

IMPROVED BOUND ON THE NUMBER OF CYCLE SETS

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Abstract. The cycle set of a graph G is the set consisting of all sizes of cycles in G . Answering a conjecture of Erdős and Faudree, Verstraëte showed that there are at most $2^{n-n^{1/10}}$ different cycle sets of graphs with n vertices. We improve this bound to $2^{n-n^{1/2-o(1)}}$. Our proof follows the general strategy of Verstraëte of reducing the problem to counting cycle sets of Hamiltonian graphs with many chords or a large maximum degree. The key new ingredients are near-optimal container lemmata for cycle sets of such graphs.

Keywords. Cycle sets, container method

Mathematics Subject Classifications. 05C30, 05C38

1. Introduction

Given a graph G with n vertices, the *cycle set* $\mathcal{S}(G) \subseteq \{3, \dots, n\}$ is defined as $\ell \in \mathcal{S}(G)$ if and only if G contains a cycle of size ℓ . The study of sufficient conditions which guarantee that $\mathcal{S}(G)$ has certain properties is one of the central topics in graph theory. For example, Dirac's theorem [Dir52] gives a sufficient condition for $n \in \mathcal{S}(G)$ (that is, G is *Hamiltonian*) in terms of the minimum degree of G ; Mantel's theorem gives a sufficient condition which ensures $3 \in \mathcal{S}(G)$ in terms of the number of edges, and so on. Closely related to Dirac's theorem, and many other similar statements about hamiltonicity of graphs, are sufficient conditions which ensure that G is *pancyclic*, that is, $\mathcal{S}(G) = \{3, \dots, n\}$ (e.g. see [BS90, Bon71, DCS24]).

Previous examples focus on either a particular value appearing in $\mathcal{S}(G)$, or $\mathcal{S}(G)$ itself being a particular set. Another influential line of research is the study of the size and structure of $\mathcal{S}(G)$. It is a simple exercise to show that if a graph G has minimum degree d , then $|\mathcal{S}(G)| \geq d - 1$. Sudakov and Verstraëte [SV08] showed that if G is a graph with average degree d and girth at least $2g + 1$, then $|\mathcal{S}(G)| = \Omega(d^g)$. Moreover, they showed that $\mathcal{S}(G)$ contains a large subset of consecutive values. Milans, Pfender, Rautenbach, Regen, and West [MPR⁺12] showed that if G contains a Hamilton cycle and has $n + p$ edges, then $|\mathcal{S}(G)| = \Omega(\sqrt{p})$. While this result is,

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in general, optimal, Bucić, Gishboliner, and Sudakov [BGS22] showed that if G is Hamiltonian with minimum degree at least 3, then $|\mathcal{S}(G)| \geq n^{1-o(1)}$. A celebrated result of Gyárfás, Komlós, and Szemerédi [GKS84] shows that if G has average degree d , then $\sum_{\ell \in \mathcal{S}(G)} 1/\ell > \varepsilon \log(d)$. Note that this is really a structural result: if a graph does not have short cycles, it has many long ones. The lower bound was recently strengthened to an almost optimal $(1/2 - o_d(1)) \log(d)$ by Liu and Montgomery [LM23].

The previous list of examples merely touches on the long list of results about cycle sets and is only meant to convey their breadth and depth. For a thorough treatment of the topic, we refer the reader to the survey by Verstraëte [Ver16]. With this in mind, problems of Erdős (see [Erd97, Problem 5]; he attributes them to Faudree and himself) of characterising subsets of $\{3, \dots, n\}$ which are the cycle set of a n -vertex graph, or how many different cycle sets of n -vertex graphs there are, are rather natural ones. Erdős conjectured that cycle sets form a vanishing fraction of all possible subsets, and this was verified by Verstraëte [Ver04]. In particular, he showed that there are at most $2^{n-n^{1/10}}$ subsets of $\{3, \dots, n\}$ which are the cycle set of a n -vertex graph. We improve this bound:

Theorem 1.1. *The number of different cycle sets of n -vertex graphs is at most $2^{n-\Omega(\sqrt{n}/\log^{3/2}(n))}$.*

The proof of Theorem 1.1 follows the overall strategy of Verstraëte [Ver04], which through a series of assumptions reduces the problem to counting cycle sets of Hamiltonian graphs with many chords or a large maximum degree. We improve upon both of these sub-problems using container-type lemmata, presented in Section 4. Both of these lemmata crucially rely on Lemma 3.1 which provides a small *fingerprnt* for a given set of chords. We postpone the discussion about possible further improvements of Theorem 1.1 to Section 6.

