High-dimensional envy-free partitions

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Abstract. A vast array of envy-free results have been found for the subdivision of one-dimensional resources, such as the interval $[0,1]$. The goal is to divide the space into n pieces and distribute them among n guests such that each receives their favorite pieces. We study high-dimensional versions of these results. We prove that several spaces of convex partitions of $\mathbb{R}^d$ allow for envy-free division among any n guests. We also prove the existence of convex partitions of $\mathbb{R}^d$ which allow envy-free divisions among several groups of n guests simultaneously.

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1. Introduction

The problem of fair division is central to both mathematical economics and topological combinatorics. Given a resource to be shared between n people, can we partition and distribute it so that each person receives a "fair" share according to their subjective measure of value? The methods for solving such problems often involve topological tools [RPS22, Živ17]. In this manuscript, we are interested in studying how two large families of fair partition problems – mas partitions and envy-free partitions – relate to each other.

In a mass partition problem, we are given a family of absolutely continuous probability measures on $\mathbb{R}^d$, and a family $\mathcal{H}$ of partitions of $\mathbb{R}^d$. We will say that a measure on $\mathbb{R}^d$ is *absolutely continuous* if it is absolutely continuous with respect to the Lebesgue measure — this will be

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the main condition on our measures. We want to know if there exists a partition in $\mathcal{H}$ that splits each measure in a predetermined way (often, we want each measure to be split into parts of equal value). The quintessential result is the ham sandwich theorem, attributed to Steinhaus.

Theorem 1.1 (Ham Sandwich Theorem[Ste38])**.** *Given d absolutely continuous probability measures $\mu_1, \dots, \mu_d$ on $\mathbb{R}^d$, there exists a hyperplane H such that its two closed half-spaces H^+, H^- satisfy*

$$\mu_i(H^+) = \mu_i(H^-) = \frac{1}{2} \qquad \textit{for } i = 1, \dots, d.$$

A measure can be interpreted as the way in which one person assigns value to different sets. This means that for any d guests, there is a way to split $\mathbb{R}^d$ into two parts by a hyperplane such that each person agrees that the two parts have equal value. We call such a partition that splits each measure into parts of equal value an *equipartition*.

In the problem of envy-free division, we want to divide a resource among k guests so that each person thinks that they got the best piece (though they may not agree that everyone received the same value). In other words, no one is envious of any other guest; every guest thinks that their share is at least as good as that of any other. The central envy-free result is the cake-splitting theorem. We denote by $[n]$ the set $\{1, 2, \dots, n\}$.

Theorem 1.2 (Cake-splitting theorem; Stromquist 1980, Woodall 1980[Str80, Woo80])**.** *Let $\mu_1, \dots, \mu_n$ be absolutely continuous probability measures in $[0, 1]$. Then, there exists a partition of $[0, 1]$ into n intervals $I_1, \dots, I_n$ and a permutation $\pi : [n] \to [n]$ such that*

$$\mu_i(I_{\pi(i)}) \geqslant \mu_i(I_{\pi(i')}) \qquad \textit{for all } i, i' \in [n].$$

Here, the permutation π designates how the pieces should be allocated between the n guests. Then, each person i prefers the piece $\pi(i)$ they receive, so this division is envy-free. More generally, we say that a partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ is an *envy-free partition* for $\mu_1, \dots, \mu_n$ if there is a permutation $\pi : [n] \to [n]$ such that $\mu_i(C_{\pi(i)}) \geqslant \mu_i(C_{\pi(i')})$ for all $i, i' \in [n]$. In other words, we can assign each piece to a measure so that each measure μ receives a piece with maximal μ-measure. There are many extensions of the cake-splitting theorem, which involve more relaxed conditions on the preferences of the players [MZ19], secretive guests [AFP⁺18], more guests than pieces in the partition [Sob22], more pieces than guests [NSZ20], and divisions of undesirable resources [Su99, AK23, PvZ23]. The algorithmic versions are also of substantial interest [BT96, Pro16]. Recent generalizations of the necklace splitting theorem by Jojić et al also involve envy-free partitions [JPŽ21].

For equipartitions, the number of measures we can split evenly is typically bounded by the dimension (or, in some cases, a function of the dimension). By relaxing the fairness condition, the number of guests that an envy-free partition can satisfy can be larger than the dimension. For example, Theorem 1.2 shows the existence of an envy-free partition of a one-dimensional resource for an *arbitrary number of guests*. In this manuscript, we will prove high-dimensional versions of this envy-free result.

A simple example of our results is as follows.

Theorem 1.3. *Given four absolutely continuous probability measures on $\mathbb{R}^2$, there exists an envy-free partition that splits $\mathbb{R}^2$ into four pieces using two lines. Moreover, the partition can be found even if we only have access to three of the four measures.*

In other words, once we split $\mathbb{R}^2$ into four pieces using two lines, we will be able to assign each piece to a measure such that each measure receives a piece with maximal measure. Since we do not have access to the fourth measure μ_4, this means that regardless of which piece has the largest value for μ_4, there is a permutation assigning that piece to μ_4 which generates an envy-free partition for all four measures. The theorem above was generalized by McGinnis and Zerbib to partitions of $\mathbb{R}^2$ into $2k$ convex regions using k lines [MZ24]*Thm 4.1.

If all measures are equal, this is a folklore result first noted by Courant and Robbins [CR41]. We prove a wide range of results of this kind, in which the spaces of partitions can be parametrized by a simplex of a sufficiently high dimension. These include nested hyperplane partitions with fixed directions and generalized Voronoi diagrams. Nested hyperplane partitions with fixed directions have been used before to establish high-dimensional versions of the necklace splitting theorem [KRPS16, BS18]. In the necklace splitting problem, the goal is to find an equipartition of several measures among k players by splitting the space into potentially more than k pieces and then distributing them among the players [GW85, Alo87, Lv08].

We also prove more general results for convex partitions of $\mathbb{R}^d$. A convex partition of $\mathbb{R}^d$ into n parts is a collection $C_1, \dots, C_n$ of closed convex sets in $\mathbb{R}^d$ such that

- the union of the n sets is $\mathbb{R}^d$, and
- the interiors of $C_1, \dots, C_n$ are pairwise disjoint.

We prove the existence of partitions which are fair partitions for several groups of measures simultaneously.

Theorem 1.4. *Let n, d be positive integers, where n is a prime power. Let μ be an absolutely continuous probability measure on $\mathbb{R}^d$, and let $(\mu_1^1, \dots, \mu_n^1), \dots, (\mu_1^{d-1}, \dots, \mu_n^{d-1})$ be $d-1$ tuples of n absolutely continuous probability measures on $\mathbb{R}^d$ each.*

Then, there exists a convex partition $C_1, \dots, C_n$ of $\mathbb{R}^d$ such that

$$\mu(C_1) = \cdots = \mu(C_n)$$

and $(C_1, \dots, C_n)$ is an envy-free partition for $(\mu_1^r, \dots, \mu_n^r)$, for all $r \in [d-1]$. Moreover, the partition can be found even if we do not have access to μ_n^r for all $r \in [d-1]$.

