COMBINATORIAL THEORY **6 (1)** (2026), #2 combinatorial-theory.org

Branching rules of minuscule representations via a new partial order

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Submitted: Feb 8, 2024; Accepted: Jul 9, 2025; Published: Apr 20, 2026

Abstract. We introduce a new partial order on the set of all antichains of a fixed size in any poset. When applied to minuscule posets, these partial orders give rise to distributive lattices that appear in the branching rules for minuscule representations of complex simple Lie algebras.

Keywords. Antichain, distributive lattice, minuscule representation, branching rule

Mathematics Subject Classifications. 06A07, 05E10, 06A11

1. Introduction

This paper introduces a new partial order $\leqslant_k$ on the set $\mathcal{A}_k(P)$ of all antichains of a fixed size k in any poset P. We assume basic familiarity with poset theory, including the notions of antichains, order ideals, order filters, covering relations, Hasse diagrams, products of posets, and distributive lattices. These notions can all be found in [Sta97, §3], whose definitions and notations we will follow. In particular, we write $a \lessdot_P b$ to indicate two elements a, b are in a covering relation in a poset P. We denote the sets of positive integers and nonnegative integers by $\mathbb{Z}_+$ and $\mathbb{N}$, respectively. For each $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, 2, \cdots, n\}$, viewed as a poset with the natural order. Unless otherwise stated, all posets in the paper will be finite.

It is well known that for a (finite) poset P, there is a bijection between the set $J(P)$ of ideals of P and the set $\mathcal{A}(P)$ of antichains of P, given by associating an ideal with its set of maximal elements. The containment order on $J(P)$ then induces a partial order on $\mathcal{A}(P)$, which we will denote by $\leqslant_J$, and which we may restrict to the set $\mathcal{A}_k(P)$ for each $k \in \mathbb{N}$. It is a classic result of Dilworth [Dil60] that when k is the *width* of P, defined as $\mathrm{width}(P) = \max\{|A| : A \in \mathcal{A}(P)\}$, the set $\mathcal{A}_k(P)$ is a distributive lattice under the restriction of $\leqslant_J$.

The new partial order $\leqslant_k$ we introduce on $\mathcal{A}_k(P)$ is defined as the reflexive transitive extension of the relation $\prec_k$, where we declare $A \prec_k B$ for $A, B \in \mathcal{A}_k(P)$ if $B = A \setminus \{a\} \cup \{b\}$ for elements $a, b \in P$ such that $a \lessdot_P b$. As we show in Section 2, the order $\leqslant_k$ is coarser than the restriction of $\leqslant_J$ in general, and $\mathcal{A}_k(P)$ might not be a distributive lattice under the order $\leqslant_k$ when $k = \mathrm{width}(P)$.

The order $\leqslant_k$ has striking properties when applied to minuscule posets in the sense of Proctor [Pro84]. Recall that a finite dimensional representation of a simple Lie algebra is called a minuscule representation if the Weyl group of the Lie algebra acts transitively on the weights of the representation. A minuscule poset is a poset P for which the poset $J(P)$ is isomorphic to the weight poset of a minuscule representation. Both minuscule representations and minuscule posets have well-known and explicit classifications. In particular, the minuscule posets are the posets of the forms $[a] \times [b]$, $J([n] \times [2])$, $J^m([2] \times [2])$, $J^2([2] \times [3])$, and $J^3([2] \times [3])$ where $a, b, n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$. The Hasse diagrams of these posets are shown in Figure 1.1.

In Theorem 5.2, we determine the posets of form $\mathcal{A}_k(P)$ for all minuscule posets P, and we show that they are distributive lattices in all cases. This is notable since, as mentioned earlier, $\mathcal{A}_k(P)$ does not have to be a distributive lattice for a general poset P. More remarkably, the sets $\mathcal{A}_k(P)$ for minuscule posets P retain intimate connections to minuscule representations: it turns out that the order $\leqslant_k$ can be used to determine the branching rules of all minuscule representations up to diagram automorphisms of Lie algebras; see Theorem 6.4 and Remark 6.2. For example, for the minuscule poset $P = [a] \times [b]$, the poset $J(P)$ is isomorphic to the weight poset of a suitable minuscule representation V for a simple Lie algebra of type A, and $\mathcal{A}_k(P)$ controls the branching of V. In this case, the poset $\mathcal{A}_k(P)$ is naturally isomorphic to the poset $\mathcal{D}_k([a] \times [b])$ of Young diagrams of Durfee length k that fit into an $a \times b$ box; see Definition 3.6 and Corollary 4.3.

Our motivation for studying the partial order $\leqslant_k$ comes from our earlier work [GX23] on Kazhdan–Lusztig cells of $\mathbf{a}$-value 2, where $\mathbf{a}$ is Lusztig's $\mathbf{a}$-function. For $\mathbf{a}(2)$-finite Coxeter groups (as defined and described in [GX23]), every element w of $\mathbf{a}$-value 2 has an associated heap poset H. A key property of H can be summarized as follows: the poset $\mathcal{A}_2(H)$ has a minimum element with respect to $\leqslant_2$, and the ideal generated by the minimal element determines the left cell of w. A similar statement holds for maximal elements and right cells.

The rest of the paper is organized as follows. We introduce the order $\leqslant_k$ and study its basic properties in Section 2. We introduce a family of posets we call binomial posets in Section 3; the section does not treat the order $\leqslant_k$ directly, but the binomial posets will provide a useful model for the subsequent parts of the paper. Sections 4 and 5 study the poset $\mathcal{A}_k(P)$ for minuscule posets of type A and of all other types, respectively, culminating in the explicit descriptions of all such posets $\mathcal{A}_k(P)$ in Theorem 5.2. Section 6 explains how the order $\leqslant_k$ relates to branching rules of minuscule representations of simple Lie algebras when applied to minuscule posets. Finally, we discuss several open questions related to the order $\leqslant_k$ in Section 7.

2. A new partial order on antichains

Throughout this section, let P be a poset, let $k \in \mathbb{N}$, and let $\mathcal{A}_k(P)$ be the set of antichains of P of cardinality k. Recall from the introduction that the containment order on ideals of P induces an order $\leqslant_J$ on the antichains of P. The order $\leqslant_J$ can be described without reference to ideals

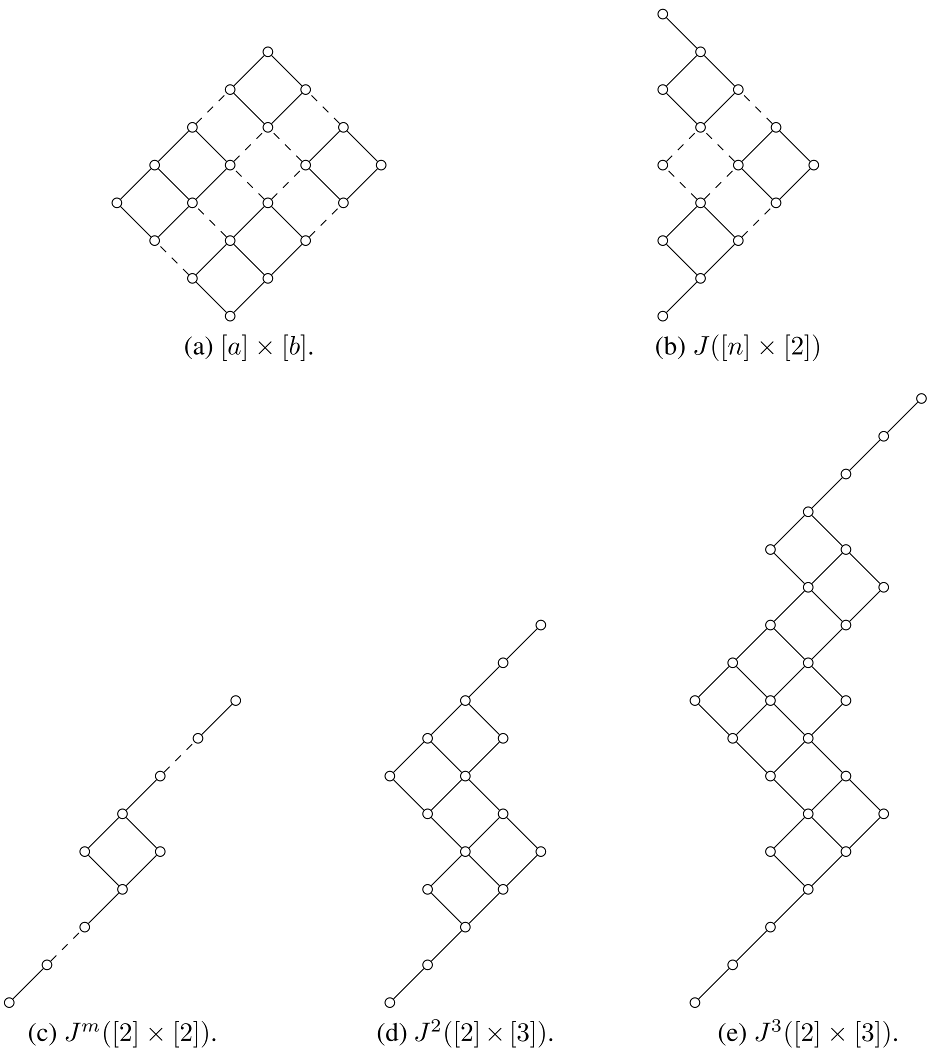


Figure 1.1: Minuscule posets.

as follows: for any $A, B \in \mathcal{A}(P)$, we have

$$A \leqslant_J B \quad \Longleftrightarrow \quad \forall\, a \in A, \exists\, b \in B : a \leqslant_P b.$$

In this section, we will show that the partial order $\leqslant_k$ defined in the introduction is indeed a partial order, study its basic properties, and discuss its relationship with $\leqslant_J$. Recall that $\leqslant_k$ is defined as follows.