At present, we do not know how far Theorem 1.1 is from the truth. The best known lower bound is a construction due to Faudree: Suppose n is even. Given $A \subseteq \{n/2 + 1, \dots, n\}$, form the graph G by taking a Hamilton path $1, 2, \dots, n-1, n$ and all the edges $\{1, a\}$ for $a \in A$. Then $\mathcal{S}(G) \cap \{n/2 + 1, \dots, n\} = A$. This shows there are at least $2^{n/2}$ different cycle sets of n -vertex graphs. Improving this to $2^{(1+c)n/2}$, for a constant $c > 0$, would already be interesting.

2. Definitions

Throughout the paper we tacitly avoid use of floors and ceilings. All stated inequalities can be made to hold with sufficient margin to accommodate for this. For brevity, we introduce a number of definitions.

- Given a family of graphs \mathcal{G} , we define $\mathcal{F}(\mathcal{G})$ to be the family of all cycle sets of graphs in \mathcal{G} :

$$\mathcal{F}(\mathcal{G}) = \{\mathcal{S}(G) : G \in \mathcal{G}\}.$$

- Given a family of cycles \mathcal{C} in some graph G , we define

$$\mathcal{S}(\mathcal{C}) = \{|\mathcal{C}| : \mathcal{C} \in \mathcal{C}\}.$$

- We refer to a graph G on the vertex set $\{1, \dots, n\}$ as *labelled*. Given an edge $e \in G$ in a labelled graph, we denote with (a_e, b_e) its endpoints such that $a_e < b_e$.
- Suppose G is a labelled graph such that $(1, 2, \dots, n - 1, n, 1)$ forms a Hamilton cycle, which we denote with H . We refer to the edges $E(G) \setminus E(H)$ as *chords*. For two distinct chords $e, e' \in E(G) \setminus E(H)$, there are either one or two cycles in G which use both e and e' , with all other edges being in $E(H)$. If there is one such cycle, we denote it with $C(e, e')$. This is the case if either $\{a_e, b_e\} \cap \{a_{e'}, b_{e'}\} \neq \emptyset$, $b_e < a_{e'}$, $b_{e'} < a_e$, or $a_e < a_{e'} \Leftrightarrow b_{e'} < b_e$. Otherwise, we have $a_e < a_{e'} < b_e < b_{e'}$ (or the same with e and e' swapped), in which case we define the cycle $C(e, e')$ to be the following:

- If $a_e < a_{e'}$, then

$$C(e, e') = a_e \rightarrow b_e \searrow a_{e'} \rightarrow b_{e'} \nearrow a_e, \tag{2.1}$$

where $b_e \searrow a_{e'}$ means “go from b_e to $a_{e'}$ along H in the decreasing order of labels“, and $b_{e'} \searrow a_e$ is defined analogously in the increasing order (see Figure 2.1).

- If $a_e > a_{e'}$, set $C(e, e') := C(e', e)$,

This definition of $C(e, e')$ will be important at one point in the proof of Lemma 4.2.

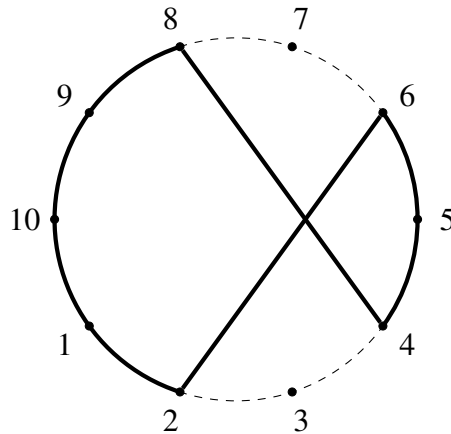


Figure 2.1: The cycle $C(e, e')$ where $e = (2, 6)$ and $e' = (4, 8)$.

Given a subset $R \subseteq E(G) \setminus E(H)$, we denote with $\mathcal{C}(e, R)$ the set of all cycles coming from interactions of e and chords in R :

$$\mathcal{C}(e, R) = \{C(e, e') : e' \in R \setminus \{e\}\}.$$

We denote with $\mathcal{C}(R)$ the set of all pairwise interactions of chords in R :

$$\mathcal{C}(R) = \{C(e, e') : e, e' \in R, e \neq e'\}.$$

- We frequently use the notion of *short-cutting*. Given an edge $e \in G$ and a cycle C in a labelled G , such that vertices a_e, \dots, b_e appear in that order in C , *short-cutting C along e* refers to the cycle obtained from C by simply removing vertices labelled $a_e + 1, \dots, b_e - 1$ (this is indeed a cycle as e connects a_e and b_e).