The case when $\mu_j^r = \mu_{j'}^r$ for each $r \in [d-1]$ and any $j, j' \in [n]$ (each n-tuple just has a single measure repeated n times) shows that for any d measures on $\mathbb{R}^d$, there exists a convex equipartition. For this equipartition result, the case for general n follows from the case with n being a prime power via a simple subdivision argument. This is a known extension of the ham sandwich theorem, with several different proofs [Sob12, KHA14, BZ14], originally motivated by the Nandakumar–Ramana-Rao problem [NRR12].

The prime power condition in Theorem 1.4 is due to the use of a much stronger theorem of Blagojević and Ziegler [BZ14] in our proof, for which this condition is essential. However, we conjecture that Theorem 1.4 should still hold for general n. We can remove the prime power condition if we are satisfied with a fairness condition that is slightly weaker than envy-freeness. Given n absolutely continuous probability measures $\mu_1, \dots, \mu_n$ on $\mathbb{R}^d$, we say that a partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ is a *proportional partition* if there is a permutation $\pi : [n] \to [n]$ such that $\mu_i(C_{\pi(i)}) \geqslant 1/n$. Note that envy-freeness implies proportionality, but the converse does not necessarily hold. In a proportional partition, each measure receives a piece with at least the average value, but this is not guaranteed to be the piece with the greatest value.

Theorem 1.5. *Let n, d be positive integers. Let μ be an absolutely continuous probability measure on $\mathbb{R}^d$, and let $(\mu_1^1, \dots, \mu_n^1), \dots, (\mu_1^{d-1}, \dots, \mu_n^{d-1})$ be $d-1$ tuples of n absolutely continuous probability measures on $\mathbb{R}^d$ each.*

Then, there exists a convex partition $C_1, \dots, C_n$ of $\mathbb{R}^d$ such that

$$\mu(C_1) = \cdots = \mu(C_n)$$

and $(C_1, \dots, C_n)$ is a proportional partition for $(\mu_1^r, \dots, \mu_n^r)$ for each $r \in [d-1]$.

We also present a generalization of Theorem 1.4 in Section 5, in which preferences are not necessarily derived from measures. We also discuss the possibility of removing the condition that n is a prime power.

2. Preliminaries

In this section, we discuss two tools that will be needed in our proofs: KKM covers and power diagrams.

Consider the $(n-1)$-dimensional simplex $\Delta^{n-1} \subset \mathbb{R}^n$ as the set

$$\Delta^{n-1} = \{(x_1, \dots, x_n) : x_1 + \cdots + x_n = 1 \quad \text{and } x_i \geqslant 0 \text{ for all } i \in [n]\}.$$

Denote the vertices of Δ^{n-1} by the numbers $1, \dots, n$ so that the equation $x_i = 0$ corresponds to the facet opposite of vertex i. For a collection of vertices $I \subset [n]$, let σ_I be the face spanned by those vertices. A *KKM cover* (named after Knaster, Kuratowski, and Mazurkiewicz [KKM29]) of Δ^{n-1} is an n-tuple of sets $(A_1, \dots, A_n)$ such that

- each A_i is a closed subset of Δ^{n-1} and
- for each $I \subset [n]$, the face σ_I is contained in $\bigcup_{i \in I} A_i$.

The second condition implies that the sets $A_1, \dots, A_n$ cover Δ^{n-1}, by taking $I = [n]$. The first condition is many times replaced by all A_i being open sets. KKM covers are commonly used to prove existence results for envy-free partitions and their duals (e.g., [Su99, MZ19, AFP+18, Sob22, MZ24] and the references therein). The main result we will use is the following theorem.

Theorem 2.1 (Asada, Frick, Psharody, Polevy, Stoner, Tsang, Wellner 2018 [AFP+18])**.** *Let n be a positive integer and $(A_1^1, \dots, A_n^1), \dots, (A_1^{n-1}, \dots, A_n^{n-1})$ be KKM covers of Δ^{n-1}. Then, there exists a point $x \in \Delta^{n-1}$ such that for any KKM cover $(A_1^n, \dots, A_n^n)$, there exists a permutation $\pi : [n] \to [n]$ such that*

$$x \in A^i_{\pi(i)} \text{ for each } i \in [n].$$

This result generalizes the classic "colorful" KKM theorem of Gale [Gal84], in which $(A^n_1, \ldots, A^n_n)$ is also given in advance.

A particular space of partitions of $\mathbb{R}^d$ in which we are interested is the family of *power diagrams*, also known as *generalized Voronoi diagrams*. Given n distinct points $x_1, \ldots, x_n$ in $\mathbb{R}^d$ (referred to as "sites") and n real numbers $\lambda_1, \ldots, \lambda_n$ (referred to as "weights"), this induces a partition of $\mathbb{R}^d$ into n sets $C_1, \ldots, C_n$ defined by

$$C_j = \{y \in \mathbb{R}^d : \operatorname{dist}^2(y, x_j) - \lambda_j \leqslant \operatorname{dist}^2(y, x_i) - \lambda_i \text{ for all } i \in [n]\}.$$

These regions are convex. Since adding the same constant to all weights does not change the partition, we may assume that $\sum_{i=1}^n \lambda_i = 0$ for simplicity. The case $\lambda_1 = \cdots = \lambda_n = 0$ corresponds to a classic Voronoi diagram.

Given an absolutely continuous probability measure μ on $\mathbb{R}^d$ and n sites $x_1, \ldots, x_n$, there exist weights $\lambda_1, \ldots, \lambda_n$ such that $\mu(C_i) = 1/n$, as first shown by Aurenhammer, Hoffman, and Aronov [AHA98]. Moreover, if $\mu(U) \neq 0$ for any non-empty open set U, then this set of weights is unique and varies continuously with $(x_1, \ldots, x_n)$. Denote by $\operatorname{EMP}(\mu, n)$ the space of "equal measure partitions" of μ by power diagrams into n parts. Formally,

$$\operatorname{EMP}(\mu, n) = \left\{ (C_1, \ldots, C_n) : \begin{array}{c} (C_1, \ldots, C_n) \text{ is a power diagram of } \mathbb{R}^d \text{ and} \\ \mu(C_i) = 1/n \text{ for all } i \in [n] \end{array} \right\}.$$

Then, this allows us to parametrize $\operatorname{EMP}(\mu, n)$ by the set of n-tuples of n distinct sites in $\mathbb{R}^d$. The space $\operatorname{EMP}(\mu, n)$ also has a natural action of the symmetric group S_n of permutations on $[n]$. Given $P = (C_1, \ldots, C_n)$ and $\pi \in S_n$, we declare $\pi P = (C_{\pi(1)}, \ldots, C_{\pi(n)})$.

A function $F : \operatorname{EMP}(\mu, n) \to \mathbb{R}^n$ can be expanded as $F = (f_1, \ldots, f_n)$, where each component is a function $f_i : \operatorname{EMP}(\mu, n) \to \mathbb{R}$. We say that F is *S_n-equivariant* if $f_i(\pi(P)) = f_{\pi(i)}(P)$ for all $i \in [n]$ and all $\pi \in S_n$. A simple way to generate such a function is to have each f_i compute some function of C_i, such as $\mu(C_i)$ for some measure μ.