Definition 2.1. For any $A, B \in \mathcal{A}_k(P)$, we write $A \prec_k B$ if $A \backslash B = \{a\}$ and $B \backslash A = \{b\}$ are singleton sets with the property that $a <_P b$. We define $\leqslant_k$ to be the reflexive transitive extension of $\prec_k$ on $\mathcal{A}_k(P)$.

Lemma 2.2. *The relation $\leqslant_k$ of Definition 2.1 is a partial order on the set $\mathcal{A}_k(P)$, and the restriction of the partial order $\leqslant_J$ to $\mathcal{A}_k(P)$ refines $\leqslant_k$.*

Proof. The definition of $\prec_k$ implies that if $A, B \in \mathcal{A}_k(P)$ satisfy $A \prec_k B$, then we have $A \leqslant_J B$. The antisymmetry of $\leqslant_J$ then implies the antisymmetry of $\leqslant_k$, and it follows in turn that $\leqslant_k$ is a partial order refined by the restriction of $\leqslant_J$ to $\mathcal{A}_k(P)$. □

Remark 2.3. It is immediate from Definition 2.1 that $\mathcal{A}_k(P)$ is nonempty if and only if $0 \leqslant k \leqslant \mathrm{width}(P)$, that $\mathcal{A}_0(P)$ is the singleton poset, and that $\mathcal{A}_1(P)$ is canonically isomorphic to P itself.

Proposition 2.4. *Let $A, B \in \mathcal{A}_k(P)$.*

(i) If $A \leqslant_k B$, then the elements of $A = \{a_1, a_2, \cdots, a_k\}$ and $B = \{b_1, b_2, \cdots, b_k\}$ can be ordered in such a way that $a_i \leqslant_P b_i$ for all $1 \leqslant i \leqslant k$.

(ii) The elements A and B are in a covering relation $A \lessdot_{\mathcal{A}_k(P)} B$ if and only if we have both (1) $A \prec_k B$ and (2) the unique elements $a \in A \backslash B$ and $b \in B \backslash A$ satisfy $a \lessdot_P b$.

Proof. If $A \prec_k B$, then (i) follows by the definition of $\prec_k$; the general case follows by induction.

To prove (ii), assume first that $A \lessdot_{\mathcal{A}_k(P)} B$. By the definition of $\leqslant_k$, we must have $A \prec_k B$. We then have $A = C \cup \{a\}$ and $B = C \cup \{b\}$, where $C = A \cap B \in \mathcal{A}_{k-1}(P)$ and $a, b \in P$ satisfy $a <_P b$. Suppose that some element $x \in P$ satisfies $a <_P x <_P b$. Then the set $C' := C \cup \{x\}$ must be an antichain in $\mathcal{A}_k(P)$ for the following reason: we cannot have $x \leqslant_P c$ for any $c \in C$ because $a <_P x \leqslant_P c$ and A is an antichain, and we cannot have $c \leqslant_P x$ for any $c \in C$ because $c \leqslant_P x <_P b$ and B is an antichain. It follows that $A <_k C' <_k B$, which is a contradiction, so we have $a \lessdot_P b$.

Conversely, assume that $A, B \in \mathcal{A}_k(P)$ satisfy $A \prec_k B$, and also that $a \lessdot_P b$, where $C = A \cap B = \{c_1, c_2, \cdots, c_{k-1}\}$, $A = C \cup \{a\}$, and $B = C \cup \{b\}$. Write $A = \{a_1, a_2, \cdots, a_k\}$, where $a_i = c_i$ for $i < k$ and $a_k = a$, and $B = \{b_1, b_2, \cdots, b_k\}$, where $b_i = c_i$ for $i < k$ and $b_k = b$. Suppose for a contradiction that there exists $X = \{x_1, x_2, \cdots, x_k\} \in \mathcal{A}_k(P)$ such that $A <_k X$ and $X <_k B$. It follows from (i) that there are permutations σ and τ of $\{1, 2, \cdots, k\}$ such that for all $1 \leqslant i \leqslant k$, we have $a_i \leqslant_P x_{\sigma(i)}$ and $x_i \leqslant_P b_{\tau(i)}$, which implies that $a_i \leqslant_P b_{\tau(\sigma(i))}$. Because A and B are antichains, we must have $\tau(\sigma(i)) = i$ for all $1 \leqslant i < k$, and this implies that $\tau = \sigma^{-1}$. By relabelling X if necessary, we may assume that σ and τ are both the identity permutation, and that $X = C \cup \{x_k\}$, where $a = a_k <_P x_k <_P b_k = b$. This contradicts the hypothesis that $a \lessdot_P b$, and (ii) follows. □

To compare $\leqslant_k$ with $\leqslant_J$, we first note that the order $\leqslant_k$ may be strictly coarser than the restriction of the order $\leqslant_J$ to $A_k(P)$. For example, consider the poset $P = \{a, b, c, d, e\}$ with covering relations $a \lessdot_P c$, $b \lessdot_P c$, $c \lessdot_P d$, and $c \lessdot_P e$. Then $\mathrm{width}(P) = 2$, and the set $\mathcal{A}_2(P)$ consists of the two antichains, $\{a, b\}$ and $\{d, e\}$. These antichains are comparable in the partial order $\leqslant_J$, but not in the order $\leqslant_k$ by Proposition 2.4 (ii). This shows that $\leqslant_2$ strictly coarsens $\leqslant_J$. The same example also shows that the converse of Proposition 2.4 (i) does not hold. On the other hand, we note that for some important classes of examples, the partial orders $\leqslant_J$ and $\leqslant_k$ on the maximal antichains of P are identical. Examples of such posets include the heaps of fully commutative elements (in the sense of [Ste96]) in finite Coxeter groups, and more generally in star reducible Coxeter groups (in the sense of [Gre06]).

Dilworth [Dil60, Theorem 2.1] proved that $\mathcal{A}_k(P)$ is a distributive lattice under the partial order $\leqslant_J$ for $k = \mathrm{width}(P)$. The poset P from the previous paragraph proves that this is not the case for the order $\leqslant_k$, since $\mathcal{A}_2(P) = \mathcal{A}_{\mathrm{width}(P)}(P)$ has no maximum or minimum element under $\leqslant_k$. Remarkably, however, all the nonempty posets $\mathcal{A}_k(P)$ are distributive lattices under the order $\leqslant_k$ if P is a minuscule poset, as we will prove in Theorem 5.2. Note that by Remark 2.3, for $\mathcal{A}_k(P)$ to be a distributive lattice for all values of k, the poset P must be a distributive lattice itself, but this is not a sufficient condition: if $P = \{a, b, c\}$ is an antichain with three elements, then $J(P)$ is a distributive lattice, but $\mathcal{A}_2(J(P))$ is not because it has no maximum or minimum element.

3. Binomial posets

In this section we study a family of posets, which we call binomial posets, that will be used extensively in the descriptions of posets of the form $\mathcal{A}_k(P)$. We show that binomial posets naturally parameterize posets of the form $J([a] \times [b])$ (Proposition 3.3) and a poset $\mathcal{D}_k([a] \times [b])$ that arises in the context of Young diagrams (Proposition 3.7).