3. The Fingerprint lemma

The following lemma is the key ingredient in the proof of results in Section 4 which, in turn, are the main new components in the proof of Theorem 1.1. We think of the set F in the lemma as a *fingerprint* – it is a rather small set that is a good representative of the whole R .

Lemma 3.1. *Let G be a labelled graph with n vertices such that $(1, 2, \dots, n-1, n, 1)$ forms a Hamilton cycle, denoted by H . Suppose a set of chords $R \subseteq E(G) \setminus E(H)$ has the property that for each two distinct $e, e' \in R$, there is at most one chord $e'' \in R \setminus \{e, e'\}$ such that $|C(e, e')| = |C(e, e'')|$. If $|R| \geq C_0$, where C_0 is a sufficiently large constant, then there exists a subset of chords $F \subseteq R$ of size $|F| = \sqrt{|R|}$ such that $|\mathcal{S}(H + F)| \geq |R|/24$.*

Proof. Let $F_1 \subseteq R$ be an arbitrary subset of size $\sqrt{|R|}/2$. Set $Y = \mathcal{S}(H + F_1)$ and $F_2 = \emptyset$, and repeat the following $\sqrt{|R|}/2$ times: Pick $e \in R \setminus (F_1 \cup F_2)$ which maximises $|\mathcal{S}(C(e, F_1)) \setminus Y|$, add it to F_2 and update $Y := Y \cup \mathcal{S}(C(e, F_1))$. We claim that if $|Y| < |R|/6$ at the beginning of some iteration, then in that iteration Y increases by at least $|F_1|/6$. This implies that taking $F := F_1 \cup F_2$ at the end of the process, we have $|\mathcal{S}(H + F)| \geq |\mathcal{S}(C(F))| \geq |R|/24$.

Consider sets Y and F_2 at the beginning of some iteration, and suppose $|Y| < |R|/3$. Let B be an auxiliary bipartite graph with vertex sets R and $\mathbb{N} \setminus Y$, and an edge between e and x if there exists $f \in F_1$ such that $|C(e, f)| = x$. Note that the vertices corresponding to $F_1 \cup F_2$ have no incident edges in B . Indeed, if $e \in F_1$ and $f \in F_1$ then $|C(e, f)| \in Y$ by the way we initialised Y ; if $e \in F_2$ and $f \in F_1$, then $|C(e, f)|$ was added to Y in the round where we added e to F_2 . For an $f \in F_1$, by the assumption of the lemma each $y \in Y$ can be realised by at most two cycles $C(e, f)$ and $C(e', f)$ with $e, e' \in R$. Therefore, there are at least $|R| - 1 - 2|Y| \geq |R|/3$ chords $e \in R \setminus \{f\}$ such that $|C(e, f)| \notin Y$. As we have just observed, these chords have to lie outside of $F_1 \cup F_2$. Again using the assumption of the lemma, for each $e \in R$ and $f \in F_1$, there is at most one other $f' \in F_1 \setminus \{f\}$ such that $|C(e, f)| = |C(e, f')|$. Therefore, B contains at least $|F_1||R|/6$ edges. The quantity $|\mathcal{S}(C(e, F_1)) \setminus Y|$ corresponds to the degree of e in B . As we pick a chord which maximises this value, it is at least as large as the average degree of vertices from R in the graph B , which is at least $|F_1|/6$. This verifies the claim. \square

4. Container lemmata

Lemmata presented in this section are the key new ingredients in the proof of Theorem 1.1. Informally, they show there exists a small family of large sets which are unavoidable by cycle sets. That is, every graph with certain properties has to contain one of these sets as a subset of its cycle set. We start with the easier of the two results.

Lemma 4.1. *Given n and $p \geq \log^3 n$, there exists a family $\mathcal{F}'(n, p)$ of subsets of $\{1, \dots, n\}$ such that*

$$\sum_{S \in \mathcal{F}'(n, p)} 2^{-|S|} = 2^{-\Omega(p)}, \quad (4.1)$$

with the property that if G is a Hamiltonian graph with n vertices and maximum degree at least p , then $S \subseteq \mathcal{S}(G)$ for some $S \in \mathcal{F}'(n, p)$.

Proof. Without loss of generality, we may assume that G a labelled graph such that $(1, 2, \dots, n - 1, n, 1)$ forms a Hamiltonian cycle, which we denote with H . Let $R \subseteq E(G) \setminus E(H)$ be the set of chords incident to a vertex v with the largest degree. Then $|R| \geq p - 2$. We may assume $|R| = p - 2$ (otherwise remove arbitrary edges from R) and $v = 1$. One easily checks that R satisfies the assumption of Lemma 3.1, thus we can apply it and obtain a subset $F \subseteq R$ of size $|F| = \sqrt{|R|}$ such that $|\mathcal{S}(H + F)| \geq |R|/24 = (p - 2)/24$.

