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Order and inverses in the monoid of misère blocking games

Alfie Davies[1] and Rebecca Milley[2]

[1,2]*Memorial University of Newfoundland, Canada*
research@alfied.xyz , rebeccamilley@mun.ca

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Abstract. This paper considers combinatorial games with the last move losing (misère play) instead of winning (normal play). Misère games form a pomonoid in which no non-zero elements are invertible; normal-play games, however, form a group with a much richer partial order. To compensate, misère researchers typically weaken the comparison relation by restricting to a subset of games, especially to a "universe" (closed under addition, conjugation, and options). For some well-studied universes (like the dicot and dead-ending universes), there exist finite comparison tests and also complete characterisations of their invertible elements; more recently, the invertible elements of every "parental" universe have been characterised. In this paper, we study the recently-defined universe of "blocking games" (containing both dicots and dead-ending games): we develop a finite comparison test, and we improve on the general invertibility characterisation by showing that, as in dead-ending, a game is invertible modulo blocking if and only if it is free of previous-win subpositions (or is equal to such a form).

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Mathematics Subject Classifications. 91A46, 06F05, 20M14, 20K27

1. Introduction

Combinatorial games have much weaker algebraic structure under *misère play* (last player loses) than under *normal play* (last player wins): compared with the well-studied group of normal play positions, the set of misère play positions yields only a monoid in which no non-zero elements are invertible (such a monoid is often called *reduced*). The game simplifications from normal play (removing dominated options, and bypassing reversible ones) do still work in misère, but they are much less likely to apply to a given game in practice. The lack of a richer, general theory had kept misère research from gaining much traction in the 1900s.

Over the past 20 years, however, there has been a relative flourish in the theory of misère games. Plambeck's introduction of impartial misère quotients [Pla05] led to the solutions of many (at-the-time) unsolved impartial games. This was later extended by Plambeck and Siegel [PS08], and Siegel [Sie15b] (Siegel [Sie19] gives a fairly extensive introduction). A partizan generalisation of the misère quotients was possible, and has been explored by Allen [All09, All15]. The key idea is that of game "equivalence" (or "distinguishability"), modulo only a subset of misère games, rather than all possible forms. The subsets we study here are called *universes* of misère games. What properties a set of games should have in order to be called a "universe" has been refined several times over the years: we follow the recent survey [DLN+25]—by Davies, Larsson, Milley, Nowakowski, Santos, and Siegel—and define a universe as a set of games closed under addition, conjugation, and options. (For a review of these and other basic game theory terms, we refer the reader to Siegel's *Combinatorial Game Theory* [Sie13].) All universes in our paper also have *parental* closure; this property, along with other newer definitions, is explained in Section 2. As is typical, we write $\mathcal{M}$ for the *full misère* universe of all games under misère play.

From 2013 to 2023, the two main universes of study were the *dicot universe* $\mathcal{D}$ and the *dead-ending universe* $\mathcal{E}$. A game is a Left (Right) *end* if Left (Right) has no option. A game is a *dicot* if the only subposition that is an end is the *zero game*, 0, in which neither player can move. A *dead* Left (Right) end not only has no immediate move for Left (Right), but also has no move for Left (Right) in any subposition; i.e. every subposition is a Left (Right) end. A *dead-ending game* is one whose ends are all dead ends. Notice that $\mathcal{D}$ is a proper subset of $\mathcal{E}$, and that many well-studied rulesets, including HACKENBUSH and DOMINEERING, satisfy the dead-ending property.

Research into the algebra of the dicot and dead-ending universes (which are naturally pomonoids, like many other sets of games studied) has been quite successful. Dead-ending games were introduced by Milley and Renault [MR13], and it was shown then that all dead ends are invertible modulo $\mathcal{E}$; later, a complete characterisation of invertible dead-ending games was found by Milley and Renault [MR22], and the invertible dicot positions were similarly categorised by Fisher, Nowakowski, and Santos [FNS22]; most recently, Davies and Yadav [DY25] have given a characterisation of the invertible elements of a general parental universe.

In general restricted misère play (i.e. modulo a proper subset of $\mathcal{M}$), there is no finite test for game comparison (there are, after all, only countably many algorithms, but uncountably many relations [Sie25]); yet computational comparison tests were developed for both $\mathcal{D}$ and $\mathcal{E}$ by Milley, Larsson, Nowakowski, Renault, and Santos [LMN+21]. There is also, in general, an issue with canonical forms: we cannot always bypass reversible options if the replacement set is empty. But this problem has been resolved for $\mathcal{D}$ by Dorbec, Renault, Siegel, and Sopena [DRSS13], and for $\mathcal{E}$ by Larsson, Milley, Nowakowski, Renault, and Santos [LMNS25] (more recently, as we will see in Section 2, Siegel has proposed a general solution to give unique *simplest* forms in each parental universe [Sie25, §5], of which there are infinitely many).

It was also recently shown, by Davies, McKay, Milley, Nowakowski, and Santos, that the pomonoid $\mathcal{E}$ is pocancellative[1], even though $\mathcal{M}$ is not [DMM+24]. (Interestingly, $\mathcal{D}$ is *not* pocancellative, either.) It was this study of pocancellativity where the titular *blocking* games were born: it was shown that blocking games are pocancellative modulo full misère[2], though

[1] If $G, H, C \in \mathcal{E}$ and $G + C \geqslant_{\mathcal{E}} H + C$ then $G \geqslant_{\mathcal{E}} H$.

[2] If G, H are blocking, and $C \in \mathcal{M}$, then $G + C \geqslant H + C$ implies $G \geqslant H$ (working modulo $\mathcal{M}$).

it is an open question whether the blocking games are themselves pocancellative (i.e. modulo blocking games).

But what is the universe of *blocking* games $\mathcal{B}$? A Left end is *blocked* if every Right option either is itself a blocked Left end, or else has a Left option to a blocked Left end. A blocked Right end is defined symmetrically. A blocking game, then, is one whose ends are all blocked. The game MAZE (see, for example, Albert, Nowakowski, and Wolfe's *Lessons In Play* [ANW19, p. 311]) is a naturally-occurring blocking ruleset. Notice that

$$\mathcal{D} \subsetneq \mathcal{E} \subsetneq \mathcal{B} \subsetneq \mathcal{M}.$$

In the present paper, we explore the algebra of this new universe, to see if blocking games are as "misère-friendly" as its subuniverses of dicot and dead-ending games.

We begin in Section 2 with notation, important definitions, and relevant background results. We will review the general comparison test developed by Larsson, Nowakowski, and Santos [LNS25], which invokes a critical property known by "Left $\mathcal{U}$-strong"[3], where $\mathcal{U}$ is a universe: a game G is Left $\mathcal{U}$-strong if Left wins playing first on $G + X$ for all Left ends X in $\mathcal{U}$. (Being "Right $\mathcal{U}$-strong" is defined symmetrically.)

In Section 3, we use the comparison test of [LNS25] to develop a computational means of checking $G \geqslant_{\mathcal{B}} H$ (i.e. comparison modulo B), including for when G and H are not even themselves in $\mathcal{B}$. We do so by first giving a recursive test for checking whether a game is Left $\mathcal{B}$-strong; this test is atypical in the sense that it does not prescribe a set of Left ends X to test against. Then, in Sections 3.1 and 3.2, we develop a more usual test by giving explicitly a set of Left ends X that one needs to check against for each rank. It is of note that one can check whether a game G is Left $\mathcal{B}$-strong just by checking against a single specific Left end X, which we will construct; fascinatingly, this X is *not* minimal among those games with the same rank. In Section 3.3, we explore the minimal elements of the set of Left ends in $\mathcal{B}$ up to a given rank, and contrast them with the test set discussed above.

In Section 4, we classify precisely which games are invertible modulo $\mathcal{B}$. It was shown previously that the invertible elements of the dead-ending universe $\mathcal{E}$ were precisely those games that were "$\mathscr{P}$-free (mod $\mathcal{E}$)"—games that are equivalent (mod $\mathcal{E}$) to a dead-ending form that has no subposition of outcome $\mathscr{P}$. Using the recent study of $\mathscr{P}$-free games by Davies, Miller, and Milley [DMM25], we are able to show a striking similarity for the blocking universe: the invertible blocking games are exactly those games that are $\mathscr{P}$-free (mod $\mathcal{B}$). This does not, however, mean that the groups of invertible elements of $\mathcal{E}$ and $\mathcal{B}$ are isomorphic, which we discuss at the end of the section.

Finally, in Section 5, we discuss possible next steps. For instance, it is hoped that the blocking universe will enjoy many of the properties of the dead-ending universe. And while we show in Section 4 that the characterisations of the invertible subgroups of $\mathcal{B}$ and $\mathcal{E}$ are almost identical, there still exist properties of $\mathcal{E}$ that are not currently known to hold for $\mathcal{B}$; in particular, whether $\mathcal{B}$ is pocancellative. We also discuss some other open problems: how one can test whether a game is Left $\mathcal{U}$-strong for a parental universe $\mathcal{U}$; whether our comparison tests can be modified to work for various subuniverses of $\mathcal{B}$; and understanding the structure of the invertible subgroup of $\mathcal{B}$ (and other parental universes, more generally).

[3]This term was defined by Siegel in [Sie25, Definition 2.3 on p. 195] as a convenience.

2. Background

A game or *position* is written as $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, where $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are the Left and Right option sets, respectively. The *subpositions* of G include G itself, along with every other position that can be reached from G (options, options of options, etc.). We write $\overline{G}$ for the *conjugate* of G, instead of the *negative* of G as in normal play (i.e. $-G$), since we do not generally have invertibility in misère. We also write $\mathrm{rank}(G)$ for the *rank* of G, which is the height of its game tree (i.e. the length of a longest run); this is typically called the *formal birthday* in normal play, written $\tilde{\mathrm{b}}(G)$. Given two games G and H, we say that they are *isomorphic*, and write $G \cong H$, if every Left (Right) option of G is isomorphic to some Left (Right) option of H, and vice versa.

We can now give a more formal definition of blocking games [DMM+24]. A Left end X is *blocked* if for every Right option X^R of X, either X^R is a blocked Left end, or else there exists a Left response X^{RL} that is again a blocked Left end. Essentially, Left has the ability to "block" moves by Right in Left ends. Symmetrically, a Right end Y is blocked if for every Left option Y^L of Y, either Y^L is a blocked Right end, or else there exists a Right response Y^{LR} that is a blocked Right end. A game is then called *blocking* if every subposition that is an end is blocked.

Recall that a universe is closed under addition, conjugation, and options. A universe $\mathcal{U}$ has *parental* closure if, for every pair of finite, non-empty subsets $\mathcal{A}, \mathcal{B} \subsetneq \mathcal{U}$, the game $\{\mathcal{A} \mid \mathcal{B}\}$ is in $\mathcal{U}$.[4] The universes $\mathcal{D}$ and $\mathcal{E}$ are parental, and it was noted in [DMM+24] that $\mathcal{B}$ is also a parental universe.

