{n-1-i}{n-1} \right) \\ =& \prod_{i=1}^{n-1} (xy + (i-1)x + n - 1 - i). \end{aligned}$$

□

Corollary 5.2. *Taking* $y = 1$ *in (5.1), we have*

$$\sum_{\pi \in \mathrm{PPF}(n)} x^{\mathrm{unl}(\pi)} = \prod_{i=1}^{n-1} (ix + (n-1-i)).$$

This agrees with the corresponding formula in Gessel and Seo [GS06, Corollary 10.3].

Corollary 5.3. *Taking* $x = 1$ *in (5.1), we have*

$$\sum_{\pi \in \mathrm{PPF}(n)} y^{\mathrm{rep}(\pi)} = (y+n-2)^{n-1}.$$

This agrees with the corresponding formula in Kalikow [Kal99, Section 1.7.1].

Theorem 5.4.

$$\sum_{\pi \in \mathrm{PPF}(n)} x^{\mathrm{unl}(\pi)} y^{\mathrm{lel}(\pi)} = y \prod_{i=1}^{n-1} (xy + (i-1)x + n - 1 - i).$$

Proof. The proof is almost identical to the proof of Theorem 5.1. We note that for $1 \leqslant i \leqslant n-1$, if the desired spot of the $(i+1)$th car coincides with that of the first car, then it must be unlucky. ☐

Remark 5.5. As in classical parking functions, prime parking functions are invariant under the action of the symmetric group $\mathfrak{S}_n$ permuting the n cars. Thus for $s \geqslant 1$,

$$\sum_{\pi \in \mathrm{PPF}(n)} y^{\#\{i:\, \pi_i = \pi_s\}} = y(y+n-2)^{n-1}.$$

Some asymptotic analysis of the above parking statistics readily follows.

Proposition 5.6. *Take* n *large and* $1 \leqslant s \leqslant n$ *any integer. Consider parking preference* π *chosen uniformly at random. Let* $U(\pi)$ *be the number of unlucky cars,* $R(\pi)$ *be the number of repeats in* π *reading from left to right, and* $L_s(\pi)$ *be the number of cars with the same preference as car* s*. Then for fixed* $j = 0, 1, \ldots,$

$$\mathbb{P}\left(R(\pi) = j \mid \pi \in \mathrm{PPF}(n)\right) \sim \frac{1}{ej!},$$

$$\mathbb{P}\left(L_s(\pi) = 1 + j \mid \pi \in \mathrm{PPF}(n)\right) \sim \frac{1}{ej!},$$

and for fixed x, $-\infty < x < \infty$,

$$\mathbb{P}\left(\frac{U(\pi) - \frac{n}{2}}{\sqrt{\frac{n}{6}}} \leqslant x \mid \pi \in \mathrm{PPF}(n)\right) \sim \Phi(x),$$

where $\Phi(x)$ *is the standard normal distribution function.*

Proof. We proceed as in the proof of Proposition 3.11. From the proof of Theorem 5.1, the probability generating function of $\boldsymbol{X} := (U(\pi), R(\pi))$ may be decomposed into $\boldsymbol{X} = \sum_{i=1}^{n-1} \boldsymbol{X}_i$, with $\boldsymbol{X}_i = (U_i(\pi), R_i(\pi))$ independent, U_i and R_i both Bernoulli, and

$$\mathbb{E}(U_i) = \frac{i}{n-1}, \qquad \mathbb{E}(R_i) = \frac{1}{n-1},$$
$$\mathbb{V}\mathrm{ar}(U_i) = \frac{i}{n-1}\left(1 - \frac{i}{n-1}\right), \qquad \mathbb{V}\mathrm{ar}(R_i) = \frac{1}{n-1}\left(1 - \frac{1}{n-1}\right).$$

We compute

$$\sum_{i=1}^{n-1} \mathbb{E}(U_i) \sim \frac{n}{2}, \qquad \sum_{i=1}^{n-1} \mathbb{E}(R_i) \sim 1, \qquad \sum_{i=1}^{n-1} \mathbb{V}\mathrm{ar}(U_i) \sim \frac{n}{6}, \qquad \sum_{i=1}^{m-1} \mathbb{V}\mathrm{ar}(R_i) \sim 1.$$

We recognize that $R(\pi)$ may be approximated by Poisson(1) while $U(\pi)$ may be approximated by normal$(0,1)$ after standardization. From Theorem 5.4 and Remark 5.5, the distribution of $L_s(\pi)$ similarly follows. ☐

Even though results concerning parking statistics for classical parking functions and prime parking functions seem largely parallel until now, the next theorem shows that unlike classical parking functions, the distributions of the leading elements statistic and the 1's statistic are not the same for prime parking functions.