With the previous discussion in mind, let $\mathcal{G}(n, p)$ consist of all graphs on the vertex set $\{1, \dots, n\}$ of the form $H + F$, where H is the Hamilton cycle $(1, 2, \dots, n - 1, n, 1)$, $|F| = \sqrt{p - 2}$ and $|\mathcal{S}(H + F)| \geq (p - 2)/24$. As just shown, for every Hamiltonian graph G with a vertex of degree at least p there exists $G' \in \mathcal{G}(n, p)$ such that $G' \subseteq G$, and so $\mathcal{S}(G') \subseteq \mathcal{S}(G)$. Therefore, we can set $\mathcal{F}'(n, p) = \mathcal{F}(\mathcal{G}(n, p))$. The bound (4.1) follows from $|\mathcal{F}'(n, p)| \leq |\mathcal{G}(n, p)| \leq \binom{n}{2}^{\sqrt{p}}$, $|\mathcal{S}| \geq (p - 2)/24$ for every $S \in \mathcal{F}'(n, p)$, and $p \geq \log^3 n$ (with room to spare). \square

The next result applies to a much broader class of Hamiltonian graphs. The proof is also significantly more involved.

Lemma 4.2. *Given a sufficiently large n and $p \geq \log^9 n$, there exists a family $\mathcal{F}(n, p)$ of subsets of $\{1, \dots, n\}$ such that*

$$\sum_{S \in \mathcal{F}(n, p)} 2^{-|S|} = 2^{-\Omega(\sqrt{p}/\log n)}, \tag{4.2}$$

with the property that if G is a Hamiltonian graph with n vertices and at least $n + p$ edges, then $S \subseteq \mathcal{S}(G)$ for some $S \in \mathcal{F}(n, p)$.

Note that there exists a Hamiltonian graph G with n vertices and $n + p$ edges such that $|\mathcal{S}(G)| = \Theta(\sqrt{p})$. For example, start with a Hamilton cycle, choose \sqrt{p} consecutive vertices and add all possible edges between them. Therefore, the property (4.2) is the best possible up to the $\log n$ factor in the exponent.

The proof of Lemma 4.2 combines ideas of Milans, Pfender, Rautenbach, Regen, and West [MPR⁺12], as well as some ideas of Verstraëte [Ver04], with Lemma 3.1. The overall theme in the proof is showing that given a graph G , one can produce a succinct encoding $\Phi(G)$ and, from it, a large set $\phi(\Phi(G)) \subseteq \mathbb{N}$ with the property $\phi(\Phi(G)) \subseteq \mathcal{S}(G)$.

Proof of Lemma 4.2. Let us denote with $\mathcal{H}(n, p)$ the family of all Hamiltonian graphs with n vertices and at least $n + p$ edges. We can suppose that all the graphs in $\mathcal{H}(n, p)$ are labelled and that $(1, 2, \dots, n - 1, n, 1)$ forms a Hamilton cycle, denoted by H . We partition $\mathcal{H}(n, p)$ into subfamilies \mathcal{H}_1 and \mathcal{H}_2 , and construct a separate family \mathcal{F}_i for each. Throughout the proof, we fix K to be a sufficiently large constant.

Graphs with many independent chords. Given a graph $G \in \mathcal{H}(n, p)$, we say that a subset $R \subseteq E(G) \setminus E(H)$ of chords is *independent* if there exists an ordering $e_1, \dots, e_{|R|}$ of the edges in R such that $b_{e_i} < a_{e_{i+1}}$ for every $i \in \{1, \dots, |R| - 1\}$. We think of the chords in R as being independent as one can decide for each chord, independently of all other chords, whether

it is used to shortcut H . We now define \mathcal{H}_1 as follows:

$$\mathcal{H}_1 = \left\{ G \in \mathcal{H}(n, p) : \begin{array}{l} \text{there exists an independent } R \subseteq E(G) \setminus E(H) \\ \text{of size } |R| \geq \sqrt{p}/(K \log n) \end{array} \right\}.$$

We show that there exists a mapping

$$\Psi: \mathcal{H}_1 \rightarrow (\{0, \dots, n-1\} \times \{0, \dots, n-1\})^t,$$

where $t = p^{1/4}$, with the following properties:

1. Consider some $\mathbf{a} = ((n_1, d_1), \dots, (n_t, d_t)) \in \Psi(\mathcal{H}_1)$, and let $D(\mathbf{a})$ be the multiset consisting of n_i many elements d_i , for each $i \in [t]$. Then the set

$$\mathcal{S}(\mathbf{a}) = \left\{ n - \sum_{d \in D'} d : D' \subseteq D(\mathbf{a}) \right\}$$

is of size at least $\sqrt{p}/(K \log n)$.