The main result we use about S_n-equivariant functions on $\operatorname{EMP}(\mu, n)$ is the following theorem of Blagojević and Ziegler.

Theorem 2.2 (Blagojević, Ziegler 2014 [BZ14])**.** *Let μ be an absolutely continuous probability measure on $\mathbb{R}^d$. If n is a prime power, then for any $d-1$ continuous S_n-equivariant functions $F_r = (f_{1,r}, \ldots, f_{n,r})$ on $\operatorname{EMP}(\mu, n)$, there exists a partition $(C_1, \ldots, C_n) \in \operatorname{EMP}(\mu, n)$ that simultaneously equalizes the components of each F_r. In other words,*

$$f_{i,r}(C_1, \ldots, C_n) = f_{i',r}(C_1, \ldots, C_n) \qquad \text{for all } i, i' \in [n], r \in [d-1].$$

In our application of this result, we will use functions where each $f_{i,r}$ depends on the entire partition $(C_1, \ldots, C_n)$ and not just C_i.

Given an absolutely continuous probability measure μ on $\mathbb{R}^d$, the space $X(\mu, n)$ of convex equipartitions $(C_1, \ldots, C_n)$ of μ is compact. To see this, consider a ball B such that $\mu(B) > (n-1)/n$. Then, for any convex equipartition $(C_1, \ldots, C_n)$, the hyperplane H_{ij} that separates C_i and C_j (which can be chosen canonically even if C_i and C_j do not share a boundary and completely determine the sets C_i) must intersect B. Let K be the set of hyperplanes that intersect B. Therefore, we can represent X as a subset of

$$Y = \underbrace{K \times K \times \cdots \times K}_{\binom{n}{2} \text{ times}}.$$

Moreover, the function

$$\begin{aligned} f_i : Y &\to \mathbb{R} \\ y &\mapsto \mu(C_i(y)) \end{aligned}$$

is continuous (from the arrangement y we first construct C_i and then evaluate μ). The set X is closed since we can write it as an intersection of closed sets $X = \bigcap_{i=1}^n f_i^{-1}(1/n)$. Additionally, X inherits a metric space structure, as Y is a product of metric spaces. This will be useful at the end of Section 4.

3. Results parametrized by simplices

We start by proving Theorem 1.3. The parametrization below is based on recent work by McGinnis and Zerbib for problems related to line transversals to finite families of convex sets [MZ22] (see also the recent extensions [GNRP25, MZ24, Zer24]).

Proof of Theorem 1.3. First, we parametrize a subset of the space of partitions of $\mathbb{R}^2$ using two lines by Δ^3. Given $(x_1, x_2, x_3, x_4) \in \Delta^3$, consider the following points in the unit circle S^1:

$$\begin{aligned} p_0 &= (1, 0), \\ p_1 &= \Big(\cos(2\pi x_1), \sin(2\pi x_1)\Big), \\ p_2 &= \Big(\cos(2\pi(x_1 + x_2)), \sin(2\pi(x_1 + x_2))\Big), \\ p_3 &= \Big(\cos(2\pi(x_1 + x_2 + x_3)), \sin(2\pi(x_1 + x_2 + x_3))\Big). \end{aligned}$$

Note that the length of the arc between p_{i-1} and p_i (indices modulo 4) is exactly $2\pi x_i$. Let B^2 be the disk bounded by S^1. We construct the lines p_0p_2 and p_1p_3, which divide the disk into four closed regions. Denote by C_i the region of the disk that contains the arc $p_{i-1}p_i$. An example construction is shown in Figure 3.1. We first prove the result for measures whose support is contained in B^2.

Given an absolutely continuous probability measure μ whose support is contained in B^2, we construct a cover of Δ^3 as following. Given $x = (x_1, x_2, x_3, x_4) \in \Delta^3$, construct C_1, C_2, C_3, C_4 as above. We say $x \in A_i$ if $\mu(C_i) \geqslant \mu(C_j)$ for all $j \in [4]$. Since the condition involves checking a finite number of non-strict inequalities, each A_i is a closed set. Moreover, if $x_i = 0$, the set C_i is the single point p_i, so $\mu(C_i) = 0$. This implies that (A_1, A_2, A_3, A_4) is a KKM cover of Δ^3. Then the four measures induce four corresponding KKM covers of Δ^3. A direct application of Theorem 2.1 finishes the proof.

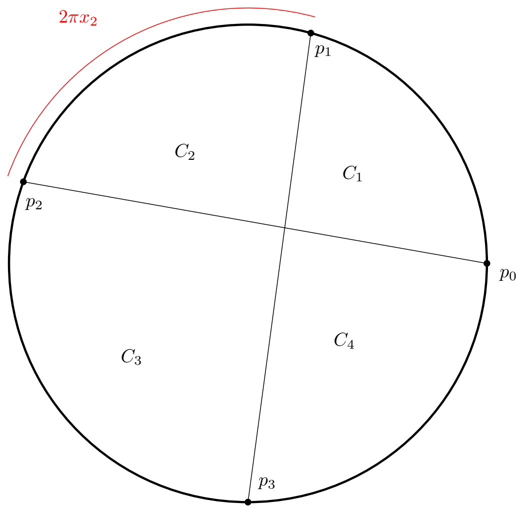

Figure 3.1: Construction of the two lines from parameters (x_1, x_2, x_3, x_4).

Now consider the case of general measures and let $\varepsilon > 0$. We may assume, without loss of generality, that for each measure $\mu_1, \dots, \mu_4$ we have $\mu_i(B^2) \geqslant 1 - \varepsilon$ by, say, scaling the plane or the disk. We can apply the result above and find two lines generating an envy-free partition for the restriction of the measures to B^2. We call this a partition that is ε-fair, as the piece assigned to each μ_i needs at most ε measure to be the maximal part. Observe that the space of pairs of lines that could represent a δ-fair partition for the four measures for any $\delta \leqslant 1/2$ is compact. A simple compactness argument by taking $\varepsilon \to 0$ gives us an envy-free partition as desired. □

In general, the technique above works for a wide range of partition spaces parametrized by a simplex, which we denote as Δ-*spaces*.

Definition 3.1. Let n, d be positive integers. We say that a space of of partitions $\mathcal{H}$ of $\mathbb{R}^d$ into n parts is a Δ-space if there exists a function $R : \Delta^{n-1} \to \mathcal{H}$ such that the following holds:

- If $R(x_1, \dots, x_n) = (C_1, \dots, C_n)$, then $x_i = 0$ implies that C_i has Lebesgue measure zero for all $i \in [n]$, and
- if μ is a finite measure absolutely continuous with respect to the Lebesgue measure, then $(x_1, \dots, x_n) \mapsto \mu(C_j)$ is a continuous function from Δ^{n-1} to $\mathbb{R}$ for all $j \in [n]$.

We will show that Δ-spaces are spaces of partitions for which envy-free distributions are easy to find. Before doing this, we explain how to obtain large families of Δ-spaces. The Δ-spaces we can obtain with the process shown below include spaces of partitions which were used in other mass partition problems by Karasev, Roldán-Pensado, and Soberón [KRPS16], and by Blagojević and Soberón [BS18].