Definition 3.1. For any $k, n \in \mathbb{N}$ such that $k \leqslant n$, we define $\mathcal{C}(n, k)$ to be the poset consisting of strictly increasing length-k sequences with entries from $[n]$, ordered by coordinatewise comparison: for sequences $\mathbf{x} = (x_1, x_2, \cdots, x_k)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_k)$ in $\mathcal{C}(n, k)$, we define $\mathbf{x} \leqslant_{\mathcal{C}(n,k)} \mathbf{y}$ if $x_i \leqslant y_i$ for all $1 \leqslant i \leqslant k$. We abbreviate the notation $\leqslant_{\mathcal{C}(n,k)}$ to $\leqslant_{\mathcal{C}}$ and $\lessdot_{\mathcal{C}(n,k)}$ to $\lessdot_{\mathcal{C}}$.

The fact that $\mathcal{C}(n, k)$ is a poset is immediate from the above definition. The name "binomial poset" is motivated by the fact that we may naturally identify $\mathcal{C}(n, k)$ with the set of all size-k subsets of $[n]$, by identifying each sequence in $\mathcal{C}(n, k)$ with its set of entries.

Lemma 3.2. *Let k, n be nonnegative integers such that $k \leqslant n$.*

(i) *The function $\rho : \mathcal{C}(n, k) \to \mathbb{N}$ defined by*

$$\rho((x_1, \cdots, x_k)) = x_1 + x_2 + \cdots + x_k$$

is a rank function on $\mathcal{C}(n, k)$. In other words, if $\mathbf{x}, \mathbf{y} \in \mathcal{C}(n, k)$ satisfy $\mathbf{x} \leqslant_{\mathcal{C}} \mathbf{y}$, then we have $\rho(\mathbf{x}) \leqslant \rho(\mathbf{y})$, with $\rho(\mathbf{y}) = \rho(\mathbf{x}) + 1$ if and only if $\mathbf{x} \lessdot_{\mathcal{C}} \mathbf{y}$.

(ii) *Two elements $\mathbf{x} = (x_1, \cdots, x_k)$ and $\mathbf{y} = (y_1, \cdots, y_k)$ in $\mathcal{C}(n, k)$ are in a covering relation $\mathbf{x} \lessdot_{\mathcal{C}} \mathbf{y}$ if and only if there exists $i \in [k]$ such that $y_i = x_i + 1$ and $y_j = x_j$ for all $j \in [k] \backslash \{i\}$.*

Proof. The definition of $\leqslant_{\mathcal{C}}$ implies that $\rho(\mathbf{x}) < \rho(\mathbf{y})$ whenever $\mathbf{x} <_{\mathcal{C}} \mathbf{y}$. It follows that if $\mathbf{x} <_{\mathcal{C}} \mathbf{y}$ and $\rho(\mathbf{y}) = \rho(\mathbf{x}) + 1$, then $\mathbf{x} \lessdot_{\mathcal{C}} \mathbf{y}$. To prove the converse, suppose that $\mathbf{x} <_{\mathcal{C}} \mathbf{y}$ for elements $\mathbf{x} = (x_1, x_2, \cdots, x_k)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_k)$ in $\mathcal{C}(n, k)$. Choose $1 \leqslant r \leqslant k$ to be the maximal index satisfying $x_r < y_r$, and let $\mathbf{z} = (\mathbf{x} \backslash \{x_r\}) \cup \{x_r + 1\}$. Note that the set $\mathbf{z}$ still consists of k distinct numbers: if $r = k$ then $\mathbf{z}$ is obtained from $\mathbf{x}$ by increasing the largest entry in $\mathbf{x}$, while if $r < k$ then we have $x_r + 1 \leqslant y_r < y_{r+1} = x_{r+1}$. Note also that regardless of whether $r = k$, we have $x \leqslant_{\mathcal{C}} z \leqslant_{\mathcal{C}} y$ and $\rho(\mathbf{z}) = \rho(\mathbf{x}) + 1$. It follows that if $\mathbf{x} \lessdot_{\mathcal{C}} \mathbf{y}$ then we must have $\mathbf{z} = \mathbf{y}$ and $\rho(\mathbf{y}) = \rho(\mathbf{x}) + 1$. This completes the proof of (i), and (ii) follows immediately. □

We now discuss two applications of binomial posets. The first application concerns ideals of $[a] \times [b]$, and is illustrated in Example 3.4. We think of $[a] \times [b]$ as embedded in the lattice $\mathbb{Z}_+^2 := \{(x, y) : x, y \in \mathbb{Z}_+\}$, where a point (i_1, j_1) is smaller than another point (i_2, j_2) if and only if (i_1, j_1) lies weakly to the southwest of (i_2, j_2). It follows that if I is an ideal in $[a] \times [b]$ and we define m_j to the maximal integer $i \in [a]$ such that $(i, j) \in I$ for each $j \in [b]$, then each m_j records the number of elements in I in Row j (i.e., in the set $\{(k, j) : k \in \mathbb{Z}_+\}$), and the sequence

$$\mathbf{x}_I = (m_b, \cdots, m_1)$$

is a weakly increasing sequence such that $0 \leqslant m_b \leqslant m_{b-1} \leqslant \cdots \leqslant m_1 \leqslant a$. The sequence

$$\mathbf{x}'_I = (m_b + 1, m_{b-1} + 2, \cdots, , m_1 + b)$$

is then an element of the binomial poset $\mathcal{C}(a + b, b)$. Furthermore, it is routine to verify that conversely every sequence in $\mathcal{C}(a+b, b)$ has the form $\mathbf{x}'_I$ for a unique ideal I of $[a] \times [b]$, and that the map $f : J([a] \times [b]) \to \mathcal{C}(a + b, a), I \mapsto \mathbf{x}'_I$ satisfies the condition that $I_1 \subseteq I_2$ if and only if $f(I_1) \leqslant_{\mathcal{C}} f(I_2)$ for all $I_1, I_2 \in J([a] \times [b])$. We have thus proved the following result (which holds trivially if either a or b is zero).

Proposition 3.3. *For any $a, b \in \mathbb{N}$, the posets $J([a] \times [b])$ and $\mathcal{C}(a + b, b)$ are isomorphic.* □

Example 3.4. If $a = 6, b = 5$, and I consists of the filled vertices in the grid $[a] \times [b]$ shown in Figure 3.1, then we have $x_I = (0, 1, 4, 6, 6)$ and $x'_I = (1, 3, 7, 10, 11) \in \mathcal{C}(11, 5)$.

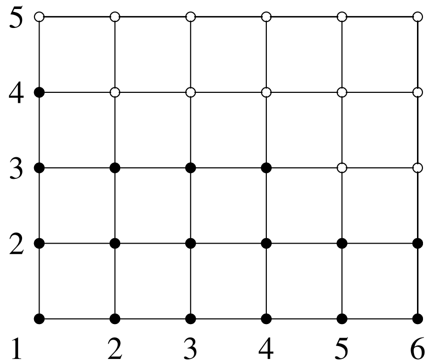


Figure 3.1: An ideal in the poset $[6] \times [5] \subset \mathbb{Z}_+^2$.

Remark 3.5. (i) Since $J(P)$ is a distributive lattice for any finite poset P, Proposition 3.3 implies that binomial posets are distributive lattices.

(ii) Given two posets P and Q, the map $P \times Q \to Q \times P, (p, q) \mapsto (q, p)$ is clearly a poset isomorphism, so Proposition 3.3 also implies that $\mathcal{C}(a+b, b) \cong J([a]\times[b]) \cong J([b]\times[a]) \cong \mathcal{C}(a + b, a)$ as posets for all $a, b \in \mathbb{N}$.

The second application of binary posets concerns Young diagrams or, equivalently, Ferrers diagrams (see [Sta97, §1.7]). Using the French notation, we may conveniently view Ferrers diagrams to be finite ideals of the infinite poset $\mathbb{Z}_+^2$. Our goal is to use binomial posets to parameterize the posets $\mathcal{D}_k(a, b)$ defined below.

Definition 3.6. (i) We define the *Durfee length* of a Ferrers diagram D to be the largest integer $k \in \mathbb{N}$ such that $[k] \times [k] \subseteq D$, i.e., the side length of the largest square grid S that fits inside D, and we call S the *Durfee square* of D.

(ii) For any $a, b, k \in \mathbb{N}$ with $k \leqslant \min(a, b)$, we define $\mathcal{D}_k(a, b)$ to be the poset of all Ferrers diagrams with Durfee length k contained in the set $[a] \times [b]$, ordered by set containment.