2. $\mathcal{S}(\Psi(G)) \subseteq \mathcal{S}(G)$ for every $G \in \mathcal{H}_1$.

Having such a mapping, we simply take $\mathcal{F}_1 = \{\mathcal{S}(\mathbf{a}) : \mathbf{a} \in \Psi(\mathcal{H}_1)\}$ which clearly satisfies desired properties.

Let us now show how to construct Ψ . Consider some $G \in \mathcal{H}_1$ and let $R \subseteq E(G) \setminus E(H)$ be a set of independent chords of size $|R| = \sqrt{p}/(K \log n)$. If we shortcut H using a chord $e \in R$, we leave out exactly $b_e - a_e - 1$ vertices. More generally, if we start with H and then successively shortcut it using chords from some $R' \subseteq R$, in an arbitrary order, we end up with a cycle of size

$$n - \sum_{e \in R'} (b_e - a_e - 1).$$

We stress that the order in which we shortcut, as well as the actual values a_e, b_e for chords in R' , is irrelevant for the final outcome. Let $D = \{b_e - a_e - 1 : e \in R\}$.

- **Case 1:** $|D| \leq p^{1/4}$. Let $d_1, \dots, d_{|D|}$ be an arbitrary ordering of the elements of the set D , and for each d_i let n_i denote the number of chords $e \in R$ such that $b_e - a_e - 1 = d_i$. Set

$$\Psi(G) := ((n_1, d_1), \dots, (n_{|D|}, d_{|D|}), (0, 0), \dots, (0, 0)),$$

where $(0, 0)$ is padded $t - |D|$ times (this is just a technicality to make Ψ well defined). Using the observations about short-cutting H with a subset of chords $R' \subseteq R$, we conclude

$$\mathcal{S}(\Psi(G)) = \left\{ n - \sum_{e \in R'} (b_e - a_e - 1) : R' \subseteq R \right\} \subseteq \mathcal{S}(G).$$

From the second set and $b_e \geq a_e + 2$, as otherwise e is not a chord, one easily observes $|\mathcal{S}(\Psi(G))| \geq |R|$.

- **Case 2:** $|D| > p^{1/4}$. Choose a subset $F \subseteq R$ of size $|F| = p^{1/4}$ such that $b_e - a_e \neq b_{e'} - a_{e'}$ for distinct $e, e' \in F$. Let $d_1 > \dots > d_{|F|}$ be the descending ordering of the values $b_e - a_e - 1$ for $e \in F$, and set

$$\Psi(G) := ((1, d_1), \dots, (1, d_{|F|})).$$

The same reasoning as in the previous case shows $\mathcal{S}(\Psi(G)) \subseteq \mathcal{S}(G)$. To see that $\mathcal{S}(\Psi(G))$ is sufficiently large, for each integer $1 \leq q \leq |F|$ consider all the sums of the form

$$d_1 + \dots + d_q + d_i$$

for $i > q$. Then all these sums are different, and there are $\binom{|F|}{2} = \Omega(\sqrt{p})$ many of them. Note that this is in fact a stronger lower bound than required.

Graphs with rich chords interaction. Let $\mathcal{H}_2 = \mathcal{H}(n, p) \setminus \mathcal{H}_1$. We show that there exists a mapping

$$\Phi: \mathcal{H}_2 \rightarrow \mathcal{G}(n),$$

where $\mathcal{G}(n)$ denotes the family of graphs on the vertex set $\{1, \dots, n\}$, with the following properties for every $F \in \Phi(\mathcal{H}_2)$:

1. $|\mathcal{S}(H + F)| \geq 2|F| \log n + \Omega(\sqrt{p}/\log n)$, and
2. If $F = \Phi(G)$, for some $G \in \mathcal{H}_2$, then $F \subseteq G$.

Set $\mathcal{F}_2 = \{\mathcal{S}(H + F) : F \in \Phi(\mathcal{H}_2)\}$. As $H + \Phi(G) \subseteq G$, and consequently $\mathcal{S}(H + F) \subseteq \mathcal{S}(G)$, we just need to verify (4.2):

$$\sum_{S \in \mathcal{F}_2} 2^{-|S|} \leq \sum_{f=0}^{\binom{n}{2}} \binom{n}{2}^f 2^{-2f \log n - \Omega(\sqrt{p}/\log n)} = 2^{-\Omega(\sqrt{p}/\log n)}.$$