Now, given two Δ-spaces of partitions of $\mathbb{R}^d$, there is a simple way to combine them. Let $\mathcal{A}$ be a Δ-space of partitions of $\mathbb{R}^d$ into n pieces, let $\mathcal{B}$ be a Δ-space of partitions of $\mathbb{R}^d$ into m pieces, and let v be a direction in S^{d-1}. Using these parameters, we can construct a Δ-space of partitions $\mathcal{A} *_v \mathcal{B}$ of $\mathbb{R}^d$ into $n + m$ pieces.

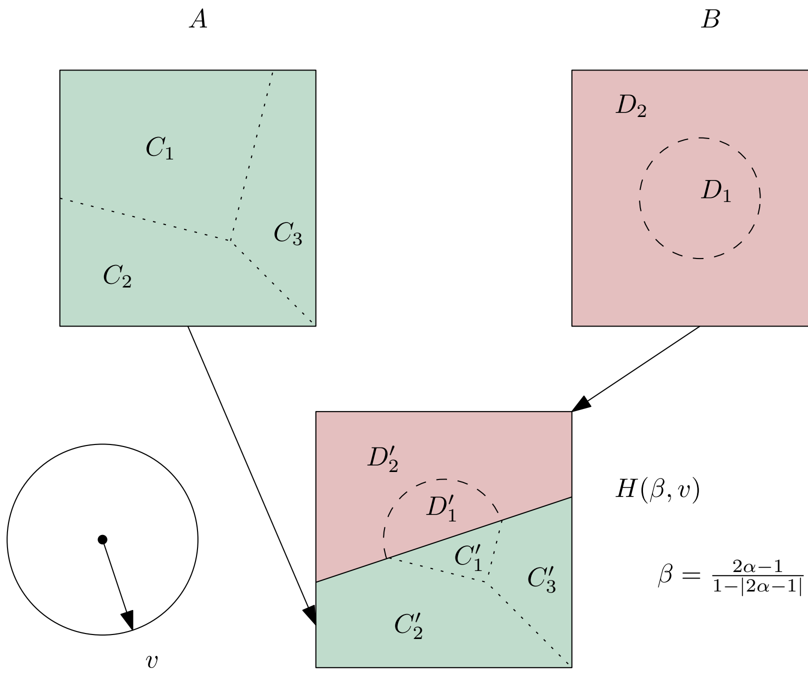

Figure 3.2: A construction of a partition of $\mathbb{R}^2$ into five parts.

Given $\beta \in \mathbb{R}$, consider the half-spaces

$$H^+(\beta, v) = \{y \in \mathbb{R}^d : \langle y, v\rangle \geqslant \beta\}$$
$$H^-(\beta, v) = \{y \in \mathbb{R}^d : \langle y, v\rangle \leqslant \beta\}.$$

We include the limiting cases

$$H^+(\infty, v) = H^-(-\infty, v) = \varnothing \qquad \text{and} \qquad H^-(\infty, v) = H^+(-\infty, v) = \mathbb{R}^d.$$

Now, given partitions $(C_1, \dots, C_n) \in \mathcal{A}$, $(D_1, \dots, D_m) \in \mathcal{B}$ and $\alpha \in [0, 1]$, we form a new partition with $n + m$ sets, of the form

$$C'_j = C_j \cap H^+\left(\frac{2\alpha - 1}{1 - |2\alpha - 1|}, v\right), j \in [n] \qquad D'_k = D_k \cap H^-\left(\frac{2\alpha - 1}{1 - |2\alpha - 1|}, v\right), k \in [m].$$

The construction is illustrated in Figure 3.2. Some parts may be empty, which is not a problem. Note that as $\alpha \to 0$, we recover only the partition in $\mathcal{A}$ and m empty pieces for $\mathcal{B}$, and as $\alpha \to 1$ we recover only the partition of $\mathcal{B}$ and n empty pieces for $\mathcal{A}$. This means that if $\mathcal{A}$ and $\mathcal{B}$ have parametrizations of Δ^{n-1} and Δ^{m-1}, respectively, then $\mathcal{A} *_v \mathcal{B}$ can be parametrized by the join $\Delta^{n-1} * \Delta^{m-1} = \Delta^{n+m-1}$. Hence $\mathcal{A} *_v \mathcal{B}$ is also a Δ-space, as pieces that must have Lebesgue measure zero in the construction preserve this property. This method of combining spaces of partitions has been used before to obtain high-dimensional generalizations of the necklace splitting theorem.

A simple example of a Δ-space is to simply take $n = 1$ and the unique partition $\{\mathbb{R}^d\}$. Then, using the join operation repeatedly with potentially different directions v, we obtain the space of *nested hyperplane partitions* with fixed directions. Alternatively, we can define the same nested hyperplane partition space of $\mathbb{R}^d$ into n parts recursively. For $n = 1$, it is simply the partition $\{\mathbb{R}^d\}$. For $n > 1$, assume we have a space $\mathcal{A}$ of nested hyperplane partitions into $n-1$ parts, a direction $v \in S^{d-1}$, and a value $j \in [n-1]$. Given a partition $(C_1, \dots, C_{n-1}) \in \mathcal{A}$, we simply cut C_j into two parts C'_j and C''_j using a hyperplane orthogonal to v (we always cut the j-th piece). This gives us a partition of $\mathbb{R}^d$ into n pieces, as desired. Note that the hyperplanes involved in the cutting have fixed directions, and by recursion the parts are convex.

This gives us the following consequence of Theorem 2.1. Note that it implies a generalization of Theorem 1.3.

Theorem 3.2. *Let $\mathcal{B}$ be a Δ-space of partitions of $\mathbb{R}^d$ into n parts. Let $\mu_1, \dots, \mu_n$ be absolutely continuous probability measures on $\mathbb{R}^d$. Then, there is a partition in $\mathcal{B}$ that is an envy-free partition for $\mu_1, \dots, \mu_n$. The partition can be found even if we do not have access to μ_n in advance.*

Proof. For each μ_j, we define a KKM cover $(A^j_1, \dots, A^j_n)$ of Δ^{n-1}. Consider the function $R : \Delta^{n-1} \to \mathcal{B}$ from the definition of a Δ-space. We define A^j_i as the set of points $(x_1, \dots, x_n) \in \Delta^{n-1}$ such that for $(C_1, \dots, C_n) = R(x_1, \dots, x_n)$, we have that C_i has the maximal μ_j-measure among all C_i. By construction, the set A^j_i is closed. The condition of Lebesgue measure zero implies that A^j_i does not intersect the facet opposite to vertex i. Consider a face σ of Δ^{n-1}, and let k be a vertex such that $k \notin \sigma$. Then, $\sigma \subset F_k$, the facet opposite to k. This means that $\sigma \cap A^j_k = \varnothing$. Therefore, σ can has to be covered by the sets A^j_i with $i \in \sigma$. Therefore, $(A^j_1, \dots, A^j_n)$ is a KKM cover of Δ^{n-1}.