As in recent papers (e.g. [DLN+25]), we use $\mathrm{o_L}(G)$ to refer to the *Left outcome* of G: the winner ($\mathscr{L}$ or $\mathscr{R}$) of G when Left plays first. Similarly, the Right outcome, $\mathrm{o_R}(G)$, is the winner when Right plays first. The usual outcome function $\mathrm{o}(G)$ is then defined by

$$\mathrm{o}(G) := \begin{cases} \mathscr{L} & \text{if } (\mathrm{o_L}(G), \mathrm{o_R}(G)) = (\mathscr{L}, \mathscr{L}), \\ \mathscr{N} & \text{if } (\mathrm{o_L}(G), \mathrm{o_R}(G)) = (\mathscr{L}, \mathscr{R}), \\ \mathscr{P} & \text{if } (\mathrm{o_L}(G), \mathrm{o_R}(G)) = (\mathscr{R}, \mathscr{L}), \\ \mathscr{R} & \text{if } (\mathrm{o_L}(G), \mathrm{o_R}(G)) = (\mathscr{R}, \mathscr{R}). \end{cases}$$

As mentioned in the introduction, a game G is *Left $\mathcal{U}$-strong* if $\mathrm{o_L}(G + X) = \mathscr{L}$ for every Left end X in $\mathcal{U}$. Symmetrically, we say G is *Right $\mathcal{U}$-strong* if $\mathrm{o_R}(G+Y) = \mathscr{R}$ for every Right end Y in $\mathcal{U}$. Notice that every Left end is Left $\mathcal{U}$-strong for every universe $\mathcal{U}$, since a sum of Left ends is necessarily a Left end, which Left of course wins playing first. (And every Right end is Right $\mathcal{U}$-strong by a symmetric argument; we will typically state such results only for "Left $\mathcal{U}$-strong".)

The concept of Left / Right $\mathcal{U}$-strong is critical for the comparison of games modulo parental universes. The comparison tests for the dicot and dead-ending universes were developed from a general comparison test called the Absolute Fundamental Theorem [LNS25], which we state for the special case of misère games here.

[4] It is important to note that this property is sometimes included in the definition of a universe, as in [Sie25]. In this paper, every universe considered is parental.

Theorem 2.1 ([LNS25, Theorem 4 on p. 103])**.** *If $\mathcal{U}$ is a parental universe and $G, H \in \mathcal{M}$, then $G \geqslant_{\mathcal{U}} H$ if and only if G and H satisfy the following properties:*

1. *for all G^R, either there is a G^{RL} such that $G^{RL} \geqslant_{\mathcal{U}} H$, or else there is an H^R such that $G^R \geqslant_{\mathcal{U}} H^R$;*

2. *for all H^L, either there is an H^{LR} such that $G \geqslant_{\mathcal{U}} H^{LR}$, or else there is a G^L such that $G^L \geqslant_{\mathcal{U}} H^L$;*

3. *if H is a Left end, then G is Left $\mathcal{U}$-strong; and*

4. *if G is a Right end, then H is Right $\mathcal{U}$-strong.*

The first two conditions in Theorem 2.1 are commonly referred to as the *maintenance property*, and the latter two the *proviso*.

In the full misère universe $\mathcal{M}$, a game is Left (Right) $\mathcal{M}$-strong if and only if it is a Left (Right) end. (To see this, consider $G + \{\cdot \mid \operatorname{rank}(G)\}$, which Left wins going first if and only if G is a Left end.) Thus, it is worth noting the following simplified corollary for comparison modulo full misère, which had previously been proven by Siegel as a special case [Sie15a].

Corollary 2.2 ([Sie15a, Theorems 5.3 and 5.4 on pp. 231–232])**.** *If $G, H \in \mathcal{M}$, then $G \geqslant H$ if and only if the maintenance property holds and both of the following are true:*

- *G is a Left end or H is not a Left end; and*

- *H is a Right end or G is not a Right end.*

For proper subuniverses of $\mathcal{M}$, notice that if we had a finite test to determine the truth of "G is Left $\mathcal{U}$-strong" and "H is Right $\mathcal{U}$-strong" in the proviso of Theorem 2.1, then the maintenance property could be checked recursively, and the entire comparison test would be computational. This has been done for dicots and dead-ending games (and doing the same for blocking games is precisely our goal in Section 3). In the dicot universe, the only end is the zero game, so a game is Left $\mathcal{D}$-strong if and only if its Left outcome is $\mathscr{L}$. A dead-ending game G of rank n is Left $\mathcal{E}$-strong if and only if Left wins playing first on both G and $G + W_n$, where W_n is the *waiting game* of rank n [LMN+21]. The waiting games are the "worst" (for Left) Left ends of each rank: they are defined recursively by

$$\begin{aligned} W_0 &:= 0, \text{ and} \\ W_n &:= \{\cdot \mid 0, W_{n-1}\} \end{aligned}$$

for $n > 0$, as illustrated in Figure 2.1.

Our final item in this background discussion is Siegel's new theory of simplest forms for parental universes. Recall that, in normal play, games can be simplified to a unique canonical form by removing dominated options and bypassing reversible ones. In full misère, the situation is just as pleasant: Siegel showed in [Sie15a] that games can be simplified to a unique canonical form. In restricted misère, however, a problem arises: while removing dominated options works as usual, bypassing a reversible option *through an end* can (often) cause problems. Recall that,

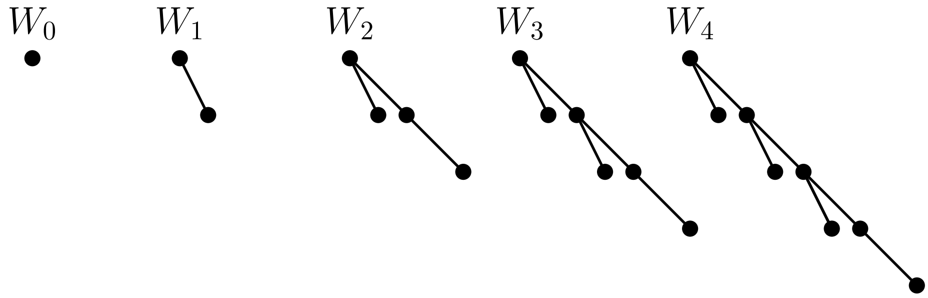

Figure 2.1: The waiting games W_n of rank up to 4.

if G^L is reversible through G^{LR} (i.e. when $G \geqslant G^{LR}$), then what we want to do is replace G^L with the Left options of G^{LR} (i.e. the *replacement set*). When this replacement set is empty, problems can arise, as has been seen in [LMNS25, §3].

Note that the issue of reversing through an end can never occur in $\mathcal{M}$ because, modulo full misère, a Left end (in particular, G^{LR}) cannot be better for Right than a game that is not a Left end (in particular, G), as shown by Siegel [Sie13, Theorem 6.6 on p. 271]. In restricted misère play, such as modulo $\mathcal{D}$, $\mathcal{E}$, or $\mathcal{B}$, it is possible that an option reverses through an end, but simply removing G^L (as we would in normal play) does not necessarily leave an equivalent position; the reduction could force the form to leave its equivalence class. Alternative reductions for reversibility through ends were developed for $\mathcal{D}$ [DRSS13] and $\mathcal{E}$ [LMNS25], and they do lead to a reasonably natural "canonical form" (which is unique).

Siegel showed in [Sie25, §5] that there exist parental universes where there is no reasonable choice for a unique canonical form; we would be forced to make arbitrary decisions that present unnecessary complications. In the same paper, he then presented a solution to this problem: the so-called "simplest forms".

Notice that, modulo a universe $\mathcal{U}$, if a game G has a reversible option G^L through a Left end G^{LR}, then G is necessarily Left $\mathcal{U}$-strong, since, for all Left ends $X \in \mathcal{U}$, we have

$$\mathrm{o_L}(G + X) \geqslant \mathrm{o_L}(G^{LR} + X) = \mathscr{L}.$$

The novel reduction presented in [Sie25, §5] is the following: if G^L is reversible through a Left end G^{LR}, then replace G^L with a "tombstone", which we denote by a typographical tombstone "∎".[5] This tombstone serves as a witness to the fact that G is Left $\mathcal{U}$-strong. What Siegel shows is that the information in such a reversible G^L is almost all superfluous; the only information we need to retain to remain in the equivalence class is that G is Left $\mathcal{U}$-strong. What is amazing is that this theory naturally extends what we have already; Siegel shows how to extend the notions of addition and comparison naturally to this larger set of games that may contain tombstones—the *augmented forms*.

Given a parental universe $\mathcal{U}$, Siegel then defines the $\mathcal{U}$-simplest form of $G \in \mathcal{M}$ as the (augmented) form obtained by performing the following simplifications to all subpositions of G:

[5] Siegel uses the notation Σ^L and Σ^R to refer to a tombstone appearing on the Left and Right respectively, but we elect for a simplified notation here.

1. remove $\mathcal{U}$-dominated options;

2. bypass $\mathcal{U}$-reversible options if the replacement set is non-empty;

3. replace $\mathcal{U}$-reversible options whose replacement sets are empty by a tombstone (if it is not already present); and

4. remove a Left (Right) tombstone if the resulting form is Left (Right) $\mathcal{U}$-strong.

As one would expect, he shows that these reductions do not ever change the $\mathcal{U}$-equivalence class of the form; i.e. every game is equal (modulo $\mathcal{U}$) to its $\mathcal{U}$-simplest form. The important result regarding these simplest forms, which is why they are a solution to the canonical form problem, is their uniqueness: if $\mathcal{U}$ is a parental universe, and $G, H \in \mathcal{M}$ satisfy $G \equiv_{\mathcal{U}} H$, then the $\mathcal{U}$-simplest form of G is isomorphic to the $\mathcal{U}$-simplest form of H.

We will use the theory of simplest forms here in a way that is mostly complementary to, rather than absolutely necessary for, our main results (although we will require them on occasion). While the reader is most strongly encouraged to refer to Siegel's paper for the formal details and all of the lovely motivation, they can take solace in the fact that we will not need all of those details here; on the few occasions that we mention simplest forms, the reader may think of them as similar to the canonical forms that we enjoy in normal play and full misère, just sometimes with the appearance of strange tombstone symbols.

We are now ready to explore the application of Theorem 2.1 to the blocking universe; in particular, we will investigate some computational means of checking whether a game is Left $\mathcal{B}$-strong.

3. Order

We would like to find a computational comparison test for arbitrary forms modulo $\mathcal{B}$. It suffices to find a computational way to verify the proviso, since the maintenance property could then be checked recursively. We waste no time in establishing exactly this in Theorem 3.1: a recursive test to determine whether a given game G is Left $\mathcal{B}$-strong. (Symmetric results for Right are omitted, as usual.) Notice that Theorem 3.1 works for *every* game form in $\mathcal{M}$—not just blocking forms.

Theorem 3.1. *If $G \in \mathcal{M}$, then G is Left $\mathcal{B}$-strong if and only if either*

1. *G is a Left end; or*

2. *G has a Left-win option G^L such that G^L and all Right responses G^{LR} are Left $\mathcal{B}$-strong.*

Proof. If G is a Left end, then the result is immediate. So, suppose that G is not a Left end.

($\Leftarrow$) Suppose G has a Left-win option G^L such that G^L and all G^{LR} are Left $\mathcal{B}$-strong. We need to show that G itself is Left $\mathcal{B}$-strong. Let $X \in \mathcal{B}$ be an arbitrary Left end. Playing first on $G + X$, Left can move on G to this G^L, leaving $G^L + X$. It suffices to show that Right loses playing next on this sum. Assume inductively that Right loses playing first on $G^L + Y$ when Y is a Left end of smaller rank than X; the base case $Y = 0$ is clear since G^L is Left-win.