Consider some $G \in \mathcal{H}_2$. By the pigeon-hole principle, there exist a set of chords $R \subseteq E(G) \setminus E(H)$ of size $|R| \geq p/\log n$ such that

$$L \leq b_e - a_e < 2L \tag{4.3}$$

for every $e \in R$, where L is some power of two. Let $I \subseteq R$ denote a largest subset of chords which is independent, that is, there is an ordering $e_1, \dots, e_{|I|}$ of the chords in I such that $b_{e_i} < a_{e_{i+1}}$ for $1 \leq i < |I|$. In case there are multiple such largest sets, choose one which minimises

$$\sum_{e \in I} b_e - a_e. \tag{4.4}$$

For each $i \in \{1, \dots, |I|\}$, let $X_i \subseteq R$ be the set consisting of all $e \in R$ such that $a_e \leq a_{e_i} \leq b_e$ or $a_e \leq b_{e_i} \leq b_e$. As I minimises (4.4), each chord belongs to at least one set X_i . Let $\ell \in \{1, \dots, |I|\}$ denote the index of a largest set X_ℓ , and note that the previous observation implies $|X_\ell| \geq |R|/|I|$. Choose $x \in \{a_{e_\ell}, b_{e_\ell}\}$ such that at least $|X_\ell|/2$ chords $e \in X_\ell$ satisfy $a_e \leq x \leq b_e$. Let us denote such chords with X . Define a relation \preceq on X as $e \preceq e'$ if $a_e \leq a_{e'}$ and $b_{e'} \leq b_e$, and note that (X, \preceq) is a partially ordered set. If $e \preceq e'$, then we say e and e' are “parallel”, and otherwise they are “intersecting”.

- **Case 1: Many parallel or intersecting chords.** Suppose X contains a chain or an anti-chain Y of size $|Y| = \sqrt{p}/(K \log n)$. In both cases, the set of chords Y satisfies the assumption of Lemma 3.1. This is easily seen from the definition of $C(e, e')$, and perhaps the only thing worth pointing out is that if Y is an anti-chain, then it is crucial here that $a_e \leq x \leq b_e$ for every $e \in X$, which forces $a_e < a_{e'} < b_e < b_{e'}$ for $e, e' \in Y$ such that $a_e < a_{e'}$. Let $F \subseteq Y$ be a subset given by Lemma 3.1: $|F| = \sqrt{|Y|}$ and $|\mathcal{S}(H + F)| \geq |Y|/24$. Set $\Psi(G) = F$ (the first property of Ψ follows from the lower bound on p).
- **Case 2: Rich combination of intersecting chords with I .** Let $Z \subseteq X$ be a largest chain in X . By the Erdős-Szekeres theorem, there exists an anti-chain $A \subseteq X$ of size at least $|X|/|Z|$. That is, no two chords in A are \preceq -comparable. Again, we stress that in this case we have $a_e < a_{e'} < b_e < b_{e'}$ for every $e, e' \in A$ such that $a_e < a_{e'}$. As the assumption of the previous case is not satisfied, we have $|Z|, |A| < \sqrt{p}/(K \log n)$. From $G \notin \mathcal{H}_1$ we have $|I| < \sqrt{p}/(K \log n)$. We show that $|\mathcal{S}(H + A + I)| \geq \sqrt{p}$, which allows us to set $\Phi(G) = A + I$. In a sense, this is technically the easier of the two cases as it does not use Lemma 3.1.

From $|A||Z| \geq |X|$, $|X| \geq |R|/(2|I|)$, and the upper bound on $|Z|$, we conclude

$$|A||I| \geq \frac{K}{2} \sqrt{p}. \quad (4.5)$$

As K is sufficiently large, it suffices to show, say, $|\mathcal{S}(H + A + I)| \geq |A||I|/12$.

Consider $f \in A$ with the smallest a_f . Then all the cycles in $\mathcal{C}(f, A)$ have different size, and so $|\mathcal{S}(\mathcal{C}(f, A))| = |A| - 1$. Moreover, by the assumption (4.3) and the definition (2.1), we have

$$\mathcal{S}(\mathcal{C}(f, A)) \subseteq \{n - 4L, \dots, n\}.$$

The main idea from Milans et al. [MPR⁺12] is that one can use chords from I to “shift” sizes of cycles in $\mathcal{C}(f, A)$ and construct new cycles of different sizes. Owing to (4.3), for every $e \in A$ we have $b_e < a_{e_i}$ for $i \geq \ell + 3$, and $b_{e_j} < a_f$ for $j \leq \ell - 3$. That means we can use any subset $S \subseteq \{e_1, \dots, e_{\ell-3}, e_{\ell+3}, \dots, e_{|I|}\} := I'$ to shortcut any cycle from $\mathcal{C}(f, A)$. Let $S_1 \subset S_2 \subset \dots \subset S_k \subseteq I'$, $k = \lfloor |I'|/5 \rfloor > |I|/6$ (using that I is sufficiently large, which follows from (4.5)), be arbitrary sets such that $|S_1| = 5$ and $|S_i \setminus S_{i-1}| = 5$ for each $i \in \{2, \dots, k\}$. By short-cutting every $\mathcal{C}(f, A) =: \mathcal{C}_0$ using S_1 , we obtain a set of cycles \mathcal{C}_1 such that

