Ryser's Theorem for symmetric $\boldsymbol{\rho}$-latin squares

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Abstract. Let L be an $n \times n$ array whose top left $r \times r$ subarray is filled with k different symbols, each occurring at most once in each row and at most once in each column. We establish necessary and sufficient conditions that ensure the remaining cells of L can be filled such that each symbol occurs at most once in each row and at most once in each column, L is symmetric with respect to the main diagonal, and each symbol occurs a prescribed number of times in L. The case where the prescribed number of times each symbol occurs is n was solved by Cruse (J. Combin. Theory Ser. A 16 (1974), 18–22), and the case where the top left subarray is $r \times n$ and the symmetry is not required, was settled by Goldwasser et al. (J. Combin. Theory Ser. A 130 (2015), 26–41). Our result allows the entries of the main diagonal to be specified as well, which leads to an extension of the Andersen–Hoffman Theorem (Annals of Disc. Math. 15 (1982) 9–26, European J. Combin. 4 (1983) 33–35).

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1. Introduction

Throughout this paper, n and k are positive integers, $[k] := \{1, 2, \dots, k\}$, and $\boldsymbol{\rho} := (\rho_1, \dots, \rho_k)$ with $1 \leqslant \rho_\ell \leqslant n \leqslant k$ for $\ell \in [k]$ such that $\sum_{\ell \in [k]} \rho_\ell = n^2$. A *$\boldsymbol{\rho}$-latin square* L of *order* n is an $n \times n$ array filled with k different *symbols*, $[k]$, each occurring at most once in each row and at most once in each column, such that each symbol ℓ occurs exactly ρ_ℓ times in L for $\ell \in [k]$, and L is *symmetric* if $L_{ij} = L_{ji}$ for $i, j \in [n]$. An $r \times s$ *$\boldsymbol{\rho}$-latin rectangle* on the set $[k]$ of symbols is an $r \times s$ array in which each symbol in $[k]$ occurs at most once in each row and in each column, and in which each symbol i occurs at most ρ_ℓ times for $\ell \in [k]$. A *latin square* (or *rectangle*) is a $\boldsymbol{\rho}$-latin square (or *rectangle*) with $\boldsymbol{\rho} = (n, \dots, n)$.

We are interested in the following problem.

Question 1.1. Let $n > \min\{r, s\}$. Find necessary and sufficient conditions that ensure that a symmetric $r \times s$ $\boldsymbol{\rho}$-latin rectangle L can be extended to a symmetric $n \times n$ $\boldsymbol{\rho}$-latin square L'.

We remark that even the case $\boldsymbol{\rho} = (n, \dots, n)$ of Problem 1.1 is far from being settled. The following result of Cruse which can be viewed as an analogue of Ryser's theorem [Rys51] for partial symmetric latin squares, resolves the case $r = s, \boldsymbol{\rho} = (n, \dots, n)$ of Problem 1.1. For $\ell \in [k]$, let e_ℓ be the number of occurrences of ℓ in L.

Theorem 1.2. *[Cru74, Theorem 1] An $r \times r$ symmetric latin rectangle L on $[n]$ can be extended to an $n \times n$ symmetric latin square if and only if*

(i) $e_\ell \geqslant 2r - n$ *for* $\ell \in [n]$*;*

(ii) $\big|\{\ell \in [n] \mid e_\ell \equiv n \pmod 2\}\big| \geqslant r$.

Let $\boldsymbol{d} := (d_1, \dots, d_k)$ be the *diagonal tail* of L' where d_ℓ is the minimum number of occurrences of ℓ in $\{L'_{ii} \mid r + 1 \leqslant i \leqslant n\}$ and $\sum_{\ell \in [k]} d_\ell \leqslant n - r$ (we refer to the diagonal tail as the *diagonal* if $r = s = 0$). Observe that if L is extended to L', then by permuting rows and columns of L', one can obtain another $\boldsymbol{\rho}$-latin square L'' containing L with the same diagonal tail as L' such that the entries of L'' in positions (i, i) for $i > r$ are in a prescribed order. Hence, in order to extend L to L' whose diagonal entries are specified, it is enough to extend L to L' with a specified diagonal tail. Andersen, and independently, Hoffman, proved the following extension of Cruse's Theorem.

Theorem 1.3. *[And82, Hof83] An $r \times r$ symmetric latin rectangle L on $[n]$ can be extended to an $n \times n$ symmetric latin square with a prescribed diagonal tail $\boldsymbol{d}$ with $\sum_{\ell \in [n]} d_\ell = n - r$ if and only if*

(i) $e_\ell \geqslant 2r - n + d_\ell$ *for* $\ell \in [n]$*;*

(ii) $e_\ell + d_\ell \equiv n \pmod 2$ *for* $\ell \in [n]$.

Bryant and Rodger [BR04] solved the case $r \in \{1, 2\}, s = n, \boldsymbol{\rho} = (n, \dots, n)$ of Problem 1.1. Goldwasser et al. [GHHO15] found necessary and sufficient conditions under which an $r \times n$ $\boldsymbol{\rho}$-latin rectangle can be extended to an $n \times n$ $\boldsymbol{\rho}$-latin square, generalizing a classical result of Hall [Hal45]. Most recently, Bahmanian [Bah22] extended this further by establishing necessary and sufficient conditions that ensure an $r \times s$ $\boldsymbol{\rho}$-latin rectangle can be extended to an $n \times n$ $\boldsymbol{\rho}$-latin square, generalizing Ryser's theorem [Rys51]. For a survey of results on embedding of latin squares and related structures, cycle systems and graph designs, we refer the reader to [Rod92].

Let L be an $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle. Let $i \in [r]$ be a row of L, $\ell \in [k]$ be a symbol, $I \subseteq [r]$ be a subset of rows, and $K \subseteq [k]$ be a subset of symbols. Then $\mu_K(i)$ and $\mu_I(\ell)$ denote the number of symbols in K that are missing in row i, and the number of rows in I where symbol ℓ is missing, respectively. Notice that

$$\begin{aligned} &\mu_{[r]}(\ell) = r - e_\ell, && 0 \leqslant \mu_I(\ell) \leqslant \min\{|I|, r - e_\ell\} && \forall \ell \in [k], I \subseteq [r], \\ &\mu_{[k]}(i) = k - r, && 0 \leqslant \mu_K(i) \leqslant \min\{|K|, k - r\} && \forall i \in [r], K \subseteq [k]. \end{aligned}$$

The complement of a set S is denoted by $\bar{S}$, and $x \dot{-} y := \max\{0, x-y\}$. If x, y, z are non-negative, then $(x \dot{-} y) \dot{-} z = (x-y) \dot{-} z$. Whenever it is not ambiguous, we write $x - y \dot{-} z$ instead of $(x-y) \dot{-} z$.