Now, in $G^L + X$, if Right replies on G, then, since G^{LR} is Left $\mathcal{B}$-strong, we know that Left wins playing next on $G^{LR} + X$. If Right instead plays on X, then either X^R is a Left end and Left wins playing next on $G^L + X^R$ because G^L is Left $\mathcal{B}$-strong, or, since $X \in \mathcal{B}$, Left can reply on X^R to a Left end, say Y. In that latter case, by induction, Right loses playing next on $G^L + Y$. Thus, Right loses playing first on $G^L + X$, and so Left wins playing first on $G + X$, as required.

($\Rightarrow$) Suppose now that G is Left $\mathcal{B}$-strong. We proceed by induction on the rank of a game: assume that if H is Left $\mathcal{B}$-strong and has rank strictly less than G, then either H is a Left end, or else it has a Left-win option H^L such that H^L and all H^{LR} are Left $\mathcal{B}$-strong. We need to show there exists a Left-win option G^L of G with the same property: G^L and every Right option G^{LR} are Left $\mathcal{B}$-strong.

Recall that we are assuming G is not a Left end, and so G has at least one Left option. It cannot be that Left's only option in G is to zero, because then Left would not win first on $G + 0$, contradicting that G is Left $\mathcal{B}$-strong. Write

$$A_1, \ldots, A_a, B_1, \ldots, B_b, C_1, \ldots, C_c$$

for the non-zero Left options of G, where the A_i are Left $\mathcal{B}$-strong and Left-win, the B_i are Left $\mathcal{B}$-strong but not Left-win, and the C_i are not Left $\mathcal{B}$-strong, and note that $a + b + c > 0$. For each C_i, let $X_i \in \mathcal{B}$ be a Left end such that $\mathrm{o_L}(C_i + X_i) = \mathscr{R}$; such an X_i must exist since C_i is not Left $\mathcal{B}$-strong.

We will first show that at least one of the non-zero Left options is Left $\mathcal{B}$-strong (i.e. $a + b > 0$). Suppose, for a contradiction, that this is not so; that is, the non-zero Left options are only $C_1, \ldots, C_c$. Consider Left playing first on the game $G + X$, where

$$X = \{\cdot \mid 1, X_1, \ldots, X_c\} \in \mathcal{B}.$$

If Left plays on G to 0, then Right will respond on X to 1, thus winning the game. If Left instead plays to some $C_i + X$, then Right will move to $C_i + X_i$, which is, by construction, winning for Right. So Left loses $G + X$ playing first, which contradicts G being Left $\mathcal{B}$-strong. Thus, there exists a non-zero Left option of G that is Left $\mathcal{B}$-strong.

The Left options that are Left $\mathcal{B}$-strong are $A_1, \ldots, A_a, B_1, \ldots, B_b$, where we now know $a + b > 0$. We aim to show $a > 0$; suppose, for a contradiction, that $a = 0$, so that none of the Left $\mathcal{B}$-strong Left options are Left-win. Note that $\mathrm{o}(B_i) = \mathscr{N}$ for each i, because Left wins playing first on a Left $\mathcal{B}$-strong game and $\mathrm{o}(B_i) \neq \mathscr{L}$. Thus, Right wins playing first on each B_i. Now consider the game $G + Y$, where

$$Y = \{\cdot \mid 1, X_1, \ldots, X_c, \{0 \mid \mathrm{rank}(G)\}\} \in \mathcal{B}.$$

We claim that Right wins when Left moves first on $G + Y$. The case of Left moving to $0 + Y$ or $C_i + Y$ are the same as previously discussed. If Left instead moves to $B_i + Y$, then Right should move to $B_i + \{0 \mid \operatorname{rank}(G)\}$. From here, if Left responds on the B_i component, then Right can respond on the other component to $\operatorname{rank}(G)$, and Left will lose the sum because $\operatorname{rank}(B_i) < \operatorname{rank}(G)$. So, Left must move from $B_i + \{0 \mid \operatorname{rank}(G)\}$ to $B_i + 0$, and, as established above, Right wins this playing first. We have shown that Right wins when Left plays first on $G + Y$, which is a contradiction because G is Left $\mathcal{B}$-strong. Therefore, at least one Left option of G is both Left $\mathcal{B}$-strong and Left-win; that is, $a > 0$.

Finally, we must show that every Right option of at least one of $A_1, \ldots, A_a$ is Left $\mathcal{B}$-strong. Suppose, for a contradiction, that, for each A_i, there exists a Right response A_i^R that is not Left $\mathcal{B}$-strong. Then, for each $i \leqslant a$, we can find a Left end $T_i \in \mathcal{B}$ such that $\mathrm{o_L}(A_i^R + T_i) = \mathscr{R}$. We remark that no T_i can be equal to zero, since that would imply $\mathrm{o_L}(A_i^R) = \mathscr{R}$, contradicting the fact that $\mathrm{o}(A_i) = \mathscr{L}$. Now consider Left playing first on the game $G + Z$, where

$$Z = \{\cdot \mid 1, X_1, \ldots, X_c, T_1^{\mathcal{R}}, \ldots, T_a^{\mathcal{R}}, \{0 \mid \operatorname{rank}(G)\}\} \in \mathcal{B},$$

and where $T_i^{\mathcal{R}}$ is the set of Right options of T_i. It follows as before that Right wins if Left plays to $0 + Z$, $C_i + Z$, or $B_i + Z$. For each Left move to $A_i + Z$, we claim that $A_i^R + Z$ is a winning move for Right. Since each T_i is non-zero, by the hand-tying principle we have $T_i \geqslant Z$ (in full misère), yielding $A_i^R + T_i \geqslant A_i^R + Z$. But Left loses $A_i^R + T_i$ going first by construction, and so Left must also lose $A_i^R + Z$ going first. We have shown there is no good first move for Left in $G + Z$, which contradicts G being Left $\mathcal{B}$-strong.

Thus, there exists some Left option G^L such that $\mathrm{o}(G^L) = \mathscr{L}$, where G^L and every Right response G^{LR} are Left $\mathcal{B}$-strong, as required. □

If one so desired, they could rewrite the second item in the conclusion of Theorem 3.1 as requiring G to admit an option G^L that is not a Right end, such that G^L and all Right responses G^{LR} are Left $\mathcal{B}$-strong.

As an example application of Theorem 3.1, we see that every game with a Left option to $\overline{1}$ is Left $\mathcal{B}$-strong, regardless of the other options for Left or Right, because $\overline{1}$ is Left-win and trivially Left $\mathcal{B}$-strong, and the only Right response in $\overline{1}$ is to 0, which is also trivially Left $\mathcal{B}$-strong. Indeed, it is straightforward to observe that the same is true for every game with a Left option to a non-zero Left dead end.

With Theorem 3.1, we have the means to computationally check the comparison modulo $\mathcal{B}$ of any two games in $\mathcal{M}$, which was our primary goal for this section. But we can say more than this: for dead-ending games, it was determined that a game G being Left $\mathcal{E}$-strong ($\mathrm{o_L}(G+X) = \mathscr{L}$ for all Left ends $X \in \mathcal{E}$) could be verified by checking just two Left ends: $X = 0$ and $X = W_n$, the "waiting game" of rank $n = \operatorname{rank}(G)$. This works for any $G \in \mathcal{M}$. Is it possible to have an analogous result for blocking games, so that

$$\mathrm{o_L}(G + X) = \mathscr{L} \quad \text{for all Left ends } X \in \mathcal{B}$$

can be verified by checking only finitely many X?

We will show that we can indeed do this for $\mathcal{B}$, and we do so by explicitly giving such X. We develop the theory in two parts: first, we will worry only about $G \in \mathcal{B}$; after this, we will zoom out and consider an arbitrary $G \in \mathcal{M}$. Interestingly, we will see that the 'optimal' tests are different depending on whether G is blocking, which was not the case for the dead-ending universe.

Before we do so, however, it will be useful for us to have a more intuitive understanding of what it means to be Left $\mathcal{B}$-strong, instead of the (perhaps difficult to internalise) recursive check of Theorem 3.1. To do this, imagine we are playing our games where the rules are modified so that Right is allowed to *pass* whenever he wishes. We will call the outcome of a game played under this modified ruleset the *Right-passing outcome*; and we will say that we are playing G *under Right-passing*. (We will only consider Right-passing, ignoring symmetric observations, and so we shorten this to *passing outcome*.) We will write $\mathrm{p_L}(G)$ for the passing outcome of G when Left plays first.

The alternative way to understand when a game is Left $\mathcal{B}$-strong is then as follows: a game $G \in \mathcal{M}$ is Left $\mathcal{B}$-strong if and only if $\mathrm{p_L}(G) = \mathscr{L}$—i.e. Left wins going first when playing G under Right-passing. We prove this assertion now. Since this idea of passing is to aid our understanding, rather than to serve as a foundation for our later arguments, we do not go into the formal details of defining such an outcome, but it can of course be done; see Appendix A for the details.

Theorem 3.2. *If $G \in \mathcal{M}$, then G is Left $\mathcal{B}$-strong if and only if $\mathrm{p_L}(G) = \mathscr{L}$.*

Proof. If G is a Left end, then we are done. So, suppose G is not a Left end.

We will first consider when G is Left $\mathcal{B}$-strong. By Theorem 3.1, there exists some Left-win option G^L such that G^L and all Right responses G^{LR} are Left $\mathcal{B}$-strong. Left playing first on G under Right-passing should play to this G^L. Note that, since G^L is Left-win, there must exist a Right option G^{LR}. Since G^L is Left $\mathcal{B}$-strong, we have $\mathrm{p_L}(G^L) = \mathscr{L}$ by induction, and so Right loses if he passes here. Now, every Right response G^{LR} is Left $\mathcal{B}$-strong by Theorem 3.1, and so $\mathrm{p_L}(G^{LR}) = \mathscr{L}$ by induction, which yields that $\mathrm{p_L}(G) = \mathscr{L}$.

Now consider instead when $\mathrm{p_L}(G) = \mathscr{L}$. Let G^L be a winning move for Left when playing under Right-passing. Since it is winning, we know that G^L admits a Right option, and that Right loses when passing here. Hence we have $\mathrm{p_L}(G^L) = \mathscr{L}$, which implies by induction that G^L is Left $\mathcal{B}$-strong. Now consider an arbitrary Right response G^{LR}. We must have $\mathrm{p_L}(G^{LR}) = \mathscr{L}$, and hence G^{LR} is Left $\mathcal{B}$-strong by induction. Since G^L is Left $\mathcal{B}$-strong, we know that $\mathrm{o_L}(G^L) = \mathscr{L}$. We also know that G^L admits at least one Right response, and every such response is Left $\mathcal{B}$-strong; hence, $\mathrm{o_R}(G^L) = \mathscr{L}$. Thus, we have that G^L is Left-win, which concludes the proof via Theorem 3.1. □

3.1. Finite test set: $G \in \mathcal{B}$

More specifically than we mentioned previously, our goal in this subsection is to find a finite test set of Left ends $\{X_i\}$ for each rank n such that, for every game $G \in \mathcal{B}$ of rank at most n, we can determine whether G is Left $\mathcal{B}$-strong simply by checking $\mathrm{o_L}(G + X_i)$ for all i. In fact, we will show that all we need is a single Left end X (for each rank).