Given any Ferrers diagram D in the poset $\mathcal{D}_k([a] \times [b])$, we may naturally decompose D into three parts: the Durfee square $S = [k] \times [k]$ of D; the part $I_1 = \{(i, j) \in D : j > k\}$ above S; and the part $I_2 = \{(i, j) \in D : i > k\}$ to the right of S. Shifting I_1 down and I_2 to the left by k, we obtain ideals $I_1' = \{(i, j - k) : (i, j) \in I_1\}$ and $I_2' = \{(i - k, j) : (i, j) \in I_2\}$ in the grids $[k] \times [b - k]$ and $[a - k] \times [k]$, respectively; see Example 3.8. It follows that we have a map

$$\varphi : \mathcal{D}_k([a] \times [b]) \to J([k] \times [b - k]) \times J([a - k] \times [k]), \quad D \mapsto (I_1', I_2').$$

Furthermore, the map φ is invertible, with the inverse φ^{-1} being the map that stacks I_1' and I_2' on top and to the right of the Durfee square. Both φ and φ^{-1} clearly respect the orders in $\mathcal{D}_k([a]\times[b])$ and $\mathcal{C}(a, k) \times \mathcal{C}(b, k)$, so we have proved the following:

Proposition 3.7. *If $a, b, k \in \mathbb{N}$ satisfy $k \leqslant \min(a, b)$, then the posets $\mathcal{D}_k([a] \times [b])$ and $\mathcal{C}(a, k) \times \mathcal{C}(b, k)$ are isomorphic.* □

Example 3.8. If we view the ideal I from Example 3.4 as a Ferrers diagram contained in the grid $[a] \times [b]$ with $a = 6$ and $b = 5$, then the Durfee length of the Ferrers diagram is $k = 3$, and the Durfee square is the $[k] \times [k] = [3] \times [3]$ square in the bottom left corner. The natural decomposition of the Ferrers diagram is illustrated in Figure 3.2: the set I_1 in the decomposition consists of the unique element in I above the Durfee square, which forms an ideal of the $[k] \times [b - k] = [3] \times [2]$ grid above S, and I_2 consists of the 7 elements in I to the right of S, which form an ideal of the $[a - k] \times [k] = [3] \times [3]$ grid to the right of S.

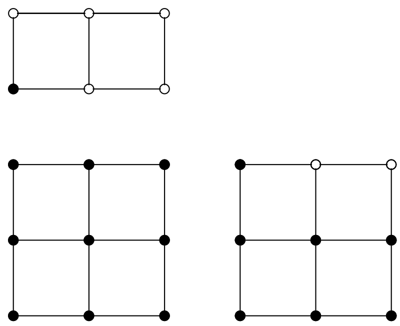


Figure 3.2: Decomposition of the ideal I from Example 3.4.

4. Minuscule posets of type A

The goal of this and the next section is to describe the posets of the form $\mathcal{A}_k(P)$ where P is a minuscule poset. We start with the posets $P = [a] \times [b]$, which correspond to minuscule representations of type A as we will explain in Section 6. Viewing P as embedded in the lattice $\mathbb{Z}_+^2$ as in Section 3, we note that two elements (i_1, j_1) and (i_2, j_2) form an antichain in $[a] \times [b]$ if and only if either (a) $i_1 < i_2$ and $j_1 > j_2$, or (b) $i_1 > i_2$ and $j_1 < j_2$. A basic induction then yields the following characterization of antichains in P, which we record for ease of reference.

Lemma 4.1. *Let $P = [a] \times [b]$. Then a subset of P forms an antichain in P if and only if it can be written in the form $A = \{(x_1, y_1), \cdots, (x_k, y_k)\}$ where $x_1 < \cdots < x_k$ and $y_1 > \cdots > y_k$.* □

The main result of the section is the following proposition, which describes $\mathcal{A}_k(P)$ as a product of two binomial posets.

Proposition 4.2. *If $a, b, k \in \mathbb{N}$ satisfy $k \leqslant \min(a, b)$, then $\mathcal{A}_k([a] \times [b]) \cong \mathcal{C}(a, k) \times \mathcal{C}(b, k)$ as posets.*

Proof. By Lemma 4.1, any antichain of P of size k can be written uniquely as

$$A = \{(x_1, y_1),\ (x_2, y_2),\ \cdots,\ (x_k, y_k)\},$$

where $1 \leqslant x_1 < x_2 < \cdots < x_k \leqslant a$ and $b \geqslant y_1 > y_2 > \cdots > y_k \geqslant 1$. It follows that there is a function $\phi : \mathcal{A}_k(P) \to \mathcal{C}(a, k) \times \mathcal{C}(b, k)$ given by

$$\phi(A) = (\phi_1(A), \phi_2(A)) = ((x_1, x_2, \cdots, x_k),\ (y_k, y_{k-1}, \cdots, y_1)).$$

Furthermore, ϕ is a bijection, because the assignment

$$((x_1, x_2, \cdots, x_k), (y_k, y_{k-1}, \cdots, y_1)) \mapsto \{(x_1, y_1), \cdots, (x_k, y_k)\}$$

takes $\mathcal{C}(a, k) \times \mathcal{C}(b, k)$ to $\mathcal{A}_k(P)$ by Lemma 4.1 and is clearly a two-sided inverse of ϕ.

We claim that ϕ is an isomorphism of posets. To see this, it is enough to prove that ϕ respects covering relations. Let $A \in \mathcal{A}_k(P)$, and write $A = \{(x_1, y_1),\ (x_2, y_2),\ \cdots,\ (x_k, y_k)\}$, where $x_1 < x_2 < \cdots < x_k$ and $y_1 > y_2 > \cdots > y_k$.

Suppose that we have $A' \lessdot_{\mathcal{A}_k(P)} A$ for some $A' \in \mathcal{A}_k(P)$. Proposition 2.4 (ii) implies that A' can be obtained from A by replacing one of the (x_i, y_i) by (x_i', y_i'), where $(x_i', y_i') \lessdot_P (x_i, y_i)$. This condition means that we either have (a) $x_i' = x_i$ and $y_i' = y_i - 1$, or (b) $x_i' = x_i - 1$, $y_i' = y_i$. In case (a), we have $\phi_1(A') = \phi_1(A)$ and $x_1 < x_2 < \cdots < x_{i-1} < x_i' < x_{i+1} < \cdots < x_k$, which implies that $y_1 > y_2 > \cdots > y_{i-1} > y_i' > y_{i+1} > \cdots > y_k$ by Lemma 4.1. Since $y_i' = y_i - 1$, it follows that $\phi_2(A') < \phi_2(A)$ is a covering relation in $\mathcal{C}(b, k)$. Similarly, in case (b), we have $\phi_2(A') = \phi_2(A)$ and $\phi_1(A') < \phi_1(A)$ is a covering relation in $\mathcal{C}(a, k)$. In either case, $\phi(A') < \phi(A)$ is a covering relation $\mathcal{C}(a, k) \times \mathcal{C}(b, k)$.

Conversely, suppose that $(X', Y') \lessdot_{\mathcal{C}(a,k) \times \mathcal{C}(b,k)} (X, Y)$. This implies that we either have (a) $Y' = Y$ and $X' \lessdot_{\mathcal{C}(a,k)} X$, or (b) $X' = X$ and $Y' \lessdot_{\mathcal{C}(b,k)} Y$. Suppose that we are in case (a).

Lemma 3.2 (ii) implies that if we write $X = \{x_1, x_2, \cdots, x_k\}$ with $x_1 < x_2 < \cdots < x_k$, then there exists $i \in [k]$ such that $X' = X \backslash \{x_i\} \cup \{x_i'\}$, $x_i' = x_i - 1$, and

$$x_1 < x_2 < \cdots < x_{i-1} < x_i' < x_{i+1} < \cdots < x_k.$$

It follows that $A = \{(x_1, y_1), (x_2, y_2), \cdots, (x_k, y_k)\}$ and $A' = (A \backslash \{(x_i, y_i)\}) \cup \{(x_i', y_i)\}$ are antichains in $\mathcal{A}_k(P)$, and that $(x_i', y_i) \lessdot_P (x_i, y_i)$. Proposition 2.4 (ii) then implies that $A' \lessdot_{\mathcal{A}_k(P)} A$. This completes the proof of case (a). Case (b) is proved using an analogous argument, which completes the proof of the proposition. □

Corollary 4.3. *If $a, b, k \in \mathbb{N}$ satisfy $k \leqslant \min(a, b)$, then we have $\mathcal{A}_k([a] \times [b]) \cong \mathcal{D}_k([a] \times [b])$ as posets.*

Proof. This follows immediately from Propositions 3.7 and 4.2. □

5. Other minuscule posets

We now investigate the posets $\mathcal{A}_k(P)$ for minuscule posets of other types. We first deal with the infinite minuscule family $P_n = J([n] \times [2])$, which correspond to the spin representations of type D_{n+2} for $n \geqslant 2$ and satisfy $\mathrm{width}(P_n) = \lfloor n + 2/2 \rfloor$.