Using the KKM covers induced by $\mu_1, \dots, \mu_{n-1}$, we can apply Theorem 2.1, and find a point $x \in \Delta^{n-1}$ such that for any other KKM-cover $(B_1, \dots, B_n)$ there exists $\pi : [n] \to [n]$ such that

$$x \in \left(\bigcap_{i=1}^{n-1} A^i_{\pi(i)} \right) \cap B_{\pi(n)}.$$

Let π_0 be the permutation we obtain for $(B_1, \dots, B_n) = (A^n_1, \dots, A^n_n)$. Then, for the partition $(C_1, \dots, C_n) = R(x)$ we have that $\mu_j(A^j_{\pi(j)}) \geqslant \mu_j(A^j_i)$ for all $i, j \in [n]$, so we have the envy-free partition for $\mu_1, \dots, \mu_n$. Moreover, the point $x \in \Delta^{n-1}$ was found without using μ_n. □

If we apply the result for nested hyperplane partitions, we get the following corollary.

Corollary 3.3. *Let $d, n \geqslant 1$ and let $\mathcal{A}$ be a space of nested hyperplane partitions of $\mathbb{R}^d$ with fixed directions. Then, for any n absolutely continuous measures, there exists an envy-free partition in $\mathcal{A}$ for the n measures. This partition can be found even if we do not have access to one of the measures in advance.*

Proof. The space $\mathcal{A}$ is a Δ-space. A direct application of Theorem 3.2 finishes the proof. □

Although power diagrams with fixed sites are not Δ-spaces since they do not meet the Lebesgue measure zero condition, they satisfy a similar condition for finite measures. To see why they may fail to meet the Lebesgue measure zero condition, note that if $(y_1, \dots, y_n)$ is a set of distinct points and y_i is a vertex of the convex hull of $\{y_1, \ldots, y_n\}$, then the set C_i will have non-empty interior for any power diagram $(C_1, \ldots, C_n)$ with sites $(y_1, \ldots, y_n)$. Therefore, for some finite measures such as a multivariate normal distribution, we will always have $\mu(C_i) > 0$.

Let $(y_1, \ldots, y_n)$ be n distinct points in $\mathbb{R}^d$. For an n-tuple $(\alpha_1, \ldots, \alpha_n)$ of real numbers with sum equal to zero, consider $(C_1, \ldots, C_n)$ the power diagram with sites $(y_1, \ldots, y_n)$ and weights $(\alpha_1, \ldots, \alpha_n)$. Given any absolutely continuous probability measure μ on $\mathbb{R}^d$ and $\varepsilon > 0$, we claim that there exists a real number $M = M(\mu, \varepsilon)$ such that $\alpha_j < M$ implies $\mu(C_j) < \varepsilon$ for all $j \in [n]$. If $\alpha_j < M$, there must be an $i \in [n] \setminus \{j\}$ such that $\alpha_i > -M/(n-1)$. Therefore, by the definition of C_j, we will have

$$
\begin{aligned}
C_j &\subset \{x \in \mathbb{R}^d : \operatorname{dist}^2(x, y_j) - \alpha_j \leqslant \operatorname{dist}^2(x, y_i) - \alpha_i\} \\
&= \left\{x \in \mathbb{R}^d : \operatorname{dist}^2(x, y_j) - \operatorname{dist}^2(x, y_i) \leqslant \alpha_j - \alpha_i < M - \frac{M}{n-1} = M\left(\frac{n-1}{n-2}\right)\right\}
\end{aligned}
$$

This last set is a half-space with boundary hyperplane orthogonal to $y_j - y_i$ containing the side in the direction $y_j - y_i$. As $M \to -\infty$, this half-space tends to the empty set, so the measure of C_j must tend to 0. Since this happens for all possible choices of i, there must be a value $M_j(\mu, \varepsilon)$ such that $\alpha_j < M_j(\mu, \varepsilon)$ implies $\mu(C_j) < \varepsilon$. Finally, we take $M(\mu, \varepsilon) = \min_j M_j(\mu, \varepsilon)$.

The set

$$
\left\{(\alpha_1, \ldots, \alpha_n) : \sum_{i=1}^{n} \alpha_i = 0, \quad \alpha_i \geqslant M \text{ for all } i \in [n]\right\}
$$

is a rescaling of Δ^{n-1}. By choosing ε appropriately (in this case, any $0 < \varepsilon < 1/n$), we can include power diagrams in Theorem 3.2. In other words, we have the following corollary.

Corollary 3.4. *Let n, d be positive integers. Given n absolutely continuous probability measures in $\mathbb{R}^d$ and a fixed set of n sites, there exist weights such that the corresponding power diagram gives an envy-free partition for the n measures.*

Proof. Suppose we are given n measures $\mu_1, \ldots, \mu_n$ and our n sites $y_1, \ldots, y_n$ in $\mathbb{R}^d$. Let $1/n > \varepsilon > 0$ be a constant. Let $M = \min_{j \in [n]} M(\mu_j, \varepsilon)$ in the description above. Now consider the simplex

$$
\Delta^{n-1} = \left\{(\alpha_1, \ldots, \alpha_n) : \sum_{i=1}^{n} \alpha_i = 0, \quad \alpha_i \geqslant M \text{ for all } i \in [n]\right\}
$$

As above, each measure μ_j induces an n-tuple of sets $(A_1^j, \ldots, A_n^j)$ where A_i^j is the points in Δ^{n-1} for which in the induced power diagram $(C_1, \ldots, C_n)$ we have $\mu_j(C_i) \geqslant 1/n$. In our current parametrization of Δ^{n-1} with weights that add to zero, the facet opposite to vertex i corresponds to the equation $\alpha_i = M$, which by construction implies $\mu(C_i) \leqslant \varepsilon < 1/n$.

This implies that A_i^j does not intersect the facet opposite to vertex i. The sets $A_i^j \subset \Delta^{n-1}$ are closed by construction, and the fact that $\sum_{i=1}^{n} \mu_j(C_i) = 1$ implies that they cover Δ^{n-1}. This means that each n-tuple $(A_1^j, \ldots, A_n^j)$ is a KKM cover of Δ^{n-1}. Finally, a direct application of Theorem 2.1 finishes the proof. □

We also prove a generalization of Levi's cone partition result. Levi's theorem is the first high-dimensional mass partition result, predating the ham sandwich theorem [Lev30].

Theorem 3.5 (Levi 1930)**.** *Let $(C_1, \dots, C_{d+1})$ be a convex partition of $\mathbb{R}^d$ such that each C_i is a convex cone, and each cone has its apex at the origin. Let μ be an absolutely continuous probability measure on $\mathbb{R}^d$. Then, there exists a vector $x \in \mathbb{R}^d$ such that the translates $x + C_i$ all satisfy*

$$\mu(x + C_i) = \frac{1}{n}.$$

This result has been generalized by Borsuk and by Vrećica and Živaljević [Bor53, VŽ01]. We prove an envy-free version of their generalizations. In this version, instead of having a single measure, we have a measure for each of the $d + 1$ pieces, each corresponding to the subjective preferences of a person, and an amount α_i that each participant wants to receive.