In light of Theorem 3.2 showing that being Left $\mathcal{B}$-strong is equivalent to Left winning first when playing under Right-passing, we will proceed by considering how we might construct an X to simulate this passing outcome.

Recall the (Right) waiting games for dead-ending comparison were defined by $W_0 \coloneqq 0$ and $W_n \coloneqq \{\cdot \mid 0, W_{n-1}\}$ for $n > 0$. It is tempting to think that these are the forms we need, that Right uses the waiting moves of W_n to essentially *pass*, but this is not correct. Consider the passing outcome of $*2$: if Left plays first to 0, then Right wins; and if Left plays first to $*$, then Right passes and wins. Similarly, if Right plays first, then he can play to $*$ and win. Thus, the passing outcome must be $\mathscr{R}$. But there is no waiting game W_n such that $*2+W_n$ has outcome $\mathscr{R}$; that is, there is no way to use waiting games to simulate this modified passing ruleset.

To justify our last claim, first note that $*2 + W_0$ has outcome $\mathscr{N}$. Then, for $n > 0$, consider $*2 + W_n$; playing first, Left can play on the $*2$ component to 0, leaving W_n; from here, Right must make a move, but what remains is necessarily a Left dead end, and so Left wins. Thus, the outcome is not $\mathscr{R}$ (it is, in fact, $\mathscr{L}$, which the reader may check).

Other than this explicit example, why exactly do the waiting games fail to simulate the passing outcome? Well, the crux of it is that the only way for Right to get rid of his moves (i.e. to play to a Right end) is by moving to 0; and if Right does this when the other component of the disjunctive compound is a Left end, then Right loses immediately. An idea to solve this would be to modify the definition of W_n so that Right somehow has the ability to play to a non-zero Right end, thus not allowing Left to claim an immediate win. And this idea is precisely what we now present as the *$\mathcal{B}$-passing games*.

Definition 3.3. The nth *(Right) $\mathcal{B}$-passing game* B_n is defined recursively as

$$\begin{aligned} B_0, B_1 &\coloneqq 0, \text{ and} \\ B_n &\coloneqq \{\cdot \mid 1, B_{n-1}\} \quad \text{for } n \geqslant 2. \end{aligned}$$

Note that $B_n \in \mathcal{B}$. Also, for all $n \neq 1$, the game B_n has rank n; and B_1 instead has rank 0. We will not discuss it until the end of the section, but the reader may like to observe that the $\mathcal{B}$-simplest form of B_n is $\{\cdot \mid B_{n-1}, \blacksquare\}$ for $n \geqslant 2$; the recursion here is just like the definition of integers, and so one could think of these games, in some sense, as Right $\mathcal{B}$-strong integers. Indeed, it is a worthwhile exercise to see how some of our arguments below would change if we were to utilise the tombstones of the simplest forms; but, since the theory of simplest forms is so new, we do not want to unnecessarily overburden the reader.

Figure 3.1 illustrates the passing games up to rank 4. Compare this family with the waiting games of the dead-ending universe in Figure 2.1.

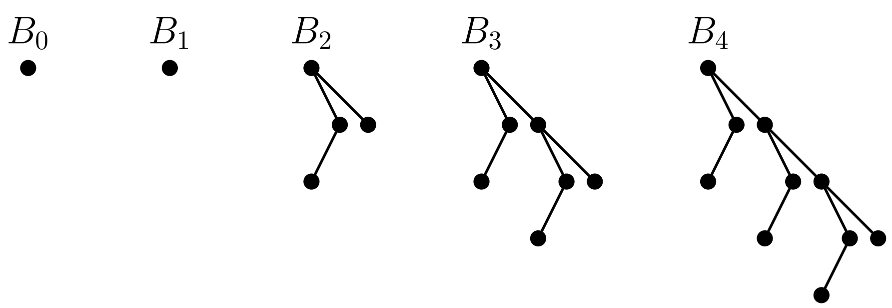

Figure 3.1: The $\mathcal{B}$-passing games B_0 to B_4.

Why have we chosen to call these B_n the '$\mathcal{B}$-passing games', instead of simply 'passing games'? It is because we will soon see that, while the $\mathcal{B}$-passing games do allow one to simulate the passing outcome, they are only guaranteed to do so for those games in $\mathcal{B}$; in particular, if $G \in \mathcal{B}$, then G is Left $\mathcal{B}$-strong if and only if $\mathrm{o_L}(G + B_n) = \mathscr{L}$, where $n = \mathrm{rank}(G)$. (In the next subsection, we will construct the more general passing games that allow us to simulate the passing outcome for arbitrary forms $G \in \mathcal{M}$; but those forms are more complicated than the B_n we are presenting now.) To get there, we need a few preliminary results, including that these $\mathcal{B}$-passing games B_n are Right $\mathcal{B}$-strong (an obvious fact if one observes the tombstones in their $\mathcal{B}$-simplest forms), that they are better for Right than 0, and that they get better for Right as the rank n increases.

We begin with a little lemma. In general, blocked Right ends are not necessarily Right-win; for example, $\{\overline{1} \mid \cdot\} \equiv_{\mathcal{B}} 0$ is next-win. But, in a sum of Right blocked ends, if one component is a non-zero Right dead end, then we can guarantee Right wins playing both first and second.

Lemma 3.4. *If Y is a blocked Right end and X is a non-zero Right dead end, then $\mathrm{o}(X+Y)=\mathscr{R}$.*

Proof. Right wins $X + Y$ going first, since it is a Right end. If $Y \cong 0$, then clearly Right wins $X + Y \cong X$ playing second, and so we will now assume that $Y \ncong 0$. We need to show that Right wins playing second.

If Left moves from $X + Y$ to some $X^L + Y$, then Right wins immediately. If Left instead moves to some $X + Y^L$, then, since $Y \in \mathcal{B}$, either: Y^L is a (blocked) Right end, and hence Right wins since he has no move; or else Right can move on Y^L to a blocked Right end Y^{LR}, and then wins the resulting $X + Y^{LR}$ by induction. □

Lemma 3.5. *For all n, the game B_n is Right $\mathcal{B}$-strong.*

Proof. If $n \leqslant 1$, then the conclusion is trivial. If $n \geqslant 2$, then Right has a move on B_n to 1. Thus, by Lemma 3.4, for any blocked Right end Y, Right wins playing first on $B_n + Y$ by moving to $1 + Y$. □

Lemma 3.6. *For all n, we have $0 \geqslant_{\mathcal{B}} B_n$.*

Proof. This is trivial if $n \leqslant 1$, so assume $n \geqslant 2$. Both maintenance conditions are vacuously true (in Theorem 2.1). For the proviso, since B_n is a Left end, we need 0 to be Left $\mathcal{B}$-strong (and it is, trivially); and since 0 is a Right end, we need B_n to be Right $\mathcal{B}$-strong, which it is by Lemma 3.5. □

Lemma 3.7. *If $k \leqslant n$, then $B_k \geqslant_{\mathcal{B}} B_n$.*

Proof. This is immediate if $n \leqslant 2$, and so assume that $n \geqslant 3$. Similarly, if $k \leqslant 1$, then this is just Lemma 3.6, so assume $k \geqslant 2$. Now, the proviso is clearly satisfied because B_k is a Left end (and hence Left $\mathcal{B}$-strong), and it is not a Right end (even if it were, we know that B_n is Right $\mathcal{B}$-strong by Lemma 3.5). For the maintenance, the Right options of B_k are 1 and B_{k-1}, and we have $0 \geqslant_{\mathcal{B}} B_n$ by Lemma 3.6 and $B_{k-1} \geqslant_{\mathcal{B}} B_{n-1}$ by induction on k. □

We are now able to prove the claimed finite test set for checking that $G \in \mathcal{B}$ is Left $\mathcal{B}$-strong. We note again that we only have to check the outcome of $\mathrm{o_L}(G+X)$ for a single Left end $X \in \mathcal{B}$.

Theorem 3.8. *If $G \in \mathcal{B}$ has rank n, then G is Left $\mathcal{B}$-strong if and only if $o_L(G + B_n) = \mathscr{L}$.*

Proof. If G is a Left end, then the result is clear, and so we will assume throughout that G is not a Left end.

If G is Left $\mathcal{B}$-strong, then, by definition, $o_L(G + X) = \mathscr{L}$ for every Left end $X \in \mathcal{B}$, which proves the forward direction. So, assume now that $o_L(G + B_n) = \mathscr{L}$. We will show that G is Left $\mathcal{B}$-strong via Theorem 3.1. Since this is straightforward when $n \leqslant 1$, we will assume $n \geqslant 2$.

To apply Theorem 3.1, we need to show there is a Left option G^L of G that is (1) Left $\mathcal{B}$-strong and (2) Left-win, and for which (3) all Right responses G^{LR} are Left $\mathcal{B}$-strong.

Since $o_L(G+B_n) = \mathscr{L}$, and since $G^{\mathcal{L}} \neq \varnothing$, there must be a G^L such that $o_R(G^L+B_n) = \mathscr{L}$, and hence $o_L(G^L + B_{n-1}) = \mathscr{L}$. By Lemma 3.7, it follows that $o_L(G^L + B_{\text{rank}(G^L)}) = \mathscr{L}$, and therefore G^L is Left $\mathcal{B}$-strong by induction (1).

We need $o(G^L) = \mathscr{L}$. Suppose not; i.e. suppose that Right wins G^L playing first (G^L is Left $\mathcal{B}$-strong, and hence Left must win playing first). Then, playing first on $G^L + B_n$, Right can ignore the B_n component and play his winning strategy on G^L. At some point, Left must play to a Right end subposition of G^L, say Y, leaving Right to play next on $Y + B_n$, which Right wins by Lemma 3.5, contradicting $o_R(G^L + B_n) = \mathscr{L}$. Thus, $o(G^L) = \mathscr{L}$ (2).

Finally, we need all Right options of G^L to be Left $\mathcal{B}$-strong. Let G^{LR} be an arbitrary Right option of G^L (there must exist one since $o(G^L) = \mathscr{L}$). We know that $o_R(G^L + B_n) = \mathscr{L}$, so $o_L(G^{LR} + B_n) = \mathscr{L}$. By Lemma 3.7, it follows that $o_L(G^{LR} + B_{\text{rank}(G^{LR})}) = \mathscr{L}$, and hence G^{LR} is Left $\mathcal{B}$-strong by induction. □

Notice that the proof of Theorem 3.8 does indeed require that G has the blocking property (we need the subposition Y of G to be blocking). In Section 3.2, we develop an analogous finite test set that works for arbitrary $G \in \mathcal{M}$.

3.2. Finite test set: $G \in \mathcal{M}$

Recall that Theorem 3.1 holds for any $G \in \mathcal{M}$, so we already have a computational way to check that a non-blocking game is Left $\mathcal{B}$-strong (and therefore a way to compute comparisons between arbitrary forms modulo $\mathcal{B}$). Our aim in this subsection is to generalise Theorem 3.8: to find a finite test set of Left ends $\{X_i\}$ for each rank n such that, for every game $G \in \mathcal{M}$ of rank at most n, we can determine whether G is Left $\mathcal{B}$-strong simply by checking $o_L(G + X_i)$ for all i. Like in Theorem 3.8, we will show that all we need is a single Left end X (for each rank), but this X is more complicated than the one we needed before (hence our separating of this into two subsections).