Proposition 5.1. *If $n, k \in \mathbb{N}$ satisfy $0 \leqslant k \leqslant \lfloor n + 2/2 \rfloor$, then we have isomorphisms of posets*

$$\mathcal{A}_k(J([n] \times [2])) \cong \mathcal{A}_k(\mathcal{C}(n+2, 2)) \cong \mathcal{C}(n+2, 2k).$$

Proof. Since $J([n] \times [2]) \cong \mathcal{C}(n+2, 2)$ by Proposition 3.3, it suffices to prove that $\mathcal{A}_k(\mathcal{C}(n+2, 2)) \cong \mathcal{C}(n+2, 2k)$. Let $P = \mathcal{C}(n+2, 2)$ and write every element of P in the form (x, y) where $x < y$ as in Definition 3.1. Then we may view P as embedded in the lattice $\mathbb{Z}_+^2$ as we did the poset $[a] \times [b]$ in the previous two sections, because two elements $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in P satisfy $p_1 \leqslant_{\mathcal{C}} p_2$ if and only if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. It follows that the characterization of antichains in $[a] \times [b]$ given in Lemma 4.1 holds for the poset $P = \mathcal{C}(n+2, 2)$ as well, so that every antichain of P of size k can be written uniquely as

$$A = \{(x_1, y_1),\ (x_2, y_2),\ \cdots,\ (x_k, y_k)\}$$

where $x_1 < x_2 < \cdots < x_k$ and $y_1 > y_2 > \cdots > y_k$. By assumption we have $x_k < y_k$, so we have

$$x_1 < x_2 < \cdots < x_k < y_k < \cdots < y_2 < y_1.$$

It follows that there is a function $\phi : \mathcal{A}_k(P) \to \mathcal{C}(n+2, 2k)$ given by

$$\phi(A) = (x_1, x_2, \cdots, x_k, y_k, \cdots, y_2, y_1).$$

Furthermore, ϕ is a bijection, because the assignment

$$(x_1, x_2, \cdots, x_k, y_k, y_{k-1}, \cdots, y_1) \mapsto \{(x_1, y_1), \cdots, (x_k, y_k)\}$$

is clearly a two-sided inverse of ϕ that takes $\mathcal{C}(n+2,2k)$ to $\mathcal{A}_k(P)$.

We claim that ϕ is an isomorphism of posets. To see this, it is enough to prove that ϕ and ϕ^{-1} preserve covering relations. Let $A \in \mathcal{A}_k(P)$, and write $A = \{(x_1, y_1),\ (x_2, y_2),\ \cdots,\ (x_k, y_k)\}$, where $x_1 < x_2 < \cdots < x_k$ and $y_1 > y_2 > \cdots > y_k$.

Suppose that we have $A' \lessdot_{\mathcal{A}_k(P)} A$ for some $A' \in \mathcal{A}_k(P)$. Proposition 2.4 (ii) implies that A' can be obtained from A by replacing one of the (x_i, y_i) by (x_i', y_i'), where $(x_i', y_i') \lessdot_P (x_i, y_i)$. By Lemma 3.2 (ii), it follows that that we either have (a) $x_i' = x_i$ and $y_i' = y_i - 1$, or (b) $x_i' = x_i - 1$ and $y_i' = y_i$. In the first case, we have $x_1 < \cdots < x_i' = x_i < \cdots < x_k$, so that we must have

$$x_1 < \cdots < x_i' < \cdots < x_k < y_k < \cdots < y_i' < \cdots < y_1$$

by the argument from the first paragraph of this proof. It follows that

$$\begin{aligned} \varphi(A') &= (x_1, \cdots, x_i, \cdots, x_k, y_k, \cdots, y_i', \cdots, y_1) \\ &\lessdot (x_1, \cdots, x_i, \cdots, x_k, y_k, \cdots, y_i, \cdots, y_1) \\ &= \varphi(A) \end{aligned}$$

in $\mathcal{C}(n+2,2k)$. Moreover, since $y_i' = y_i - 1$, the relation $\varphi(A') < \varphi(A)$ is a covering relation in $\mathcal{C}(n+2,2k)$ by Lemma 3.2 (ii). Similarly, a symmetric argument shows that in the second case we have

$$\begin{aligned} \varphi(A') &= (x_1, \cdots, x_i', \cdots, x_k, y_k, \cdots, y_i, \cdots, y_1) \\ &\lessdot (x_1, \cdots, x_i, \cdots, x_k, y_k, \cdots, y_i, \cdots, y_1) \\ &= \varphi(A) \end{aligned}$$

in $\mathcal{C}(n+2,2k)$.

Conversely, suppose that $\varphi(A') \lessdot_{\mathcal{C}} \varphi(A)$ for some $A' \in \mathcal{A}_k(P)$. Lemma 3.2 (ii) implies that one of the following conditions must hold for some $i \in [k]$:

1. we have $\varphi(A') = (x_1, \cdots, x_i' = x_i - 1, \cdots, x_k, y_k, \cdots, y_1)$;
2. we have $\varphi(A) = (x_1, \cdots, x_k, y_k, \cdots, y_i' = y_i - 1, \cdots, y_1)$.

In the first case, we can use the inverse of φ mentioned earlier to recover A' as the set

$$A' = \{(x_j, y_j) : j \in [k], j \neq i\} \cup \{(x_i', y_i)\},$$

where $(x_i', y_i) \lessdot_P (x_i, y_i)$ by Lemma 3.2 (ii); therefore $A' \lessdot_{\mathcal{A}_k(P)} A$ by Proposition 2.4 (ii). A similar argument shows that $A' \lessdot_{\mathcal{A}_k(P)} A$ in the second case, and we are done. □

Theorem 5.2. *Let P be a minuscule poset, let $k \in \mathbb{N}$, and suppose $k \leqslant \mathrm{width}(P)$.*

(i) *If $P \cong [a] \times [b]$, then we have $\mathcal{A}_k(P) \cong \mathcal{C}(a,k) \times \mathcal{C}(b,k)$ as posets.*

(ii) *If $P \cong J([n] \times [2])$, then we have $\mathcal{A}_k(P) \cong \mathcal{A}_k(\mathcal{C}(n+2,2)) \cong \mathcal{C}(n+2,2k)$ as posets.*

(iii) *If* $P \cong J^m([2] \times [2])$ *where* $m \in \mathbb{N}$*, then* $\mathcal{A}_k(P)$ *is a singleton if* $k \in \{0, 2\}$*, and* $\mathcal{A}_k(P)$ *is isomorphic to* P *if* $k = 1$*.*

(iv) *If* $P \cong J^2([2] \times [3])$*, then* $\mathcal{A}_k(P)$ *is a singleton if* $k = 0$*,* $\mathcal{A}_k(P)$ *is isomorphic to* P *if* $k = 1$*, and* $\mathcal{A}_k(P)$ *is isomorphic to* $J^3([2] \times [2])$ *if* $k = 2$*.*

(v) *If* $P \cong J^3([2] \times [3])$*, then* $\mathcal{A}_k(P)$ *is a singleton if* $k = 0$ *or* $k = 3$*, and* $\mathcal{A}_k(P)$ *is isomorphic to* P *if* $k = 1$ *or* $k = 2$*.*

(vi) *The poset* $\mathcal{A}_k(P)$ *is a distributive lattice under the order* $\leqslant_k$*.*

Proof. Part (i) is a restatement of Proposition 4.2, and (ii) is a restatement of Proposition 5.1. Induction shows that the Hasse diagram of the poset $J^m([2] \times [2])$ for $m \in \mathbb{N}$ is as depicted in Figure 1.1 (c), and the assertions in (iii), (iv) and (v) then all follow by direct computation. In particular, we have $\mathcal{A}_2(P) \cong P$ for $P = J^3([2] \times [3])$, as illustrated in Figure 5.1.