Theorem 3.6. *Let $(C_1, \dots, C_{d+1})$ be a convex partition of $\mathbb{R}^d$ such that each C_i is a convex cone and all cones have their apex at the origin, and let $\mu_1, \dots, \mu_{d+1}$ be absolutely continuous measures on $\mathbb{R}^d$. Let $\alpha_1, \dots, \alpha_{d+1}$ be positive real numbers that sum to 1. Then, there exists a vector x and a permutation $\pi : [d + 1] \to [d + 1]$ such that each translate $x + C_i$ satisfies*

$$\mu_{\pi(i)}(x + C_i) \geqslant \alpha_i.$$

Moreover, the vector x can be found even if we do not have access to μ_{d+1} in advance.

If $\mu_1 = \cdots = \mu_{d+1}$, the result above is the generalization by Borsuk and by Živaljević and Vrećica.

Proof of Theorem 3.6. For each C_i, we can find a half-space H_i^- such that $C_i \subset H_i^-$ and its boundary hyperplane contains the origin. We also assume that any d of the normal vectors to the boundary hyperplanes are linearly independent. Since the union of all C_i covers $\mathbb{R}^d$, so does the union of the half-spaces H_i^-. Let G_i^- be a translate of H_i^- so that $\mu_j(G_i^-) < \alpha_i$ for all j. Let G_i^+ be the closed half-space opposite to G_i^-. Now consider the simplex $\bigcap_{i=1}^{d+1} G_i^+$. Figure 3.3 illustrates the construction in the plane. We denote by F_i the facet contained in G_i^-, and v_i the vertex opposite to F_i. Now, each measure μ_j induces a KKM cover $(A_1^j, \dots, A_{d+1}^j)$ of Δ in the following way. We have $x \in A_i^j$ if and only if $\mu_j(x + C_i) \geqslant \alpha_i$. Note that for $x \in F_i$, by construction we have $x + C_i \subset G_i^-$, so $\mu_j(C_i) \leqslant \mu_j(G_i^-) < \alpha_i$, implying that $x \notin A_i^j$. Also, since $\sum_{i=1}^{d+1} \mu_j(x + C_i) = 1 = \sum_{i=1}^{d+1} \alpha_i$, then for at least one i we must have $\mu_j(x + C_i) \geqslant \alpha_i$, so $(A_1^j, \dots, A_{d+1}^j)$ indeed covers Δ. Now we simply apply Theorem 2.1 to the $d + 1$ covers to obtain the desired result. □

4. Convex partitions in high dimensions

We prove the existence of simultaneous proportional and envy-free partitions for several groups of measures on $\mathbb{R}^d$. First, if the number of convex pieces n is a prime power, we prove the existence of simultaneous envy-free partitions of $\mathbb{R}^d$. This is the stronger fairness constraint in our results.

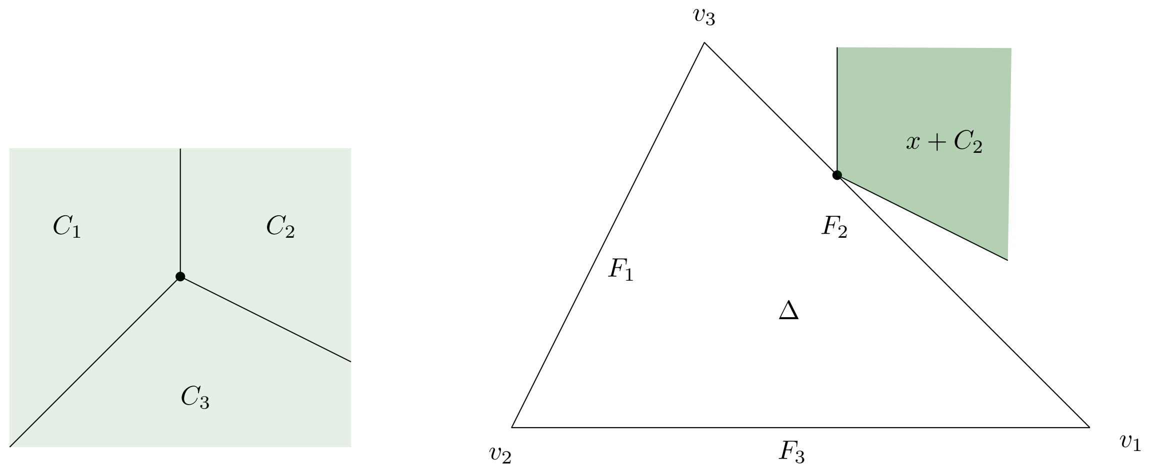

Figure 3.3: A construction of Δ in $\mathbb{R}^2$.

Proof of Theorem 1.4. Let us briefly explain the structure of the proof. Given a partition $P = (C_1, \ldots, C_n) \in \mathrm{EMP}(\mu, n)$, for each n-tuple $(\mu_1^r, \ldots, \mu_n^r)$ of measures we will construct an $n \times n$ matrix $M^r(P)$. Theorem 2.2 will guarantee that for some $P \in \mathrm{EMP}(\mu, n)$, each of the $d-1$ matrices we constructed will be doubly stochastic, which we will use to finish the proof. We have to be careful that our construction of P does not depend on μ_n^r for $r \in [d-1]$ to fulfill the requirements of Theorem 1.4.

Let $\varepsilon > 0$ be a real number. Given a partition $P = (C_1, \ldots, C_n) \in \mathrm{EMP}(\mu, n)$, and $i \in [n], j \in [n-1]$

$$g_{ij}^r(P) = \begin{cases} 0 & \text{if } \mu_j^r(C_i) < \max\{\mu_j^r(C_{i'}) \text{ for } i' \in [n]\} - \varepsilon \\ \mu_j^r(C_i) - (\max\{\mu_j^r(C_{i'}) \text{ for } i' \in [n]\} - \varepsilon) & \text{otherwise} \end{cases}$$

and for $j = n$ we define

$$g_{in}^r(P) = \frac{1}{n}.$$

Note that $g_{ij}^r(P)$ is non-negative and varies continuously with P. We have that $g_{ij}^r(P) > 0$ if and only if $\mu_j^r(C_i) > \mu_j^r(C_{i'}) - \varepsilon$ for all $i' \in [n]$. If we fix j, there exists at least one value $i \in [n]$ such that $g_{ij}^r(P) \geqslant \varepsilon > 0$. We can consider the values g_{ij}^r as the entries of an $n \times n$ matrix and normalize the columns so that they sum to 1. In explicit terms, we define

$$M_{ij}^r(P) = \frac{g_{ij}^r(P)}{\sum_{k=1}^n g_{kj}^r(P)}.$$

The value M_{ij}^r is the entry at the i-th row and j-th column in the $n \times n$ matrix $M^r(P)$. Note that $M^r(P)$ changes continuously as P changes in $\mathrm{EMP}(\mu, n)$. This gives us $d-1$ matrices $M^1(P), \ldots, M^{d-1}(P)$.

Let

$$f_i^r(P) = \sum_{j=1}^n M_{ij}^r(P)$$

be the sum of the i-th row of $M^r(P)$. By construction, the function $F^r = (f_1^r, \dots, f_n^r) : \mathrm{EMP}(\mu, n) \to \mathbb{R}^n$ is a continuous S_n-equivariant function.