Suppose that an $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle L is extended to an $n \times n$ symmetric $\boldsymbol{\rho}$-latin square L' with a prescribed diagonal tail $\boldsymbol{d}$, and let us fix a symbol $\ell \in [k]$. Without loss of generality we assume that $L'_{ii} = \ell$ for $r+1 \leqslant i \leqslant r+d_\ell$. On the one hand, there are $\rho_\ell - e_\ell - d_\ell$ occurrences of ℓ outside the top left $(r+d_\ell) \times (r+d_\ell)$ subsquare. On the other hand, there are at most $n-r-d_\ell$ occurrences of ℓ in the last $n-r-d_\ell$ rows, and at most $n-r-d_\ell$ occurrences of ℓ in the last $n-r-d_\ell$ columns. Therefore, we must have

$$\rho_\ell - e_\ell + d_\ell \leqslant 2(n-r) \quad \forall \ell \in [k]. \tag{1.1}$$

Due to the symmetry of L and L', off-diagonal entries occur in pairs, and so, if for some $\ell \in [k]$, $\rho_\ell - e_\ell - d_\ell$ is odd, then ℓ must occur at least one more time on the diagonal, and consequently,

$$\big|\{\ell \in [k] \mid \rho_\ell - e_\ell \not\equiv d_\ell \pmod 2\}\big| \leqslant n - r - \sum_{\ell \in [k]} d_\ell. \tag{1.2}$$

Moreover,

$$n - r - \sum_{\ell \in [k]} d_\ell \equiv \sum_{\ell \in [k]} (\rho_\ell - e_\ell - d_\ell) \equiv \big|\{\ell \in [k] \mid \rho_\ell - e_\ell \not\equiv d_\ell \pmod 2\}\big| \pmod 2 \quad \forall \ell \in [k]. \tag{1.3}$$

An $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle L is $(\boldsymbol{\rho}, \boldsymbol{d})$-*admissible* if it meets conditions (1.1)–(1.3). Here is our main result.

Theorem 1.4. *An $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle L can be completed to an $n \times n$ symmetric $\boldsymbol{\rho}$-latin square with a prescribed diagonal tail $\mathbf{d}$ if and only if L is $(\boldsymbol{\rho}, \mathbf{d})$-admissible and*

$$(n-r)(r-|I|) + \sum_{\ell \in \bar{K}} \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor \geqslant \sum_{\ell \in K} \Big(\rho_\ell - e_\ell - n + r \dot{-} \mu_I(\ell)\Big) + \sum_{i \in I} \Big(n - r \dot{-} \mu_K(i)\Big) \quad \forall I \subseteq [r], K \subseteq [k]. \tag{1.4}$$

Remark 1.5. As we shall see in Section 3, Condition (1.4) can be replaced by the following two conditions.

$$(n-r)|I| \leqslant \sum_{\ell \in [k]} \min\left\{ \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor, \mu_I(\ell) \right\} \qquad \forall I \subseteq [r], \tag{1.5}$$

$$\sum_{\ell \in K} (\rho_\ell - e_\ell + r \dot{-} n) \leqslant \sum_{i \in [r]} \min\Big\{ n - r, \mu_K(i) \Big\} \qquad \forall K \subseteq [k]. \tag{1.6}$$

At first sight, it may not be obvious why Theorem 1.4 implies Theorems 1.2 and 1.3; see Remarks 4.5. Both Corollaries 4.2 and 4.4 offer simpler generalizations of Theorems 1.2 and 1.3.

Using detachments we reduce the completion of the desired partial ρ-latin square A to finding a subgraph with prescribed degree sequences of an auxiliary graph associated with A. We complete the proof by applying the celebrated Lovász's (g, f)-factor Theorem. Intuitively speaking, we use matching theory to decide which $n-r$ symbols among $k-r$ available symbols of each row should be chosen, and then we use detachment theory to arrange the chosen symbols in such a way that the latin property is maintained.

Further terminology along with the two main tools are discussed in Section 2. Theorems 1.4 is proven in Section 3. We conclude the paper with a few corollaries and open problems.

2. Prerequisites

For a graph $G=(V,E)$, $u\in V, e\in E$, and $K\subseteq V$, $\deg_G(u), \text{mult}_G(e), \text{mult}_G(uS)$ denote the number of edges incident with u, the multiplicity of the edge e, and the number of edges between u and S, respectively. If the edges of G are colored with k colors (the set of colors is always $[k]$), then $G(\ell)$ is the color class ℓ of G for $\ell\in[k]$. A bigraph G with bipartition $\{X,Y\}$ will be denoted by $G[X,Y]$, and if $S\subseteq X$, then $\bar{S}$ means $X\backslash S$.

For $i=1,2$, an edge that is incident with one vertex only is called an i*-loop* if it contributes i to the degree of that vertex. If e is a 1-loop incident with vertex u, we write $e=u$, and if e is a 2-loop incident with vertex u, we write $e=u^2$. We denote the set of 1-loops of a graph G by $E^1(G)$, and $E^2(G):=E(G)\backslash E^1(G)$.

Let $\mathbb{K}_n$ denote the n-vertex graph in which every pair of distinct vertices are adjacent and each vertex is incident with a 1-loop. Observe that $\text{mult}_{\mathbb{K}_n}(u)=\text{mult}_{\mathbb{K}_n}(uv)=1$ for $u,v\in V(\mathbb{K}_n)$ with $u\neq v$. There is a one-to-one correspondence between symmetric latin squares of order n and 1-factorizations of $\mathbb{K}_n$.

Let G be a graph whose edges are colored, and let $\alpha\in V(G)$. By *splitting* α into $\alpha_1,\dots,\alpha_p$, we obtain a new graph F whose vertex set is $(V(G)\backslash\{\alpha\})\cup\{\alpha_1,\dots,\alpha_p\}$ so that each edge αu in G becomes $\alpha_i u$ for some $i\in[p]$ in F, and each 1-loop α in G becomes α_i for some $i\in[p]$ in F. Intuitively speaking, when we *split* a vertex α into $\alpha_1,\dots,\alpha_p$, we share the edges incident with α among $\alpha_1\dots,\alpha_p$. In this manner, F is a *detachment* of G, and G is an *amalgamation* of F obtained by *identifying* $\alpha_1,\dots,\alpha_p$ by α. We need the following detachment lemma. Here, $x\approx y$ means $\lfloor y\rfloor\leqslant x\leqslant\lceil y\rceil$.