Theorem 5.7.

$$\begin{aligned}\sum_{\pi \in \mathrm{PPF}(n+1)} & x^{\mathrm{lel}(\pi)} y^{\mathrm{ones}(\pi)} \\ &= x(n-1+x+y)^{n-1}\left((n-1)y - x - (n-1)\right) + x(n-1+x)^n + x^2y^2(n+xy)^{n-1}.\end{aligned} \tag{5.2}$$

Proof. In Kalikow [Kal99, Section 1.7.2], a bijection between (p,i) where $p \in \mathrm{PPF}(n+1)$ and $p(i) = 1$ and (q,i) where $q \in \mathrm{PF}(n)$ and $i \in \{1,\ldots,n+1\}$ was identified. The bijection is easy to describe: we insert an extra 1 anywhere into a parking function q to obtain a prime parking function p, and then divide by the number of 1's in the prime parking function p so that we do not overcount.

Borrowing notation from Proposition 4.6, this implies that

$$\begin{aligned}\sum_{\pi \in \mathrm{PPF}(n+1)} & x^{\mathrm{lel}(\pi)} y^{\mathrm{ones}(\pi)} \\ =& \sum_{s=0}^{n-2}\sum_{t=0}^{n-s-2} \frac{n}{s+2} \#\{\pi \in \mathrm{PF}(n) \colon \pi_1 \neq 1,\ \mathrm{lel}(\pi) = t+1 \text{ and } \mathrm{ones}(\pi) = s+1\} x^{t+1} y^{s+2} \\ &+ \sum_{s=0}^{n-2}\sum_{t=0}^{n-s-2} \frac{1}{s+2} \#\{\pi \in \mathrm{PF}(n) \colon \pi_1 \neq 1,\ \mathrm{lel}(\pi) = t+1 \text{ and } \mathrm{ones}(\pi) = s+1\} x^{s+2} y^{s+2} \\ &+ \sum_{s=0}^{n-1} \frac{n+1}{s+2} \#\{\pi \in \mathrm{PF}(n) \colon \pi_1 = 1,\ \mathrm{lel}(\pi) = s+1 \text{ and } \mathrm{ones}(\pi) = s+1\} x^{s+2} y^{s+2}.\end{aligned}$$

Here for $\pi \in \mathrm{PF}(n)$ with $\pi_1 \neq 1$, we classify into two cases as inserting the extra 1 in front of all elements in π will change the leading element while inserting the extra 1 anywhere else will keep the leading element.

We perform generating function calculations for the summands on the right. From the proof of Proposition 4.6,