Recall that the singleton poset, posets of the form $J(P)$ where P is a finite poset, and products of distributive lattices are distributive lattices, as are the binomial posets $\mathcal{C}(n, k)$ by Remark 3.5. By (i)–(v), every poset of the form $\mathcal{A}_k(P)$ where P is a minuscule poset has one of the forms described above, and (vi) follows. ☐

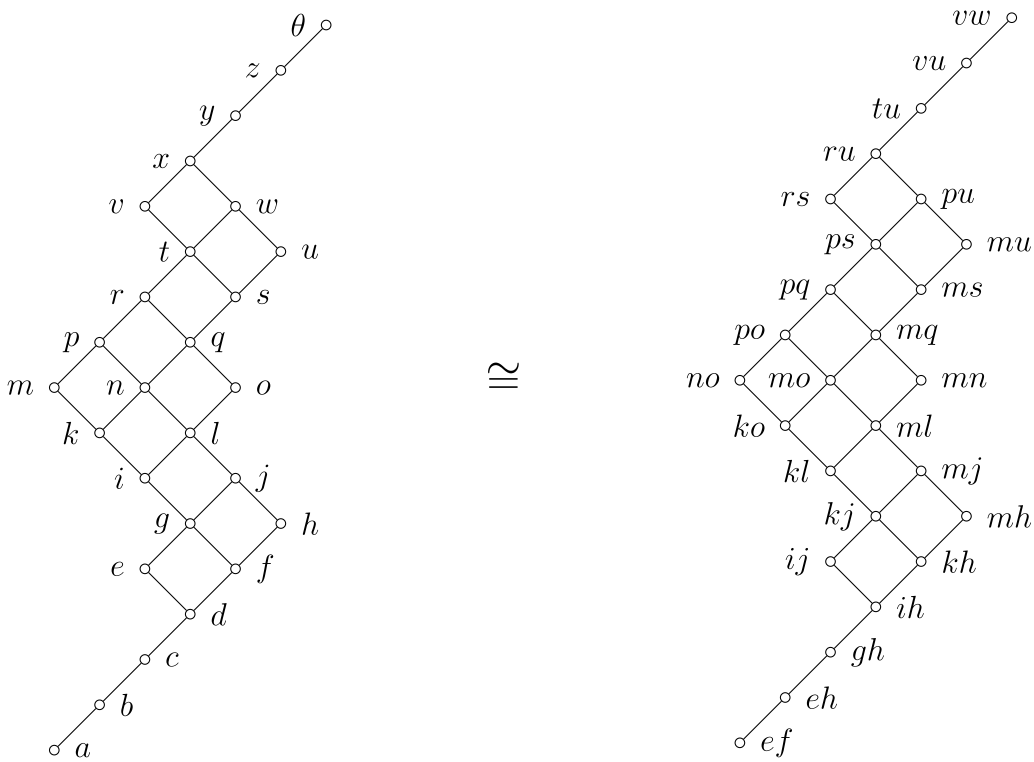


Figure 5.1: Isomorphism between $P = J^3([2] \times [3])$ and $\mathcal{A}_2(P)$, with each element $\{\alpha, \beta\} \in \mathcal{A}_2(P)$ written as $\alpha\beta$.

6. Branching rules of minuscule representations

Recall from the introduction that the minuscule posets are precisely the posets P such that the poset $J(P)$ appears as the weight poset of a minuscule representation of a simple Lie algebra. The goal of this section is to explain a remarkable connection in this setting between the sets $\mathcal{A}_k(P)$ and branching rules of suitable restrictions of the associated minuscule representations. We fix $\mathbb{C}$ as the ground field throughout the section.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of Dynkin type X and rank n. It is known that a minuscule representation exists for $\mathfrak{g}$ if and only if $\mathfrak{g}$ has type $A_n (n \geqslant 1), D_n (n \geqslant 4), E_6$, or E_7, and that in these types each minuscule representation has a fundamental weight ω_p for some $1 \leqslant p \leqslant n$ as its highest weight; we will denote such a minuscule representation by $L(X, \omega_p)$. The Dynkin diagrams of the aforementioned types are shown in Figure 6.1 below, where we adopt the labelling conventions of [Kac90, Chapter 4]. The precise values of p corresponding to the possible minuscule representations are listed in Table 6.1, along with the isomorphism type of the minuscule poset P for which the weight poset of $L(X, \omega_p)$ is isomorphic to $J(P)$; see also Figure 1.1 and [Gre13, Theorem 8.3.10 (v)]. We denote the minuscule poset P corresponding to $L(X, \omega_p)$ by $P(X, \omega_p)$ from now on.

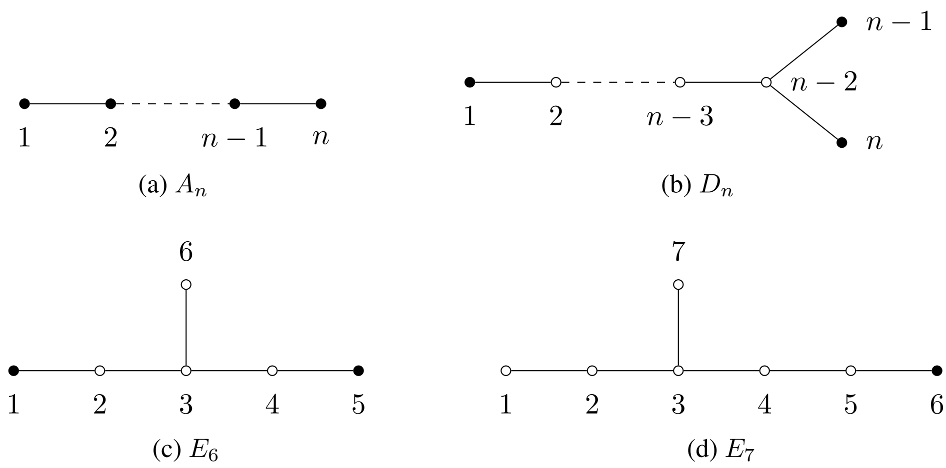


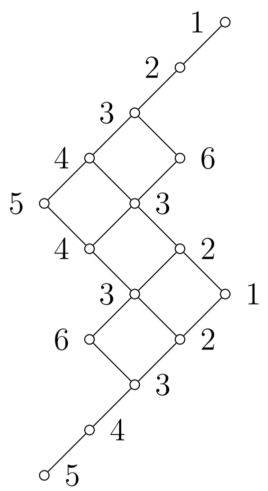
Figure 6.1: Simple Lie algebras admitting minuscule representations.

X	p	$P(X, \omega_p)$
A_n	$k (k \in [n])$	$[k] \times [n+1-k]$
D_n	1	$J^{n-3}([2] \times [2])$
D_n	$n-1$ or n	$J([n-2] \times [2])$
E_6	1 or 5	$J^2([2] \times [3])$
E_7	6	$J^3([2] \times [3])$

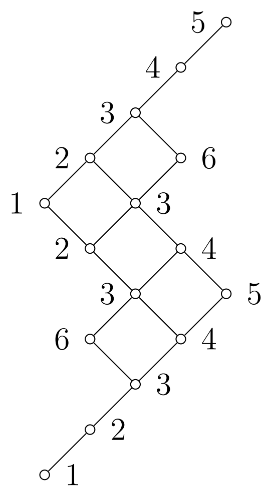
Table 6.1: Classification of minuscule posets.

In addition to recovering the poset structure of the weight poset of $L(X,\omega_p)$, the minuscule poset $P(X,\omega_p)$ can in fact also be used to simultaneously construct $L(X,\omega_p)$ itself and the Lie algebra $\mathfrak{g}$; see [Wil03, Gre20]. The construction involves turning the poset $P(X,\omega_p)$ into a so-called heap (in the sense of Viennot [Vie86]) and then defining linear operators X_i, Y_i, H_i $(1 \leqslant i \leqslant n)$ on the free vector space V spanned by the ideals of the heap; we denote this heap by $H(X,\omega_p)$. The linear operators $X_i, Y_i, H_i (1 \leqslant i \leqslant n)$ generate an isomorphic copy of $\mathfrak{g}$ inside the Lie algebra $\mathfrak{gl}(V)$, and the construction endows V with the structure of a $\mathfrak{g}$-module affording the minuscule representation $L(X,\omega_p)$. In addition, the construction has the convenient feature that p will appear as the label of the unique maximal element of $P(X,\omega_p)$ in the heap $H(X,\omega_p)$, allowing us to quickly identify the highest weight of the minuscule representation associated with the heap as ω_p. For more details about the construction, we refer the reader to [Wil03, Gre20, Gre13].

Example 6.1. Figure 6.2 shows the heaps corresponding to the minuscule representations $L(E_6,\omega_1)$ and $L(E_6,\omega_5)$, which are not isomorphic as heaps because of their different labellings but have isomorphic underlying posets $P(E_6,\omega_1) \cong P(E_6,\omega_5) \cong J^2([2] \times [3])$.



(a) $H(E_6,\omega_1)$



(b) $H(E_6,\omega_5)$

Figure 6.2: Heaps for the minuscule representations $L(E_6,\omega_1)$ and $L(E_6,\omega_5)$.

Remark 6.2. Example 6.1 illustrates the fact that in general a minuscule representation $L(X,\omega_p)$ (or, equivalently, the corresponding heap $H(X,\omega_p)$) contains strictly more data than those of the associated minuscule poset $P(X,\omega_p)$ due to choices in labelling. On the other hand, we note that the only cases where distinct values p, p' give rise to isomorphic minuscule posets $P(X,p) \cong P(X,p')$ are as follows: when $X = A_n$ and $p + p' = n$, or when $X = D_4$ and $2 \notin \{p,p'\}$, or when $X = D_n$ for some $n > 4$ and $\{p,p'\} = \{n-1,n\}$ as sets, or when $X = E_6$ and $\{p,p'\} = \{1,5\}$ as sets. In all these cases, p and p' are conjugate by an involutive automorphism of the Dynkin diagram, and consequently the minuscule representations $L(X,\omega_p)$ and $L(X,\omega'_p)$ differ only by the corresponding automorphism of the Lie algebra $\mathfrak{g}$.