Lemma 2.1. *[Bah12, Theorem 4.1] Let G be a graph whose edges are colored with k colors, and let $\alpha\in V(G)$. There exists a graph F obtained by splitting α into $\alpha_1,\dots,\alpha_p$ such that*

(i) $\deg_{F(\ell)}(\alpha_i)\approx\deg_{G(\ell)}(\alpha)/p$ *for* $i\in[p],\ell\in[k]$;

(ii) $\text{mult}_F(\alpha_i)\approx\text{mult}_G(\alpha)/p$ *for* $i\in[p]$;

(iii) $\text{mult}_F(\alpha_i u)\approx\text{mult}_G(\alpha u)/p$ *for* $i\in[p], u\in V(G)\backslash\{\alpha\}$;

(iv) $\text{mult}_F(\alpha_i\alpha_j)\approx\text{mult}_G(\alpha^2)/\binom{p}{2}$ *for* $i,j\in[p], i\neq j$.

Using this detachment lemma, it is easy to construct symmetric $\boldsymbol{\rho}$-latin squares with prescribed diagonals.

Theorem 2.2. *For every* $n, k, \mathbf{d} = (d_1, \dots, d_k), \boldsymbol{\rho} = (\rho_1, \dots, \rho_k)$ *with* $1 \leqslant \rho_1, \dots, \rho_k \leqslant n \leqslant k$, $\sum_{\ell \in [k]} \rho_\ell = n^2$ *and* $\sum_{\ell \in [k]} d_\ell \leqslant n$*, there exists a symmetric* $\boldsymbol{\rho}$*-latin square of order* n *with diagonal* $\mathbf{d}$ *if and only if*

$$\big|\{\ell \in [k] \mid \rho_\ell \not\equiv d_\ell \pmod 2\}\big| \leqslant n - \sum_{\ell \in [k]} d_\ell,$$
$$\big|\{\ell \in [k] \mid \rho_\ell \not\equiv d_\ell \pmod 2\}\big| \equiv n - \sum_{\ell \in [k]} d_\ell \pmod 2.$$

Proof. The proof of necessity is very similar to those of (1.2) and (1.3). To prove the sufficiency, let G be a graph with $V(G) = \{\alpha\}$, and $\text{mult}_G(\alpha) = n, \text{mult}_G(\alpha^2) = \binom{n}{2}$. First, we color the 1-loops of G such that $\text{mult}_{G(\ell)}(\alpha) = d_\ell$ for $\ell \in [k]$. Then we color an additional 1-loop with ℓ if $\rho_\ell - d_\ell$ is odd for $\ell \in [k]$. Since $|\{\ell \in [k] \mid \rho_\ell \not\equiv d_\ell \pmod 2\}| \leqslant n - \sum_{\ell \in [k]} d_\ell$, this is possible. The number of uncolored 1-loops is $n - \sum_{\ell \in [k]} d_\ell - |\{\ell \in [k] \mid \rho_\ell \not\equiv d_\ell \pmod 2\}|$ which is even. Therefore, we can color the uncolored 1-loops by coloring an even number of 1-loops with each color. Then, we color the remaining edges of G such that $\text{mult}_{G(\ell)}(\alpha^2) = \big(\rho_\ell - \text{mult}_{G(\ell)}(\alpha)\big)/2$ for $\ell \in [k]$. This is possible, because the coloring of 1-loops ensures $\rho_\ell - \text{mult}_{G(\ell)}(\alpha)$ is even for $\ell \in [k]$, and

$$\sum_{\ell \in [k]} \big(\rho_\ell - \text{mult}_{G(\ell)}(\alpha)\big)/2 = (n^2 - n)/2 = \text{mult}_G(\alpha^2).$$

Applying the detachment lemma with $p = n$ yields the graph $F \cong \mathbb{K}_n$ whose colored edges correspond to symbols in the desired $\boldsymbol{\rho}$-latin square L. More precisely, L is obtained by placing symbol ℓ in L_{ij} and L_{ji} whenever the edge $\alpha_i\alpha_j$ is colored ℓ for $i \neq j$, and placing symbol ℓ in L_{ii} whenever the 1-loop α_i is colored ℓ. □

Let f, g be integer functions on the vertex set of a graph G such that $0 \leqslant g(x) \leqslant f(x)$ for all x. A (g, f)*-factor* is a spanning subgraph F of G with the property that $g(x) \leqslant \deg_F(x) \leqslant f(x)$ for each x. Let $G[X, Y]$ be a bigraph. By [CK16, Theorem 5] and [HHKL90, Theorem 1], G has a (g, f)-factor if and only if either one of the following two conditions hold.

$$\sum_{a \in A} g(a) \leqslant \sum_{a \in N_G(A)} \min\Big\{f(a), \text{mult}_G(aA)\Big\} \qquad \forall A \subseteq X, A \subseteq Y,$$
$$\sum_{a \in A} f(a) \geqslant \sum_{a \notin A} \Big(g(a) \dot{-} \deg_{G-A}(a)\Big) \qquad \forall A \subseteq X \cup Y.$$

Here, $N_G(A)$ is the neighborhood of A in G. We remark that both of these results are special cases of the Lovász's (g, f)-factor Theorem [Lov72].

3. Proof of Theorem 1.4

We established the necessity of (1.1)–(1.3) in the introduction. The necessity of the remaining conditions will be evident at the end of the proof. To prove the sufficiency, suppose that L is a $(\boldsymbol{\rho}, \boldsymbol{d})$-admissible $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle. Let $F = \mathbb{K}_n$ with $V(F) = \tilde{X} := \{x_1, \dots, x_n\}$, and let $X = \{x_1, \dots, x_r\}$. In F, the 1-loop x_i is colored ℓ if $L_{ii} = \ell$ for $i \in [r]$, and the edge $x_i x_j$ is colored ℓ if $L_{ij} = \ell$ for distinct $i, j \in [r]$. Since L is symmetric, this coloring is well-defined. Observe that some edges of F are uncolored. For $\ell \in [k]$, we color d_ℓ arbitrary uncolored 1-loops (incident with the vertices of $\tilde{X} \backslash X$) with color ℓ. We have $\deg_{F(\ell)}(u) \leqslant 1$ for $u \in \tilde{X}, \ell \in [k]$. Let G be the graph obtained by amalgamating $x_{r+1}, \dots, x_n$ of F into a single vertex α, so $\text{mult}_G(\alpha) = \text{mult}_G(\alpha x_i) = n - r$ for $i \in [r]$, $\text{mult}_G(\alpha^2) = \binom{n-r}{2}$, and $\text{mult}_{G(\ell)}(\alpha) = d_\ell$ for $\ell \in [k]$.