$$\begin{aligned}
&\sum_{s=0}^{n-2}\sum_{t=0}^{n-s-2}\frac{n}{s+2}\binom{n-2}{s,t,n-s-t-2}(n-1)^{n-s-t-1}x^{t+1}y^{s+2}\\
=&\sum_{s=0}^{n-2}\sum_{t=0}^{n-s-2}\frac{1}{n+1}\left((s+2)(t+1)-(t+1)\right)\binom{n+1}{s+2,t+1,n-s-t-2}(n-1)^{n-s-t-2}x^{t+1}y^{s+2}\\
=&\frac{1}{n+1}\left(xy\frac{\partial^2}{\partial x\partial y}(n-1+x+y)^{n+1}-x\frac{\partial}{\partial x}(n-1+x+y)^{n+1}+x\frac{d}{dx}(n-1+x)^{n+1}\right)\\
=&x(n-1+x+y)^{n-1}\left((n-1)y-x-(n-1)\right)+x(n-1+x)^{n}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{s=0}^{n-2}\sum_{t=0}^{n-s-2}\frac{1}{s+2}\binom{n-2}{s,t,n-s-t-2}(n-1)^{n-s-t-1}(xy)^{s+2}\\
=&\sum_{s=0}^{n-2}\frac{n-1}{s+2}\binom{n-2}{s}(xy)^{s+2}\sum_{t=0}^{n-s-2}\binom{n-s-2}{n-s-t-2}(n-1)^{n-s-t-2}\\
=&\sum_{s=0}^{n-2}\frac{n-1}{s+2}\binom{n-2}{s}(xy)^{s+2}n^{n-s-2}\\
=&\sum_{s=0}^{n-2}\frac{(s+2)-1}{n}\binom{n}{s+2}(xy)^{s+2}n^{n-s-2}\\
=&\frac{1}{n}\left(xy\frac{d}{d(xy)}(n+xy)^{n}-(n+xy)^{n}\right)+n^{n-1}\\
=&\frac{1}{n}(n+xy)^{n-1}((n-1)xy-n)+n^{n-1}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{s=0}^{n-1}\frac{n+1}{s+2}\binom{n-1}{s}(s+1)n^{n-s-2}(xy)^{s+2}\\
=&\sum_{s=0}^{n-1}\frac{(s+2)(s+1)-(s+2)+1}{n^2}\binom{n+1}{s+2}n^{n-s-1}(xy)^{s+2}\\
=&\frac{1}{n^2}\left(x^2y^2\frac{d^2}{d^2(xy)}(n+xy)^{n+1}-xy\frac{d}{d(xy)}(n+xy)^{n+1}+(n+xy)^{n+1}\right)-n^{n-1}\\
=&\frac{1}{n}(n+xy)^{n-1}(nx^2y^2-(n-1)xy+n)-n^{n-1}.
\end{aligned}$$

Putting all this together, the desired result is obtained. □

Corollary 5.8. *Taking $x = 1$ in (5.2), we have*

$$\sum_{\pi \in \mathrm{PPF}(n+1)} y^{\mathrm{ones}(\pi)} = (n+y)^n(y-1) + n^n.$$

This agrees with the corresponding formula in Kalikow [Kal99, Section 1.7.2].

5.2. Unit interval parking functions

Unit interval parking functions are a subset of classical parking functions where each car is displaced by at most 1 spot [MHJ$^+$23]. Here we consider the deterministic model where a car moves forward when its desired spot is occupied. Denote the set of unit interval parking functions of length n by $\mathrm{UPF}(n)$. If a car is displaced by 1 after parking then it is unlucky; otherwise it is lucky. Let $P_0(y) = 1$ and

$$P_n(y) = \sum_{\pi \in \mathrm{UPF}(n)} y^{\#\{i\colon \text{ displacement of car } i \text{ is } 1\}}.$$

We recognize that $P_n(y) = \sum_{k=1}^{n} S(n,k)k!y^{n-k}$, where $S(n,k)$ is a Stirling number of the second kind. One way to see this is by partitioning the parking preference sequence $\pi \in \mathrm{UPF}(n)$ into $k \leqslant n$ noninteracting consecutive blocks while reading π from left to right. The parking outcomes of different blocks have no effect on one another, and the first number assigned to each block corresponds to a lucky car. For example, for $\pi = (4,1,1,2,6,4)$, there are three blocks: $\{4,4\}$, $\{1,1,2\}$, and $\{6\}$. We see that car 1 (with the preference 4), car 2 (with the preference 1), and car 5 (with the preference 6) are lucky. Note that $P_n(1)$ gives the Fubini numbers (ordered Bell numbers, or number of ordered set partitions).

Theorem 5.9. *$P_n(y)$ satisfies the following recurrence relation:*

$$P_n(y) = (y+1)\sum_{i=0}^{n-1}\binom{n-1}{i}P_i(y)P_{n-i-1}(y) - yP_{n-1}(y).$$

Proof. As in the proof of Theorem 2.1, we focus on the last car. The parking protocol implies that $(\pi_1, \ldots, \pi_{n-1})$ may be decomposed into two unit interval parking functions $\alpha \in \mathrm{UPF}(i)$ and $\beta \in \mathrm{UPF}(n-1-i)$, where $0 \leqslant i \leqslant n-1$, β is formed by subtracting $i+1$ from the relevant entries in π, and α and β do not interact with each other. For $1 \leqslant i \leqslant n-1$, this open spot $i+1$ could be either the same as j, the preference of the last car, in which case the car parks directly. Or, $i+1$ could be equal to $j+1$, in which case the car travels forward to park. Combined, parking the last car contributes $y+1$ to the generating function. We note that when $i = 0$ (and so $i+1 = 1$), the preference of the last car has to coincide with the open spot. Subtracting $yP_{n-1}(y)$ from the generating function takes care of this special case. ☐