The first thing that we should point out is the unfortunate fact that the $\mathcal{B}$-passing forms do indeed fail for arbitrary $G \in \mathcal{M}$. Consider the game

$$G = \left\{ \left\{ *, \{\overline{2} \mid \cdot\} \;\middle|\; \overline{1} \right\} \;\middle|\; \cdot \right\}.$$

We leave it to the reader to verify that $o_L(G + B_n) = \mathscr{L}$ for all n, but that G is *not* Left $\mathcal{B}$-strong. This counterexample is actually stronger than it appears; it also shows that modifying the recursive definition of B_n to $\{\cdot \mid 0, 1, B_{n-1}\}$ is insufficient, too, but this observation will be irrelevant for our purposes.

So, we really do need a new idea; we want to try and modify our $\mathcal{B}$-passing games so that they work for such strange games as the G just discussed. It is not obvious at this stage that such a modification is even possible, but indeed it is!

Recall that the adjoint of G was defined by Siegel [Sie15a, Definition 3.2 on p. 228] as the partizan analogue to Conway's *mate* in the impartial theory:

$$G^\circ := \begin{cases} * & \text{if } G \cong 0, \\ \{(G^{\mathcal{R}})^\circ \mid 0\} & \text{if } G \not\cong 0 \text{ is a Left end,} \\ \{0 \mid (G^{\mathcal{L}})^\circ\} & \text{if } G \not\cong 0 \text{ is a Right end,} \\ \{(G^{\mathcal{R}})^\circ \mid (G^{\mathcal{L}})^\circ\} & \text{otherwise.} \end{cases}$$

Its usefulness to us here will be due to the fact that $\mathrm{o}(G + G^\circ) = \mathscr{P}$ for every partizan misère game G [Sie15a, Proposition 3.3 on p. 228].

Definition 3.9. For all $n \geqslant 0$, let $\mathcal{J}'_n = \{G^\circ : G \in \mathcal{M} \text{ and } \mathrm{rank}(G) \leqslant n\}$; i.e. the set of adjoints of all forms of rank at most n. (Since every adjoint is dicotic, clearly $\mathcal{J}'_n \subseteq \mathcal{B}$.) Now let $\mathcal{J}_n$ be the set of minimal elements (in full misère) of $\mathcal{J}'_n$.

Now define the nth *(Right) passing game* A_n recursively by

$$\begin{aligned} A_0, A_1, A_2 &:= 0, \\ A_3 &:= \{\cdot \mid 0, 1\}, \text{ and} \\ A_n &:= \{\cdot \mid 0, 1, \{0 \mid \mathcal{J}_{n-3}\}, A_{n-1}\} \quad \text{for } n \geqslant 4. \end{aligned}$$

Note that $A_n \in \mathcal{B}$ for all n. Also, for all $n \geqslant 4$, the nth passing game has rank n; the games A_0, A_1, and A_2 instead have rank 0, and A_3 has rank 2. (It would be prudent to remind oneself that $\mathrm{rank}(G^\circ) = \mathrm{rank}(G) + 1$ for all games G.) The reader may wish to confirm that

$$A_4 = \{\cdot \mid 0, 1, \{0 \mid *, \{0 \mid *\}, \{* \mid 0\}, \{* \mid *\}\}, A_3\};$$

that is, $\mathcal{J}_1$ is the set $\{*, \{0 \mid *\}, \{* \mid 0\}, \{* \mid *\}\}$. Note that $\mathcal{J}_2$ already contains 256 elements.

We need only a small lemma, and then we can build our generalised test.

Lemma 3.10. *If* $3 \leqslant k \leqslant n$, *then* $A_k \geqslant A_n$.

Proof. It should be clear that it will suffice to check $\{0 \mid \mathcal{J}_{k-3}\} \geqslant \{0 \mid \mathcal{J}_{n-3}\}$ (the rest is trivial and follows by induction). But this is also immediate, since, for every element in $\mathcal{J}_{k-3}$ (of which there is at least one), there exists an element in $\mathcal{J}_{n-3}$ that is at least as good for Right. □

Theorem 3.11. *If* $G \in \mathcal{M}$ *has rank* n, *then* G *is Left* $\mathcal{B}$*-strong if and only if* $\mathrm{o_L}(G + A_{n+1}) = \mathscr{L}$.

Proof. If G is a Left end, then the result is clear, and so we will assume throughout that G is not a Left end.

If G is Left $\mathcal{B}$-strong, then it is immediate that $o_L(G+A_{n+1}) = \mathscr{L}$. So, for the other direction, assume now that $o_L(G + A_{n+1}) = \mathscr{L}$. We will show that G is Left $\mathcal{B}$-strong via Theorem 3.1. This is straightforward to verify for $n \leqslant 1$. So, suppose that $n \geqslant 2$.

We first show that there exists an option G^L with outcome $\mathscr{L}$. Left must play first on $G+A_{n+1}$ to some $G^L + A_{n+1}$. If G^L has outcome $\mathscr{N}$ or $\mathscr{R}$, then Right can respond with $G^L + \{0 \mid \mathcal{J}_{n-2}\}$. Left cannot move to G^L+0, else she would lose, and hence has to play again on the G component to some G^{LL}. But then Right can play on $\{0 \mid \mathcal{J}_{n-2}\}$ to leave a game at least as good for Right as $G^{LL} + (G^{LL})^{\circ}$, thus winning the game, and contradicting the hypothesis. Instead, if G^L has outcome $\mathscr{P}$, then Right may play on A_{n+1} to 0, necessarily leaving himself a winning position. Therefore, G^L must have outcome $\mathscr{L}$.

We now show that this G^L, along with every Right response G^{LR}, is Left $\mathcal{B}$-strong. Since $o_L(G + A_{n+1}) = \mathscr{L}$ by hypothesis, it follows that $o_L(G^L + A_n) = \mathscr{L}$ (for some G^L with $o(G^L) = \mathscr{L}$ by the previous discussion). If $\operatorname{rank}(G^L) \leqslant 1$, then, since $o(G^L) = \mathscr{L}$, it must be that $G^L \cong \overline{1}$, and hence G^L and every Right option G^{LR} is Left $\mathcal{B}$-strong. Otherwise, $2 \leqslant \operatorname{rank}(G^L) \leqslant n-1$, and we may conclude by induction that G^L is Left $\mathcal{B}$-strong (using Lemma 3.10). It also follows from the hypothesis that $o_L(G^{LR} + A_{n+1}) = \mathscr{L}$ for every Right option G^{LR}. Similar to before, if $\operatorname{rank}(G^{LR}) \leqslant 1$, then, since $o(G^L) = \mathscr{L}$, it follows that $o_L(G^{LR}) = \mathscr{L}$, and so $G^{LR} \cong 0$ or $\overline{1}$; in either case, it is Left $\mathcal{B}$-strong. Otherwise, $2 \leqslant \operatorname{rank}(G^{LR}) \leqslant n-2$, and we may conclude by induction that G^{LR} is Left $\mathcal{B}$-strong (using Lemma 3.10 again). Therefore, by Theorem 3.1, we can conclude that G is Left $\mathcal{B}$-strong. □

3.3. The minimal Left ends

Harking back to the dead-ending universe $\mathcal{E}$, recall that the waiting games W_n are used to determine whether a game is Left $\mathcal{E}$-strong. (Specifically, for a game G of rank n, we need to calculate $o_L(G+W_0)$ and $o_L(G+W_n)$.) It so happens that, among the Left dead ends of rank at most n, the games 0 and W_n are the two minimal elements (of course, when $n = 0$, they are the same). This makes sense; if we want to check whether $o_L(G+X) = \mathscr{L}$ for all Left ends X in a universe, then of course we need only check those X that are minimal, and it miraculously turns out that we need only check two of them in the case of the dead-ending universe: the two minimal elements with ranks at most the rank of the game we are checking. Given this, one might reasonably seek to prove that the passing games A_n, which we used in the previous subsection to determine whether a game is Left $\mathcal{B}$-strong, are each minimal in the set of blocked Left ends of rank at most n. Surprisingly, this turns out not to be the case: among all the Left blocked ends of rank at most n, the game A_n is *not* (in general) minimal. In this section, we will explore this idea, and, in particular, find the true minimal elements.

The blocking universe $\mathcal{B}$, much like $\mathcal{E}$, has particularly well-behaved ends, by which we mean that we have a lot of flexibility in the types of ends we can construct. In $\mathcal{E}$, for instance, we can construct a Left dead end by putting any (finite) number of Left dead ends that we want as options for Right, and what results is necessarily in $\mathcal{E}$. This sort of behaviour is much more restricted in other universes. In our case, in $\mathcal{B}$, we can do a similar thing: we can construct what we might call the *super ends*, which one can think of intuitively as simply giving Right all possible options while keeping the form in $\mathcal{B}$. (This is how one can construct the waiting games in $\mathcal{E}$, too; it just so happens that they have many reductions that result in their simple definition.) It will be clear that the super end of a given rank n must be minimal over the non-zero Left blocked ends of rank at most n (just by the hand-tying principle). We construct these games now.

Definition 3.12. For all $n \geqslant 0$, let $\mathcal{B}_n$ be the set of all blocking forms of rank at most n; let $\mathcal{K}_n$ denote the subset of positions of $\mathcal{B}_n$ that are Left ends; and let $\mathcal{T}_n$ denote the subset of positions of $\mathcal{B}_n$ that are not Left ends but which have at least one Left option to a Left end. The *blocked super end* of rank n is defined by

$$\begin{aligned} S_0 &:= 0, \\ S_1 &:= \overline{1}, \text{ and} \\ S_n &:= \{\cdot \mid {S_n}^{\mathcal{R}}\} \quad \text{for } n \geqslant 2, \end{aligned}$$

where the Right options of S_n are ${S_n}^{\mathcal{R}} = \mathcal{K}_{n-1} \cup \mathcal{T}_{n-1}$; i.e. all possible options that keep S_n in $\mathcal{B}$.

For example, $S_2 = \{\cdot \mid 0, \overline{1}, 1, *\}$. And, even this quickly, we can begin to see why the passing games A_n are not the minimal Left blocked ends of their respective ranks; recalling that $A_3 = \{\cdot \mid 0, 1\}$, is it true that $S_2 \geqslant A_3$? Absolutely not! For one, the Right option $\overline{1}$ of S_2 has no match in A_3, and so clearly the maintenance is not satisfied. Nearer to the end of this section, we will discuss why, looking back, it should not be so surprising that this occurred. But first, we seek to further explore these super ends. Indeed, we will find their $\mathcal{M}$- and $\mathcal{B}$-simplest forms.

Lemma 3.13. *If X is a Left blocked end of rank n, then $X \geqslant S_n$.*

Proof. If $n = 0$, then $X \cong 0 \cong S_0$, so the conclusion is immediate. Otherwise, the comparison $X \geqslant S_n$ (which is modulo $\mathcal{M}$) follows immediately from the definition of S_n, by hand-tying, because every Right option of X is necessarily a Right option of S_n. □

Lemma 3.14. *If $n > 0$, then S_n is incomparable with 0.*

Proof. Consider first the inequality $0 \geqslant S_n$. Since $n > 0$, it is clear that S_n is not a Right end by definition, and hence it is not Right $\mathcal{M}$-strong. Thus, the proviso fails.