Let $\Gamma[X, [k]]$ be the simple bigraph whose edge set is

$$\{u\ell \mid u \in X, \ell \in [k], \deg_{F(\ell)}(u) = 0\}.$$

For $i \in [r]$, $\sum_{\ell \in [k]} \deg_{F(\ell)}(x_i) = r$, and so we have

$$\begin{cases} \deg_\Gamma(x_i) = k - r & \text{if } i \in [r], \\ \deg_\Gamma(\ell) = r - e_\ell & \text{if } \ell \in [k]. \end{cases} \tag{3.1}$$

Observe that L can be completed to an $n \times n$ symmetric $\boldsymbol{\rho}$-latin square if and only if the uncolored edges of F can be colored so that

$$\forall \ell \in [k], \quad \begin{cases} \deg_{F(\ell)}(u) \leqslant 1 & \text{if } u \in \tilde{X}, \\ |E^1(F(\ell))| + 2|E^2(F(\ell))| = \rho_\ell. \end{cases} \tag{3.2}$$

We show that the coloring of F can be completed such that (3.2) holds if and only if the coloring of G can be completed such that

$$\forall \ell \in [k] \qquad \begin{cases} \deg_{G(\ell)}(u) \leqslant 1 & \text{if } u \in X, \\ \deg_{G(\ell)}(\alpha) \leqslant n - r, \\ |E^1(G(\ell))| + 2|E^2(G(\ell))| = \rho_\ell. \end{cases} \tag{3.3}$$

To see this, first assume that the coloring of F can be completed so that (3.2) holds. Identifying all the vertices in $\tilde{X} \backslash X$ by α, we will get the graph G satisfying (3.3). Conversely, suppose that we have a coloring of G such that (3.3) holds. Applying the detachment lemma to G, we get a graph F' obtained by splitting α into $\alpha_1, \dots, \alpha_{n-r}$, such that

(i) $\deg_{F'(\ell)}(\alpha_i) \approx \deg_{G(\ell)}(\alpha)/(n - r) \leqslant 1$ for $i \in [n - r], \ell \in [k]$;

(ii) $\text{mult}_{F'}(\alpha_i) = \text{mult}_G(\alpha)/(n - r) = 1$ for $i \in [n - r]$;

(iii) $\text{mult}_{F'}(\alpha_i u) = \text{mult}_G(\alpha u)/(n - r) = 1$ for $i \in [n - r], u \in X$;

(iv) $\text{mult}_{F'}(\alpha_i\alpha_j) = \text{mult}_G(\alpha^2)/\binom{n-s}{2} = 1$ for distinct $i, j \in [n-r]$.

Since $F' \cong F$ and the coloring of F' satisfies (3.2), we are done.

Since L is $(\boldsymbol{\rho}, \boldsymbol{d})$-admissible, for $\ell \in [k]$ we have $2(n-r) \geqslant \rho_\ell - e_\ell + d_\ell$, and so $\rho_\ell - e_\ell - n + r \leqslant (\rho_\ell - e_\ell - d_\ell)/2$. Moreover, $\rho_\ell - e_\ell - n + r \leqslant r - e_\ell$ for $\ell \in [k]$. We show that the coloring of G can be completed such that (3.3) is satisfied if and only if there exists a subgraph Θ of Γ with $r(n-r)$ edges so that

$$\begin{cases} \deg_\Theta(x_i) = n - r & \text{if } i \in [r], \\ \rho_\ell - e_\ell - n + r \leqslant \deg_\Theta(\ell) \leqslant \dfrac{\rho_\ell - e_\ell - d_\ell}{2} & \text{if } \ell \in [k]. \end{cases} \tag{3.4}$$

To prove this, suppose that the coloring G can be completed such that (3.3) is satisfied. Let $\Theta[X, [k]]$ be the bigraph whose edge set is

$$\{u\ell \mid u \in X, \ell \in [k], \alpha u \in E(G(\ell))\}.$$

It is clear that $\Theta \subseteq \Gamma$. For $i \in [r]$, $\deg_\Theta(x_i) = \text{mult}_G(\alpha x_i) = n - r$, and for $\ell \in [k]$,

$$\begin{aligned} \rho_\ell &= |E^1(G(\ell))| + 2|E^2(G(\ell))| \\ &= e_\ell + \text{mult}_{G(\ell)}(\alpha) + 2\,\text{mult}_{G(\ell)}(\alpha, X) + 2\,\text{mult}_{G(\ell)}(\alpha^2) \\ &\geqslant e_\ell + d_\ell + 2\deg_\Theta(\ell). \end{aligned}$$

Thus, $\deg_\Theta(\ell) \leqslant (\rho_\ell - e_\ell - d_\ell)/2$ for $\ell \in [k]$. Moreover,

$$\begin{aligned} n - r \geqslant \deg_{G(\ell)}(\alpha) &= \deg_\Theta(\ell) + \text{mult}_{G(\ell)}(\alpha) + 2\,\text{mult}_{G(\ell)}(\alpha^2) \\ &= \deg_\Theta(\ell) + \text{mult}_{G(\ell)}(\alpha) + \big(\rho_\ell - e_\ell - \text{mult}_{G(\ell)}(\alpha) - 2\deg_\Theta(\ell)\big) \\ &= \rho_\ell - e_\ell - \deg_\Theta(\ell), \end{aligned}$$

and so $\deg_\Theta(\ell) \geqslant \rho_\ell - e_\ell - n + r$. Conversely, suppose that $\Theta \subseteq \Gamma$ satisfying (3.4) exists. For each $\ell \in [k]$, if $\ell x_i \in E(\Theta)$ for some $i \in [r]$, we color an αx_i-edge in G with ℓ. Since $\deg_\Theta(x_i) = n - r$ for $i \in [r]$, all the edges between α and X can be colored this way. Since Θ is simple, $d_{G(\ell)}(u) \leqslant 1$ for $\ell \in [k]$ and $u \in X$. Let O be the set of colors $\ell \in [k]$ such that $\rho_\ell - e_\ell - d_\ell$ is odd. By (1.2) and (1.3), $(n - r - |O| - \sum_{\ell\in[k]} d_\ell)/2$ is a non-negative integer. By (3.4), $(\rho_\ell - e_\ell - d_\ell)/2 - \deg_\Theta(\ell) \geqslant 0$ for $\ell \in [k]\backslash O$ and $(\rho_\ell - e_\ell - d_\ell - 1)/2 - \deg_\Theta(\ell) \geqslant 0$ for $i \in O$. Since