Let $Q(y,t) = \sum_{n=0}^{\infty} P_n(y)\frac{t^n}{n!}$ and $Q(y,0) = 1$. From Theorem 5.9, differentiating with respect to t yields

$$\frac{\partial Q}{\partial t} = (y+1)Q^2 - yQ.$$

This is known as a Bernoulli differential equation. To solve it, we follow standard techniques.

$$Q^{-2}\frac{\partial Q}{\partial t}+Q^{-1}y=y+1.$$

We make a substitution $R=Q^{-1}$. Then

$$-\frac{\partial R}{\partial t}+Ry=y+1.$$

Hence

$$R(y,t)=1+\frac{1}{y}\left(1-\exp(yt)\right) \text{ and } Q(y,t)=\frac{y}{y+1-\exp(yt)}.$$

Setting $y=1$, $Q(1,t)=1/(2-\exp(t))$ gives the exponential generating function of Fubini numbers.

5.3. Generic $\boldsymbol{u}$-parking functions

We defined $\boldsymbol{u}$-parking functions in Section 3. Recurrence relations for generalized displacement in $\boldsymbol{u}$-parking functions have been widely studied; see Yan [Yan15, Section 1.4.3] for a summary of results. Let us derive the recurrence relations for some other parking statistics under the deterministic model.

Let

$$A_{\boldsymbol{u}}(x)=\sum_{\pi\in\mathrm{PF}(\boldsymbol{u})}x^{\mathrm{unl}(\pi)},\quad B_{\boldsymbol{u}}(y)=\sum_{\pi\in\mathrm{PF}(\boldsymbol{u})}y^{\mathrm{ones}(\pi)},\quad \text{and } C_{\boldsymbol{u}}(z)=\sum_{\pi\in\mathrm{PF}(\boldsymbol{u})}z^{\mathrm{lel}(\pi)}.$$

Theorem 5.10. *$A_{\boldsymbol{u}}(x)$, $B_{\boldsymbol{u}}(y)$, and $C_{\boldsymbol{u}}(z)$ respectively satisfy the following recurrence relations:*

$$A_{\boldsymbol{u}}(x)=\sum_{i=0}^{m-1}\binom{m-1}{i}(ix+(u_{i+1}-i))A_{\boldsymbol{u}_1}(x)A_{\boldsymbol{u}_2}(x),$$

$$B_{\boldsymbol{u}}(y)=\sum_{i=0}^{m-1}\binom{m-1}{i}(y+(u_{i+1}-1))B_{\boldsymbol{u}_1}(y)B_{\boldsymbol{u}_2}(1),$$

$$C_{\boldsymbol{u}}(z)=\sum_{i=0}^{m-1}\left(\binom{m-2}{i-1}(z+(u_{i+1}-1))C_{\boldsymbol{u}_1}(z)C_{\boldsymbol{u}_2}(1)+\binom{m-2}{i}u_{i+1}C_{\boldsymbol{u}_1}(1)C_{\boldsymbol{u}_2}(z)\right),$$

where $\boldsymbol{u}_1=(u_1,\dots,u_i)$ and $\boldsymbol{u}_2=(u_{i+2}-u_{i+1},\dots,u_m-u_{i+1})$.

Proof. As in the proof of Theorem 2.1, we focus on the last car. We assume π_m is the maximal parking completion for $\pi_1,\dots,\pi_{m-1}$, i.e., $\pi=(\pi_1,\dots,\pi_{m-1},\pi_m)$ is a $\boldsymbol{u}$-parking function but $\pi'=(\pi_1,\dots,\pi_{m-1},\pi_m+1)$ is not a $\boldsymbol{u}$-parking function. Let $\lambda=(\lambda_1,\dots,\lambda_m)$ be the increasing rearrangement of π. Then for some i where $0\leqslant i\leqslant m-1$, we have $\pi_m=\lambda_{i+1}\leqslant u_{i+1}$. If there are multiple entries in π (and hence λ) that take the same value as π_m, we may assume that $i+1$ is the maximum such index, which implies that $\lambda_{i+1}<\lambda_{i+2}$ if $i+1<m$. We claim