By Theorem 2.2, there exists a partition $P \in \mathrm{EMP}(\mu, n)$ that simultaneously equalizes the components of each of $F^1, \dots, F^{d-1}$:

$$f_1^r(P) = \dots = f_n^r(P) \text{ for } r = 1, \dots, d-1.$$

Then for this partition, $M^1(P), \dots, M^{d-1}(P)$ are $d-1$ doubly stochastic matrices. By Birkhoff's theorem, each $M^r(P)$ is a convex combination of permutation matrices. Recall we still have not used μ_n^r. For any $i' \in [n]$, since $M_{i'n}^r(P) = 1/n > 0$ and $M^r(P)$ is a convex combination of permutation matrices, there exists a permutation $\pi_r^{i'} : [n] \to [n]$ such that $M_{\pi_r^{i'}(j)j}^r(P) > 0$ for all $j \in [n]$ and $\pi_r^{i'}(n) = i'$. This implies that for $j \in [n]$, we have $\mu_j^r(C_{\pi_r^{i'}(j)}) > \mu_j^r(C_i) - \varepsilon$ for all $i \in [n]$. The space of convex equipartitions of μ into n parts (not necessarily from a power diagram) is compact, and the number of possible permutations $\pi_r^{i'}$ is finite. Therefore, if we take a sequence ε_m such that $\varepsilon_m \to 0$, we can assume without loss of generality that for each ε_m each of the permutations $\pi_r^{i'}$ is the same, and that as $\varepsilon_m \to 0$ the partition P approaches a convex equipartition $\overline{P} = (C_1, \dots, C_n)$ of μ. This equipartition satisfies

$$\mu_j^r(C_{\pi_r^{i'}(j)}) \geqslant \mu_j^r(C_i) \text{ for all } i \in [n]$$

for all $r \in [d-1]$ and $i \in n$. Finally, for each $r \in [d-1]$, we check which part C_{i_r} has the largest μ_n^r value. The permutations $\pi_r^{i_r}$ for $r \in [d-1]$ give us the desired allocation. □

For general n, we use a subdivision argument to show the existence of simultaneous proportional partitions of $\mathbb{R}^d$. Subdivision arguments are standard in mass partition results. In this case, the subdivision argument is more nuanced than usual, as it requires a slightly stronger version of Theorem 1.4. Given ms participants, each with their own measure, we seek to divide $\mathbb{R}^d$ into m sets so that each part is preferred by exactly s participants, without overlaps. Moreover, we want to do this simultaneously for $d-1$ groups of ms participants each.

Lemma 4.1. *Let m be a prime power, n be a multiple of m and μ be an absolutely continuous probability measure on $\mathbb{R}^d$. For any $d-1$ n-tuples of absolutely continuous probability measures $(\mu_1^1, \dots, \mu_n^1), \dots, (\mu_1^{d-1}, \dots, \mu_n^{d-1})$ on $\mathbb{R}^d$, there is a partition of $\mathbb{R}^d$ into m convex regions $C_1, \dots, C_m$ and $d-1$ functions $\pi^r : [n] \to [m]$ for $r \in [d-1]$ such that*

$$\mu(C_1) = \dots = \mu(C_m) = \frac{1}{m}$$
$$\mu_j^r(C_{\pi^r(j)}) \geqslant \frac{1}{m} \qquad \text{for all } i \in [m]$$

and such that for each $i \in [m], r \in [d-1]$, the preimage $(\pi^r)^{-1}(i)$ has exactly n/m elements.

The proof below can be modified to have the stronger conclusion $\mu_j^r(C_{\pi^r(j)}) \geqslant \mu_j^r(C_i)$ for all $i \in [n]$, but it does not give us any advantage for the subdivision argument. This gives a slightly stronger version of Theorem 1.4 if we have access to all measures in advance. With the condition $\mu_j^r(C_{\pi^r(j)}) \geqslant 1/m$, we have a slightly simpler matrix construction.

Proof. Let $\varepsilon > 0$ be a real number. Given a partition $P = (C_1, \dots, C_m) \in \mathrm{EMP}(\mu, m)$, define

$$g_{ij}^r(P) = \begin{cases} 0 & \text{if } \mu_j^r(C_i) < \frac{1}{m} - \varepsilon \\ \mu_j^r(C_i) - (\frac{1}{m} - \epsilon) & \text{otherwise.} \end{cases}$$

Each of these values is non-negative, and changes continuously as P changes. We have that $g_{ij}^r(P) > 0$ if and only if $\mu_i^r(C_j) > \frac{1}{m} - \epsilon$.

Let

$$M_{ij}^r(P) = \frac{m \cdot g_{ij}^r(P)}{n \cdot \sum_{k=1}^m g_{kj}^r(P)}$$

which is well-defined since $g_{kj}^r(P) > 0$ if and only if $\mu_j^r(C_k) > 1/m - \varepsilon$, which holds for at least one $k \in [m]$. These values are continuous functions of P and $M_{ij}^r(P) > 0$ if and only if $\mu_j^r(C_i) > \frac{1}{m} - \varepsilon$.

Construct a $m \times n$ matrix $M^r(P)$ such that the entry in the i-th row and and j-th column is $M_{ij}^r(P)$. The entries of $M^r(P)$ are non-negative and each of its columns sum to m/n.

Let

$$f_i^r(P) = \sum_{j=1}^n M_{ij}^r(P)$$

be the sum of the i-th row of $M^r(P)$. Note that $F^r = (f_1^r, \dots, f_m^r) : \mathrm{EMP}(\mu, m) \to \mathbb{R}^m$ is an S_m-equivariant continuous function.

By Theorem 2.2, there exists a partition $P \in \mathrm{EMP}(\mu, m)$ that simultaneously equalizes the components of each of $F^1, \dots, F^{d-1}$:

Now consider the matrix $M^r(P)$ for any $r \in [d-1]$. Its column sum is constant and equal to m/n, and its row sum is constant. Therefore, the sum of each row is 1. We can construct an $n \times n$ matrix $N^r(P)$ by placing n/m copies of $M^r(P)$ on top of each other. This gives us an $n \times n$ matrix with non-negative entries whose column sum and row sum are all 1. Applying Birkhoff's theorem again, there is a permutation $\sigma : [n] \to [n]$. Such that $N_{\sigma(j)j}^r > 0$. We now define $\pi^r(j)$ the unique integer in $[m]$ such that $\pi^r(j)$ is congruent to $\sigma^r(j)$ modulo m. We conclude by a compactness argument taking $\varepsilon \to 0$ as in the proof of Theorem 1.4. □

With Lemma 4.1, we now prove Theorem 1.5.

Proof of Theorem 1.5. We proceed via strong induction on n. The base case $n = 1$ holds by taking $C_1 = \mathbb{R}^d$. Suppose $n \geqslant 2$ and the theorem holds for all $1 \leqslant n' < n$.

First, if n is a prime power, the theorem holds by Theorem 1.4.