$$\begin{aligned} &\frac{n - r - |O| - \sum_{\ell\in[k]} d_\ell}{2} \\ &\leqslant \frac{(n-r)^2 - |O| - \sum_{\ell\in[k]} d_\ell}{2} = \frac{n^2 - r^2 - (n-r)}{2} - r(n-r) - \frac{|O|}{2} \\ &\leqslant \sum_{\ell\in[k]} \frac{\rho_\ell - e_\ell - d_\ell}{2} - \sum_{\ell\in[k]} \deg_\Theta(\ell) - \frac{|O|}{2} \\ &= \sum_{\ell\in[k]\backslash O} \left(\frac{\rho_\ell - e_\ell - d_\ell}{2} - \deg_\Theta(\ell)\right) + \sum_{i\in O} \left(\frac{\rho_\ell - e_\ell - d_\ell - 1}{2} - \deg_\Theta(\ell)\right), \end{aligned}$$

there exists a sequence of integers $a_1, \dots, a_k$ such that

$$\begin{cases} a_1 + \cdots + a_k = \dfrac{n - r - |O| - \sum_{\ell \in [k]} d_\ell}{2}, \\ 0 \leqslant a_\ell \leqslant \dfrac{\rho_\ell - e_\ell - d_\ell}{2} - \deg_\Theta(\ell) & \text{for } \ell \in [k] \backslash O, \\ 0 \leqslant a_\ell \leqslant \dfrac{\rho_\ell - e_\ell - d_\ell - 1}{2} - \deg_\Theta(\ell) & \text{for } \ell \in O. \end{cases}$$

Now let

$$\overline{d_\ell} = \begin{cases} 2a_\ell & \text{for } \ell \in [k] \backslash O, \\ 2a_\ell + 1 & \text{for } \ell \in O. \end{cases}$$

The non-negative sequence $\overline{d_1}, \dots, \overline{d_k}$ satisfies the following.

$$\begin{cases} \overline{d_1} + \cdots + \overline{d_k} = n - r - \sum_{\ell \in [k]} d_\ell, \\ \overline{d_\ell} \equiv \rho_\ell - e_\ell - d_\ell \pmod 2 & \text{for } \ell \in [k], \\ \deg_\Theta(\ell) \leqslant \dfrac{\rho_\ell - e_\ell - d_\ell - \overline{d_\ell}}{2} & \text{for } \ell \in [k]. \end{cases} \tag{3.5}$$

We color the uncolored loops of G such that there are $\overline{d_\ell}$ 1-loops colored ℓ for $\ell \in [k]$, (so $\text{mult}_{G(\ell)}(\alpha) = d_\ell + \overline{d_\ell}$ for $\ell \in [k]$) and

$$\text{mult}_{G(\ell)}(\alpha^2) = \frac{1}{2}\left(\rho_\ell - e_\ell - d_\ell - \overline{d_\ell}\right) - \deg_\Theta(\ell) \quad \forall \ell \in [k].$$

This is possible for (3.5) and

$$\begin{aligned} \sum_{\ell \in [k]} \left(\rho_\ell - e_\ell - 2\deg_\Theta(\ell) - d_\ell - \overline{d_\ell})\right) &= n^2 - r^2 - 2r(n-r) - (n-r) \\ &= 2\binom{n-r}{2} = 2\,\text{mult}_G(\alpha^2). \end{aligned}$$

For $\ell \in [k]$,

$$|E^1(G(\ell))| + 2|E^2(G(\ell))| = e_\ell + 2\deg_\Theta(\ell) + \text{mult}_{G(\ell)}(\alpha) + 2\,\text{mult}_{G(\ell)}(\alpha^2) = \rho_\ell.$$

Finally, we have the following for $\ell \in [k]$, and so (3.3) holds.

$$\begin{aligned} \deg_{G(\ell)}(\alpha) &= \text{mult}_{G(\ell)}(\alpha, X) + \text{mult}_{G(\ell)}(\alpha) + 2\,\text{mult}_{G(\ell)}(\alpha^2) \\ &= \deg_\Theta(\ell) + d_\ell + \overline{d_\ell} + (\rho_\ell - e_\ell - 2\deg_\Theta(\ell) - d_\ell - \overline{d_\ell}) \\ &= \rho_\ell - e_\ell - \deg_\Theta(\ell) \\ &\leqslant n - r. \end{aligned}$$

Let

$$\begin{cases} g, f : V(\Gamma) \to \mathbb{N} \cup \{0\}, & \\ g(u) = f(u) = n - r & \text{for } u \in X, \\ g(\ell) = \rho_\ell - e_\ell + r \dot{-} n & \text{for } \ell \in [k], \\ f(\ell) = \lfloor (\rho_\ell - e_\ell - d_\ell)/2 \rfloor & \text{for } \ell \in [k]. \end{cases}$$

Clearly, $\Theta \subseteq \Gamma$ exists if and only if Γ has a (g, f)-factor, but by [CK16, Theorem 5], Γ has a (g, f)-factor if and only if the following conditions hold.

$$\sum_{i \in I} g(i) \leqslant \sum_{\ell \in [k]} \min \Big\{ f(\ell), \text{mult}_\Gamma(\ell I) \Big\} \qquad \forall I \subseteq X,$$
$$\sum_{\ell \in K} g(\ell) \leqslant \sum_{i \in [r]} \min \Big\{ f(i), \text{mult}_\Gamma(iK) \Big\} \qquad \forall K \subseteq [k].$$

Equivalently, we must have

$$(n - r)|I| \leqslant \sum_{\ell \in [k]} \min \left\{ \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor, \mu_I(\ell) \right\} \qquad \forall I \subseteq [r],$$
$$\sum_{\ell \in K} (\rho_\ell - e_\ell + r \dot{-} n) \leqslant \sum_{i \in [r]} \min \Big\{ n - r, \mu_K(i) \Big\} \qquad \forall K \subseteq [k].$$