that $\lambda_{i+1} = u_{i+1}$. Suppose otherwise and that $\lambda_{i+1} < u_{i+1}$; then the increasing rearrangement of π' is $(\lambda_1, \dots, \lambda_i, \lambda_{i+1} + 1, \lambda_{i+2}, \dots, \lambda_m)$, making π' a $\boldsymbol{u}$-parking function, which contradicts the assumption that π_m is the maximal parking completion. Since $u_1 < \cdots < u_m$, we further have π_m is the unique entry in π with $\pi_m = u_{i+1}$. Hence π may be decomposed into two parking functions: $\alpha \in \mathrm{PF}(\boldsymbol{u}_1)$ and $\beta \in \mathrm{PF}(\boldsymbol{u}_2)$ where $\boldsymbol{u}_1 = (u_1, \dots, u_i)$ and $\boldsymbol{u}_2 = (u_{i+2} - u_{i+1}, \dots, u_m - u_{i+1})$, β is formed by subtracting u_{i+1} from the relevant entries in π, and α and β do not interact with each other.

Now in the general case, π_m is a parking completion for $\pi_1, \dots, \pi_{m-1}$ but not necessarily maximal, so $1 \leqslant \pi_m \leqslant u_{i+1}$. We have

$$\mathrm{unl}(\pi) = \mathrm{unl}(\alpha) + \mathrm{unl}(\beta) + (\text{either } 1 \text{ or } 0),$$

where it is 1 if π_m coincides with any of the i spots already parked in by the i cars that constitute α and 0 otherwise. We also have

$$\mathrm{ones}(\pi) = \#\{i \colon \alpha_i = 1\} + (\text{either } 1 \text{ or } 0),$$

where it is 1 if $\pi_m = 1$ and 0 otherwise. For both the unluckiness and the 1's statistics, the binomial coefficient $\binom{m-1}{i}$ chooses an i-element subset of $[m-1]$ to constitute the index set of α. For the leading elements statistics, depending on whether $\pi_1 < u_{i+1}$ or $\pi_1 > u_{i+1}$ (note that it is impossible for $\pi_1 = u_{i+1}$), the count on $\mathrm{lel}(\pi)$ would be different, as π_m has the possibility of equalling π_1 in the former case but no possibility of equalling π_1 in the latter case. In the former case, the binomial coefficient $\binom{m-2}{i-1}$ chooses an $(i-1)$-element subset of $[m-1] \setminus \{1\}$, and together with the index 1, they constitute the index set of α, and

$$\mathrm{lel}(\pi) = \#\{i \colon \alpha_i = \pi_1\} + (\text{either } 1 \text{ or } 0),$$

where it is 1 if $\pi_m = \pi_1$ and 0 otherwise. In the latter case, the binomial coefficient $\binom{m-2}{i}$ chooses an i-element subset of $[m-1] \setminus \{1\}$ to constitute the index set of α, and

$$\mathrm{lel}(\pi) = \#\{i \colon \beta_i = \pi_1 - u_{i+1}\}. \qquad \square$$

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References

[AR12] Drew Armstrong and Brendon Rhoades. The Shi arrangement and the Ish arrangement. *Trans. Amer. Math. Soc.*, 364(3):1509–1528, 2012. `doi:10.1090/S0002-9947-2011-05521-2`.

[DH17] Persi Diaconis and Angela Hicks. Probabilizing parking functions. *Adv. in Appl. Math.*, 89:125–155, 2017. `doi:10.1016/j.aam.2017.05.004`.

[DHH+23] Irfan Durmić, Alex Han, Pamela E. Harris, Rodrigo Ribeiro, and Mei Yin. Probabilistic parking functions. *Electron. J. Combin.*, 30(3):Paper No. 3.18, 25, 2023. `doi:10.37236/11649`.

[FR74] Dominique Foata and John Riordan. Mappings of acyclic and parking functions. *Aequationes Math.*, 10:10–22, 1974. `doi:10.1007/BF01834776`.

[GS06] Ira M. Gessel and Seunghyun Seo. A refinement of Cayley's formula for trees. *Electron. J. Combin.*, 11(2):Research Paper 27, 23, 2004/06. `doi:10.37236/1884`.