It follows that although a minuscule poset generally contains less information than the representation it arises from, one can still recover minuscule representations from minuscule posets up to diagram automorphisms of $\mathfrak{g}$ in all cases.

Let $L = L(X, \omega_p)$ be a minuscule representation of a simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ generated by the set $\{e_i, f_i, h_i : 1 \leqslant i \leqslant n, i \neq p\}$, let Y be the Dynkin type of $\mathfrak{k}$, and consider the restriction $L\downarrow_{\mathfrak{k}}$ of the module L to $\mathfrak{k}$. By [Gre13, Proposition 8.2.9 (iv)], the decomposition of $L\downarrow_{\mathfrak{k}}$ into simple components can be described in a uniform way via the heap $H = H(X, \omega_p)$, as follows. Denote the elements in H with label p by $\pi_1, \cdots, \pi_k$, ordered so that $\pi_1 \leqslant \pi_2 \leqslant \cdots \leqslant \pi_k$ in the minuscule poset $P = P(X, \omega_p)$. For each $0 \leqslant i \leqslant k$, let H_i' be the subheap of H consisting of all elements α such that $\alpha \nleqslant \pi_i$ and $\pi_{i+1} \nleqslant \alpha$, where the former condition holds vacuously for $i = 0$ and the latter condition holds vacuously for $i = k$. In other words, we take H_i' to be the subheap of H obtained by removing all elements in the order ideal generated by π_i (which is considered empty if $i = 0$) or in the order filter generated by π_{i+1} (which is considered empty if $i = k$). The subheap H_i' will not contain any element labelled by p, and it turns out that H_i' will coincide with the heap of a minuscule representation, L_i', of $\mathfrak{k}$. Moreover, we have $L\downarrow_{\mathfrak{k}} = \oplus_{i=0}^{k} L_i'$ as $\mathfrak{k}$-modules, where L_i' is the trivial representation of $\mathfrak{k}$ if H_i' is empty ([Gre13, Proposition 8.2.9 (iv)]). Note that as remarked in the paragraph above Example 6.1, it is easy to identify which minuscule representation of $\mathfrak{k}$ each L_i' is by reading the label of the top element in H_i' (whenever H_i' is nonempty).

Example 6.3. Figure 6.3 shows the heap $H = H(E_7, \omega_7)$ associated to the minuscule representation L of fundamental weight ω_7 for the Lie algebra $\mathfrak{g}$ of type E_7. We can determine the branching rule for the restriction of L to the type-E_6 subalgebra $\mathfrak{k}$ generated by $\{e_i, f_i, h_i : i \in [7], i \neq 6\}$ as follows. The heap H contains $k = 3$ elements with label 7, which are located at the top, bottom, and the middle left of the heap, so the restriction $L\downarrow_{\mathfrak{k}}$ contains $k + 1 = 4$ simple components, L_0', L_1', L_2' and L_3'. The sets H_1' and H_3' are empty, so the components L_0' and L_3' both afford the trivial representation. The subheap H_1' is colored in red, and may be characterized as the interval $[a, b] = \{x \in H : a \leqslant x \leqslant b\}$ where a is the unique minimal element with label 5 in H and b is the unique maximal element with label 1 in H. Since the top element of H_1' has label 1, the module L_1' has highest weight ω_1 and thus affords the minuscule representation $L(E_6, \omega_1)$. (Note that in the labelling convention for type E_6, we should identify the vertex 7 from the E_7 Dynkin diagram as the vertex 6 in the E_6 diagram; see Figure 6.1.) Similarly, the subheap H_2' contains the interval in H from the unique minimal element labelled by 1 to the unique maximal element labelled by 5, so that L_3' is a copy of the minuscule representation $L(E_6, \omega_5)$ of $\mathfrak{k}$. The minuscule posets underlying the heaps H_1' and H_2' are both isomorphic to $J^2([2] \times [3])$, so it follows from Theorem 5.2 (iv) that for all $0 \leqslant i \leqslant k$, the weight poset of L_i' is isomorphic to $\mathcal{A}_i(P)$ for the minuscule poset $P = J^2([2] \times [3])$.

The main theorem of this section asserts that the poset isomorphisms observed at the end of Example 6.3 in fact hold for all minuscule representations. The assertion is remarkable in the sense that it shows that on the level of the minuscule posets the branching rules of minuscule representations can be described purely in terms of the partial order $\leqslant_k$, even though the branching rules described via heaps depend heavily on the labelling of the minuscule posets by the vertices of the Dynkin diagram. In light of the last sentence of Remark 6.2, this means that the order $\leqslant_k$

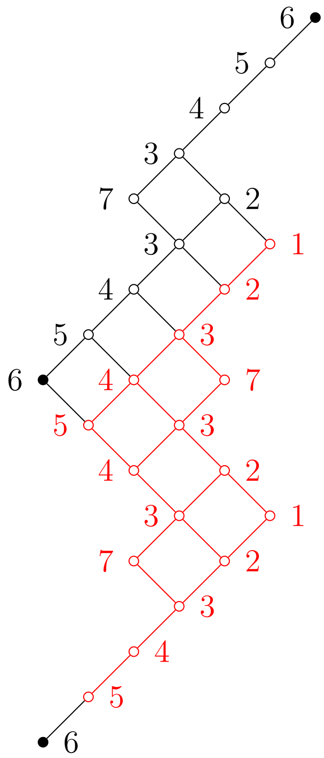


Figure 6.3: The heap of the minuscule representation $L(E_7, 6)$.

provides an efficient algorithm for computing the branching rule of minuscule representations up to diagram automorphisms of Lie algebras that does not require the use of heaps.

Theorem 6.4. *Let $L = L(X, \omega_p)$ be a minuscule representation of a simple Lie algebra $\mathfrak{g}$ and let $P = P(X, \omega_p)$ be the corresponding minuscule poset, so that the poset of weights of L is isomorphic to $J(P)$. Let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ generated by the set $\{e_i, f_i, h_i : 1 \leqslant i \leqslant n, i \neq p\}$. Then the $\mathfrak{k}$-module $L\downarrow_{\mathfrak{k}}$ decomposes into a direct sum*

$$L\downarrow_{\mathfrak{k}} \cong \bigoplus_{i=0}^{k} V_i,$$

where $k = \mathrm{width}(P)$ and V_i is a minuscule representation of $\mathfrak{k}$ whose weight poset is isomorphic to $\mathcal{A}_i(P)$ for each integer $0 \leqslant i \leqslant k$.

Proof. For the unique minuscule representation $L = L(E_7, \omega_7)$ in type E_7, the minuscule poset P is isomorphic to $J^3([2] \times [3])$ by Table 6.1 (or Figure 6.3), and the theorem follows from the discussions in Example 6.3. For the other minuscule representations listed in Table 6.1, the theorem can be proved similarly case by case, by working out the direct summands of $L\downarrow_{\mathfrak{k}}$ via heaps using [Gre13, Proposition 8.2.9 (iv)] and then verifying the facts that $k = \mathrm{width}(P)$ and $P_i \cong \mathcal{A}_i(P)$ for all $0 \leqslant i \leqslant k$. The direct sum decompositions have also been described in [Gre13, Section 8.2], so we will simply sketch the key facts for each minuscule representation below. We denote the Dynkin type of $\mathfrak{k}$ by Y in each case.

If $X = E_6$ and $p = 5$, then we have $Y = D_5$ and $P = J^2([2] \times [3])$, where P has width 2 by Figure 1.1 (d). Inspecting the heaps correponding to L given in Figure 6.2, we see that the restriction $L\downarrow_{\mathfrak{k}}$ decomposes into the direct sum of the trivial module, the module $L(D_5, \omega_1)$ (affording the natural representation with dimension 10), and the module $L(D_5, \omega_4)$ (affording a half spin representation with dimension 16); see also [Gre13, Exercise 8.2.17]. The minuscule posets corresponding to these summands are the empty poset, $J([2] \times [3])$, and $J^2([2] \times [2])$, so that the weight poset of the summands are the singleton poset, $J^2([2] \times [3])$, and $J^3([2] \times [2])$, respectively. The conclusions of the theorem now follow from Theorem 5.2 (iv), and a similar argument shows that they also hold if $X = E_6$ and $p = 1$.