By [HHKL90, Theorem 1], Γ has a (g, f)-factor if and only if

$$\sum_{a \notin A} f(a) \geqslant \sum_{a \in A} \Big(g(a) \dot{-} \deg_{G - \bar{A}}(a) \Big) \quad \forall A \subseteq X \cup Y,$$

or equivalently,

$$\sum_{i \in \bar{I}} (n - r) + \sum_{\ell \in \bar{K}} \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor \geqslant \sum_{\ell \in K} \Big(\rho_\ell - e_\ell - n + r \dot{-} \mu_I(\ell) \Big) + \sum_{i \in I} \Big(n - r \dot{-} \mu_K(i) \Big) \quad \forall I \subseteq [r], K \subseteq [k].$$

4. Corollaries and open problems

Recall that in order to embed an $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle L to to an $n \times n$ symmetric $\boldsymbol{\rho}$-latin square with a prescribed diagonal tail $\boldsymbol{d}$, it is necessary that

$$e_\ell \geqslant \rho_\ell + d_\ell + 2r - 2n \quad \forall \ell \in [k], \text{ and}$$
$$(n - r - D - q)/2 \text{ is a non-negative integer},$$

where $D := \sum_{\ell\in[k]} d_\ell$ and q is the number of symbols $\ell \in [k]$ such that $\rho_\ell + e_\ell + d_\ell$ is odd. We show that imposing slightly stronger assumptions will lead to much simpler conditions than those of Theorem 1.4. In our first application of Theorem 1.4, we assume that $e_\ell \geqslant \rho_\ell + r - n$ for $\ell \in [k]$; this in particular implies that $k \geqslant n + r$ for

$$k(n-r) = \sum_{\ell\in[k]}(n-r) \geqslant \sum_{\ell\in[k]}(\rho_\ell - e_\ell) = (n+r)(n-r).$$

Corollary 4.1. *An $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle with $e_\ell \geqslant \rho_\ell + r - n$ for $\ell \in [k]$ can be embedded in an $n \times n$ symmetric $\boldsymbol{\rho}$-latin square with a prescribed diagonal tail $\mathbf{d}$ if and only if $(n - r - D - q)/2$ is a non-negative integer, and any of the following conditions is satisfied.*

$$(n-r)|I| \leqslant \sum_{\ell\in[k]} \min\left\{ \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor, \mu_I(\ell) \right\} \qquad \forall I \subseteq [r];$$

$$\sum_{\ell\in K} \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor \geqslant \sum_{i\in[r]} \Big(n - r \dot{-} \mu_{\bar{K}}(i)\Big) \qquad \forall K \subseteq [k].$$

Proof. Suppose that $e_\ell \geqslant r - n + \rho_\ell$ for $\ell \in [k]$. Since $d_\ell \leqslant n - r$ for $\ell \in [k]$, (1.1) holds. Moreover, $\rho_\ell - e_\ell + r \dot{-} n = 0$ for $\ell \in [k]$, and consequently, (1.6) is satisfied. Modifying the proof of [HHKL90, Theorem 1], implies that the graph $\Gamma[X, [k]]$ of the proof of Theorem 1.4 has a (g, f)-factor with $g(\ell) = 0$ for $\ell \in [k]$ if and only if

$$\sum_{\ell\in K} f(\ell) \geqslant \sum_{i\in[r]} \Big(g(i) \dot{-} \deg_{G-K}(i)\Big) \quad \forall K \subseteq [k],$$

or equivalently,

$$\sum_{\ell\in K} \left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor \geqslant \sum_{i\in[r]} \Big(n - r \dot{-} \mu_{\bar{K}}(i)\Big) \quad \forall K \subseteq [k]. \qquad \square$$

In our next corollary, we assume that $e_\ell \geqslant 2r + d_\ell - \rho_\ell$ for $\ell \in [k]$; this in particular implies that $k \leqslant (n^2 + r^2 - n + r)/(2r)$ for

$$r^2 = \sum_{\ell\in[k]} e_\ell \geqslant \sum_{\ell\in[k]}(2r + d_\ell - \rho_\ell) = 2kr + (n-r) - n^2.$$

The following offers a simpler generalization of Theorems 1.2 and 1.3; see Remarks 4.5.

Corollary 4.2. *If $e_\ell \geqslant 2r + d_\ell - \rho_\ell$ for $\ell \in [k]$, then an $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle can be embedded in an $n \times n$ symmetric $\boldsymbol{\rho}$-latin square with a prescribed diagonal tail $\mathbf{d}$ if and only if $(n - r - D - q)/2$ is a non-negative integer, and any of the following conditions is satisfied.*

$$\sum_{\ell\in K}(\rho_\ell - e_\ell + r \dot{-} n) \leqslant \sum_{i\in[r]} \min\Big\{n - r, \mu_K(i)\Big\} \qquad \forall K \subseteq [k];$$

$$(n-r)(r-|I|) \geqslant \sum_{\ell\in[k]} \Big(\rho_\ell - e_\ell - n + r \dot{-} \mu_I(\ell)\Big) \qquad \forall I \subseteq [r].$$

Proof. Suppose that $e_\ell \geqslant 2r + d_\ell - \rho_\ell$ for $\ell \in [k]$. Since $\rho_\ell \leqslant n$ for $\ell \in [k]$, (1.1) is satisfied. Moreover, $r - e_\ell \leqslant (\rho_\ell - e_\ell - d_\ell)/2$ for $\ell \in [k]$. Thus, the following confirms that (1.5) holds.

$$\begin{aligned}(n-r)|I| \leqslant (k-r)|I| &= \sum_{u\in I} \deg_\Gamma(u) = \sum_{\ell\in N_\Gamma(I)} \mathrm{mult}_\Gamma(\ell I)\\ &= \sum_{\ell\in N_\Gamma(I)} \min\Big\{\mathrm{mult}_\Gamma(\ell I), \deg_\Gamma(\ell)\Big\}\\ &\leqslant \sum_{\ell\in N_\Gamma(I)} \min\Big\{\mathrm{mult}_\Gamma(\ell I), \Big\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2}\Big\rfloor\Big\}\\ &= \sum_{\ell\in [k]} \min\Big\{\Big\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2}\Big\rfloor, \mu_I(\ell)\Big\} \qquad \forall I \subseteq [r].\end{aligned}$$