[Hou16] Qing-Hu Hou. An insertion algorithm and leaders of rooted trees. *European J. Combin.*, 53:35–44, 2016. `doi:10.1016/j.ejc.2015.10.008`.

[Kal99] Louis H. Kalikow. *Enumeration of parking functions, allowable permutation pairs, and labeled trees*. PhD thesis, Brandeis University, 1999. URL: `https://www.proquest.com/docview/304494728`.

[Knu98] Donald E. Knuth. *The art of computer programming. Vol. 3*. Addison-Wesley, Reading, MA, second edition, 1998. Sorting and searching.

[KW66] Alan G. Konheim and Benjamin Weiss. An occupancy discipline and applications. *SIAM J. Appl. Math.*, 14(6):1266–1274, 1966. `doi:10.1137/0114101`.

[KY23] Richard Kenyon and Mei Yin. Parking functions: from combinatorics to probability. *Methodol. Comput. Appl. Probab.*, 25(1):Paper No. 32, 30, 2023. `doi:10.1007/s11009-023-10022-5`.

[MHJ+23] Lucas Chaves Meyles, Pamela E. Harris, Richter Jordaan, Gordon Rojas Kirby, Sam Sehayek, and Ethan Spingarn. Unit-interval parking functions and the permutohedron, 2023. `arXiv:2305.15554`.

[Pyk59] Ronald Pyke. The supremum and infimum of the Poisson process. *Ann. Math. Statist.*, 30(2):568–576, 1959. `doi:10.1214/aoms/1177706269`.

[Rio69] John Riordan. Ballots and trees. *J. Combinatorial Theory*, 6(4):408–411, 1969. `doi:10.1016/s0021-9800(69)80039-6`.

[Seo06] Seunghyun Seo. A combinatorial proof of Postnikov's identity and a generalized enumeration of labeled trees. *Electron. J. Combin.*, 11(2):Note 3, 9, 2004/06. `doi:10.37236/1890`.

[SS07] Seunghyun Seo and Heesung Shin. A generalized enumeration of labeled trees and reverse Prüfer algorithm. *J. Combin. Theory Ser. A*, 114(7):1357–1361, 2007. `doi:10.1016/j.jcta.2007.01.010`.

[Sta97] Richard P. Stanley. Parking functions and noncrossing partitions. *Electron. J. Combin.*, 4(2):R20 (14 pages), 1997. The Wilf Festschrift (Philadelphia, PA, 1996). `doi:10.37236/1335`.

[Sta07] Richard P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007. `doi:10.1090/pcms/013/08`.

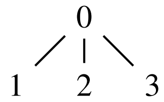
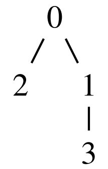
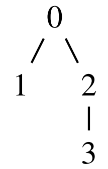
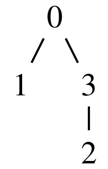
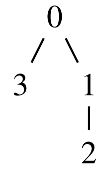
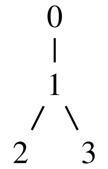
[Sta24] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 208 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, [2024] ©2024. With an appendix by Sergey Fomin.

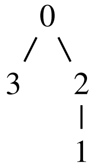
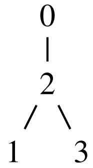
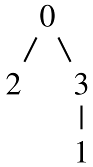
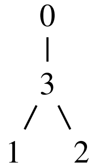
[SW18] Richard P. Stanley and Yinghui Wang. Some aspects of (r, k)-parking functions. *J. Combin. Theory Ser. A*, 159:54–78, 2018. `doi:10.1016/j.jcta.2018.05.003`.

[TTn22] Melanie Tian and Enrique Treviño. Generalizing parking functions with randomness. *Electron. J. Combin.*, 29(3):Paper No. 3.13, 13, 2022. `doi:10.37236/10884`.

[Yan15] Catherine H. Yan. Parking functions. In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 835–893. CRC Press, Boca Raton, FL, 2015.

Table A: Correspondence table for $n = 3$.

Prüfer code	labelled tree	parking function (repeats)	parking function (leading elements)	parking function (1's)
00		111	111	111
01		112	112	112
02		113	113	113
03		221	221	131
10		122	121	121
11		123	122	122
12		231	123	123
13		121	231	132

20		311	131	211
21		312	132	213
22		131	311	212
23		132	312	312
30		211	212	311
31		212	213	231
32		213	321	321
33		321	211	221