If $X = D_n$ and $p \in \{n-1, n\}$, then we have $Y = A_{n-1}$ and $P = J([n-2] \times [2])$, where P has width $\lfloor n/2 \rfloor$ by Figure 1.1 (b). In this case it is known that L is one of the two half spin representations of type D_n, and that (see [Gre13, Exercise 8.2.15]) we have an isomorphism of $\mathfrak{k}$-modules

$$L\downarrow_{\mathfrak{k}} \cong \bigoplus_{i=0}^{\lfloor n/2 \rfloor} L(A_{n-1}, \omega_{2i}).$$

The minuscule poset corresponding to each summand $L(A_{n-1}, \omega_{2i})$ is $[n-2i] \times [2i]$. The weight posets of the summands are thus the posets $J([n-2i] \times [2i])$ for $0 \leqslant i \leqslant 2k$. On the other hand, by Proposition 5.1 and Proposition 3.3 we have

$$\mathcal{A}_i(P) \cong \mathcal{C}(n, 2i) \cong J([n-2i] \times [2i])$$

for each i, so the conclusions of the theorem also hold in this case.

If $X = D_n$ and $p = 1$, then we have $Y = D_{n-1}$ and $P = J^{n-3}([2] \times [2])$, which has width 2 by Figure 1.1 (c). In this case, it is known that L_p is the natural representation of dimension $2n$ in type D_n, and that (see [Gre13, Exercise 8.2.16 (iii)]) $L\downarrow_{\mathfrak{k}}$ decomposes as the direct sum of three simple $\mathfrak{k}$-modules: two copies of the trivial representation, and one copy of the natural representation of $\mathfrak{k}$. The conclusions of the theorem now follow from Theorem 5.2 (iii).

Finally, if $X = A_n$ and $p \in [n]$, then we have $P = [p] \times [n+1-p]$, where P has width $k = \min(p, n+1-p)$ by Figure 1.1 (a). In this case, we have $Y = A_{p-1} \times A_{n-p}$, so that $\mathfrak{k}$ is a simple Lie algebra of type A_{n-1} if $p \in \{1, n\}$ and is the direct sum of two simple Lie algebras, one of type A_{p-1} and one of type A_{n-p}, otherwise. As illustrated by Figure 6.4, when the Hasse diagram of the heap is drawn as in Figure 1.1 (a), the heap $H = H(A_n, \omega_p)$ has its leftmost element labelled by 1, and the labels of the elements increase every time one moves northeast or southeast by one step in the grid; see also [Gre13, Section 6.2]. The heap contains k elements labelled by p, with the maximal such element π_k being the top element of H and the other $k-1$ elements $\pi_{k-1}, \cdots, \pi_1$ being the elements directly below π_k in the Hasse diagram. It follows that as i ranges over the list $k, k-1, \cdots, 1, 0$, the subheaps H_i' of $H(A_n, \omega_p)$ corresponding to the summands of $L\downarrow_{\mathfrak{k}}$ satisfy natural poset isomorphisms $J(H_i') \cong \mathcal{D}_{k-i}([p] \times [n+1-p]) = \mathcal{D}_{k-i}(P)$ for all i, where $\mathcal{D}_i([p] \times [n+1-p])$ is as defined in Definition 3.6. The assertions of the theorem now follow from Theorem 5.2 (i), completing the proof. □

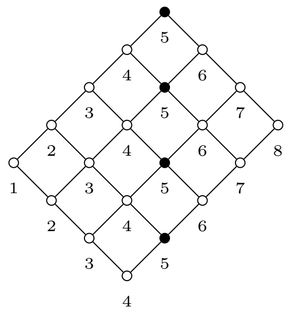


Figure 6.4: The heap of $L(A_8, \omega_5)$.

7. Concluding remarks

We discuss a few open problems concerning the partial order $\leqslant_k$ in this section. First, it would be interesting to have a conceptual, case-free proof of Theorem 6.4. Secondly, apart from the minuscule poset setting, it may be interesting to study the structure of posets of the form $\mathcal{A}_k(\Phi^+)$ where Φ^+ is a root poset, i.e., the poset of positive roots of a Weyl group W. Thirdly, if W has rank r, then the so-called Narayana numbers $|\mathcal{A}_k(\Phi^+)|$ are symmetric in the sense that $|\mathcal{A}_k(\Phi^+)| = |\mathcal{A}_{r-k}(\Phi^+)|$ for any $0 \leqslant k \leqslant r$ (see [DH21]), so it would also be interesting to know whether this symmetry can be realized by a poset isomorphism (with respect to the orders $\leqslant_k$ and $\leqslant_{r-k}$) between $\mathcal{A}_k(\Phi^+)$ and $\mathcal{A}_{r-k}(\Phi^+)$.

Bijections realizing the symmetry $|\mathcal{A}_k(\Phi^+)| = |\mathcal{A}_{r-k}(\Phi^+)|$ have been studied before. In [Pan04, Conjecture 6.1], Panyushev conjectures that for all root systems Φ, there is a natural involution $*$ on $\mathcal{A}(\Phi^+)$ satisfying a certain list of properties, one of which is that it should restrict to bijections between $\mathcal{A}_k(\Phi^+)$ and $\mathcal{A}_{n-1-k}(\Phi^+)$ for all $0 \leqslant k \leqslant r$. Panyushev also constructs ([Pan04, Section 4]) such an involution for the root poset $\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i < j \leqslant n\} \subseteq \mathbb{R}^n$ of type A_{n-1} $(n \geqslant 2)$ as follows: write $[i, j] = \varepsilon_i - \varepsilon_j$ for all $1 \leqslant i < j \leqslant n$, and for each antichain $A = \{[i_1, j_1], \cdots, [i_k, j_k]\}$ in $\mathcal{A}_k(\Phi^+)$, define A^* to the unique antichain in $\mathcal{A}_{n-1-k}$ consisting of elements $[i'_1, j'_1], \cdots, [i'_{n-1-k}, j'_{n-1-k}]$ where

$$\{i'_1, \cdots, i'_{n-1-k}\} = \{1, 2, \cdots, n-1\} \setminus \{j_1 - 1, \cdots, j_k - 1\},$$

$$\{j'_1, \cdots, j'_{n-1-k}\} = \{2, 3, \cdots, n\} \setminus \{i_1 + 1, \cdots, i_k + 1\}$$

as sets and

$$i'_1 < \cdots < i'_{n-1-k}, \quad j'_1 < \cdots < j'_{n-1-k}.$$

Using Proposition 2.4 (ii) and the fact that $[i, j] \lessdot [l, m]$ in Φ^+ if and only if either $i = l, m = j+1$ or $l = i-1, j = m$, it is straightforward to verify that if $A' \lessdot A$ for some antichain $A' \in \mathcal{A}_k(\Phi^+)$, then $A'^* \lessdot A^*$ in $\mathcal{A}_{n-1-k}(\Phi^+)$. Since the map $*$ is an involution, we deduce the following:

Proposition 7.1. *For the root poset Φ^+ of type A_{n-1}, Panyushev's natural involution $*$ on the set $\mathcal{A}(\Phi^+)$ restricts to poset isomorphisms between $\mathcal{A}_k(\Phi^+)$ and $\mathcal{A}_{n-1-k}(\Phi^+)$ for all $1 \leqslant k \leqslant n-1$.* □

More generally, Defant and Hopkins prove in [DH21] that for root systems of types A, B, C and D, a so-called rowvacuation operator satisfies Panyushev's desired properties and recovers the map $*$ in type A. However, we note that while rowvacuation provides bijections between $\mathcal{A}_k(\Phi^+)$ and $\mathcal{A}_{r-k}(\Phi^+)$, it does not give a poset isomorphism between these posets in types B, C and D. To see this, recall that Φ^+ is a ranked poset. Let R be the rank of Φ^+ and let Φ_i^+ be the antichain in Φ^+ consisting of all elements of rank i for each $0 \leqslant i \leqslant R$. Then in types B, C and D, both Φ_R^+ and Φ_{R-1}^+ are singletons satisfying $\Phi_{R-1}^+ \lessdot \Phi_R^+$ in $\mathcal{A}_1(\Phi^+)$. On the other hand, Proposition 2.9 of [DH21] implies that rowvacuation sends Φ_{R-1}^+ and Φ_R^+ to Φ_2^+ and Φ_1^+, respectively, yet Φ_2^+ and Φ_1^+ are elements of $\mathcal{A}_{r-1}(\Phi^+)$ that are not in a covering relation.

Acknowledgements

We thank Darij Grinberg and Hugh Thomas for helpful discussions. We also thank the anonymous referees for reading our paper carefully and suggesting many improvements.

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