Let

$$\begin{cases} g, f : V(\Gamma) \to \mathbb{N} \cup \{0\}, \\ g(u) = f(u) = n - r & \text{for } u \in X, \\ g(\ell) = \rho_\ell - e_\ell + r \dot- n & \text{for } \ell \in [k], \\ f(\ell) = \beta & \text{for } \ell \in [k], \end{cases}$$

where β is a sufficiently large number. The graph Γ of the proof of Theorem 1.4 has a (g, f)-factor if and only if

$$\begin{aligned}(n-r)(r-|I|) + \sum_{\ell\in \bar K} \beta \geqslant &\sum_{\ell\in K} \Big(\rho_\ell - e_\ell - n + r \dot- \mu_I(\ell)\Big)\\ &+ \sum_{i\in I} \Big(n - r \dot- \mu_K(i)\Big) \qquad \forall I \subseteq X, K \subseteq [k]. \qquad (4.1)\end{aligned}$$

For $K \neq [k]$, (4.1) is trivial, and for $K = [k]$, it simplifies to the following.

$$\begin{aligned}(n-r)(r-|I|) &\geqslant \sum_{\ell\in [k]} \Big(\rho_\ell - e_\ell - n + r \dot- \mu_I(\ell)\Big) + \sum_{i\in I}\Big(n - r \dot- \mu_{[k]}(i)\Big)\\ &= \sum_{\ell\in [k]} \Big(\rho_\ell - e_\ell - n + r \dot- \mu_I(\ell)\Big) + \sum_{i\in I}\Big(n - r \dot- (k-r)\Big)\\ &= \sum_{\ell\in [k]} \Big(\rho_\ell - e_\ell - n + r \dot- \mu_I(\ell)\Big) \qquad \forall I \subseteq [r].\end{aligned}$$

□

Corollary 4.3. *If*

$$2r + d_\ell - e_\ell \leqslant \rho_\ell \leqslant n - r + e_\ell \quad \forall \ell \in [k],$$

then an $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle can be embedded in an $n \times n$ symmetric $\boldsymbol{\rho}$-latin square with a prescribed diagonal tail $\mathbf{d}$ if and only if $(n - r - D - q)/2$ is a non-negative integer.

Proof. Since (1.1) holds, and the inequality in (3.4) is trivial, $\Theta \subseteq \Gamma$ always exists. □

Finally, the following result offers a very simple generalization of Theorems 1.2 and 1.3; see Remarks 4.5.

Corollary 4.4. *Suppose that*

$$(k-r)\left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor \geqslant (n-r)(r-e_\ell) \geqslant (k-r)(\rho_\ell - e_\ell - n + r) \qquad \forall \ell \in [k];$$
$$(r-e_{\ell'})\left\lfloor \frac{\rho_\ell - e_\ell - d_\ell}{2} \right\rfloor \geqslant (r-e_\ell)(\rho_{\ell'} - e_{\ell'} - n + r) \qquad \forall \ell, \ell' \in [k].$$

Then an $r \times r$ symmetric $\boldsymbol{\rho}$-latin rectangle can be completed to an $n \times n$ symmetric $\boldsymbol{\rho}$-latin square with a prescribed diagonal tail $\mathbf{d}$ if and only if $(n-r-D-q)/2$ is a non-negative integer.

Proof. Using the first inequality, we have $(\rho_\ell - e_\ell - d_\ell)/2 \geqslant \lfloor (\rho_\ell - e_\ell - d_\ell)/2 \rfloor \geqslant \rho_\ell - e_\ell - n + r$ which implies (1.1). By [HHKL90, Corollary 2], if for all pairs of vertices x, y of the bigraph $\Gamma[X, [k]]$,

$$f(x)\deg_\Gamma(y) \geqslant g(y)\deg_\Gamma(x), \tag{4.2}$$

then Γ has a (g, f)-factor (and so $\Theta \subseteq \Gamma$ satisfying (3.4) exists). Recall that $\deg_\Gamma(u) = k - r$, $g(u) = f(u) = n - r$ for $u \in X$, and $\deg_\Gamma(\ell) = r - e_\ell$, $g(\ell) = \rho_\ell - e_\ell + r \dot{-} n$, $f(\ell) = \lfloor (\rho_\ell - e_\ell - d_\ell)/2 \rfloor$ for $\ell \in [k]$. Since for $x, y \in X$ (4.2) is trivial, the proof is complete. □

Remark 4.5. Both Corollaries 4.2 and 4.4 generalize the Andersen–Hoffman Theorem as well as Cruse's Theorem. To see this, let $\rho_1 = \cdots = \rho_k = n = k$. Both inequalities in Corollary 4.4 simplify to

$$\left\lfloor \frac{n - e_\ell - d_\ell}{2} \right\rfloor \geqslant r - e_\ell \quad \forall \ell \in [n],$$

which is equivalent to the hypothesis of Corollary 4.2, $e_\ell \geqslant 2r + d_\ell - n$ for $\ell \in [n]$. Since

$$\begin{aligned}\sum_{\ell \in K}(\rho_\ell - e_\ell + r \dot{-} n) &= \sum_{\ell \in K}(n - e_\ell + r \dot{-} n) = \sum_{\ell \in K}(r - e_\ell)\\ &= \sum_{i \in [r]} \mu_K(i) = \sum_{i \in [r]} \min\left\{n - r, \mu_K(i)\right\} \quad \forall K \subseteq [n],\end{aligned}$$

$$\begin{aligned}\sum_{\ell \in [k]}\left(\rho_\ell - e_\ell - n + r \dot{-} \mu_{\bar{I}}(\ell)\right) &= \sum_{\ell \in [k]}\left(r - e_\ell \dot{-} \mu_{\bar{I}}(\ell)\right) = \sum_{\ell \in [k]}\left(r - e_\ell - \mu_{\bar{I}}(\ell)\right)\\ &= \sum_{\ell \in [n]} \mathrm{mult}_\Gamma(\ell I) = |I|(n-r) \quad \forall I \subseteq [r],\end{aligned}$$

the long inequalities in Corollary 4.2 are trivial. Recall that $D = \sum_{\ell \in [n]} d_\ell$ and q is the number of symbols $\ell \in [n]$ such that $n + e_\ell + d_\ell$ is odd. If $D = n - r$, the condition that

$(n-r-D-q)/2 = -q/2$ is a non-negative integer is equivalent to $q = 0$, and consequently, $e_\ell + d_\ell + n$ is even for $\ell \in [n]$. If $D = 0$, the condition that $(n-r-D-q)/2 = (n-r-q)/2$ is a non-negative integer is equivalent to $q \leqslant n-r$, and consequently, the number of symbols $\ell \in [n]$ such that $e_\ell - n$ is even, is at least r.