Consider now the inequality $S_n \geqslant 0$. Since $n > 0$, Right must have an option from S_n to 0 by definition of the super end S_n; but now the maintenance property fails, as there is neither a Right option in 0, nor a Left option we can reverse through. □

Corollary 3.15. *The set of minimal elements of $\mathcal{K}_n$ is $\{0, S_n\}$ for all $n \geqslant 0$.*

Proof. This follows immediately from Lemmas 3.13 and 3.14. □

So, we have constructed our super ends in $\mathcal{B}$ and shown fairly trivially that the super end and 0 are, for Right, the best Left blocked ends of each rank. We now want to see how to reduce these super ends: what are their $\mathcal{M}$-simplest forms? (We will then find their $\mathcal{B}$-simplest forms, and compare with the passing games of the previous subsections.) Recall that, because one cannot reverse through an end in $\mathcal{M}$, these $\mathcal{M}$-simplest forms are also $\mathcal{M}$-canonical forms, so no tombstones will appear here.

Proposition 3.16. *The $\mathcal{M}$-simplest form of the blocked super end S_n is*

$$S_n \equiv \begin{cases} 0 & \text{if } n = 0, \\ \overline{1} & \text{if } n = 1, \\ \left\{ \cdot \mid 0, 1, \{0 \mid \mathcal{B}_{n-2}^{\leqslant}\}^*, S_{n-1}^* \right\} & \text{if } n \geqslant 2, \end{cases}$$

where $\mathcal{B}_k^{\leqslant}$ is the set of minimal elements of $\mathcal{B}_k$, and $\{0 \mid \mathcal{B}_{n-2}^{\leqslant}\}^*$ is the $\mathcal{M}$-simplest form of $\{0 \mid \mathcal{B}_{n-2}^{\leqslant}\}$, and S_{n-1}^* is the $\mathcal{M}$-simplest form of S_{n-1}.

Proof. The result is immediate for $n \leqslant 1$. So, suppose that $n \geqslant 2$.

Recall from the definition of S_n ($n \geqslant 2$) that the Right options range over (1) all Left ends in $\mathcal{B}_{n-1}$ and (2) all non-Left-ends in $\mathcal{B}_{n-1}$ that have a Left option to a Left end. By Corollary 3.15, the options of type (1) are dominated by 0 and S_{n-1}, and so the simplest form will have only these two options of type (1). Next, consider the options of type (2).

We claim that each non-Left-end Right option of S_n is dominated by one of the following four options:

$$1,\ \{S_{n-2} \mid \cdot\},\ \{0 \mid \mathcal{B}_{n-2}^{\leqslant}\},\ \text{or } \{S_{n-2} \mid \mathcal{B}_{n-2}^{\leqslant}\}.$$

Let $J = \{J^{\mathcal{L}} \mid J^{\mathcal{R}}\}$ be a Right option of S_n with $J^{\mathcal{L}} \neq \varnothing$ (i.e. one of type (2)). We know by the definition of S_n that $J^{\mathcal{L}}$ contains a Left end—either 0 or some non-zero Left end X. Observe the following cases:

(i) If $0 \in J^{\mathcal{L}}$ and $J^{\mathcal{R}} = \varnothing$, then $J \geqslant 1$.

This is immediate by the hand-tying principle.

(ii) If $0 \in J^{\mathcal{L}}$ and $J^{\mathcal{R}} \neq \varnothing$, then $J \geqslant \{0 \mid \mathcal{B}_{n-2}^{\leqslant}\}$.

The proviso passes because $\{0 \mid \mathcal{B}_{n-2}^{\leqslant}\}$ is not a Left end and J is not a Right end. The maintenance holds simply by the hand-tying principle.

(iii) If $0 \notin J^{\mathcal{L}}$ and $J^{\mathcal{R}} = \varnothing$, then $J \geqslant \{S_{n-2} \mid \cdot\}$.

The proviso passes since $\{S_{n-2} \mid \cdot\}$ is not a Left end, but is a Right end. We know that there exists a non-zero Left end $X \in J^{\mathcal{L}}$, and so the maintenance holds because $X \geqslant S_{n-2}$ by Lemma 3.13 (note that $\operatorname{rank}(J) \leqslant n-1$, so $\operatorname{rank}(X) \leqslant n-2$).

(iv) If $0 \notin J^{\mathcal{L}}$ and $J^{\mathcal{R}} \neq \varnothing$, then $J \geqslant \{S_{n-2} \mid \mathcal{B}_{n-2}^{\leqslant}\}$.

The proviso passes because neither are ends, and the maintenance holds similarly to (ii) and (iii).

As it stands, we have proven that, for $n \geqslant 2$, we can reduce S_n to

$$S_n \equiv \left\{ \cdot \;\middle|\; 0, 1, \left\{0 \;\middle|\; \mathcal{B}_{n-2}^{\leqslant}\right\}, S_{n-1}, \left\{S_{n-2} \;\middle|\; \cdot\right\}, \left\{S_{n-2} \;\middle|\; \mathcal{B}_{n-2}^{\leqslant}\right\} \right\}.$$

Note that, for $n = 2$, this is exactly the form $\{\cdot \mid 0, \overline{1}, 1, *\}$, which is indeed in $\mathcal{M}$-simplest form because the options are pairwise incomparable modulo $\mathcal{M}$. So, assume $n \geqslant 3$.

Now, from Lemma 3.13, it is clear that both $\{S_{n-2} \mid \cdot\}$ and $\{S_{n-2} \mid \mathcal{B}^{\leqslant}_{n-2}\}$ are reversible through S_{n-2}. So, we may replace each of these games with the Right options of S_{n-2}.

If $n = 3$, then $S_{n-2} = S_1 \cong \overline{1}$, and the only Right option of S_1 is 0. If $n \geqslant 4$, then we may assume by induction that the Right options of S_{n-2} are 0, 1, S_{n-3}, and $\{0 \mid \mathcal{B}^{\leqslant}_{n-4}\}$. Now, by Corollary 3.15, S_{n-3} is dominated either by 0 or S_{n-1}, and, by hand-tying, $\{0 \mid \mathcal{B}^{\leqslant}_{n-4}\}$ is dominated by $\{0 \mid \mathcal{B}^{\leqslant}_{n-2}\}$. Thus, in either case ($n = 3$ or $n \geqslant 4$), we obtain

$$S_n \equiv \left\{ \cdot \,\middle|\, 0, 1, \left\{ 0 \,\middle|\, \mathcal{B}^{\leqslant}_{n-2} \right\}, S_{n-1} \right\}.$$

The options $1 \cong \{0 \mid \cdot\}$ and $\{0 \mid \mathcal{B}^{\leqslant}_{n-2}\}$ cannot be $\mathcal{M}$-reversible because $0 \not\geqslant S_n$ (use Lemma 3.14, noting that $n > 0$). Likewise, it is straightforward to verify (via Corollary 2.2) that none of these four options dominates another. If we now write $\{0 \mid \mathcal{B}^{\leqslant}_{n-2}\}^*$ for the $\mathcal{M}$-simplest form of $\{0 \mid \mathcal{B}^{\leqslant}_{n-2}\}$, and S^*_{n-1} for the $\mathcal{M}$-simplest form of S_{n-1}, then we finally obtain

$$S_n \equiv \left\{ \cdot \,\middle|\, 0, 1, \left\{ 0 \,\middle|\, \mathcal{B}^{\leqslant}_{n-2} \right\}^*, S^*_{n-1} \right\}.$$

as the $\mathcal{M}$-simplest form of S_n, as claimed. □

Of course, it is not clear exactly what $\mathcal{B}^{\leqslant}_n$ looks like (we do not know what the minimal elements of $\mathcal{B}_n$ are), and similarly for $\{0 \mid \mathcal{B}^{\leqslant}_{n-2}\}^*$. The problem is likely intractable, but perhaps it would be worthwhile to try and investigate further. Regardless of what the latter form looks like, however, we still know, by Proposition 3.16, what the $\mathcal{M}$-simplest form of S_n roughly looks like: for $n \geqslant 2$, it has exactly four options, and we know what equivalence class each of those options is in.

Contrasting the $\mathcal{M}$-simplest forms of the super ends with the passing games A_n, we see that they are (perhaps unsurprisingly) very similar. The core difference lies in the option $\{0 \mid \mathcal{B}^{\leqslant}_{n-2}\}^*$ of the super end, in comparison to the option $\{0 \mid \mathcal{J}_{n-3}\}$ of A_n. This is because the games in $B^{\leqslant}_{n-2}$ that are not in $\mathcal{J}_{n-3}$ are simply not needed to simulate the passing outcome; they are superfluous in that regard. But their inclusion in the super ends does create an equivalence class distinct from the passing games.

We now calculate the $\mathcal{B}$-simplest forms of the super ends. In this case, we can be more concrete than we were able to be in Proposition 3.16. Recall that, per Siegel's theory of simplest forms [Sie25, §5], we replace options that reverse through an end with a tombstone "∎", and we remove this Left / Right tombstone if, after its removal, the resulting form would still be still Left / Right $\mathcal{B}$-strong—the tombstone just serves as a marker to preserve the information that the game is strong. Before our calculations, we make an observation.

Recall that the $\mathcal{B}$-passing forms are Right $\mathcal{B}$-strong (Lemma 3.5), and this led to the inequality $0 \geqslant_{\mathcal{B}} B_n$ for all n (Lemma 3.6). We have seen that $B_n \geqslant S_n$ for all $n \neq 1$ (by Lemma 3.13), and so trivially $B_n \geqslant_{\mathcal{B}} S_n$ for all $n \neq 1$, from which we conclude that $0 \geqslant_{\mathcal{B}} S_n$ for all $n \neq 1$ (this also means that each such S_n is Right $\mathcal{B}$-strong).

Theorem 3.17. *The $\mathcal{B}$-simplest form of the blocked super end S_n is*

$$S_n \equiv_{\mathcal{B}} \begin{cases} 0 & \text{if } n = 0, \\ \overline{1} & \text{if } n = 1, \\ \{\cdot \mid 0, \overline{1}, \blacksquare\} & \text{if } n = 2, \text{ and} \\ \{\cdot \mid S^*_{n-1}, \blacksquare\} & \text{otherwise}, \end{cases}$$

where S^*_{n-1} refers to the $\mathcal{B}$-simplest form of S_{n-1}.

Proof. We may assume S_n to be in $\mathcal{M}$-simplest form, as described by Proposition 3.16.

For $n \leqslant 1$, the result is immediate. Consider $n = 2$: since $0 \geqslant_{\mathcal{B}} S_2$, we know that 1 and $*$ are $\mathcal{B}$-reversible options of S_2, and hence $S_2 \equiv_{\mathcal{B}} \{\cdot \mid 0, \overline{1}, \blacksquare\}$. Observe that there are no dominated options, and also that $\{\cdot \mid 0, \overline{1}\}$ is not Right $\mathcal{B}$-strong (since Right loses going first), and hence we cannot erase the tombstone. It is thus clear that $\{\cdot \mid 0, \overline{1}, \blacksquare\}$ is the $\mathcal{B}$-simplest form of S_2.