$$\mathcal{S}(\mathcal{C}_1) \subseteq \{n - 4L - d_1, n - d_1\},$$

where $d_1 = \sum_{s \in S_1} b_s - a_s - 1 > 4L$. This implies

$$\mathcal{S}(\mathcal{C}_1) \cap \mathcal{S}(\mathcal{C}_0) = \emptyset.$$

Moreover, short-cutting two cycles of different size using S_1 produces two new cycles, again of different sizes. Therefore, $|\mathcal{S}(\mathcal{C}_1)| = |A| - 1$.

Repeating the previous idea, short-cutting cycles in \mathcal{C}_0 using S_i we obtain a set of cycles \mathcal{C}_i such that

$$\mathcal{S}(\mathcal{C}_i) \subseteq \{n - 4L - d_i, n - d_i\},$$

where $d_i = \sum_{s \in S_i} b_s - a_s - 1$. For $0 \leq i < j \leq k$ we have $d_i < d_j - 4L$ (where $d_0 := 0$), thus

$$\mathcal{S}(\mathcal{C}_i) \cap \mathcal{S}(\mathcal{C}_j) = \emptyset.$$

Since $|\mathcal{S}(\mathcal{C}_i)| = |A| - 1$, we conclude

$$|\mathcal{S}(H + A + I)| \geq (k + 1)(|A| - 1) > |I||A|/12,$$

with some room to spare. □

5. Proof of Theorem 1.1

The proof of Theorem 1.1 follows the approach of Verstraëte [Ver04] of reducing the problem to counting cycle sets of graphs containing a large induced Hamiltonian subgraph with many chords or a large maximum degree. The latter is taken care of by lemmata from Section 4. This is the part that is done more efficiently than in [Ver04], and is the sole source of the improvement.

Proof of Theorem 1.1. The quantity we are interested in is the size of the family $\mathcal{S}(\mathcal{G}(n))$, where $\mathcal{G}(n)$ denotes the family of all non-isomorphic n vertex labelled graphs (that is, graphs on the vertex set $\{1, \dots, n\}$). Following Verstraëte [Ver04], consider the following families of graphs:

$$\mathcal{G}_1 = \{G \in \mathcal{G}(n) : \text{a largest cycle in } G \text{ is of size at most } n - \sqrt{n}/(4 \log n)\},$$

$$\mathcal{G}_2 = \{G \in \mathcal{G}(n) \setminus \mathcal{G}_1 : G \text{ has at most } n + n/(4 \log n) \text{ edges}\},$$

$$\mathcal{G}_3 = \left\{ G \in \mathcal{G}(n) : \begin{array}{l} G \text{ contains an induced Hamiltonian subgraph} \\ \text{with maximum degree at least } \sqrt{n}/4 \end{array} \right\},$$

$$\mathcal{G}_4 = \left\{ G \in \mathcal{G}(n) : \begin{array}{l} G \text{ contains an induced Hamiltonian subgraph } G' \\ \text{with at least } v(G') + n/(8 \log n) \text{ edges} \end{array} \right\}.$$

We first show $\mathcal{G}(n) = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$. Consider some $G \in \mathcal{G}(n) \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$. Then G contains a cycle $C = (c_1, c_2, \dots, c_\ell)$ of size $\ell \geq n - \sqrt{n}/(4 \log n)$, and has at least $n + n/(4 \log n)$ edges. Suppose there exists a vertex $v \in V(G) \setminus V(C)$ with at least $\sqrt{n}/4$ neighbours in C . Let i denote the smallest index of a vertex c_i which is neighbour of v , and j the largest index. Then $C' = (v, c_i, \dots, c_j)$ forms a cycle, and v has degree at least $\sqrt{n}/4$ in the induced subgraph $G[C']$. Therefore, $G \in \mathcal{G}_3$. Otherwise, if no vertex $v \in V(G) \setminus V(C)$ has this property, then the induced subgraph $G[C]$ has at least

$$n + \frac{n}{4 \log n} - \frac{\sqrt{n}}{4 \log n} \left(\frac{\sqrt{n}}{4 \log n} + \frac{\sqrt{n}}{4} \right) > n + \frac{n}{8 \log n}$$

edges, with room to spare. As $|C| \leq n$, we conclude $G \in \mathcal{G}_4$.