Otherwise if n is not a prime power, it can be factored as $n = ms$ with $m, s < n$ and m a prime power. By Lemma 4.1, there is a partition of $\mathbb{R}^d$ into m convex regions $C_1, \dots, C_m$ and $d-1$ partitions of $[n]$ into m sets $S_1^r, \dots, S_m^r$ of s indices each such that for $r = 1, \dots, d-1$ and $j = 1, \dots, m$, we have $\mu_i^r(C_j) \geqslant \frac{1}{m} = \frac{s}{n}$ for $i \in S_j^r$. Then for each C_j, we apply the theorem to the s measures $\mu_i|_{C_j}$ for $i \in S_j^r$, and obtain a partition of $\mathbb{R}^d$ into s regions $C_{j,1}, \dots, C_{j,s}$ and $d-1$ permutations $\pi_{j,1}, \dots, \pi_{j,d-1} : [s] \to [s]$ such that $\mu_i^r(C_{j,\pi_r(i)}) \geqslant \frac{1}{n}$. Combining these completes the proof.

Concretely, let $D_{s(j-1)+h} = C_j \cap C_{j,h}$, which is convex. Define

$$\begin{aligned}\tilde{\pi}_r &: [n] \to [n] \\ &i \mapsto \pi_{j,r}(i - s(j-1)) \text{ if } s(j-1) < i \leqslant sj \text{ for } j = 1, \dots, m.\end{aligned}$$

Then $\mu_i^r(D_{\tilde{\pi}_r(i)}) \geqslant \frac{1}{n}$, so $D_1, \dots, D_n$ and $\tilde{\pi}_1, \dots, \tilde{\pi}_{d-1}$ are the convex partition and permutations, respectively, that we need. □

Note that using a subdivision argument weakens our result to be proportional rather than envy-free. An envy-free result would require coordination across all subdivisions at once, to ensure that the piece that one person receives from a given subdivision contains more measure than any other piece produced from the other subdivisions.

In our proof, we used very little information about the measures. We can prove a result for preferences that do not come from measures in the following way. Given an absolutely continuous probability measure μ on $\mathbb{R}^d$, let $X(\mu, n)$ be the space of convex equipartitions of μ into n pieces. Note that the space of convex equipartitions of $\mathbb{R}^d$ can be defined by $\binom{n}{2}$ separating hyperplanes, so it inherits the topology of an $\binom{n}{2}$-fold product of the corresponding affine Grassmanian. It is in fact a metric space. A KKM-like cover $\mathcal{U}$ of $X(\mu, n)$ will be an n-tuple $(A_1, \dots, A_n)$ of sets of $X(\mu, n)$ such that

- the set $A_i \subset X(\mu, n)$ is open for all $i \in [n]$,
- the sets $A_1, \dots, A_n$ cover $X(\mu, n)$, and
- if $(C_1, \dots, C_n) \in A_i$ and $\pi : [n] \to [n]$ is a permutation, then $(C_{\pi(1)}, \dots, C_{\pi(n)}) \in A_{\pi^{-1}(i)}$. In other words, shuffling the partition does not change the preferences.

As mentioned in the introduction, classic KKM covers are formed of closed sets. The assumption on the sets A_i above being open is mostly for convenience in the proof, but there is no loss of generality. A standard technique to work reduce variations of the KKM theorem to covers of open sets is as follows. If we were given an n-tuple of closed sets $(B_1, \dots, B_n) \subset X(\mu, n)$, we can take a value $\varepsilon > 0$ and construct the sets $A_i = \{x : \operatorname{dist}(x, B_i) < \varepsilon\}$. Then, if we prove an intersection theorem related to covers with open sets, taking $\varepsilon \to 0$ implies the result for the families of closed sets.

Given a set $(A_1^1, \dots, A_n^1), \dots, (A_1^n, \dots, A_n^n)$ of n KKM-like covers of $X(\mu, d)$, we say that a partition $(C_1, \dots, C_n)$ is an envy-free partition for the covers if there exists a permutation $\pi : [n] \to [n]$ such that $(C_1, \dots, C_n) \in A_{\pi(i)}^i$ for all $i \in [n]$. Then, we have the following theorem:

Theorem 4.2. *Let n, d be positive integers, where n is a prime power. Let μ be an absolutely continuous probability measure on $\mathbb{R}^d$, and let $(\mathcal{U}_1^1, \dots, \mathcal{U}_n^1), \dots, (\mathcal{U}_1^{d-1}, \dots, \mathcal{U}_n^{d-1})$ be $d-1$ KKM-like covers of $X(\mu, n)$.*

Then, there exists a convex partition $C_1, \dots, C_n$ of $\mathbb{R}^d$ such that

$$\mu(C_1) = \cdots = \mu(C_n)$$

and $(C_1, \dots, C_n)$ is an envy-free partition for $(\mathcal{U}_1^r, \dots, \mathcal{U}_n^r)$ for all $r \in [d-1]$. Moreover, the partition can be found even if we do not have access to $\mathcal{U}_n^r$ for all $r \in [d-1]$.

Proof. The proof is identical to the proof of Theorem 1.4 except for the definition of $g_{ij}^r(P)$ and without the need for the compactness argument at the end (so ε is not used). For a fixed r, j, let $\mathcal{U}_j^r = (A_1, \dots, A_n)$. Recall that $A_i \subset X(\mu, d)$ is an open set, and that $X(\mu, d)$ is a metric space. Then, for $j \in [n-1]$

$$g_{ij}^r(P) = \mathrm{dist}(A_i^c, P),$$

where A_i^c denotes the complement of A_i. Once we have constructed g_{ij}^r, the rest of the proof follows. □

5. Remarks and open problems

We conjecture that Theorem 1.4 should hold for general n. Moreover, we also believe that the measure μ could be replaced by another set of n measures. Concretely, we have the following conjecture:

Conjecture 5.1. Let n, d be positive integers and let $(\mu_1^1, \dots, \mu_n^1), \dots, (\mu_1^d, \dots, \mu_n^d)$ be d tuples of n absolutely continuous probability measures each. Then, there exists a convex partition $(C_1, \dots, C_n)$ of $\mathbb{R}^d$ that is an envy-free partition for $(\mu_1^r, \dots, \mu_n^r)$ for all $j = 1, \dots, d$.

Akopyan, Avvakumov, and Karasev have presented a proof of the Nandakumar–Ramana-Rao problem for non-prime powers when $d = 2$ [AAK18], which leads to the question of whether their results or methods confirm that Theorem 1.4 holds for $d = 2$ and any n. The nuance lies in that in the proof presented in 4 of Theorem 1.4, the non-additive functions we want to equalize over a space of partitions are the sums of rows of several matrices. Each of these matrices depends on the entire partition, and not the individual convex sets in them. For example, these functions evaluated in the partitions (C_1, C_2, C_3) and (C_1, C_2', C_3') could be completely different. In the cases analyzed by Akopyan et al, the functions would at least preserve the value corresponding to C_1. The Blagojević–Ziegler theorem that is at the center of the proof of Theorem 1.4 (Theorem 2.2) fails if n is not a prime power. Moreover, as explained by Akopyan et al in their manuscript, their methods would be able to equalize in higher dimension only one non-additive function, so the current versions of Theorem 1.4 for n not a prime power would need a different approach. The reason why despite this we are still inclined to conjecture that Theorem 1.4 holds for all values of n is that at the core this is a problem about subdividing measures. Even though the functions we constructed for the proof are non-additive, they are based on additive functions, so perhaps there is a way to make use of this structure.

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