Consider now $n \geqslant 3$. Since $0 \geqslant_{\mathcal{B}} S_n$ (for $n \neq 1$), we know that 1 and $\{0 \mid \mathcal{B}^{\leqslant}_{n-2}\}^*$ are $\mathcal{B}$-reversible options of S_n, and also that $0 \geqslant_{\mathcal{B}} S_{n-1}$. Hence, $S_n \equiv_{\mathcal{B}} \{\cdot \mid S^*_{n-1}, \blacksquare\}$ (where S^*_{n-1} now refers to do the $\mathcal{B}$-simplest form of S_{n-1}). This form cannot have any dominated options, nor can it admit any reversible options (since its only ordinary option is a Left end). Also, Right loses $\{\cdot \mid S^*_{n-1}\}$ playing first, and so we cannot erase the tombstone. Thus, $\{\cdot \mid S^*_{n-1}, \blacksquare\}$ is the $\mathcal{B}$-simplest form of S_n. □

It is interesting to compare these forms both to the $\mathcal{B}$-passing forms and to the waiting games W_n. Indeed, the only difference between these forms and the $\mathcal{B}$-passing forms B_n, aside from $n = 1$, is the base of the recursion B_2; the $\mathcal{B}$-simplest form of S_2 simply has an extra option $\overline{1}$, which is never necessary for Right to use in order to disprove that a blocking form is Left $\mathcal{B}$-strong, and hence B_2 did not need it as an option to simulate the passing outcome.

Perhaps more strikingly, we can observe that the $\mathcal{B}$-simplest forms of S_n, for $n \leqslant 1$, are exactly the waiting games W_n, and, for $n \geqslant 2$, they are (in a sense) Right $\mathcal{B}$-strong waiting games: add a Right tombstone to W_2, and then recurse as usual in the definition of W_n while adding a tombstone at each step, and you will obtain the $\mathcal{B}$-simplest forms of the S_n.

4. Inverses

Given a monoid M, it is well-known that the set of invertible elements form a subgroup of the monoid, and it is commonly referred to as the *invertible subgroup* of M (this is also sometimes called the *core* of M, and is the maximal subgroup of the monoid). Recall from [DY25] that a game G is said to be *$\mathcal{U}$-invertible* if there exists some $H \in \mathcal{U}$ such that $G + H \equiv_{\mathcal{U}} 0$. (Note that this definition does not require G to be in $\mathcal{U}$.) Recalling that universes are monoids, we will also refer to the set of all $\mathcal{U}$-invertible elements inside a universe $\mathcal{U}$ as the *invertible subgroup* of $\mathcal{U}$.

In full misère play, we have $G + \overline{G} \not\equiv 0$ for all non-zero G, due to Mesdal and Ottaway [MO07, Theorem 7 on p. 5]. In restricted misère play, it becomes possible not only that $G + \overline{G}$ is equivalent to zero, but perhaps even that $G + H$ is equivalent to zero, with H not equivalent to $\overline{G}$! In [DY25], however, it is shown that the latter cannot happen in a parental universe; that is, every parental universe $\mathcal{U}$ has the *conjugate property*, so that $G \in \mathcal{U}$ is $\mathcal{U}$-invertible if and only if $G + \overline{G} \equiv_{\mathcal{U}} 0$. So, to determine whether a blocking form G is $\mathcal{B}$-invertible, we need to check whether $G + \overline{G} \equiv_{\mathcal{B}} 0$. Since $G + \overline{G}$ is a symmetric form, it clearly suffices to check only whether $G + \overline{G} \geqslant_{\mathcal{B}} 0$.

A characterisation of the elements of the invertible subgroup of every parental universe is given in [DY25].

Theorem 4.1 (cf. [DY25, Theorem 17 on p. 11])**.** *If $\mathcal{U}$ is a parental universe, then $G \in \mathcal{U}$ is $\mathcal{U}$-invertible if and only if $G + \overline{G}$ is Left $\mathcal{U}$-strong and G' is $\mathcal{U}$-invertible for every proper subposition G' of the $\mathcal{U}$-simplest form of G.*

Since the theorem above applies for every parental universe $\mathcal{U}$, it of course applies to our blocking universe $\mathcal{B}$. So, what is there left to do? Well, what we will see in the remainder of this section is that we can give an alternative characterisation of the $\mathcal{B}$-invertible blocking forms, much in the same vein as the characterisation for the dead-ending universe $\mathcal{E}$ given in [MR22]. In particular, just as for $\mathcal{E}$, we will show that a blocking form is $\mathcal{B}$-invertible if and only if it is $\mathscr{P}$-free (modulo $\mathcal{B}$). Of course, we need to define what it means to be $\mathscr{P}$-free; it will be useful to us to introduce two variants of the term: one for strict forms, and another for equivalence classes (the latter of which was done in [MR22]).

We say that a game is *(strictly) $\mathscr{P}$-free* if no subposition of the game has outcome $\mathscr{P}$. Given a set of games $\mathcal{A}$, we say that a game G is *$\mathscr{P}$-free modulo $\mathcal{A}$* if there exists a strictly $\mathscr{P}$-free game $H \in \mathcal{A}$ such that $G \equiv_{\mathcal{A}} H$.

Work by Davies, Miller, and Milley [DMM25] generalises the tipping-point arguments of [MR22] so that similar results hold for far more than just the dead-ending games. In particular, they prove that every $\mathscr{P}$-free blocking game is $\mathcal{B}$-invertible, which we state here.

Theorem 4.2 (cf. [DMM⁺24, Theorem 4.12 on p. 28])**.** *If $G \in \mathcal{B}$ is $\mathscr{P}$-free, then G is $\mathcal{B}$-invertible.*

We will now show that every $\mathcal{B}$-invertible blocking form is also $\mathscr{P}$-free, thus obtaining an alternative characterisation. We begin with a lemma.

Lemma 4.3. *If $G \in \mathcal{B}$ has outcome $\mathscr{P}$, then G is not $\mathcal{B}$-invertible.*

Proof. Since $\mathrm{o}(G) = \mathscr{P}$, we also have $\mathrm{o}(\overline{G}) = \mathscr{P}$, and so both of these have Left options. We know G being $\mathcal{B}$-invertible would imply that $G + \overline{G}$ is Left $\mathcal{B}$-strong; we will show that this conclusion is false, by showing that Left does not win playing first on $G + \overline{G} + B_n$, where B_n is the $\mathcal{B}$-passing game of rank $n = \mathrm{rank}(G) + \mathrm{rank}(\overline{G}) \geqslant 2$, and hence G cannot be $\mathcal{B}$-invertible.

When Left moves first in $G + \overline{G} + B_n$, it is in either G or $\overline{G}$. Right should respond locally with the $\mathscr{P}$-win strategy in each of G and $\overline{G}$. At some point, Left moves in one of these to a Right end; without loss of generality, say this happens in G first. At this point, Right should play in B_n to B_{n-1}. From now on, Right follows this strategy:

- if Left plays in G to a Right end, including possibly 0, then Right "passes" by moving in the passing game;

- if Left plays in G to a non-Right-end, then Right "blocks" and returns this component to a Right end; and
- if Left plays in $\overline{G}$, Right continues to respond locally with the previous-win strategy.

At some point, Left moves $\overline{G}$ to a Right end. If Right has no moves left in the passing game, then Right wins immediately. Otherwise, Right should play on the passing game to 1, leaving $Y+1$, where Y is a blocked Right end subposition of $G+\overline{G}$. By Lemma 3.4, Right wins. □

Note that, in the final paragraph of the above proof, it is indeed possible that Right could have no moves left in the passing game. Consider, for example, the game $*$, which has outcome $\mathscr{P}$ and is blocking. Left playing first on $*+*+B_2$ will necessarily move to $*+B_2$, at which point the strategy given for Right dictates that he move to $*+B_1$, which is isomorphic to $*$ (recall that $B_1 \cong 0$). When Left then moves on $*$ to 0, Right no longer has any passing moves available, but he wins since Left played to a Right end.

Before we can prove our main theorem, we need one final lemma: it might not be immediately obvious that the $\mathcal{B}$-simplest form of a blocking game being strictly $\mathscr{P}$-free would imply that the blocking game is $\mathscr{P}$-free modulo $\mathcal{B}$, but this is indeed the case. We must effectively show that the tombstones can be converted back into end-reversible blocking forms that are $\mathscr{P}$-free.

Lemma 4.4. *If the $\mathcal{B}$-simplest form of a blocking game G is strictly $\mathscr{P}$-free, then G is $\mathscr{P}$-free modulo $\mathcal{B}$.*

Proof. It suffices to show that every Left end-reversible option can be replaced with a $\mathscr{P}$-free end-reversible option (without changing the equivalence class of the game).

So, let G be a blocking game with a Left end-reversible option X; this means that there exists some Left end X^R such that $G \geqslant_{\mathcal{B}} X^R$. By Lemma 3.13, we know that there exists an n such that $X^R \geqslant S_n$ (recall that S_n is our super end of rank n). Now, if S_n is not strictly $\mathscr{P}$-free (which occurs when $n>1$), then we will modify it in the following way: take each subposition S' of S_n that has outcome $\mathscr{P}$ and give Right an additional option to the game 1. It then clearly follows by hand-tying that the resulting form, which we will call Y, is a blocking form that is strictly $\mathscr{P}$-free and satisfies $X^R \geqslant S_n \geqslant Y$.

Now consider the game $G' = \{G^{\mathcal{L}}, \{\cdot \mid Y\} \mid G^{\mathcal{R}}\}$. Clearly $G' \geqslant G$, and so $G' \geqslant_{\mathcal{B}} X^R$ and $G' \geqslant_{\mathcal{B}} Y$. Thus, both X and $\{\cdot \mid Y\}$ are Left end-reversible options of G', and so we may remove either one without changing the $\mathcal{B}$-equivalence class. By removing $\{\cdot \mid Y\}$, we see that $G' \equiv_{\mathcal{B}} G$. And, by removing X, it follows that $G \equiv_{\mathcal{B}} \{G^{\mathcal{L}} \setminus \{X\}, \{\cdot \mid Y\} \mid G^{\mathcal{R}}\}$. Thus, we have shown that we can replace the end-reversible option X of G with a strictly $\mathscr{P}$-free end-reversible option, yielding the result. □

We have now built enough tools to easily prove our main result.

Theorem 4.5. *The set of $\mathscr{P}$-free blocking games is equal to the invertible subgroup of $\mathcal{B}$.*

Proof. By Theorem 4.2, we know that, if G is $\mathscr{P}$-free modulo $\mathcal{B}$, then G is $\mathcal{B}$-invertible. To conclude the proof, it remains to show that every $\mathcal{B}$-invertible blocking game is $\mathscr{P}$-free modulo $\mathcal{B}$.

Consider a $\mathcal{B}$-invertible blocking form G, and let G' be its $\mathcal{B}$-simplest form. By Theorem 4.1, we know that every subposition of G' is $\mathcal{B}$-invertible. By Lemma 4.3, it follows that no subposition of G' has outcome $\mathscr{P}$, and hence G' is strictly $\mathscr{P}$-free. Then, by Lemma 4.4, we conclude that G is $\mathscr{P}$-free modulo $\mathcal{B}$. □

This result is also equivalent to saying that a blocking game is $\mathcal{B}$-invertible if and only if its $\mathcal{B}$-simplest form is strictly $\mathscr{P}$-free.