We estimate the size of each $\mathcal{S}(\mathcal{G}_i)$ separately, and then use an upper bound

$$|\mathcal{S}(\mathcal{G}(n))| \leq |\mathcal{S}(\mathcal{G}_1)| + |\mathcal{S}(\mathcal{G}_2)| + |\mathcal{S}(\mathcal{G}_3)| + |\mathcal{S}(\mathcal{G}_4)|.$$

- From $\mathcal{S}(G) \subseteq \{3, \dots, n - \sqrt{n}/(4 \log n)\}$ for $G \in \mathcal{G}_1$, we trivially have

$$|\mathcal{S}(\mathcal{G}_1)| \leq 2^{n - \sqrt{n}/(4 \log n)}.$$

- We use $|\mathcal{S}(\mathcal{G}_2)| \leq |\mathcal{G}_2|$. Every graph in \mathcal{G}_2 can be obtained by starting with a cycle of size $n \geq \ell \geq n - x$, for some $x < \sqrt{n}/(4 \log n)$, and then adding remaining $r \leq n/(4 \log n) + x < n/(3 \log n)$ edges. This can be done in at most $\binom{n}{2}^r < n^{2r} < 2^{2n/3}$ ways, which gives us

$$|\mathcal{S}(\mathcal{G}_2)| < n^2 2^{2n/3}.$$

- Set $p = \sqrt{n}/4$, and let $\mathcal{F}' = \bigcup_{m=1}^n \mathcal{F}'(m, p)$, where \mathcal{F}' 's are given by Lemma 4.1. Then for every $G \in \mathcal{G}_3$ there exists $S \in \mathcal{F}'$ such that $S \subseteq \mathcal{S}(G') \subseteq \mathcal{S}(G)$, where G' is a Hamiltonian subgraph of G with maximum degree at least p . Therefore, the family of up-sets of sets in \mathcal{F}' contains all possible cycle sets of graphs in \mathcal{G}_3 . This gives us the following:

$$|\mathcal{S}(\mathcal{G}_3)| \leq \sum_{m=1}^n \sum_{S \in \mathcal{F}'(m, p)} 2^{n-|S|} \stackrel{(4.1)}{\leq} n 2^{n-\Omega(\sqrt{n})}.$$

Note that this is actually a stronger bound than needed.

- The bound on $\mathcal{S}(\mathcal{G}_4)$ is obtained analogously, with Lemma 4.1 replaced by Lemma 4.2:

$$|\mathcal{S}(\mathcal{G}_4)| \leq \sum_{m=1}^n \sum_{S \in \mathcal{F}(m, \frac{n}{8 \log n})} 2^{n-|S|} \stackrel{(4.2)}{\leq} 2^{n-\Omega(\sqrt{n}/\log^{3/2}(n))}. \quad \square$$

6. Concluding remarks

We believe that the logarithmic factor in Theorem 1.1 can be somewhat optimised. It would be interesting to fully remove it, which at present seems difficult. Even if one could get an optimal bound $2^{-\Omega(\sqrt{p})}$ in Lemma 4.2, which is a challenge on its own, it is not clear how to make use of it to reach $2^{n-\Omega(\sqrt{n})}$. In particular, the main bottleneck to further improvements seems to be the family \mathcal{G}_2 . Let us briefly describe the issue.

Like Verstraëte [Ver04], we take care of \mathcal{G}_2 by using a simple bound $|\mathcal{S}(\mathcal{G}_2)| \leq |\mathcal{G}_2|$. To make this upper bound non-trivial, the bound on the number of edges of the form $n + O(n/\log n)$ is crucial in the definition of \mathcal{G}_2 . Raising the threshold on the number of edges in \mathcal{G}_4 to $n + \varepsilon n$ would, with an optimal version of Lemma 4.2, indeed give us a desired bound on the size of $\mathcal{S}(\mathcal{G}_4)$. However, this requires raising the threshold in the definition of \mathcal{G}_2 as well, at which point handling \mathcal{G}_2 becomes difficult – too few edges to effectively use our container lemma, but enough to make the size of \mathcal{G}_2 too large.

As we have seen in Lemma 4.1, having a few edges is not per se a bottleneck if we know something about the structure of a graph. Unfortunately, the assumption of Lemma 4.1 is rather specific and does not provide any ideas how to proceed further. To summarise, we see a better handling of the family consisting of graphs with a long cycle and (only) linear number of edges as a likely necessary step for any further advancement, at least for approaches along the lines of the presented proof.

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