A ρ-latin square L is *diagonal* if each $\ell \in [k]$ occurs at most once on the diagonal, and is *idempotent* if $L_{ii} = i$ for $i \in [n]$. Completing partial idempotent latin squares has a rich history (see [AR12]). Theorem 1.4 in particular settles the necessary and sufficient conditions that ensure an $r \times r$ symmetric diagonal (or idempotent) $\boldsymbol{\rho}$-latin rectangle L can be embedded in an $n \times n$ symmetric diagonal (or idempotent) $\boldsymbol{\rho}$-latin square, but the following problem remains unsolved.

Question 4.6. Find necessary and sufficient conditions that ensure that an idempotent $r \times s$ $\boldsymbol{\rho}$-latin rectangle can be extended to an idempotent $n \times n$ $\boldsymbol{\rho}$-latin square.

We remark that Problem 4.6 is open even for latin squares [AR12]. The most general result up to date is Rodger's theorem that settles the problem for latin squares when $r = s$, $n \geqslant 2r$ [Rod83, Rod84]. Two other notable results are [AHR82] and [ABHR22].

Let $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_k)$ with $1 \leqslant \rho_\ell \leqslant n \leqslant k$ for $\ell \in [k]$ such that $\sum_{\ell \in [k]} \rho_\ell = n^2$. A *partial $\boldsymbol{\rho}$-latin square* L of order n is an $n \times n$ array that is partially filled using k different symbols each occurring at most once in each row and at most once in each column, such that each symbol i occurs at most ρ_ℓ times in L for $\ell \in [k]$. We say that L is *critical* if it can be extended to exactly one $\boldsymbol{\rho}$-latin square of order n, but removal of any entry of L destroys the uniqueness of the extension, and the number of non-empty cells of L is the *size* of the critical partial $\boldsymbol{\rho}$-latin square.

Question 4.7. Find good bounds for the smallest and largest sizes of critical partial $\boldsymbol{\rho}$-latin squares.

For latin squares, it is known that the size of the largest critical set is between and $n^2 - O(n^{5/3})$ and $n^2/-7n/2+o(n)$, and it is conjectured that the size of the smallest critical set is $\lfloor n^2/4 \rfloor$; see [CD07, Section 1.8].

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References

[ABHR22] Lars D. Andersen, Stacie Baumann, Anthony J. W. Hilton, and C. A. Rodger. On completing partial Latin squares with prescribed diagonals. *Electron. J. Combin.*, 29(3):Paper No. 3.47, 14, 2022. doi:10.37236/10705.

[AHR82] L. D. Andersen, A. J. W. Hilton, and C. A. Rodger. A solution to the embedding problem for partial idempotent Latin squares. *J. London Math. Soc. (2)*, 26(1):21–27, 1982. doi:10.1112/jlms/s2-26.1.21.

[And82] Lars Døvling Andersen. Embedding Latin squares with prescribed diagonal. In *Algebraic and geometric combinatorics*, volume 65 of *North-Holland Math. Stud.*, pages 9–26. North-Holland, Amsterdam, 1982. doi:10.1016/S0304-0208(08)73250-X.

[AR12] Atif A. Abueida and C. A. Rodger. Completing a solution of the embedding problem for incomplete idempotent Latin squares when numerical conditions suffice. *Discrete Math.*, 312(22):3328–3334, 2012. doi:10.1016/j.disc.2012.07.026.

[Bah12] Amin Bahmanian. Detachments of hypergraphs I: The Berge-Johnson problem. *Combin. Probab. Comput.*, 21(4):483–495, 2012. doi:10.1017/S0963548312000041.

[Bah22] Amin Bahmanian. Ryser's theorem for ρ-Latin rectangles. *J. Combin. Theory Ser. A*, 190:Paper No. 105632, 21, 2022. doi:10.1016/j.jcta.2022.105632.

[BR04] Darryn Bryant and C. A. Rodger. On the completion of Latin rectangles to symmetric Latin squares. *J. Aust. Math. Soc.*, 76(1):109–124, 2004. doi:10.1017/S1446788700008739.

[CD07] Charles J. Colbourn and Jeffrey H. Dinitz, editors. *Handbook of combinatorial designs*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, second edition, 2007.

[CK16] R. Cymer and Mikio Kano. Generalizations of marriage theorem for degree factors. *Graphs Combin.*, 32(6):2315–2322, 2016. doi:10.1007/s00373-016-1699-6.

[Cru74] Allan B. Cruse. On embedding incomplete symmetric Latin squares. *J. Combinatorial Theory Ser. A*, 16(1):18–22, 1974. doi:10.1016/0097-3165(74)90068-5.

[GHHO15] J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman, and Sibel Özkan. Hall's theorem and extending partial Latinized rectangles. *J. Combin. Theory Ser. A*, 130:26–41, 2015. doi:10.1016/j.jcta.2014.10.007.

[Hal45] Marshall Hall. An existence theorem for Latin squares. *Bull. Amer. Math. Soc.*, 51:387–388, 1945. doi:10.1090/S0002-9904-1945-08361-X.

[HHKL90] Katherine Heinrich, Pavol Hell, David G. Kirkpatrick, and Gui Zhen Liu. A simple existence criterion for $(g < f)$-factors. *Discrete Math.*, 85(3):313–317, 1990. doi:10.1016/0012-365X(90)90387-W.

[Hof83] D. G. Hoffman. Completing incomplete commutative Latin squares with prescribed diagonals. *European J. Combin.*, 4(1):33–35, 1983. doi:10.1016/S0195-6698(83)80006-7.

[Lov72] L. Lovász. The factorization of graphs. II. *Acta Math. Acad. Sci. Hungar.*, 23:223–246, 1972. doi:10.1007/BF01889919.

[Rod83] C. A. Rodger. Embedding incomplete idempotent Latin squares. In *Combinatorial mathematics, X (Adelaide, 1982)*, volume 1036 of *Lecture Notes in Math.*, pages 355–366. Springer, Berlin, 1983. doi:10.1007/BFb0071529.

[Rod84] C. A. Rodger. Embedding an incomplete Latin square in a Latin square with a prescribed diagonal. *Discrete Math.*, 51(1):73–89, 1984. doi:10.1016/0012-365X(84)90025-6.

[Rod92] Chris A. Rodger. Recent results on the embedding of Latin squares and related structures, cycle systems and graph designs. *Matematiche (Catania)*, 47(2):295–311 (1993), 1992. Combinatorics 92 (Catania, 1992).

[Rys51] H. J. Ryser. A combinatorial theorem with an application to latin rectangles. *Proc. Amer. Math. Soc.*, 2:550–552, 1951. doi:10.2307/2032005.