There is an important clarifying remark to be made here: it was shown in [MR22] that a dead-ending game is $\mathcal{E}$-invertible if and only if it is $\mathscr{P}$-free (modulo $\mathcal{E}$), but this does *not* mean that the invertible subgroups of $\mathcal{B}$ and $\mathcal{E}$ coincide. Not only are there $\mathcal{B}$-invertible blocking forms that are not $\mathcal{E}$-invertible dead-ending forms, but there also exist $\mathcal{E}$-invertible dead-ending forms that are not $\mathcal{B}$-invertible blocking forms. Since the examples of rank at most 2 are relatively small in number, we will give a full list.

In the following table we give, up to conjugation, all elements of the invertible subgroup of $\mathcal{B}$ of rank at most 2 that are not $\mathcal{E}$-invertible dead-ending forms (and this is precisely because the forms are not dead-ending; they are $\mathcal{E}$-invertible since $\mathcal{E} \subseteq \mathcal{B}$). Note that the second and third columns are identical.

form	$\mathcal{B}$-simplest form	$\mathcal{E}$-simplest form
$\{\cdot \mid 1\}$	0	0
$\{\cdot \mid 0, 1\}$	$\{\cdot \mid 0, \blacksquare\}$	$\{\cdot \mid 0, \blacksquare\}$
$\{\cdot \mid 1, \overline{1}\}$	$\{\cdot \mid \overline{1}, \blacksquare\}$	$\{\cdot \mid \overline{1}, \blacksquare\}$
$\{\cdot \mid 0, 1, \overline{1}\}$	$\{\cdot \mid 0, \overline{1}, \blacksquare\}$	$\{\cdot \mid 0, \overline{1}, \blacksquare\}$
$\{\cdot \mid 1, *\}$	0	0
$\{\cdot \mid 0, 1, *\}$	$\{\cdot \mid 0, \blacksquare\}$	$\{\cdot \mid 0, \blacksquare\}$
$\{\cdot \mid 1, \overline{1}, *\}$	$\{\cdot \mid \overline{1}, \blacksquare\}$	$\{\cdot \mid \overline{1}, \blacksquare\}$
$\{\cdot \mid 0, 1, \overline{1}, *\}$	$\{\cdot \mid 0, \overline{1}, \blacksquare\}$	$\{\cdot \mid 0, \overline{1}, \blacksquare\}$

We now give, up to conjugation, all elements of the invertible subgroup of $\mathcal{E}$ of rank at most 2 that are not in the invertible subgroup of $\mathcal{B}$ (and this is precisely because they fail the invertibility condition; they must be blocking since $\mathcal{E} \subseteq \mathcal{B}$). Note that the first and second columns are identical.

form	$\mathcal{B}$-simplest form	$\mathcal{E}$-simplest form
$\{\overline{1} \mid 0, *\}$	$\{\overline{1} \mid 0, *\}$	$\{\overline{1} \mid 0, \blacksquare\}$
$\{\overline{1} \mid 0, \overline{1}, *\}$	$\{\overline{1} \mid 0, \overline{1}, *\}$	$\{\overline{1} \mid 0, \overline{1}, \blacksquare\}$

What is worthwhile to note with these two examples is that the forms are not $\mathscr{P}$-free, since they have $*$ as a subposition (which has outcome $\mathscr{P}$). Then, because there are no $\mathcal{B}$-reductions, the $\mathcal{B}$-simplest forms are not $\mathscr{P}$-free either, and hence are not $\mathcal{B}$-invertible. But the trick that $\mathcal{E}$ pulls off is that $*$ is an $\mathcal{E}$-reversible option here, and bypassing it to leave tombstones results in forms that are then strictly $\mathscr{P}$-free, which means they are $\mathcal{E}$-invertible! (Note that the simplest forms are not actually required to observe that the examples are $\mathcal{E}$-invertible here; there must exist a game in each of their $\mathcal{E}$-equivalence classes which is strictly $\mathscr{P}$-free, too.)

We finish this section with some summary statistics regarding the forms of rank at most 2. Recall that there are exactly 256 forms of rank at most 2, and, note that, when we refer to invertible (and simplest) forms below, we mean modulo the respective universe.

number of ... in	$\mathcal{B}$	$\mathcal{E}$
forms	256	232
invertible forms	72	60
simplest forms	220	196
invertible, simplest forms	52	46

Of course, counting the number of simplest forms is equivalent to counting the number of equivalence classes. All of these computations can be confirmed either by hand (by the intrepid reader), or by using the `gemau` software [Dav24].

5. Final remarks

In Section 3, we sought to understand how to compare two forms in $\mathcal{M}$ modulo the blocking universe $\mathcal{B}$. Given the remarkable Theorem 2.1 [LNS25], it sufficed to understand what makes a game Left $\mathcal{B}$-strong. By definition, for all universes $\mathcal{U}$, to check whether a game G is Left $\mathcal{U}$-strong requires one to check $\mathrm{o_L}(G + X) = \mathscr{L}$ for all Left ends $X \in \mathcal{U}$. Since, for every universe—other than the dicot universe $\mathcal{D}$—there are always infinitely many Left ends in $\mathcal{U}$, this is difficult! We were able to forgo this infinite number of calculations for the blocking universe, proving that, for each rank n, there exists a Left end X such that a game G of rank at most n is Left $\mathcal{B}$-strong if and only if $\mathrm{o_L}(G + X) = \mathscr{L}$. Indeed, this is what was done in [LMN+21] and more generally in [Sie25]: it was shown in the latter paper that finite sets of Left ends could be computed for a large number of dead-ending universes that serve as checks for being Left $\mathcal{U}$-strong in those universes. (Quite astonishingly, the Left ends in the test sets did not always reside within the universe in question.)

The tests we gave involving explicit Left ends to check against (as in Sections 3.1 and 3.2) were markedly different from the first test we gave in Theorem 3.1: a recursive check on the options of the game form. We were also able to show that this could be viewed alternatively as determining the outcome of the game when Right was allowed to pass (see Theorem 3.2). Our first question concerns this dichotomy: if there is some way we can compute whether games are Left $\mathcal{U}$-strong, can we always find finite test sets of Left ends?

Problem 5.1. Does there exist a universe $\mathcal{U}$ such that we can compute whether games are Left $\mathcal{U}$-strong (in finite time), but there exists no finite set of Left ends $\mathcal{X}_n = \{X_i\}$, for some integer n, such that: if G has rank at most n, then G is Left $\mathcal{U}$-strong if and only if $\mathrm{o_L}(G + X_i) = \mathscr{L}$ for all i?

We just mentioned it briefly above, but Siegel's work [Sie25] gave comparison tests for a large number of subuniverses of the dead-ending universe. Since the blocking universe appears to share many nice properties of the dead-ending universe, it would be an interesting direction to explore whether Siegel's arguments could generalise to yield more comparison tests.

Problem 5.2. Can the results of [Sie25] be generalised to subuniverses of the blocking universe $\mathcal{B}$?

In [DMM+24], it was shown that the blocking universe is pocancellative modulo $\mathcal{M}$; that is, if $G, H \in \mathcal{B}$, $J \in \mathcal{M}$, and $G + J \geqslant H + J$, then $G \geqslant H$. Indeed, the blocking universe was introduced in the first place to try and tame pocancellativity. But it is not known whether $\mathcal{B}$ itself is pocancellative (i.e. modulo $\mathcal{B}$): if we have $G, H, J \in \mathcal{B}$ and $G + J \geqslant_{\mathcal{B}} H + J$, can we conclude that $G \geqslant_{\mathcal{B}} H$? In that same paper, it was shown that the dead-ending universe is pocancellative. So, if $\mathcal{B}$ is to overthrow $\mathcal{E}$ as a larger universe that enjoys the same behaviours, then it must prove itself to be pocancellative.

Problem 5.3. Is $\mathcal{B}$ pocancellative?

An unmistakeable similarity between $\mathcal{B}$ and $\mathcal{E}$ that results from our work in this paper is the characterisations of their invertible subgroups: the invertible subgroups of both universes contain precisely those elements that are $\mathscr{P}$-free modulo $\mathcal{B}$ and $\mathcal{E}$ respectively.

At the end of Section 4, we gave some summary statistics comparing the blocking and dead-ending forms of rank at most 2. It would be interesting to investigate relevant bounds for those blocking forms of rank at most 3: how many equivalence classes; how many invertible elements? Indeed, understanding the structure of the invertible subgroups of $\mathcal{B}$ and $\mathcal{E}$, which must be countably generated abelian groups (like the invertible subgroups of all universes), is a tantalising (but difficult) question. A first step could be to analyse the subgroups generated by those invertible forms of rank at most 2; these would necessarily be finitely generated abelian groups, and hence might be relatively straightforward to classify. Indeed, the explorer wishing to embark upon this quest need not limit themselves only to $\mathcal{B}$ and $\mathcal{E}$.

Problem 5.4. Investigate the invertible subgroups of universes: what can be said about their group structure?

As mentioned in relation to $\mathcal{B}$ and $\mathcal{E}$ above, starting with the subgroups generated by those invertible elements of rank at most 2 would perhaps be wise; looking for torsion elements would also likely be fruitful.

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A. The passing outcome

Definition A.1. We define the *Right-passing* outcome, $\mathrm{p}(G)$, as

$$\mathrm{p}(G) \coloneqq \begin{cases} \mathscr{L} & \text{if } (\mathrm{p_L}(G), \mathrm{p_R}(G)) = (\mathscr{L}, \mathscr{L}), \\ \mathscr{N} & \text{if } (\mathrm{p_L}(G), \mathrm{p_R}(G)) = (\mathscr{L}, \mathscr{R}), \\ \mathscr{P} & \text{if } (\mathrm{p_L}(G), \mathrm{p_R}(G)) = (\mathscr{R}, \mathscr{L}), \\ \mathscr{R} & \text{if } (\mathrm{p_L}(G), \mathrm{p_R}(G)) = (\mathscr{R}, \mathscr{R}), \end{cases}$$

where, recursively,

$$\begin{aligned} \mathrm{p_L}(G) &\coloneqq \max(\mathrm{p_R}(G^L) : G^L \text{ is a Left option of } G), \text{ and} \\ \mathrm{p_R}(G) &\coloneqq \min(\mathrm{p_L}(G), \min(\{\mathrm{p_L}(G^R) : G^R \text{ is a Right option of } G\})), \end{aligned}$$

with the base case $\mathrm{p_L}(X) \coloneqq \mathscr{L}$ for all Left ends X, and the overriding $\mathrm{p_R}(Y) \coloneqq \mathscr{R}$ for all Right ends Y (since Right is not forced to pass).

Note that this Right-passing outcome is well-defined: the recursion always terminates since play alternates between the players and Left is forced to reduce the rank of the position on her turn.

It is simple to observe that the Right-passing outcome must be better for Right than the usual outcome: playing G under Right-passing, Right can simply follow the same strategy as he would on G without Right-passing, and then, if it is Right's turn to move on a Right end, it is defined to be Right win, which agrees with the usual outcome.

Proposition A.2. *For all $G \in \mathcal{M}$, we have* $\mathrm{o}(G) \geqslant \mathrm{p}(G)$.

Right's being able to pass should not be thought of as adding another option for Right; it is an 'optional' option. So, if it is Right's turn to move on a Right end, then he will elect to 'forget' he can pass, leaving him with no options, hence winning the game.